Chapter 2

Networks and Dynamics: The Structure of the World We Live In

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Abstract Many complex networks of systems have structures with similar topological properties: e.g. clustering, small world effect, and scale free structure. The dynamics of a system – either static or dynamic – are effected by the topological structure of the underlying network. Some examples of static and dynamic systems that act on networks include social, epidemiological, and transportation systems. This chapter gives an introduction to the analysis of nonlinear dynamics as it applies to such systems.

Keywords: network science; complex networks; nonlinear dynamics

1. Introduction

In this chapter we seek to explain the basic knowledge that is needed to understand and apply the results of network science and nonlinear science. Familiarity with the material of this chapter will allow the reader to better understand subsequent chapters of this book. The tone of this chapter is quite informal, and the mathematical background required is that of elementary calculus, an introduction to ordinary differential equations, and a working knowledge of the essentials of graph theory. It is also desirable to have some familiarity with transportation networks and the key issues arising in the design and operation of infrastructure systems – although this last mentioned familiarity is not essential.

2. Network Science

Everyday, each one of us travels to work. Some people wake up at dawn in order to make it to their New York or Los Angeles job on time. These people have to leave at the break of day in order to beat the traffic. If there is an accident on the highway on the way to work, it will take longer to get to other side of the highway or they may even have to take an alternate route. If they have to travel over a bridge, alternate routes may be very inconvenient. Some roads may have construction or a street light may go out, all of which slow down the traffic passing by. It may be curious to note that sometimes even if nothing has gone wrong, that is, with no accidents and no construction, some roads may still be highly congested simply because too many people are trying to use the same road.

Why is it that some roads may break down and very few people even notice and yet other roads can break down and the effects may be catastrophic? It may not be at the front of the average driver's mind, but on the way to work, we each travel through a transportation network. This network is a web of streets, highways, and intersections that we have to travel through to get from our starting point (home) to our destination (work). Networks like this have various topological properties, these are characteristics of the structure of the network. Learning about the structure of the network will help us to be able to answer questions like "The failure of which roads would cause the greatest harm to the network?" Maybe more importantly, "What can we do to prevent catastrophe?" Network Science is the study of the topology of networks and its role in the functionality of the network.

2.1 Social Networks

Networks have been used quite a bit to model social relationships and this easy model will help acquaint the reader with networks. In these models, each person is represented as a vertex or a node. A relationship between two people is represented as an arc or edge between their nodes. For example, in the very small social network in Figure 1 Alice, Bob, and Charlie are all represented as vertices. Since Alice is connected to Bob and Charlie via an arc, this means that Alice knows Bob and Charlie. However, since Bob and Charlie are not connected through an arc, they do not know each other.

From this simple representation of a social network, new questions may come to mind. Who is connected to the most number of people? How many people would you have to go through in order to connect a particular pair of people? Which two people are the furthest away from one another? Which people are the most crucial to the network? To begin to answer these questions, some basic definitions will help clarify the language of the following pages.

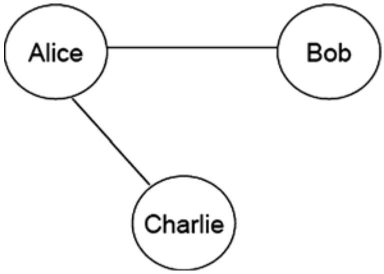


Figure 1. Social Network.

2.2 Basic Definitions

We shall have cause to employ the following notions central to graph theory and network science:

vertex	The connection of arcs, often referred to as a node.
edge	The link or arc connecting two vertices.
degree of a vertex	The number of edges connected to the vertex.
component of a vertex	The set of vertices reached from the given vertex.
geodesic path	The shortest path between two vertices in a network.
diameter	The longest geodesic path of a network.

The definitions presented above have been taken from the review article by Newman (2003).

2.3 Graph Theory is Born

The Pregel river splits into two paths around the island Kneiphof in the city of Konigsberg, Prussia (now Kaliningrad, Russia). In Konigsberg, the people of the city used to spend their Sunday afternoons walking over the seven bridges that connected the island to the pieces of land surrounding the island. The question of whether there was a path that allowed a person to walk over all seven bridges exactly once arose among the patrons of coffee shops near the bridges; it later became known in the scholarly literature as "the seven bridges of Konigsberg" problem. Figure 2, gives an abstract depiction of the configuration of the Konigsberg bridges at issue. The historical record also indicates this question soon came to the attention of Leonhard Euler (1707–1783), who isolated the underlying graph to obtain a depiction like that of Figure 3 taken from Barabási (2002).

How did Euler approach this problem? Euler reasoned that if a traveler wants to pass through a node (land mass), then he has to enter and leave, so there must be an even number of arcs for each node that the traveler passes through. Similarly, there must be an odd number of arcs for a node if a traveler

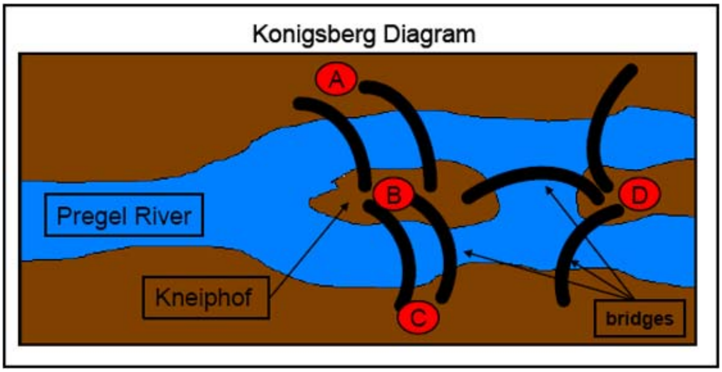


Figure 2. Konigsberg Bridge.

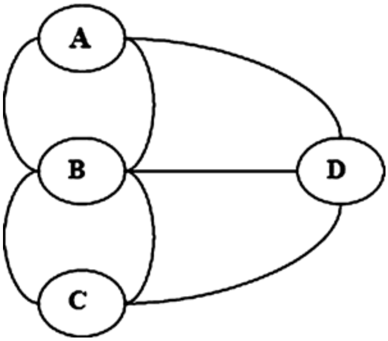


Figure 3. Abstract Representation of the Konigsberg Bridge Problem.

is to begin or end his path at that node, but if he begins and ends his path at the same node then there must be an even number of arcs at that node. So, for what networks does there exist a path such that one can visit each arc once and only once? For a solution to exist, there must be either zero (start and end at same node) or two (start and end at different nodes) nodes with an odd number of arcs and the rest of the nodes must have an even number of arcs. Any other graph has no solution to the question posed.

For the Konigsberg Bridge question, we can easily verify that each node has an odd number of arcs and since there are four nodes, there is no solution to the question. However, one may also easily verify that adding or subtracting any one bridge would make it a solvable problem. It just so happens that in 1875, another bridge was erected and it was indeed possible to visit each bridge once and only once. The appropriate path would be to begin and end at the two land masses not connected by this new bridge.

Another version of the question is to ask whether there exists a path beginning and ending at the same land mass, that will cross all seven bridges.

This version will also be answered in the same way, but solution is simply that a path exists only if there are no nodes with an odd number of arcs. If this is the case, one can start at any node and there is a path from that node that is a solution. Euler's solution gave birth to a new field of math known as graph theory, which would provide many rich methods used to analyze such problems. In addition, the field of network science eventually emerged from the field of graph theory.

2.4 Random Graphs

Paul Erdös and Alfréd Rényi made some of the first breakthroughs in network science. They modeled networks as random graphs, by first selecting a number, n, of nodes. They then connected every pair of nodes with a probability p, forming what came to be known as random graphs or what are sometimes called the "Poisson Random Graph" or "Bernoulli Graph."

Paul Erdös and Alfréd Rényi were the first to thoroughly study the structure of graphs. Mathematicians began to ask questions like, on a graph with n nodes where each arc had a probability p of being connected, what is the probability that the entire graph is connected? Where a connected graph is defined as one in which any node can be accessed by any other node. In other words, for all pairs of nodes, there exists a path between them. Certain properties of graphs may change as n, the number of nodes, increases and mathematicians often looked at the properties in the limit as n went to infinity.

More generally, it was found that random graphs often experience a phase change. A component of a graph is a set of nodes that are connected. The degree of a given node, is the number of arcs attached to it. This often is the same as the number of other nodes attached to it, but in some networks, there may be multiple arcs attaching two nodes. It was found that by varying the average degree of the nodes in a graph, the structure of the graph may change. Specifically, as the average degree increases, the random graph will experience a rather quick phase transition. Before this transition, the graph will be composed of many rather small components, the arc density will be very low, and the graph will not be connected. However, after the quick transition, the graph will have one "giant component" containing most nodes. The graph still may not be connected, but it will be close, in that most nodes will be connected to one another.

While it was found that these random graphs did exhibit some features such as the "small world effect" mentioned in the next section, these random graphs failed to display many of the other topological characteristics of real networks (to be discussed in the coming sections). Hence, they were left behind for newer models more capable of modeling the real networks around us. However, the approach of Erdös and Rényi to study topological properties of graphs would have lasting effects in network science because they built the foundation of how we fundamentally think about networks.

2.5 It's a Small World

Surely, at one time or another, everyone has been told "you are within six degrees of Kevin Bacon" [or something with essentially the same implication]. This phrase is one of the most popular examples of the small world effect that is within networks. The idea of the small-world effect was first published by Frigyes Karinthy in the short story Chains (Karinthy, 1929), where a character claimed that all people in the world were connected via at most five people.

The scientific history of the small world effect dates back to the research of Stanley Milgram (1967). Milgram randomly chose people in a seemingly far off place like Wichita, Kansas and in a second study, Omaha, Nebraska. He then sent them a letter with instructions on how to participate in his study. He told them to send it to a divinity student in Sharon, Massachusetts in the first study and to a Boston Stockbroker in the second study. However, there was a catch. The participants were only allowed to give the letter to a person they knew by their first name. The lists were mailed back to Milgram, so that he could trace the progress of the letters. Milgram found that the completed letters passed through an average of just under six people. While the methodology and results of Milgram's experiment have been contested by some, it is commonly believed that the social network that we all live in, can be characterized by a small world. At first glimpse it may seem surprising that random people chosen could be connected by an average of less than six degrees. However, if we consider how many people are within six degrees of us, then it may no longer be surprising. Each person easily has fifty friends connected to them, most people have hundreds.

Suppose we consider the random graph of Erdös and Rényi. Suppose each node has on average one hundred links. Thus a given node X, has one hundred other nodes within one degree of it, but each of these nodes also has an average of one hundred nodes linked to it. So there are 100^2 nodes within two degrees of X and 100^3 nodes with in three degrees and so on. Thus, there are 100^6 nodes within six degrees of X. This amounts to 1,000,000,000,000 nodes within six degrees of X. This surely cannot be an argument that identifies the number of people within six degrees of any person in the real world, since the current world population is only at about 6.5 billion. The error in thinking intrinsic to the argument just presented is fairly simple to find. Denote the group of nodes within one degree of X as X_1. Now each node in X_1 also has on average 100 nodes within one degree of it. However, some of those nodes within one degree of it are also in X_1. So there is a significant overlap in the links, causing there to be far fewer than 100^6 nodes within six degrees of X. This may be a bit

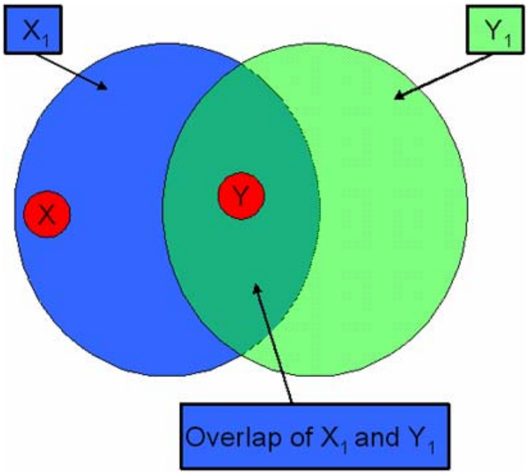


Figure 4. Overlapping Sets and the Small World Effect.

confusing, so let us denote node Y as a node in X and denote Y_1 as the set of nodes within one degree of Y. Now, some of the nodes in Y_1 may also be in X_1, as depicted in Figure 4.

This overlap in the links causes there to be far fewer than 100 nodes within six degrees of X. This may seem crippling to the small world effect, but it should be noticed that the current American Population is about 300,000,000 which is less than $100^{3.25}$ nodes. The degree of overlap between the nodes within one link of X_1 and those nodes in X_1 will surely affect the number of nodes within 6 degrees of X. However, the important thing to realize is that as the number of links L grows, the number of nodes connected to X within L links grows with a power relationship. The number of nodes within L links should be able to be modeled approximately as $\beta^{\alpha L}$where β is some way of estimating the average links per node and α is some parameter to estimate the amount of overlap in links between nodes.

This may make you think, do all nodes have the same number of links? If not, how could we estimate β? Is there a distribution for the number of links L that some node X has? These questions will be looked into a bit more when Hubs are considered. In addition, this model may bring new questions of how exactly do we describe or find α? This is one of the questions that could be answered by the clustering model presented next.

2.6 Clustering

The random graphs of Erdös and Rényi easily show the small world effect found in social networks by Milgram. However, network applications such as social networks often display other topological properties not captured by random graphs. One of the most obvious characteristics of social networks that random graphs cannot model is social cliques or what Watts and Strogatz call clustering (1998).

Random graphs connect any two nodes with a probability p. This means that if Adam is friends with Bob and Adam is also friends with Cathy, then Bob and Cathy have a probability p of being friends with each other (refer to Figure 5). David, not a friend of Adam, also has a probability p of being friends with Bob and a probability p of being friends with Cathy. So David is just as likely to be friends with Bob or Cathy as Cathy and Bob are of being friends with one another, even though Bob and Cathy have a mutual friend (Adam). Random graphs do not appropriately model social networks because this scenario is simply not realistic. Since Bob and Cathy are both friends with Adam, they are more likely to be friends with each other than with David, who is not friends with Adam.

Often people are a member of a social clique or group of friends where everyone is friends with everyone else or almost everyone else. Two people who are friends with Adam are more likely to know each other because they may meet at a common social gathering, including birthday parties, family events, work, or class. So this implies that if Adam is friends with Bob with a probability p and also with Cathy, with a probability p, then Bob and Cathy should be friends with a probability greater than p and David should be friends with Bob and Cathy, each with a probability lower than p. In Figure 5 below, this is shown by the thick solid lines representing the highest probability and the thin small dotted lines representing the lowest probability. Watts and Strogatz (1998) provided the first model that incorporated this property of clustering into the topology of the network, while still maintaining the small world effect (short path between any two nodes). They developed a model which began with the network in a ring, where each node was connected to the nodes close to it, but not those far away. In Figure 6, taken from Watts and Strogatz (1998), the network on the left has 20 nodes, each of which is connected to its 4 neighbors.

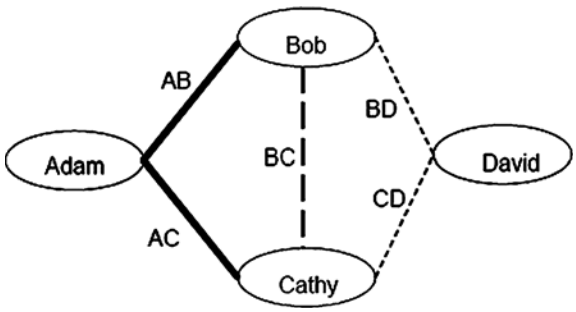


Figure 5. Small World Social Network.

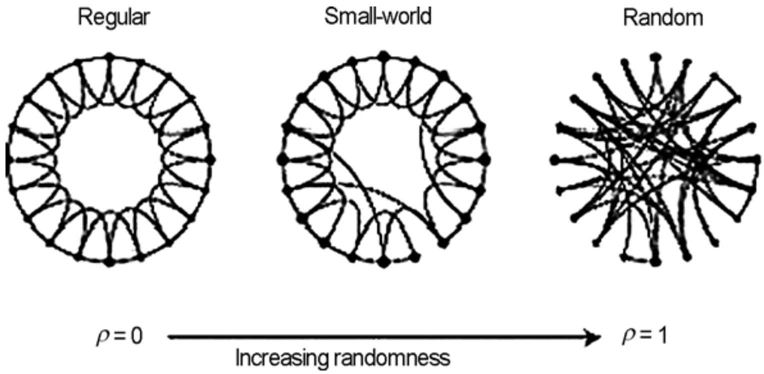


Figure 6. Randomness and Clustering.

Then the method proceeds by starting with a vertex and the edge that connects that vertex to the closest vertex clockwise to it. With a probability p we rewire it to another vertex chosen uniformly at random (without duplication allowed). Then we move to the next vertex in clockwise order. When the lap is completed we move on to the edge with the next shortest link without ever considering the same edge twice, again rewiring it with probability p. For small p Watts and Strogatz (1998) found that the small world shown in the middle diagram emerged. For this small world, the path length was rather short on average due to some arcs going across the circle, yet the clustering coefficient was very high because most neighbors were connected within one or two links. Since each neighbor is connected within one or two links, the neighbor of a node's neighbor is also very close allowing clustering to become a part of the topology of the network.

2.7 Hubs

Watts and Strogatz (1998) still did not fully capture all of the topological properties of many of the networks around us. In their model, each node begins with the same number of links. Even as p increases, since the links are rewired with a uniform distribution, the distribution of the number of links that a node has will not follow a power law distribution. Instead it will follow a peaked distribution. One may think this is not a significant issue, seeing as we should expect some type of bell shaped distribution from a random network. However, Albert-László Barabási has found that the distribution of links in various networks follow a power law distribution not a bell shape distribution.

Barabási found that various networks displayed a power law distribution. This means that the great majority of nodes have very few if any links, while just a few have the great majority of links. This seems intuitive when looking at networks around us. Google has millions of links while most personal web pages simply have one or two. In fact, the ten largest airports in the United States have flights to virtually any airport in the United States, if not the world. For example, a recent check revealed that Philadelphia International Airport had 255 flights arriving from 108 different airports. Yet, a local rural airport typically only offers flights to two or three other airports. In University Park, for example, the local airport has 8 flights arriving from 4 airports, all of which are major airports. There are about twenty to thirty very large airports (in the US) that most people fly out of when they need to travel a significant distance. Yet, there are over 19,000 airports in the United States.[1] From University Park, Pennsylvania, there are 10 airports within 110. Of these 10 airports, 9 are small and 1, Harrisburg, is a bit larger. But, even Harrisburg had only 42 flights coming in from 12 cities on a recent day when we checked the published schedules. When compared to Philadelphia, Harrisburg was linked to less than 1/6 the number of airports and had 1/9 the number of flights serving these other airports. Harrisburg is the largest of the 10 closest airports to University Park (within 110 miles) and Philadelphia is a large airport, but surely not the largest. In 2004, Philadelphia was the 16th largest airport in the United States. So for a network of airport connections in the U.S., there are more than 10 times the number of small nodes than there are large ones and the large nodes often have far more than 10 times the number of links. This means that the distribution of links of the networks around us surely are not bell shaped because of this small number of nodes that have often more than 80% of the total links incident upon them. Barabási refers, quite appropriately, to these popular nodes as "hubs". Barabási also began to call complex networks that displayed the power distribution scale free networks. Since there is no single node which can be chosen to characterize the population of nodes, there was no scale in these networks, and hence the term scale free network was coined.

2.8 Modeling Hubs

So if the Watts-Strogatz model cannot portray a power distribution of links, then how can we model it? Let's start by looking a bit deeper at the Watts-Strogatz model and try to find why it was bell-shaped to begin with. The model presented by Watts and Strogatz (1998) begins with each node having the same number of neighbors, denoted by k. Then through the rewiring process, p, the probability of an edge being rewired is the same for all nodes on the ring. Hence, when the rewiring is finished, each node has a minimum of half the number of arcs that it began with. In addition, since the arcs are rewired to a

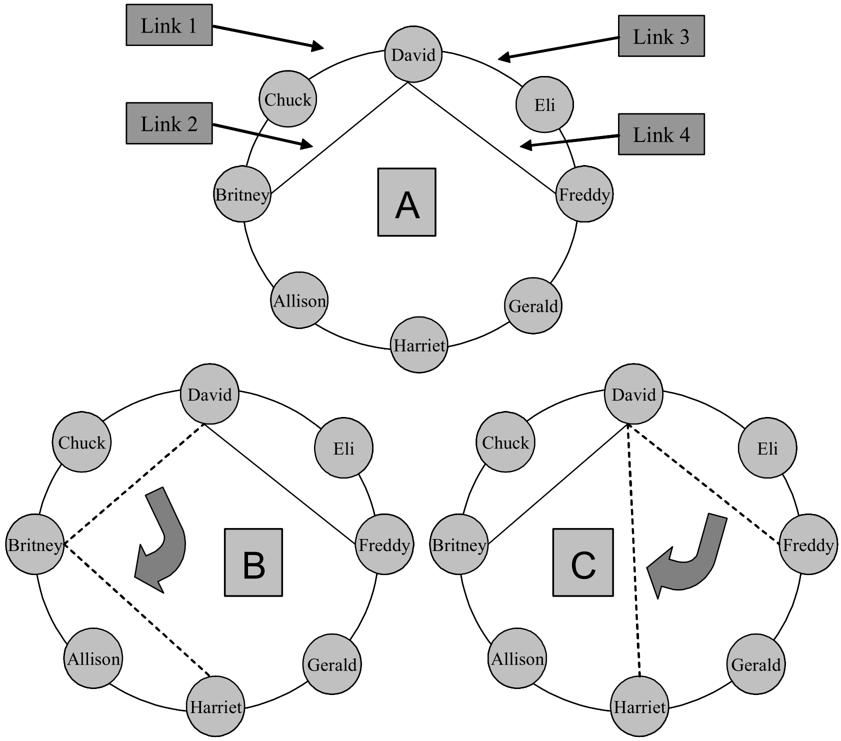
[1] In 2003, there were 19,581 US airports, but only 5,286 were public airports according to the Bureau of Transportation Statistics. Airline travel data can be obtained from the Bureau of Transportation Statistics (2006).

node with a uniform distribution, no node will gain that many more arcs than the rest.

The number of arcs gained by any node will have a normal distribution, while the number of arcs lost will have a binomial distribution. This will create a unique distribution for the number of arcs a node will have. More importantly, an upper and lower bound was just put on the number of arcs a node may have. The lower bound was created because each node must keep $k/2$ arcs. Since this may not be obvious, Figure 7 illustrates the process by which hubs are formed. Suppose we have a particular node David that is connected to two neighbors before it, Britney and Charles as well as two neighbors after it, Eli and Freddy (this makes $k = 4$). Now each of David's links to these other people may be rewired with probability p. Now, let's assume that in clockwise order, the order is Britney, Charles, David, Eli, and Freddy. Then we will randomly decide if Britney's link to David is rewired when it is Britney's turn, so this means if this link is rewired then Britney will be re-linked to someone else (chosen uniformly) and David will lose one of his four links (Britney is re-linked to Harriet). Similarly, Charles' link will go through the same process. However, when we consider Freddy's link to David, if it is rewired, then David will be re-linked to another node (chosen uniformly). David will obviously have to keep this link regardless of whether it is re-linked, while Freddy will be losing a link if it is re-linked. Figure 7 also shows David being re-linked to Harriet (chosen uniformly). In addition, Eli's link will go through the same process. Thus David must keep 2 links no matter what value of p or what random outcome occurs. In the general case, this is one half of the starting links or $k/2$.

In Figure 7, not only is there a lower bound on the number of links, but there is also an upper bound on the number of links a node can obtain. No node can obtain more than $N - 1$ links, where N is the number of nodes in a network. In a large complex network, this practically is not an upper bound. But there is a limiting upper bound, one that is more of a statistical upper bound. There are only $\frac{1}{2}NK$ links in the network and on average only $\frac{1}{2}pNK$ links will be rewired. Since there are N nodes in the network and each will receive a rewired link with equal probability, each node will receive rewired links according to a binomial distribution. Each node will keep its original links according to a binomial distribution as well. So, the number of links a node will have is the sum of two binomial distributions. This resulting distribution is bell-shaped and not a power distribution as the real complex network link distribution is.

At the core of his inquiries, Barabási (2002) asked two related questions: (i) why is the random graph model insufficient to model the power distributions of real networks? and (ii) how are real networks built? The first step in answering these questions is to recognize that a real network is not built by first making nodes and then randomly connecting them as Erdös and Rényi assumed, nor is a real network created by configuring nodes in a certain way



A) Original wiring for David in the Watts Strogatz Model
B) If the link between Britney and David is rewired (this occurs with probability *p*) so that Britney is linked with Harriet (Harriet is chosen uniformly). Now David lost a link, he would have 3 not 4.
C) If the link between David and Freddy is rewired (this occurs with probability *p*) so that David is linked with Harriet (Harriet is chosen uniformly. Now David keeps the link and Freddy is the one who lost a link.
Note also that Link 1 and Link 2 will both be lost by David, if rewired (with probability *p*) and Link 3 and Link 4 will both be kept by David, even if rewired. Thus David, must have a minimum of 2 Links in the end.

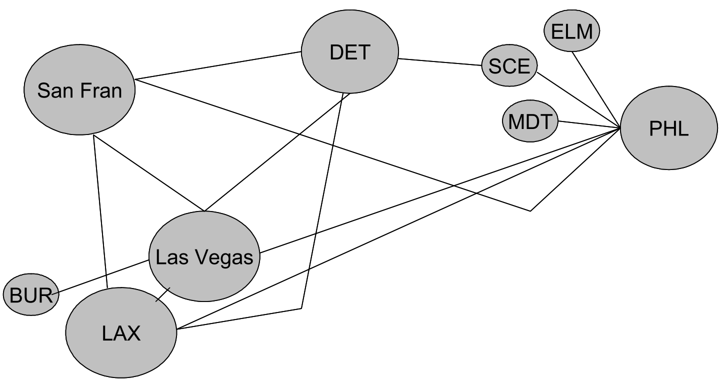
Figure 7. Emergence of Hubs.

and then changing the links as Watts and Strogatz (1998) assumed. When a company, say an airline decides which cities to link with a flight, do they flip a coin to decide whether to offer a certain flight? When David is trying to decide who will be in his social circle, does he flip a coin to decide whether he should ditch one friend for another? Does the airline company flip a coin to decide whether to cancel the Los Angeles to Philadelphia flight and replace it with a Philadelphia to State College flight?

The truth of the matter is that most real networks are formed by just a few nodes and a few links. The original links may be random or may not, but – as the network grows – more and more links are added. However, they are not added completely randomly or at least not uniformly so. Barabási hypothesized that they are added with preferential attachment. That is, each new link would be more likely to be added to those nodes which already had more links. Thus, the bigger cities would be more likely to have new flights added. This allows the rich to get richer, so to speak, and the big nodes to get huge, while the small ones stay small. The combination of allowing a network to grow and allowing it to grow with preferential attachment, forms the power distribution characteristic of scale free networks.

2.9 Vulnerability and Epidemiology

As we have seen, the model of network evolution proposed by Barabási often favors older more established nodes, but it does not preclude younger ones. When a node is first created, it will only have a few links and those links will be more likely to be connected to older more popular nodes. This means that many of the smaller nodes are linked to the popular nodes, that Barabási calls hubs. These hubs allow the existence of the small world effect and clustering. This can be seen in the airline example of Figure 8, fairly easily. The University Park airport is a very small node, connected to only four other airports.



University Park, PA (SCE) is connected to:

1. Philadelphia, PA (PHL) and Detroit, MI (DET) within one link
2. Elmira, NY (ELM) and Harriburg, PA (MDT) within two links
3. Las Vegas, Nevada and Los Angeles, CA (LAX) within two links
4. Burbank, Ca (BUR) within three links

Figure 8. Airline Network Connecting University Park to the World.

Recently the largest of these four was Detroit (9th) and the smallest was Dulles in Washington, DC (23rd).[2] So basically, this very small node was connected to four hubs. These hubs are surely connected to all the other hubs, even on the other side of the county, like San Francisco or Dallas. So, University Park is connected within one link to 4 hubs, two in the Midwest and two in the East. From these hubs, University Park is connected to all of the hubs in the United States. So University Park is within 2 links of any hub in the country. Then each small airport in the country is connected to a hub, so University Park, is connected to virtually any other commercial airport in the US by only 3 links. This results in quite a small world, from the perspective of airport connectivity. In addition, of the 10 airports within 110 miles of University Park, most of them will have flights to Philadelphia as well; making them within 2 links of University Park and also causing the topological property we earlier called clustering.

It is precisely the emergence of hubs that allows the small world effect to occur. This topology induces a few other properties as well – including quick diffusion of information and innovations as well as increased vulnerability. In particular, Thadakamalla et al. (2004) have investigated what they called the survivability of a network. According to Thadakamalla et al. there are four aspects of survivability. The first of these is called the robustness of a network: robustness can be measured by measuring the connectedness or the size of the largest component, after a number of nodes are removed. A robust network should be able to survive both random node breakdowns as well as targeted attacks. Some networks will breakdown to various components that cannot communicate with one another, after only a few nodes breakdown. A second aspect is the responsiveness of a network, which is a measure of how quickly the network can be traveled through. It is usually measured by averaging the shortest paths between all pairs of nodes. A third aspect, flexibility of a network, measures how often there exists an alternate path between any two nodes. The clustering coefficient discussed earlier, is a good measure of the flexibility of a network. The fourth aspect, the adaptivity of a network, is a measure of how easily paths can be created and destroyed in order to change a topological characteristic of the network.

Each of the four network characteristics identified by Thadakamalla et al. is important in measuring how well a network will perform in the event that nodes or arcs breakdown. In fact Thadakamalla et al. ran simulations on several networks showing that scale-free networks were quite robust when random nodes broke down. This means that quite a few nodes can fail and yet the rest will still be connected to each other. Yet, scale-free networks are quite vulnerable to attacks on their hubs. By contrast, random networks and small

[2] Airline Statistics from the Bureau of Transportation Statistics (2006).

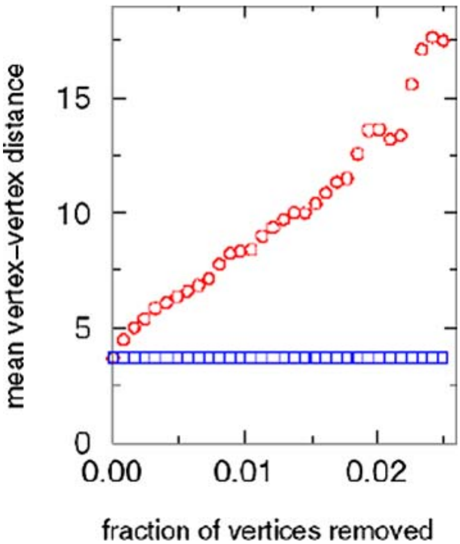


Figure 9. The Effect of Vertex Removal.

world networks are less vulnerable to attacks, but more vulnerable to random breakdowns. Albert, Jeong, and Barabási (2000) studied the vulnerability of scale free networks. They did this by measuring the average vertex to vertex distance as vertices are removed by random failures and by targeted attacks. As seen in Figure 9 taken from Newman (2003), the mean vertex-to-vertex distance is hardly affected by random attacks (squares), yet the mean vertex-to-vertex distance significantly increases for targeted attacks (circles). Thus, we can see that scale free networks survive random failures quite well, but do not easily survive targeted attacks. These two studies used different measurements, but in fact they show similar results.

The vulnerability of supply chain networks and the stubbornness of some epidemics can be analyzed in a similar way with network structure. In particular, the SIR (susceptible/infective/removed) model is one mathematical model of epidemics. It breaks up the total population into three sets of people: those who are susceptible (S), infected (I), and removed (R). In the SIR model, the population of each group is governed by coupled differentials equations relating the populations. The basic SIR model given by Newman (2002) is the system of equations in (1). Most mathematical models of epidemics, including the SIR model assume that each person is equally likely to come into contact with an infected person. However, this is simply not the case in most real networks; so a rather challenging task arises – namely that of imposing a hub-based topology on the simple set of ordinary differential equations comprising the above model. Thus, the mathematical theory of epidemiology – as well as that of population migration and innovation diffusion – may need to be re-thought.

$$\frac{ds}{dt} = -\beta is \tag{1}$$
$$\frac{ds}{dt} = \beta is - \gamma i$$
$$\frac{dr}{dt} = \gamma i$$

The above challenges not withstanding, it is constructive to consider what might be the insights on epidemic management afforded by network science. By turning the vulnerability of scale free networks on its head, it can be seen that if a social network has a scale-free nature then the spread of a virus can be stopped by targeting key people. If we vaccinate certain key people in a social network, it is the same as making a targeted attack to break down the connectivity of the social network. However, since random attacks on a scale-free network, make a negligible impact on connectivity, this implies that random vaccinations on a social network will make a negligible impact on stopping the spread of a virus. The topology of a network can grossly effect the spread of an outbreak over a network. Watts (2002) discusses cases in which for one type of topological structure an outbreak may simply die out, yet in another it will spread to every site (epidemic), while in other topological structures it will depend on how quickly the infection can spread. This is one way in which topological properties such as degree distribution and clustering can effect the spread of disease.

Similarly, an epidemic's transmission network may be compartmentalized in such a way that an infection cannot spread quickly. Often the term community structure is used to describe the property of networks where the nodes may be split into different "classes" which have higher clustering coefficients within each class.The spread of a Sexually Transmitted Disease will depend on a social network. However, most people are more likely to interact sexually with others of similar age and financial status, the same ethnicity, and different sex. So, people do not simply interact uniformly with all other people allowing the structure to play an integral role in the spread of an STD. For this reason, since the SIR model assumes "full-mixing", it is not always a good approximation. Often in social networks, an epidemic may spread more slowly.

Power distribution networks and communication networks are similar to epidemics viewed as networks, in that it is important for the network not to breakdown if only one node or arc suffers failure or interdiction. However, in these networks when an arc or node breaks down, the remaining load is redistributed to the rest of the network. This may result in overloading more nodes or arcs, which then breakdown and the load is redistributed again. Hence there are cascading failures until the network stabilizes. Boccaletti et al. (2006) discuss the effects that topological properties, such as path redundancy, load and capacity, can have on cascading network failures.

When it comes to viruses spreading through computer networks, however, Balthrop et al. (2004) argue that this targeted vaccination technique may not work as well. Viruses can "choose" a network topology by choosing their mode of transmission: email, IP addresses, etc. Some of these networks, such as the network created by IP addresses, are quite uniform while others such as the one created by email traffic are close to scale-free. A virus can spread through any of various networks created via a mode of transmission. So, if a virus is attacked by vaccinating the 10 percent of the population most at risk, little may be achieved if there is unknown scale-freeness. Balthrop et al. argue that any vaccination technique that requires knowledge of the topology or assumes a certain topology will be ineffective because networks topologies are constantly changing and the virus may change the network it is acting on by changing its method of infection.

Thus, Balthrop et al. (2004) propose instead a method called "throttling" to thwart the spread of computer viruses. Many viruses depend on contacting other machines hundreds of times a second in order to cause an epidemic. However, people can at most contact a few computers a second and usually much less. Throttling proposes to take advantage of this disparity by slowing down the contact rate of computers. By only letting a computer contact one other computer per second, people's legitimate information flow is not slowed down because this does not limit most people. However, a virus which may need to make hundreds of contacts per second, will be slowed down by a factor of 100. This means, however, that the epidemic propagates much more slowly. With "new" time created in this fashion, traditional anti-virus software can be updated to destroy a new virus. If throttling can slow a virus down to doubling once per day or week, ordinary users can with – a near certainty equivalent – update their antivirus software and stop the epidemic. Throttling in theory will work on any network, regardless of topology.

2.10 Network Games

The topology of complex networks has been discussed in prior sections, but now I turn to look at the routing of flows over a given network. The most common example, is the traffic network that most of us travel through each day on the way to work. Each person travels from their own home to their workplace along a route that they expect to minimize their travel time. The problem that each of us has surely encountered, is that of traffic or congestion. A road can only hold so many people and as the number of people on a road increases, the time it takes to travel across it increases in some way also. Additionally, each person does not consider the fact that for each road they choose to take on the way to work, they are increasing the travel time for every other person who needs to take that road. That is, they act selfishly in noncooperation with others.

This network can be modeled in game theory as a complex network of nodes and arcs, with infinite agents. Each agent has its own source and destination (not necessarily unique) and controls a small fraction of the flow over the network. Further, we assume that each person acts in a selfish manner, that is, they do not consider the effect on others, when selecting their route. They simply try to minimize their travel time. The cost that each agent incurs over an arc is called the latency and the total flow over an arc is called the load. It is generally assumed that the latency is a nondecreasing function of the load.

The Nash Equilibrium is the set of routing or set of flows that is reached when each person travels selfishly. For the Nash Equilibrium, each agent will incur the same cost for a given source and destination. That is, any two agents that have the source destination pair, will incur the same latency regardless of the route they took. This may not be immediately intuitive. To clear things up, let us ponder the case if it were not true. Suppose agent A incurs a latency of x and agent B incurs a latency of y for the same route and $x < y$. Then soon enough, agent B would discover agent A's route that is less costly and would then switch to agent A's route. But, then the original solution would not be at equilibrium. Thus, in order for an equilibrium to occur, all agents must incur the same cost for a given source and destination.

Since each person chooses their route in such a way as to minimize their own travel time, the total travel time traveled by all (Social Cost) is higher than it would be if this were minimized in a regulated network. Now, let me call the socially optimal solution that which is achieved if the social cost is minimized in a regulated network. This leads to the question, "How much worse is the Nash Equilibrium then the social optimal?" The difference between the Nash Equilibrium and the Social Optimal is commonly referred to as the "Price of Anarchy."

Roughgarden (2002) gives a simple example created by Pigou (1920) and given in Figure 10, to display the difference between the Nash Equilibrium and Social Optimal. Pigou's example consists of one unit of flow that has to travel

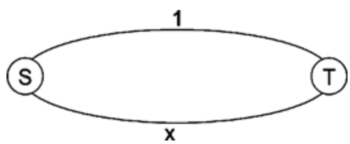


Figure 10. Pigou's Example.

from s to t. The cost or latency of the top arc is 1 regardless of the load over the arc, whereas the bottom arc has a latency equal to the flow over it. If there is more than one unit of flow over the network, we can think of x as the fraction of the total flow.

Now the Nash Equilibrium consists of the full unit of flow going on the bottom arc, that is $x = 1$. Why? Well suppose it were just a bit less, so $x < 1$. Then this implies that some agents are using the top arc and incurring a cost of 1 unit while others are going on the bottom arc and incurring a cost less than 1. So, clearly those agents incurring the cost of 1 on the top arc, would switch to the bottom to save a bit. This would continue, until the cost of the top arc equals that of the bottom, which happens to occur in this example when there are no more agents using the top arc and all are on the bottom.

Yet this Nash Equilibrium is clearly not the social optimal. The social cost is simply the total cost incurred which equals:

$$SC = f_{top} \cdot l_{top}(f_{top}) + f_{bottom} \cdot l_{bottom}(f_{bottom})$$

where f is the flow over the arc and l is the load dependent latency over the arc. So

$$\begin{aligned} l_{top}(f_{top}) &= 1 \\ l_{bottom}(f_{bottom}) &= f_{bottom} \end{aligned}$$

lets denote f_{bottom} as x so

$$\begin{aligned} f_{bottom} &= x \\ f_{top} &= 1 - x \\ l_{bottom}(f_{bottom}) &= l_{bottom}(x) = x \\ l_{top}(f_{top}) &= l_{top}(1-x) = 1 \\ \Longrightarrow SC &= (1-x)\cdot(1) + (x)\cdot(x) \end{aligned}$$

then to find the x that minimizes SC is elementary calculus:

$$\begin{aligned} SC &= x^2 - x + 1 \\ \frac{\partial SC}{\partial x} &= 2x - 1 \\ \frac{\partial SC}{\partial x} &= 0 \Longrightarrow 2x - 1 = 0 \\ \Longrightarrow x^* &= \frac{1}{2} \\ \Longrightarrow SC^* &= \left(\frac{1}{2}\right)^2 - \frac{1}{2} + 1 = \frac{3}{4} \end{aligned}$$

So, the social cost is minimized with a flow of $\frac{1}{2}$ on each arc giving a Social Cost of $\frac{3}{4}$. Note that this is different from the Nash Equilibrium, which had all of the flow on the bottom arc ($x = 1$), but had equal costs for each arc ($l_{bottom}(f_{bottom}) = l_{top}(f_{top}) = 1$). The arc latencies for the Social Optimal are 1 for the top and $\frac{1}{2}$ for the bottom. So those who take the top arc spend just as much time, while those on the bottom save. Indeed, selfish routing does not gain anything for anybody but simply results in some people being more hurt than they need be if the socially optimal solution were used.

Tim Roughgarden and Eva Tardos (2002) found bounds for the price of anarchy. They found when the arc latencies are a linear function of the arc flows, the total latency of flows for the Nash Equilibrium will be no more than $\frac{4}{3}$ of the total latency incurred by the optimal regulated routing. However, when the latencies are not linear, but instead simply monotonic with respect to flows, the price of anarchy can still be bounded. Now the flow of selfish routing is bounded by the total latency achieved by routing twice as many units though the regulated network.

For any Nash Equilibrium solution of flows in a network, each path from a particular source to a particular destination must have the same total latency (commodity). If this were not true, then there would exist an alternate route for the same source and destination, with a lower latency. But, if this were the case and each agent acts selfishly, they would surely prefer to switch to such a lower latency route implying the non-optimality of such a solution. Beckman et al. (1956) first showed such properties of the Nash Equilibrium Solution.

Braess's Paradox has captured much of the work done in the game theoretic applications in networks. The basic idea of Braess's paradox is that for a given network it is possible that adding an additional arc may result in increasing the total latency as well as the latency of each agent for the Nash Equilibrium Solution. At first blush, one may wonder how this could be true? Hence why it is called a paradox. Roughgarden (2002) gave the example in Figure 11, which should help demystify the paradox:

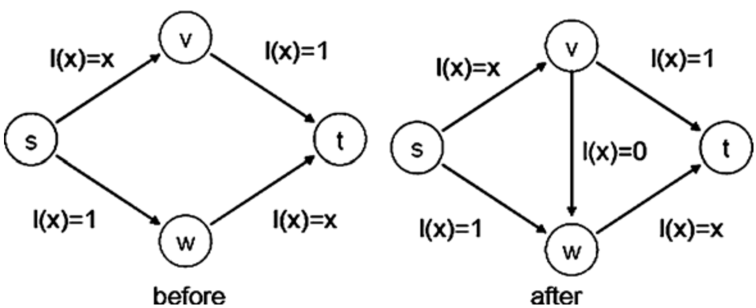


Figure 11. Braess's Paradox.

As Roughgarden explains, in the first network (before), the Nash Equilibrium for one unit of flow is for half to go on the top route and half on the bottom, each paying a total latency of $\frac{3}{2}$. In this example, the solution also happens to be the socially optimal solution as well. Now, if another arc with 0 latency is added, the Nash equilibrium will change. In the new network (after), the Nash Equilibrium solution will be for the entire unit of flow to take the path s, v, w, t with a total latency of 2, which is greater than before. Meanwhile, the socially optimal solution stays the same. This again, may not be intuitive, but suppose a small fraction or a single agent decides to deviate from this path (s, v, w, t), then it must take either (s, v, t) or (s, w, t). Now, if the agent decides to instead take (s, v, t), then it will travel along (s, v) with all the other agents, saving nothing. However, it will then have to travel across (v, t) for a cost of 1 while all the others travel (v, w) for 0 and then (w, t) for less than 1 (since this agent just left this path). Seeing that all the agents traveling (s, v, w, t) have a lower total latency, this agent should not make such a change in path. A similar argument can be used to show that (s, w, t) is also not a good choice. So, the Nash Equilibrium results in each agent having a higher total latency in this example, but how much higher?

Suppose we route two units through the (after) network, with half (1 unit) taking (s, v, t) and half (1 unit) taking (s, w, t). Then the total latency is 4. According to Roughgarden, in the general case, the flow of selfish routing would be bounded by 4. Since, the latency functions are linear though, the total latency of selfish routing is bounded by $\frac{4}{3}$ that of the social optimal of $\frac{3}{2}$, which is 2. Thus, the selfish routing in this network is bounded by 2, which is exactly what the Nash Equilibrium total latency is, so it is a case of the worst case scenario.

The cases considered by Roughgarden assume that the latencies of each user are independent of each other, that is the latency functions are separable. Perakis found bounds for the considerably more complex case of non-separable latency function Perakis (2004). Refer to Perakis (2004) to read more about these more general cases. This is simply one extension of Braess's paradox, there has been a host of literature published in recent years on various aspects of this topic.

It is important to keep the results of Braess's paradox in mind because sometimes it may be tempting to think that adding arcs or loads to a network will increase flexibility or clustering coefficients or some other property, making the network perform better, but while one characteristic may have been made better it may be adversely effecting the performance of the network. For very large networks, such adverse effects are difficult to find and analyze, so this presents a potential danger in many large networks in communication, supply chain, power supply, traffic, and others.

3. Nonlinear Dynamics

Chaos is defined by Merriam-Webster's Dictionary as "the inherent unpredictability in the behavior of a natural system." However, chaos is often misunderstood in its everyday use. Often, people mistake the fact that chaos is unpredictable for the idea that it is random. In fact, a chaotic system is deterministic, yet unpredictable. A chaotic system is one in which a very small difference in an initial condition, so small it may not be able to be measured accurately, can result in significantly different results. So, the system is unpredictable, but is not random.

Chaos is just one section of a broader field known as dynamics or nonlinear dynamics. Dynamical systems are systems governed by deterministic laws, however, sometimes even deterministic laws can result in unpredictability. The discovery of these unpredictable, chaotic systems raised the interest in the field of Dynamics.

3.1 A Brief History of Dynamics

In 1887, Henri Poincaré entered a contest on which he was supposed to show that the solar system was dynamically stable according to Newton's Mechanics. Although, he could not do so, his work was revolutionary and the judges (including Weierstrass) awarded him the prize anyway. As it turns out, instead he was the first person to stumble upon a chaotic dynamical system. Poincaré argued that even if the equations governing the system are known and deterministic that "small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible..." (Nijkamp and Reggiani, 1992).

The basic idea is that systems exist such that a small perturbation in initial conditions can cause differences later that are not only numerically significant, but in some applications, qualitatively significant as well. In such a system, the two paths are obviously quite close at the initial time and thus it is easily conceivable that they can again come close and even look quite similar for long lengths of time. However, they are not the same and they lead to quite different results. So, no matter how precisely such a system is measured, within that precision, the error in measurement could result in quite different states at a later time. This makes it impossible to predict the future state of a system, even if the laws governing it are perfectly known and perfectly deterministic. This lack of ability to predict the future state is seemingly random, but to call it such would be a mistake because the system is governed by deterministic laws. Systems such as these that are governed by deterministic laws are known as dynamical systems and chaos is simply one class of dynamical systems.

Unfortunately, although the work of Poincaré was immediately recognized as valuable, like many other discoveries, it was neglected for many years. This was most likely because such chaotic effects took place in infinite iterations and empirical experimentation was not practical for the time. His foresight would have to wait until the invention of the computer to be recognized for its full value.

In 1961, a meteorologist Edward Lorenz created a simulation model of weather conditions. One time he wanted to see the simulation a second time, so he attempted to rerun his model from a point somewhere in the middle, instead of the original conditions. To do this, he took the computer printout of the state conditions at the new time he wanted to start at and plugged them into his computer. To his surprise, the results were drastically different from the original results that he wanted to repeat. Figure 12 from Stewart (1989) shows how the two trajectories diverge.

His model was composed of deterministic equations. How could the results be so different the second time? He entered the data that he had from the same path and the equations were deterministic and yet the results were different.

He was eventually able to track it back to the fact that the computer used six-digit numbers and his computer printout cut off the numbers with three digits. So when he reentered the data for the second simulation, he only entered three digits, but when the computer ran the simulation the first time, it had all six. The conventional wisdom would tell most people that the last three digits are not very significant. In most calculations, it would be considered accurate to have three digits and six is above and beyond necessary. Yet, in this case the perturbation caused by the fourth digit at some time in the middle of the simulation was able to cause drastic changes in the results.

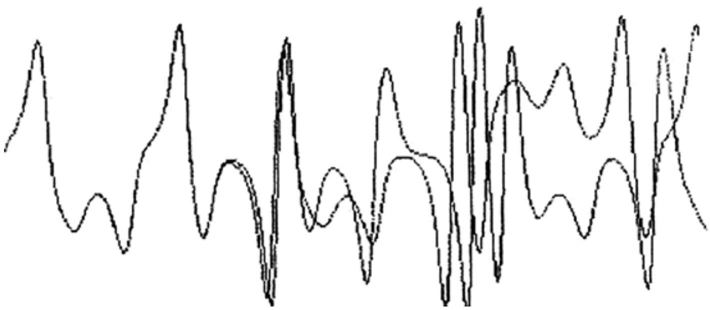


Figure 12. Lorenz Simulation.

3.2 The Basics of Dynamics

Dynamical systems are systems that have a state (represented as a variable or vector of variables) that changes over time in a manner which is governed by deterministic laws or equations. That is, the system is dynamic in a deterministic sense. Below are two examples:

$$x_{t+1} = f(x_t) \tag{2}$$

$$\begin{aligned} \dot{x} &= f_1(x, y) \\ \dot{y} &= f_2(x, y) \end{aligned} \tag{3}$$

The dynamics are often given as a system of recurrence relations (as in 2) or differential equations (as in 3). The recurrence relation can be iterated infinite times to find the trajectory of the system as shown in Figure 13. A similar trajectory can be found for the differential equations by using small discrete steps to approximate a solution.

To "solve" the system, an equation must be found to directly find the state at any time in the future (find an equation $x(t)$ where x is the state of the system). Dynamical systems can be linear or nonlinear, that is the functions f, f_1, and f_2 in (2) and (3) may be linear or nonlinear functions. Linear systems have the property of being able to be easily solved, yet they often do not provide the array of behavior that can be found in nonlinear systems.

The dynamics of a system often involve parameters, which may or may not change over time, but do effect the nature of the dynamics either way. Often these parameters are responsible for the character of the system, causing it to be stable, unstable, or even chaotic for different values of the parameters. For example, the system shown above in (3) could have a solution for x and y

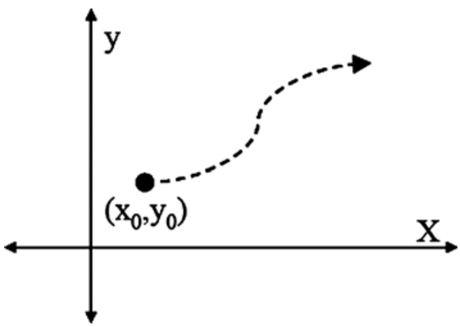


Figure 13. Trajectory.

(trajectory whose x and y components are determined parametrically by these equations):

$$x(t) = x_0 e^{at}$$
$$y(t) = y_0 e^{bt}$$

Now, the values of a and b will greatly affect the trajectories of the system. Think about what may happen if $a > 0$ as opposed to $a < 0$ and similarly for b.

3.3 Phase Diagram or Phase Space

In Dynamics, it is often easier to see the character of a system by looking at the phase diagram or phase space. In a phase diagram, each state of a system is plotted by its dimensions (so time is not included), so it is a simple point in the plot, sometimes called a phase point. Each state is then connected to the state preceding it and succeeding it. For example, a system with two variables x and y, would have a phase diagram with one axis for x and one axis for y. Then each state $S_1, S_2, S_3, \ldots$ are plotted as $(x_{1,} y_1), (x_{2,} y_2), (x_{3,} y_3), \ldots,$ where x_i is the value of x at time i and y_i is the value of y at time i. For a discrete time dynamical system we must then connect the points $(x_{1,} y_1), (x_{2,} y_2), (x_{3,} y_3), \ldots$ to see the trajectory, but in a continuous dynamic system the transition between states are continues curves in the plane. The trajectory of Figure 13 is an example of a continuous time phase diagram. In a discrete system, only the points are plotted and the line connecting them is only to visualize a path between the states as shown in Figure 14.

It should also be noted that the particular trajectory that a system follows is dependent upon the initial starting point. That is, the trajectory may be significantly different if it begins from a different starting point.

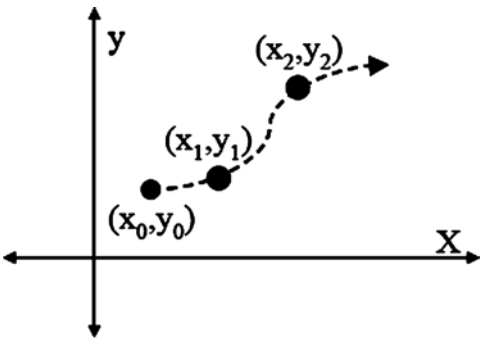


Figure 14. Discrete Trajectory.

3.4 Fixed Points or Equilibrium Points

Given the system

$$\dot{x} = f_1(x, y)$$
$$\dot{y} = f_2(x, y)$$

we can see that at any given point (x, y) we can calculate and find $\dot{x}$ and $\dot{y}$. In fact, $v = (\dot{x}, \dot{y})$ is the velocity vector for any given point (x, y). The vectory field V of velocity vectors shows the flow of the system. Those points where $v = (\dot{x}, \dot{y}) = (0, 0)$ have no flow and are thus called fixed points or equilibrium points.

There are two types of fixed points, those that are stable and those that are unstable. The stable points are fixed points whose neighboring points have velocity vectors pointing towards them, showing that the flow moves into them, and thus they are often called sinks or attractors. Similarly, the unstable fixed points have neighboring points whose velocity vectors point away from them, showing that the flow moves away from it, giving them the name repellers or sources. Take the system

$$\dot{x} = x$$
$$\dot{y} = y$$

whose vector field is shown in Figure 15.

At (0, 0), $\dot{x} = 0$ and $\dot{y} = 0$, which means that the system will not move from $(0, 0)$. Hence it is a fixed point. As we can see all of the vectors point away from the fixed point $(0, 0)$ meaning that the flow of the phase point beginning near $(0, 0)$ will flow away from it. Thus, this point is unstable and is often referred to as a source or repeller. A very similar system is:

$$\dot{x} = -x$$
$$\dot{y} = -y$$

whose vectory field is shown in Figure 16.

Again at $(0, 0)$, $\dot{x} = 0$ and $\dot{y} = 0$, so $(0, 0)$ is again a fixed point. As we can see all of the vectors point toward the fixed point $(0, 0)$, meaning that the flow of the phase point beginning near $(0, 0)$ will flow toward it. Thus, this point is stable and is often referred to as a sink or attractor.

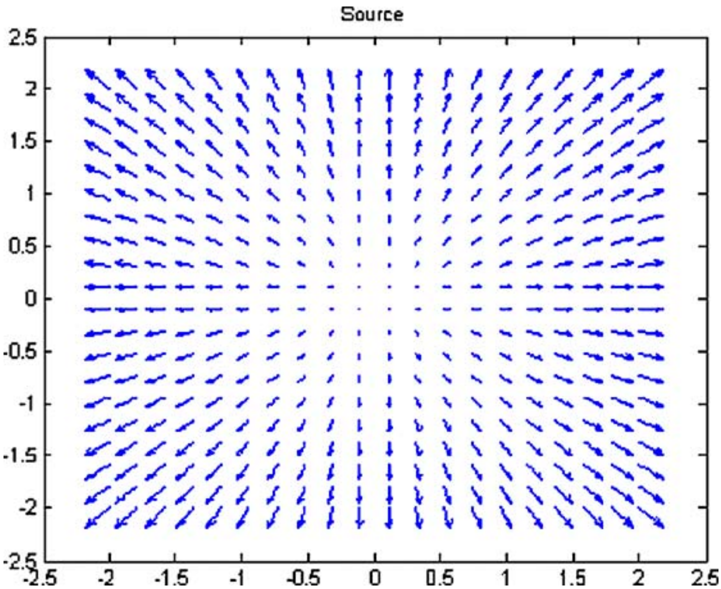


Figure 15. Source.

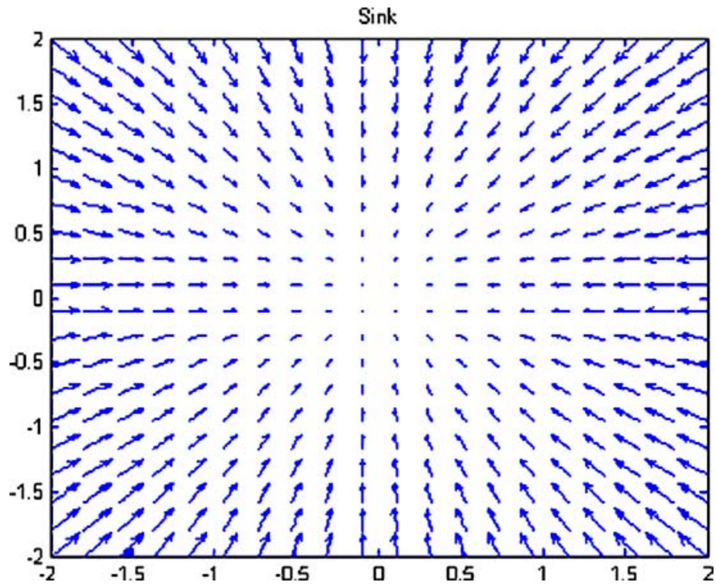


Figure 16. Sink.

There is yet a third type of equilibrium point, called a saddle point. A saddle point has two axes or manifolds going through it. The trajectories

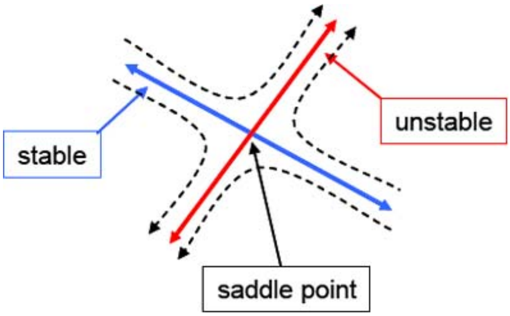


Figure 17. Saddle Point.

are asymptotic to one of the axes, known as the unstable manifold (refer to Figure 17) in forward time. The other axis is the stable manifold, the trajectories asymptotically come from this axes, that is in reverse time or the reverse trajectory is asymptotic to the stable manifold. At first blush it may seem odd that the trajectories approach the unstable manifold. So, it may help to think of it this way: in the direction of the stable manifold, the trajectories approach the saddle point as time goes to infinity, making it stable in this direction. However, in the direction of the unstable manifold, the trajectories go away from the saddle point, they are infinitely far away as time goes to infinity. It should now seem much more reasonable to label the stable manifold as stable and the unstable manifold as unstable.

The examples of systems in Figure 17, can once more be slightly modified to:

$$\dot{x} = -x$$

$$\dot{y} = y$$

whose vectory field is shown in Figure 18.

Now we can see that the vector field forms a saddle point at $(0,0)$ and the x-axis is the stable manifold and the y-axis is the unstable manifold.

An important note to make is that in the above three systems I have simply added a negative sign in one or two places and the topology of the vector field has changed significantly. So, now we can see that if we plot the vector field for the system and then we are given a starting position, we can follow the vectors to form a trajectory from that starting position. This type of analysis does not involve time, so we do not know how long it will take to follow a trajectory, but we know it will eventually follow the trajectory. The important think to realize is that minor changes in the differential equation governing the dynamics can significantly alter the topology of the vector field and thus the solution paths.

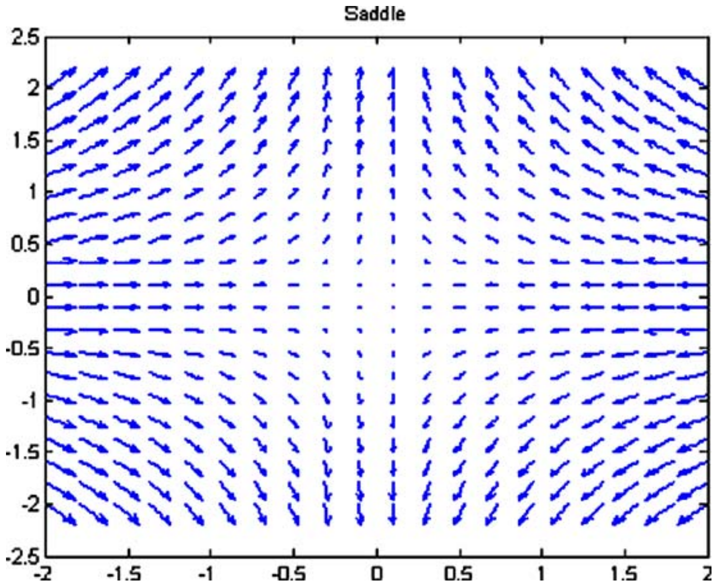


Figure 18. Saddle Vector Field.

3.5 Closed Orbits and Oscillators

A closed orbit is a loop or orbit, in which a system will periodically cycle through. That is, if the system starts on the cycle, then it will continue to go around it infinitely many times. At any time, it will visit each state infinitely many more times in the future. An example is:

$$\dot{x} = -y$$

$$\dot{y} = x$$

whose trajectory is shown in Figure 19.

The vector field for the system is shown in Figure 20.

3.6 Attractors and Repellers

By plotting the trajectory from different initial points, one can then see the behavior of the system. Many systems are periodic and will have a cycle or orbit in the phase diagram, no matter where they begin from. Others will have an asymptotic behavior where they will be attracted to a point or line. Still other systems may show a spiral towards a single point in the center, like a hurricane or water flushing down a toilet. Systems may show that different trajectories are attracted to an orbit from nearby locations. The term attractor refers precisely to these occurrences. The attractor may be a point, loop, or multidimensional

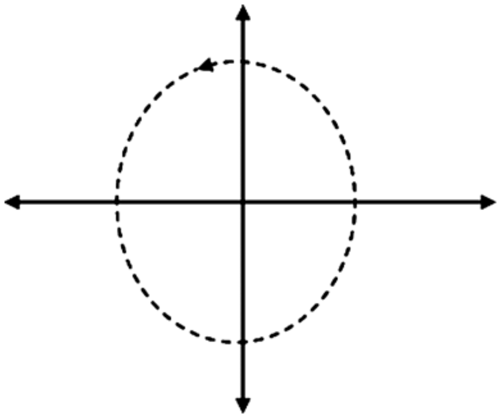


Figure 19. Center.

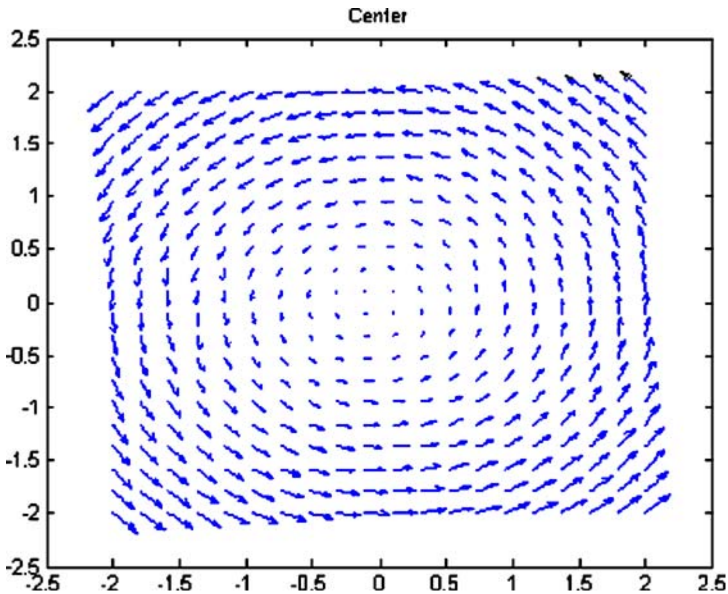


Figure 20. Center Vector Field.

loop that the trajectory moves towards as time or iterations progress. In a more mathematical view, the attractors are points or sets of points that the system approaches in infinite time.

A given fixed point, $\bar{x}$, is a local attractor if it is approached from starting points, x_0, that are within a certain neighborhood of $\bar{x}$. Similarly, $\bar{x}$ is a

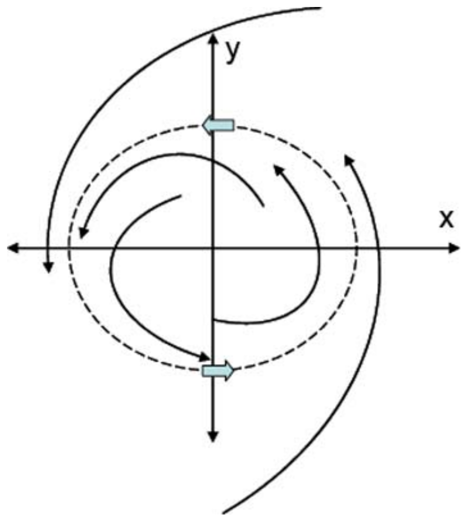


Figure 21. Limit Cycle.

globally attracting fixed point, if it is approached in infinite time by all starting points, x_0.[3]

There are various types of attractors and repellers. A fixed point is a single phase point which can either attract or repel trajectories (e.g. a stable fixed point, unstable fixed point, saddle point). There may also be attractors and repellers that are limit cycles. A limit cycle is a closed orbit, that is surrounded by trajectories that are not closed. Figure 21 is an example where the dotted trajectory is closed, but all the trajectories on the inside spiral toward it and all of the trajectories on the outside spiral toward it as well. This is called an attracting limit cycle. Another type of limit cycle will have a closed orbit, but the trajectories are repelled from it, such a limit cycle is a repelling limit cycle. Another case in where the system has a closed orbit with trajectories on the inside that are attracted, yet on the outside they are repelled or vice versa. This is neither attracting nor repelling in the sense described above.

Alternatively, in the case of a purely oscillating system, the phase portrait will consist of infinitely many concentric loops shown in Figure 22. The vector field for such a case is shown in the vector field for the center shown in Figure 20.

There are several types of attractors and repellers in two dimensions including fixed points and centers. Additionally, systems in three dimensions will increase the variety of attractors and repellers to include what are called

[3] The above definition of attractor is based on the definition of assymptotic stability. Liapunov Stability is another type of stability.

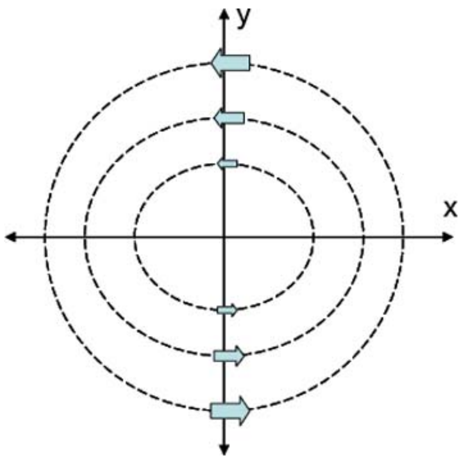


Figure 22. Center Trajectories.

strange attractors. These are significantly more complex and will be briefly discussed later in this chapter.

3.7 Stability: Attracting, Liapunov, and Asymptotic

There are a few types of stability that are often used. The first is known as attracting, where by a fixed point p or a path $p(t)$, is attracting if all trajectories within a given neighborhood of p approach it as time goes to infinity. The second definition of stability is Liapunov Stability, by which a fixed point p or solution path $p(t)$, is Liapunov Stable if all trajectories starting within one neighborhood of p, remain within a different neighborhood of p, for all time. Now, a point or path p is Asymptotically stable if it is both attracting and Liapunov Stable. It would be a good idea to stop here for a moment and ponder the differences.

If p is attracting, this means that all trajectories within a distance δ of p will approach p, as time proceeds. So, given any distance ε, the trajectory will eventually be within ε of p, however, along the way, it may be infiniteley far away. On the other hand, if p is Liapunov Stable, then this means that if a trajectory began within δ of p, then it will be within ε for all time forward. However, I should note that in the case of Liapunov Stability, this means that given an ε there exists a δ such that if a path begins within a distance δ, it remains within ε of the point or path p. However, δ may depend on ε, that is if a smaller ε is given, then a smaller δ may be necessary. This is important because p may be Liapunov Stable, but this does not imply that all paths in a sufficiently small neighborhood of p will approach it in infinite time. They may simply stay a distance of exactly $\varepsilon_1 < \varepsilon$ away for infinite time. For example, paths may form concentric circles around a fixed point p.

A small analogy may sum up the differences. Suppose we have a boy flying a kite. The center (where the boy is standing) would be Liapunov Stable, as long as the string on the kite remained the same length where ε is the distance of the string. This does not mean the kite will ever get closer to the center (the boy), but it does mean that the kite will never get further than the length of the string. It is similar to a fly trapped inside a beach ball, there is no guarantee it will ever reach the center, but surely it cannot leave. On the other hand, suppose the kite started out on an infinitely long string, so it was nowhere near the boy at the center. Then suppose, the string is steadily shortened, the kite will eventually be pulled to the center. However, a large gust of wind may come and it gets really close and then blows away again and the boy lets the string go for a moment and it gets a bit further away, but then continues to be pulled in. In this case, the center is attracting. Now, if the boy had never let the string go, so that the kite continued to get closer and if we made the ε distance equal to the length of the string, then the kite approaches the boy and it never leaves the ε ball, so it is asymptotically stable.

In general, if a point or path is attracting, this means the trajectory will approach it in infinite time. This does not mean that it necessarily gets closer as time moves forward, a trajectory's distance to the fixed point or path need not be monotonically decreasing. This means, a trajectory can get really far away before it comes back. In fact it can come close, then go far away, and then close again, as many times as it wants before it eventually approaches the fixed point or path.

Whereas in the case of Liapunov Stability, the trajectory must stay within some distance of the fixed point or path, the trajectory is trapped by a finite ball around the fixed point or path. However, that is all, it must stay in the ball, but it never has to get any closer to the center than the edge of the ball. In fact, it can simply orbit around the point on the perimeter of the ball and still be Liapunov Stable.

The strongest definition of stability is asymptotic stability which requires that a path approaches the fixed point or stable path and that trajectories with a given distance δ do not get further than ε away for any given $\varepsilon > 0$. This means that the paths have to approach the stable solution and that the paths can not get infinitely far away along the way to approaching the fixed point or stable path. For more mathematically rigorous definitions refer to Strogatz (1994) or Jordan and Smith (1999).

3.8 Linear Stability Analysis

Suppose we have a two dimensional linear system, that is:

$$\dot{x} = f_1(x, y)$$
$$\dot{y} = f_2(x, y)$$

with f_1 and f_2 being linear. Suppose our system is:

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

then we let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that we now have:

$$\dot{\mathbf{x}} = \mathbf{Ax}$$

This system is solved as it is solved in most first courses in linear algebra, by first finding the eigenvalues and eigenvectors. In order for $\mathbf{v}$ to be an eigenvector and λ to be an eigenvalue, $\mathbf{Av} = \lambda\mathbf{v}$ must be true, so

$$\begin{aligned} \mathbf{Av} &= \lambda\mathbf{v} \\ &\Rightarrow \mathbf{Av} - \lambda\mathbf{v} = \mathbf{0} \\ &\Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \\ \mathbf{v} &\in \mathbf{N}(\mathbf{A} - \lambda\mathbf{I}) \end{aligned}$$

where $\mathbf{N}(\mathbf{A} - \lambda\mathbf{I})$ is the null space of $\mathbf{A} - \lambda\mathbf{I}$. However, if $\mathbf{v} \neq \mathbf{0}$ then $\mathbf{A} - \lambda\mathbf{I}$ must be noninvertible and $\mathbf{A} - \lambda\mathbf{I}$ is noninvertible if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$.

The characteristic equation $f(\lambda)$ is defined as:

$$f(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$$

So then λ is an eigenvalue of $\mathbf{A}$ if and only if $f(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$.

$$\begin{aligned} f(\lambda) &= \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= ad - a\lambda - d\lambda + \lambda^2 - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \end{aligned}$$

The trace of a matrix $\mathbf{A}$ is defined as the sum of its diagonal elements and has the notation, $\operatorname{tr}(\mathbf{A})$. Substituting in the Trace and Determinant of A, we have the common results:

$$\begin{aligned} f(\lambda) &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - \operatorname{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) \end{aligned}$$

Now, the characteristic equation is simply a quadratic equation and can be solved using the quadratic rule:

$$\lambda = \frac{\operatorname{tr}(\mathbf{A}) \pm \sqrt{(\operatorname{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A})}}{2}$$

The two eigenvalues are specifically:

$$\begin{aligned} \lambda_1 &= \frac{\operatorname{tr}(\mathbf{A}) + \sqrt{(\operatorname{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A})}}{2} \\ \lambda_2 &= \frac{tr(\mathbf{A}) - \sqrt{(tr(\mathbf{A}))^2 - 4\det(\mathbf{A})}}{2} \end{aligned}$$

The solution to the system is then:

$$x(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$

where $\mathbf{v}_1$ is the corresponding eigenvector to λ_1, that is $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and similarly $\mathbf{v}_2$ for λ_2. The values of c_1 and c_2 can be found with the initial starting point. At $t = 0$:

$$\begin{aligned} x(t) &= c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} \\ &\Rightarrow x(0) = c_1\mathbf{v}_1 e^{\lambda_1 \cdot 0} + c_2\mathbf{v}_2 e^{\lambda_2 \cdot 0} \\ &\Rightarrow x(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \end{aligned}$$

This system of equations can be solved with some basic row operations from linear algebra. For a review of this material, refer to almost any first year linear algebra text. Now, we can see that as $t \to \infty$ the behavior of $x(t)$ will be dependent upon the values of λ_1 and λ_2, which in turn are dependent on $\operatorname{tr}(\mathbf{A})$ and $\det(\mathbf{A})$. Let me define the discriminant D as $D = (\operatorname{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A})$, so that the eigenvalues are:

$$\lambda = \frac{\operatorname{tr}(\mathbf{A}) \pm \sqrt{D}}{2}$$

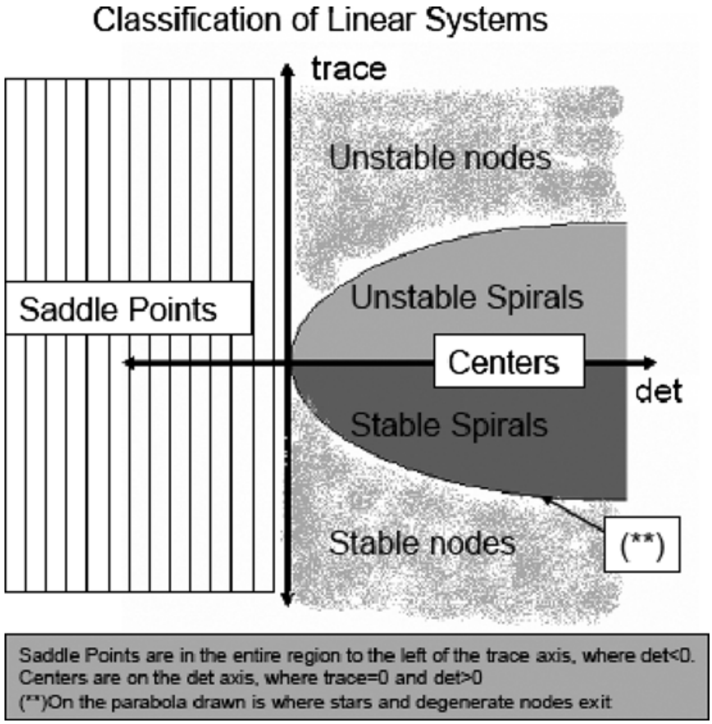


Figure 23. Fixed Point Classification.

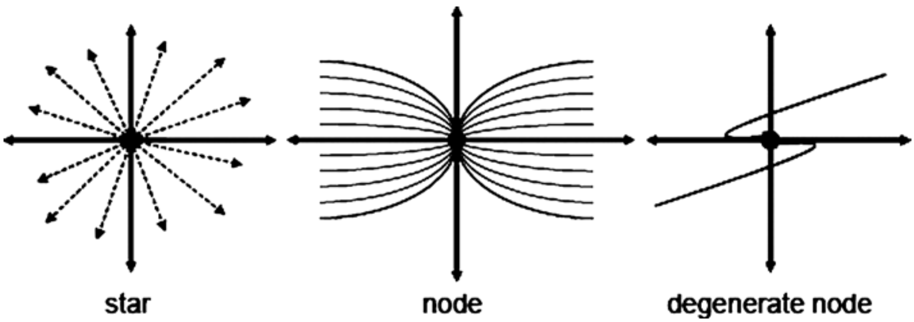


Figure 24. Star, Node, and Degenerate Node.

Figure 23 is often used in order to classify linear systems,where the spirals inside the parabola have $D < 0$ (or $(\mathrm{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A}) < 0$) and the nodes outside the parabola have $D > 0$ (or $(\mathrm{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A}) > 0$). On the parabola itself is where $D = 0$ (or $(\mathrm{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A}) = 0$) and this is where the stars and degenerate nodes exist. In Figure 24, trajectories for a star, node, and degenerate node are given.

4. Linearizing the Nonlinear

Now suppose that we again have a two dimensional system, but now it is nonlinear. That is, we have:

$$\dot{x} = f_1(x, y)$$
$$\dot{y} = f_2(x, y)$$

with f_1 and f_2 now being nonlinear. Suppose, we have a fixed point $a = (x_0, y_0)$ to analyze the stability of a, we will see how a small perturbation from a, effects the trajectory of the system. Our analysis closely follows Strogatz (1994), but this is commonly done using a Taylor series approximation at the fixed point. Thus, using the Taylor series approximation to find $\dot{x}$ at the point perturbed from the fixed point:

$$\begin{aligned}
\dot{x}\,(x_0 + \Delta x, y_0 + \Delta y) &= f_1(x_0 + \Delta x, y_0 + \Delta y) \\
&\approx f_1(x_0, y_0) + \frac{\partial f_1}{\partial x}\Delta x + \frac{\partial f_1}{\partial y}\Delta y \\
&\quad + \frac{1}{2!}\left(\frac{\partial^2 f_1}{\partial x^2}(\Delta x)^2 + 2\frac{\partial f_1}{\partial x}\frac{\partial f_1}{\partial y}\Delta x \Delta y + \frac{\partial^2 f_1}{\partial y^2}(\Delta y)^2\right) + \ldots \\
&\approx f_1(x_0, y_0) + \frac{\partial f_1}{\partial x}\Delta x + \frac{\partial f_1}{\partial y}\Delta y
\end{aligned} \tag{4}$$

where the final line is an approximation if we assume that since Δx and Δy are small that any terms smaller than them are not significant (these include terms of $(\Delta x)^2$, $(\Delta y)^2$, $\Delta x \Delta y$, and all smaller terms). Similarly, for the second equation:

$$\begin{aligned}
\dot{y}\,(x_0 + \Delta x, y_0 + \Delta y) &= f_2(x_0 + \Delta x, y_0 + \Delta y) \\
&\approx f_2(x_0, y_0) + \frac{\partial f_2}{\partial x}\Delta x + \frac{\partial f_2}{\partial y}\Delta y
\end{aligned}$$

By noting that $\dot{x}\,(x_0, y_0) = f_1(x_0, y_0)$, the above equations can be manipulated as:

$$\begin{aligned}
\dot{x}\,(x_0 + \Delta x, y_0 + \Delta y) - \dot{x}\,(x_0, y_0) &= f_1(x_0 + \Delta x, y_0 + \Delta y) - f_1(x_0, y_0) \\
&\approx \frac{\partial f_1}{\partial x}\Delta x + \frac{\partial f_1}{\partial y}\Delta y
\end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned}\dot{y}\,(x_0+\Delta x, y_0+\Delta y) - \dot{y}\,(x_0, y_0) &= f_2(x_0+\Delta x, y_0+\Delta y) - f_2(x_0, y_0)\\ &\approx \frac{\partial f_2}{\partial x}\Delta x + \frac{\partial f_2}{\partial y}\Delta y \end{aligned} \tag{6}$$

From 5 and 6, comes the following linearized system of equations:

$$\begin{aligned}\dot{x}\,(x_0+\Delta x, y_0+\Delta y) - \dot{x}\,(x_0, y_0) &= \dot{\Delta x} = \frac{d}{dt}(\Delta x) = \frac{\partial f_1}{\partial x}\Delta x + \frac{\partial f_1}{\partial y}\Delta y\\ \dot{y}\,(x_0+\Delta x, y_0+\Delta y) - \dot{y}\,(x_0, y_0) &= \dot{\Delta y} = \frac{d}{dt}(\Delta y) = \frac{\partial f_2}{\partial x}\Delta x + \frac{\partial f_2}{\partial y}\Delta y\\ \Rightarrow \begin{pmatrix}\dot{\Delta x}\\ \dot{\Delta y}\end{pmatrix} &= \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}\end{aligned}$$

Now we have a linear approximation of the system and we can do the same analysis as before, but with some dangers to beware of! Since we removed the higher order terms from the taylor expansion, sometimes the linearization is not correct. The cases that are "borderline" in the classification diagram above are those where the linearization cannot be trusted. So, the linearization is good for nodes, spirals, and saddles but not centers, stars, or degenerate nodes.

This should make sense because the borderline cases are delicate. A center is like a spiral that lines up just perfectly and a star is like a node that is perfectly straight. However, if the spiral needs to line up perfectly or the node must be perfectly straight, then the small difference of the higher order terms may destroy these rare cases. A rigorous explanation of the failure of the linearization is quite complex and beyond the scope of this introductory chapter, but refer to Strogatz (1994) for a more detailed explanation.

5. Strange Attractors

Most systems give rise to an attractor that is a fixed point or loop. However, some attractors are far more complex and are known as strange attractors. One of the most popular and also the first strange attractor to be discovered is the Lorenz Attractor shown in Figure 25. Strange attractors can occur in continuous or in discrete time dynamical systems. However, it should be noted that in the continuous time case, they can only occur if the dimensionality of the system is

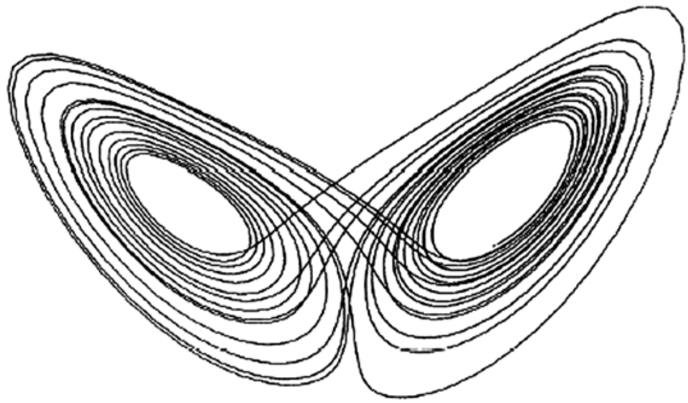


Figure 25. Strange Attractor (Stewart, 1989).

three or greater, whereas in the discrete case, they can occur in even the single dimension case (according to the Poincare-Bendixson theorem explained later).

6. Trajectories

It should be noted that a trajectory is simply the path taken in the phase diagram, given an initial starting position. So if a system begins in a different position, then it will have a different trajectory. Precisely those systems that are interesting are those that are chaotic. These systems have trajectories that are very sensitive to initial conditions. That is if the system starts at x, by some time t, it will be radically different than if it had started at y, even if x and y are very close.

When we say "very close" or "radically different," this often means numerically close or different, but for different systems, "close" may take on different definitions. For example, Lorenz coined the term, "Butterfly Effect."The basic idea was that if a butterfly flaps its wings then in some far off time, it may cause a tornado in Texas. In this example, the state of the world without the butterfly flapping its wings is "close" to the state where it does. Similarly, the state without the Tornado is far from the state with it simply because in this example we perceive the butterfly flapping its wings to be a small event and the Tornado to be a large one. A different metric may be used on different state spaces.

In a more practical sense, we may measure several variables such as temperature, humidity, and wind speed in order to find the current state of the weather. From, this initial state we may predict it will rain five days from now. But, instead it snows because our measurements today were slightly off.

So, two states that are very similar today may lead to states that are not only numerically different but are qualitatively different as well.

Since different trajectories could lead to different states, it may be difficult to understand the behavior of some systems by simply looking at one or two trajectories. Some physical systems may be modeled by equations that are not known perfectly, that is the parameters are estimates or the equations are not exact. In this case, if the trajectories are not close enough for a set of initial conditions, then it is difficult to describe the behavior because these model approximations may be enough to cause radically different trajectories. For such systems, it may be difficult to predict the topology of trajectories or their stability.

Some systems may have different sets of trajectories exhibiting different properties. One set could lead to a fixed point and another to a loop, for example. In a more complex system, you may be missing a lot if you simply look at one trajectory. Some systems are very complex and make it difficult to describe or even know all of the classes of trajectories.

Other systems may have classes of trajectories that radically change as the parameters change. So, looking at one trajectory and one parameter value will just be one case. This gives rise to the investigation of bifurcations, where a change in a parameter may cause a significant change in the behavior of the system. As a system's dimensionality increases, the complexity of its behavior can get harder to describe because there can exist more and more classes of trajectories which may act differently for different parameter values. Since we can only plot a system in two dimensions on paper and three on a computer, it gets very hard to imagine fully the behavior of higher dimension systems.

7. Bifurcations

Sometimes the trajectory or solution of a dynamical system is dependent on a parameter. As the parameter changes, the trajectories will surely change, however, the behavior or classification of the trajectory could change as well. For example, if a trajectory depends on a parameter α and when $\alpha < \alpha_o$ the structure of the phase space is a fixed point, but then when $\alpha \geqslant \alpha_o$, the phase space changes its structure to a limit cycle, then the system has experienced a bifurcation at α_o. Bifurcations can occur by creating or destroying an attractor or repeller, or by changing an attractor or repeller from one type to another (e.g. fixed point to limit cycle, or attractor to repeller).

7.1 Saddle-Node Bifurcation

The simplest example of a bifurcation is the saddle-node bifurcation where equilibrium points are either created or destroyed. A nice example taken from

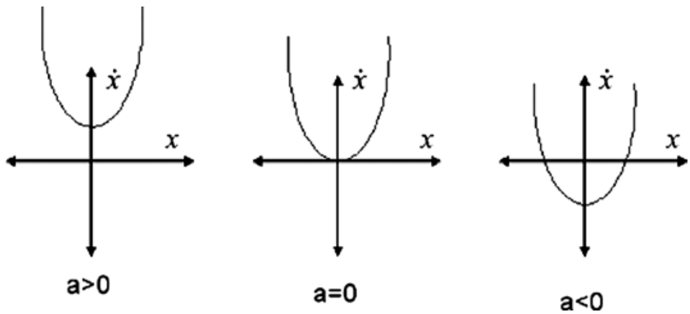


Figure 26. Saddle Node Bifurcation.

Strogatz (1994) is:

$$\dot{x} = a + x^2$$

Remember that a fixed point or equilibrium point occurs where there is no flow, that is where $\dot{x} = 0$. This means the equilibrium points may be found by finding the roots of the equation $0 = a + x^2$. However, the number of roots depends on the value of a:

$a < 0$	There are two real roots, $x = \sqrt{-a}$ and $x = -\sqrt{-a}$
$a = 0$	There is one real root, $x = 0$
$a > 0$	There are no real roots

This means that if we let $a > 0$, then there are no equilibrium points, but if we decrease a, then an equilibrium point is created when $a = 0$. Then, as a is decreased further, the equilibrium point immediately splits into two points, when $a < 0$. Thus there is a bifurcation at $a = 0$ because two equilibrium points are created or destroyed, depending on which way you look at.

Since a fixed point exists when $\dot{x} = 0$, we can see that the roots in Figure 26 are the fixed points. None exist for $a > 0$, one at $a = 0$, and two for $a < 0$. Refer to Figure 26 to see the graph of a saddle node bifurcation.

7.2 Transcritical Bifurcation

A transcritical bifurcation is one in which the equilibrium point changes is stability from stable to unstable or vise versa as a parameter varies. A good example of a transcritical bifurcation again from Strogatz (1994):

$$\dot{x} = ax - x^2$$

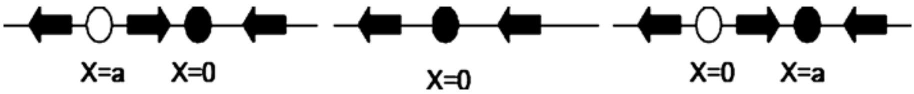


Figure 27. Transcritical Bifurcation: Stability Swap.

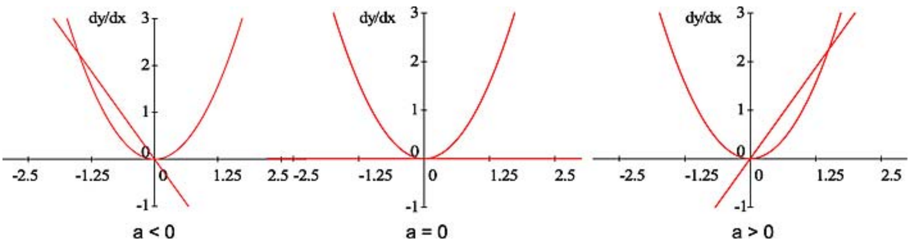


Figure 28. Transcritical Bifurcation.

Now, when:

$a < 0$	$x = a$ is an unstable fixed point and $x = 0$ is a stable fixed point
$a = 0$	$x = 0$ is a single half stable fixed point because the two fixed points merged
$a > 0$	$x = a$ is a stable fixed point and $x = 0$ is an unstable fixed point

Thus as a increased, $x = 0$ was a fixed point that lost its stability. The fixed point $x = a$ moved to the right as a increased and when it crossed $x = 0$, they merged briefly as a half stable fixed point and then when it passed $x = 0$, it took $x = 0$'s stability with it and left $x = 0$ as an unstable fixed point. Hopefully, Figure 27 will help clarity this point.

In Figure 27, again we can see there are two roots for $a < 0$ as well as $a > 0$, but only one for $a = 0$. Now, in order to analyze the stability we must look a bit deeper at this diagram. Since $\dot{x} = ax - x^2$, when the line is above the parabola $ax > x^2$ and $\dot{x} > 0$, so the flow is to the right, yet when the line is below parabola, the flow is to the left.

From the Figure 28, we can see that when $a < 0$, the flow is away from $x = a$, making this the unstable point, where as the flow is toward $x = 0$, making it the stable point. When $a = 0$, the flow is to the left on both sides, here $x = 0$ is called semi-stable. Then when $a > 0$, the fixed points swap stabilities making $x = 0$ unstable and $x = a$ stable.

There are many other bifurcations studied in dynamics. Virtually any parameterized system could have a bifurcation and many can be unique in there own way. We gave a single example of a saddle-node bifurcation and a transcritical bifurcation. These are simply classes of bifurcations, but there are many types which may not even be classified. In any event, two other types of bifurcations not discussed were the Pitchfork Bifurcation and Hopf Bifurcation. Refer to Strogatz (1994) for a more in depth explanation of more bifurcations.

8. Poincare-Bendixson Theorem

The broad implications of this theorem are that all trajectories in a two dimensional continuous time system must converge to either a fixed point or a limit cycle. So chaotic behavior (strange attractors) can only occur in systems with more than two dimensions or discrete time systems.

More specifically, the theorem says that if there exists a closed and bounded region R, often called a trapping region, such that R contains no fixed points, then all trajectories inside R must either be a closed orbit or spiral toward one.

9. Logistic Map

This is a population model first used by Pierre Francois Verhulst. The biologist Robert May was the first one to shown the applications to Dynamical Systems and Chaos Theory in his 1976. The logistic map is very simple in that there is only one equation in the dynamics. Yet this single nonlinear dynamic equation can give rise to very chaotic behavior.

The logistic map can be written as:

$$x_{t+1} = rx_t(1 - x_t)$$

where:

$$x_t = \text{population in year } t \text{ (between 0 and 1)}$$

$$x_0 = \text{population in year 0 (between 0 and 1)}$$

$$r = \text{rate of starvation and reproduction (positive number)}$$

The fate of the population is quite dependent on r:

$0 \leqslant r \leqslant 1$	Population will die regardless of initial population ($x_t \to 0 \forall x_0$)
$1 < r \leqslant 2$	Population will quickly converge to $\frac{r-1}{r}$
$2 < r \leqslant 3$	Population will converge to $\frac{r-1}{r}$, but at a slower rate
$3 \leqslant r \leqslant 1+\sqrt{6}$	Population will oscillate between two values (dependent on r but not x)

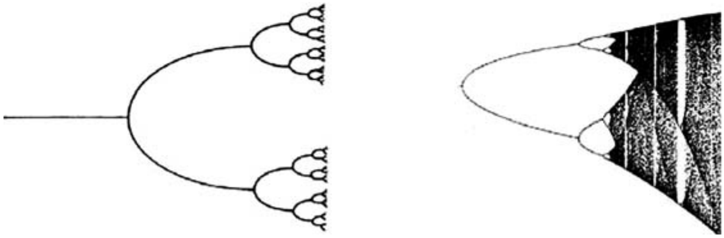


Figure 29. The Logistic Map (Stewart, 1989).

As the ranges increase the number of values x_t oscillates between, keeps doubling. It starts with two values and goes to 4 then 8 and 16, and so on. Until about $r \geqslant 3.57$ where the logistic map become chaotic. In this region, a slight difference in the initial condition can cause drastically different trajectories. Then at $r \geqslant 4$, x_t will diverge for almost all initial values.[4]

The logistic map is thus sensitive to the initial conditions for some values of r and not sensitive for others. The logistic map is an example of a dynamical system that goes through bifurcations multiple times. One of the amazing things to note is that such an interesting example appears as a one dimensional problem. This is because it is an iterative map (as opposed to a system of differential equations), which can display much more complex behavior with less dimensions.

10. Routes To Chaos

Most systems for some parameter values have nonchaotic behavior, that is limit cycles, closed orbits, and fixed points. Then as a parameter is changed, the system becomes chaotic. The way in which it becomes chaotic is referred to as a “route to chaos.” There are various routes to chaos that a system may take. In fact, for different parameter values, the system make take different routes. The important thing to realize is that systems become chaotic in similar ways, that is, there are classes called “routes to chaos.” The study of chaos theory is still young and quite open to research, so there still may be many routes yet to be discovered but nonetheless systems can be grouped by how they become chaotic. For further reading on the routes to chaos, refer to Hilborn (1994).

Hilborn (1994) describes *period doubling* as a route to chaos, in which a system has a limit cycle that becomes unstable as a parameter changes. Then, at some point the period doubles, then it doubles again, and keeps doubling

[4] Figure 29 taken from Stewart (1989) should bring clarity to this chaotic system.

until the period becomes infinite so that the trajectory never makes a second trip. At this point the system is chaotic.

Hilborn (1994) describes *intermittency* as a route to chaos when a system has periodic behavior with irregular chaotic (non periodic) bursts of behavior. Then, as a parameter is changed, the bursts of irregular behavior become longer and more frequent, until the behavior is completely irregular at which time it is said to be chaotic.

These are just two ways in which a system may become chaotic. For a more rigorous description of these routes to chaos as well as others, refer to Hilborn (1994) or any other text in nonlinear dynamics.

11. Concluding Remarks

After reading this chapter, the reader should certainly be convinced that networks are all around us. The topology of these networks can certainly effect the network's vulnerability as well as its capacity to transport commodities. While investigating any objective over a network, it is important to consider the network's topology.

The study of a network's topology and its effects is a study in its own. However, topology can also effect the dynamics of systems. Dynamical Systems and the theory of chaos has certainly gained much attention in recent years. Now, there may be some systems governed by dynamics such as epidemics that will certainly be effected by the topology of the network that it is acting on.

References

A. L. Barabási. Linked: The New Science of Networks, Perseus, Cambridge, MA, 2002.

R. Albert, H. Jeong, and A.-L. Barabási. Attack and error tolerance of complex networks. Nature, 406 (2000), pp. 378–382.

J. Balthrop, S. Forrest, M. Newman, and M. Williamson. Technological networks and the spread of computer viruses. Science, 304, (2004) pp. 527–529.

M. Beckman, C. B. McGuire, and C. B. Winsten. Studies in the Economics of Transportation. Yale University Press, 1956.

N. L Biggs, E. K. Lloyd, and R. J. Wilson. *Graph Theory: 1736–1936*. Oxford, England. Clanderon Press. 1976.

S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D. U. Hwang. Complex networks: Structure and dynamics. Phys Rep 424 (2006) pp. 175–308.

P. Erdős and A. Rényi. *On random graphs*, Publicationes Mathematicae Debrecen, 6 (1959), pp. 290–297.

P. Erdős and A. Rényi. *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl., 5 (1960), pp. 17–61.

P. Erdős and A. Rényi. *On the strength of connectedness of a random graph*, Acta Math. Acad. Sci. Hungar., 12 (1961), pp. 261–267.

S. Friedberg, A. Insel, L. Spence. Linear Algebra, Prentice Hall, Englewood Cliffs, NJ. 2003.

J. Guare. Six Degrees of Separation: A Play, Vintage, New York, 1990.

R. C. Hilborn. Chaos and Nonlinear Dynamics Oxford University Press, New York, NY. 1994.

D. W. Jordan and P. Smith. Nonlinear Ordinary Differential Equations, Oxford University Press, New York, NY, 1999.

F. Karinthy. Chains, in Everything is Different, Budapest, 1929.

S. Milgram. The small world problem, Psych. Today, 2 (1967), pp. 60–67.

M. E. J. Newman. The Structure and Function of Complex Networks, Society for Industrial and Applied Mathematics, 45 (2003), pp. 167–256.

M. E. J. Newman. Spread of epidemic disease on networks, Phys. Rev. E, 66 (2002).

P. Nijkamp and A. Reggiani. *Interaction, evolution and chaos in space*. Springer-Verlag, Berlin, 1992.

A. C. Pigou. *The economics of welfare*. Macmillan, 1920.

G. Perakis. The price of anarchy when costs are non-separable and asymmetric. In Proceedings of the 10th Conference on Integer Programming and Combinatorial Optimization (IPCO), volume 3064 of Lecture Notes in Computer Science, pages 46–58, 2004.

T. Roughgarden. É. Tardos, How bad is selfish routing? J. ACM 49 (2) (2002) pp. 236–259.

T. Roughgarden (2002), Selfish Routing, PhD dissertation, Cornell University.

I. Stewart. Does God Play Dice?, Basil Blackwell Ltd, Malden, MA, 1989.

S. H. Strogatz. Nonlinear Dynamics and Chaos, Addison–Wesley, Reading, MA, 1994.

H. P. Thadakamalla, U. N. Raghavan, S. Kumara, and R. Albert. Survivability of multiagent-based supply networks: A topological perspective. IEEE Intell. Syst., (2004), 19(5), 24–31.

D. J. Watts and S. H. Strogatz. Collective dynamics of “small-world” networks, Nature, 393 (1998), pp. 440–442.

D. J. Watts. Proc. Natl. Acad. Sci. USA 99 (2002) 5766.

Bureua of Transportation Statistics: National Transportation Statistics. Online. October 2006 http://www.bts.gov/publications/national_transportation_statistics/

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