2 SUPPLY CHAIN GAMES: MODELING IN A STATIC FRAMEWORK

A supply chain can be defined as “a system of suppliers, manufacturers, distributors, retailers, and consumers where materials flow downstream from suppliers to customers and information flows in both directions” (Geneshan et. al. 1998). The system is typically decentralized which implies that its participants are independent firms each with its own frequently conflicting goals spanning production, service, purchasing, inventory, transportation, marketing and other such functions. Due to these conflicting goals a decentralized supply chain is generally much less efficient than the corresponding centralized or integrated chain with a single decision maker. Efficiency suffers from both vertical (e.g., buyer-vendor competition) and horizontal (e.g., a number of vendors competing for the same buyer) conflicts of interest.

How to manage competition in supply chains is a challenging task which comprises a variety of problems. The overall target is to make, to the extent possible, the decentralized chain operate as efficiently as its benchmark, the corresponding centralized chain. This particular aspect of supply chain management is referred to as coordination. This chapter addresses simple static supply chain models, competition between supply chain members and their coordination.

2.1 STATIC GAMES IN SUPPLY CHAINS

In research and management literature where supply chain problems and related game theoretic applications have gained much attention in recent years, we see extensive reviews focusing on such aspects as taxonomy of supply chain management (Geneshan et. al. 1998); integrated inventory models (Goyal and Gupta 1989); game theory in supply chains (Cachon and Netessine 2004); operations management (Li and Whang 2001); price quantity discounts (Wilcox et. al. 1987); and competition and coordination (Leng and Parlar 2005).

In the literature, supply chains are distinguished by various features such as: types of decisions; operations; competition and coordination; incentives; objectives; and game theoretic concepts. In this chapter we deal with three essential features of static supply chains, i.e., the supply chains with decisions independent of time: customer demand, competition and risk. In this sense we distinguish between

- deterministic and random demands; endogenous and exogenous demands
- vertical and horizontal competition within supply chains
- no risk involved, risk incurred by only one of the parties and risk shared between the parties.

In this chapter, supply chain games are combined into three groups. The first group of games represents classical horizontal production and vertical pricing competition under endogenous demands. These games involve decisions about either product prices or quantities with respect to two types of endogenous demands: (i) the quantity demanded for a product as a function of price set for the product and (ii) an inverse demand function with price as a function of the quantity produced or sold. In both cases the demands are deterministic, which implies that all produced/supplied products are sold and thus there is no risk involved.

Random exogenous demand for products characterizes the second group of games which is related to the classical newsvendor problem. The parties vertically compete by deciding on a price to offer and a quantity to order for a particular price. Since the demand is uncertain, the downstream party, which faces the demand, runs the risk of overestimating or underestimating it. The risk involves costs incurred due to choosing the quantity to order and stock before customer demand is realized. We refer to this group of games as stocking / pricing competition with random demand.

The third group of games represents classical risk-sharing interactions between supply chain members. Similar to the second group, the competition is vertical and the demand is exogenous and random. Unlike the second group, however, incentives to mitigate risk may be offered to a party which faces uncertain customer demands. Since the incentives include buyback and urgent purchase options, some of the uncertainty is transferred from one party to another. In such a case, the risk associated with random demand is shared and the inventories of all involved parties are affected when deciding on what quantities to stock.

Motivation

We describe a few production, pricing and inventory-stock related problems which have been found in various service and industry-related supply chains. Most of these problems have been extensively studied and can be found virtually in every survey devoted to supply chain management including those mentioned above. It is worth noting that, in general, the number of basic supply chain problems is significant and selecting just a few of them for an introductory purpose is not a simple matter.

Our selection criterion is based on one of the overall goals of this book– to show how optimal pricing and inventory policies evolve when static operation conditions become dynamic. Under such conditions, we find particularly interesting the static problems which allow for straightforward and, yet natural, dynamic extensions. The problems which we discuss in this chapter will be discussed again in the following chapters to show the effect of production and service dynamics on managerial decisions.

The static feature of the problems we select implies that the period of time that the problems encompass is such that no change in system parameters is observed. Since all products are delivered at once by the end of the period and then instantly sold, these problems ignore the intermediate inventories (and associated costs) before and during the selling season. Due to the focus on *stock and pricing policies*, shortages as well as leftovers are avoided, as much as possible, by the end of the period. In all the problems that we consider, it is assumed that the information needed for decision-making is available and transparent to the supply chain participants and that the overall order lead-time is smaller than the length of the period so that all deliveries are provided on time.

This chapter introduces and discusses basic models of horizontal and vertical competition between supply chain members, the effect of uncertainty and risk sharing as well as basic tools for coping with the competition by coordinating supply chains. The analysis which we employ includes (i) formal statements of problems of each non-cooperative party involved as well as the corresponding centralized formulations where only one decision-maker is responsible for all managerial decisions in the supply chain; (ii) system-wide optimal and equilibria solution for competing parties; (iii) analysis of the effect of competition on supply chain performance and of coordination for improving the performance. In analyzing the problems we use Nash and Stackelberg equilibria which we briefly present next.

Nash and Stackelberg equilibria

Game theory is concerned with situations involving conflicts and cooperation between the players. Our focus is on two important concepts of Nash and Stackelberg equilibria intended respectively for dealing with simultaneous and sequential non-cooperating decision-making by multiple players. Consider a game, with the strategies y_i, i=1,..,N being feasible actions which the N players may undertake. All possible strategies of a player, i, form a strategy set Y_i of the player. A payoff (objective function), $J_i(y_1, y_2,..,y_N,)$, i=1,..,N is evaluated when each player i selects a feasible strategy, $y_i \in Y_i$. We assume that the games are played on the basis that complete information is available to all players. Since two-player games can be straightforwardly extended to multiple players and to simplify the presentation, we further assume that there are only two players A and B.

Each player's goal is to maximize his own payoff. The following definition presents the concept of a Nash equilibrium (Nash 1950)

Definition 2.1

A pair of strategies (y_A*, y_B*) *is said to constitute a Nash equilibrium if the following pair of inequalities is satisfied for all* $y_A \in Y_A$, *and* $y_B \in Y_A$

$$J_A(y_A*, y_B*) \geq J_A(y_A, y_B*) \text{ and } J_B(y_A*, y_B*) \geq J_B(y_A*, y_B).$$

The definition implies that the Nash solution is

$$y_A* = \arg\max_{y_A \in Y_A}\{J_A(y_A, y_B*)\} \text{ and } y_B* = \arg\max_{y_B \in Y_B}\{J_B(y_A*, y_B)\},$$

and a unilateral deviation from this solution results in a loss. If this problem is static, strategy sets are not constrained and the payoff functions are continuously differentiable. The first-order (necessary) optimality condition results in the following system of two equations in two unknowns y_A*, y_B*:

$$\frac{\partial J_A(y_A, y_B*)}{\partial y_A}\bigg|_{y_A=y_A*} = 0 \text{ and } \frac{\partial J_B(y_A*, y_B)}{\partial y_B}\bigg|_{y_B=y_B*} = 0.$$

In addition, the second order (sufficient) optimality condition which ensures that we maximize the payoffs is

$$\frac{\partial^2 J_A(y_A, y_B*)}{\partial y_A^2}\bigg|_{y_A=y_A*} < 0 \text{ and } \frac{\partial^2 J_B(y_A*, y_B)}{\partial y_B^2}\bigg|_{y_B=y_B*} < 0.$$

Equivalently, one may determine $y_A^R(y_B) = \arg\max_{y_A \in Y_A}\{J_A(y_A, y_B)\}$ for each $y_B \in Y_B$ to find the best response function, $y_A = y_A^R(y_B)$, of player A and of player B, $y_B = y_B^R(y_A)$ which constitute a system of two equations in two unknowns.

The examples we shall consider here will be elaborated later in this and subsequent chapters.

Example 2.1

Consider a supply chain consisting of one supplier, s, and one retailer r. The supplier offers products at wholesale price w and the retailer buys q product units and sets retail price $p=w+m$. This is the classical pricing game where the two firms want to maximize their profits. Let the supplier and retailer costs be negligible and the demand is linear and downward in price, $d=a-bp=a-b(w+m)$, $a>0$, $b>0$. Then the retailer's optimization problem is

$$J_r(m,w)= m(a-b(w+m)) \to \max,$$
$$0 \le m \le \frac{a}{b} - w$$

and the suppliers problem is

$$J_s(m,w)=w(a-b(w+m)) \to \max,$$
$$w \ge 0.$$

First we observe that both objective functions are strictly concave in their decision variables. Thus, the first-order optimality condition is necessary and sufficient. Using the first-order optimality condition we have

$$a-bw-2bm=0 \text{ and } a-2bw-bm=0.$$

If our constraints are not binding, the two best response functions are

$$m=m^R(w)=\frac{a-bw}{2b} \text{ and } w= w^R(m)=\frac{a-bm}{2b}.$$

Solving these two equations (or equivalently the previous two) we find a unique Nash equilibrium

$$m^n=\frac{a}{3b} \text{ and } w^n=\frac{a}{3b}.$$

The equilibrium is evidently feasible and all constraints are met, as $\frac{a}{3b}>0$, hence, $m^*>0$, $w^*>0$, and $\frac{a}{3b} < \frac{a}{b} - w^n = \frac{2a}{3b}$, hence, $m^n < \frac{a}{b} - w^n$.

Stackelberg strategy is applied when there is an asymmetry in power or in moves of the players. As a result, the decision-making is sequential rather than simultaneous as is the case with Nash strategy. The player who first announces his strategy is considered to be the Stackelberg leader. The follower then chooses his best response to the leader's move. The leader thus has an advantage because he is able to optimize his objective function subject to the follower's best response. Formally this implies that if, player A, for example, is the leader, then $y_B = y_B^R(y_A)$ is the same best response for player B as determined for the Nash equilibrium. Since the leader is aware of this response, he then optimizes his objective function subject to $y_A = y_A^R(y_B) = y_A^R(y_B^R(y_A))$.

Definition 2.2

In a two-person game with player A as the leader and player B as the follower, the strategy $y_A^* \in Y_A$ is called a Stackelberg equilibrium for the leader if, for all y_A,

$$J_A(y_A^*, y_B^R(y_A^*)) \geq J_A(y_A, y_B^R(y_A)),$$

where $y_B = y_B^R(y_A)$ is the best response function of the follower.

Definition 2.2 implies that the leader's Stackelberg solution is

$$y_A^* = \arg\max_{y_A \in Y_A}\{J_A(y_A, y_B^R(y_A))\}.$$

That is, if the strategy sets are unconstrained and the payoff functions are continuously differentiable, the necessary optimality condition for the leader is

$$\frac{\partial J_A(y_A, y_B^R(y_A))}{\partial y_A}\Big|_{y_A=y_A^*} = 0.$$

To make sure that the leader maximizes his profits, we check also the second-order sufficient optimality condition

$$\frac{\partial^2 J_A(y_A, y_B^R(y_A))}{\partial y_A^{\,2}}\Big|_{y_A=y_A^*} < 0.$$

Example 2.2

Consider again Example 2.1 but assume that the supplier is the leader. That is, the supplier sets first his wholesale price. In response, the retailer, in setting his retail price, determines the product quantity he orders. Then, to find the Stackelberg solution, we substitute the best retailer's response $m = m^R(w) = \frac{a - bw}{2b}$ (see Example 2.1) into the supplier's objective function.

$$\max_w J_s(m,w) = \max_w w(a - b(w + \frac{a - bw}{2b})) = \max_w (\frac{aw}{2} - \frac{bw^2}{2}).$$

The supplier's objective function is evidently strictly concave. Consequently, the first-order optimality condition results in

$$w^s = \frac{a}{2b}, \; m^s = m^R(w^s) = \frac{a}{4b}.$$

The found equilibrium is evidently unique and feasible, as $\frac{a}{2b}>0$, $\frac{a}{4b}>0$ and $\frac{a}{b} - w^s = \frac{a}{2b}$ and, thus, $m^s = \frac{a}{4b} < \frac{a}{b} - w^s = \frac{a}{2b}$, i.e., all constraints are met.

For comparative reasons we shall also consider a centralized supply chain with no competition (game) involved. The centralized problem can be viewed as a single-player game.

Example 2.3

Consider again Example 2.1 but assume that there is only one decision-maker in the system. Then the centralized objective function is

$$\max_{m,w} J(m,w) = \max_{m,w} [\, J_r(m,w) + J_s(m,w)] = \max_{m,w} (w+m)(a-b(w+m)).$$

Applying the first-order optimality condition we get two identical equations for m and n. This implies that there is only one decision variable p, so that the system-wide optimal solution is, $m^*+w^* = p^* = \frac{a}{2b}$.

2.2 PRODUCTION/PRICING COMPETITION

We discuss here two classical problems arising in supply chains characterized by deterministic demands and either vertical supplier-retailer or horizontal supplier-supplier competition. The competition is represented by games. We first analyze pricing equilibrium based on Bertrand's competition model and then production equilibrium according to Cournot's competition model. Since the problems are deterministic, they can be viewed as both single-period and continuous review models.

2.2.1 THE PRICING GAME

Consider a two-echelon supply chain consisting of a single supplier selling a product type to a single retailer over a period of time. The supplier has ample capacity and the period is longer than the supplier's leadtime which implies that the supplier is able to deliver on time any quantity q ordered by the retailer. The retailer faces a concave endogenous demand, $q=q(p)$, which decreases as product price p increases, i.e., $\frac{\partial q}{\partial p}<0$ and $\frac{\partial^2 q(p)}{\partial p^2}\leq 0$. The supplier incurs unit production cost c and sells at unit wholesale price w, i.e., the supplier's margin is w-c. Note that this formulation is an extension of that employed in Example 2.1, where a specific, linear in price, demand was considered.

Let the retailer's price per unit be $p=w+m$, where m is the retailer's margin. Both players, the supplier and the retailer, want to maximize their profits – margin times demand which are expressed as $J_s(w)=(w-c)q(w+m)$ and $J_r(p)=mq(w+m)$ respectively (see Figure 2.1). This leads us to the following problems.

The supplier's problem

$$\max_{w} J_s(w,m)=\max_{w}\,(w-c)q(w+m) \tag{2.1}$$

s.t.

$$w\geq c. \tag{2.2}$$

The retailer's problem

$$\max_{m} J_r(w,m)=\max_{m}\, mq(w+m) \tag{2.3}$$

s.t.

$$m\geq 0, \tag{2.4}$$

$$q(w+m)\geq 0. \tag{2.5}$$

Note that from $w\geq c$ and $m\geq 0$, it immediately follows that $p=w+m\geq c$. In contrast to the vertical competition between the two decision-makers as determined by (2.1)-(2-5), the supply chain may be vertically integrated or centralized. Such a chain is characterized by a single decision-maker who is in charge of all managerial aspects of the supply chain. We then have the following single problem as a benchmark.

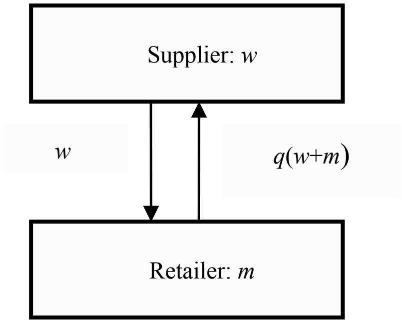

Figure 2.1. Vertical pricing competition

The centralized problem

$$\max_{m,w} J(m,w) = \max_{m,w} [\, J_r(m,w) + J_s(m,w)] = \max_{m,w} (w+m-c)q(w+m) \qquad (2.6)$$

s.t.

$$m \geq 0,\ q(w+m) \geq 0.$$

To distinguish between different optimal strategies, we will use below superscript n for Nash solutions, s for Stackelberg solutions and * for centralized solutions.

System-wide optimal solution

We first study the centralized problem by employing the first-order optimality conditions

$$\frac{\partial J(m,w)}{\partial m} = q(w+m) + (w+m-c)\frac{\partial q(p)}{\partial p} = 0,$$

$$\frac{\partial J(m,w)}{\partial w} = q(w+m) + (w+m-c)\frac{\partial q(p)}{\partial p} = 0.$$

Since both equations are identical, only the optimal price matters in the centralized problem, p^*, while the wholesale price $w \geq 0$ and the retailer's margin $m \geq 0$ can be chosen arbitrarily so that $p^*=w+m$. This is because w and m represent internal transfers of the supply chain. Thus, the proper notation for the payoff function is $J(p)$ rather than $J(m,w)$ and the only optimality condition is

$$q(p^*) + (p^* - c)\frac{\partial q(p^*)}{\partial p} = 0. \qquad (2.7)$$

Let $q(P)=0$, $P>c$. Then it is easy to verify that,

$$\frac{\partial^2 J(p)}{\partial p^2} = \frac{\partial q(p)}{\partial p} + \frac{\partial q(p)}{\partial p} + (p-c)\frac{\partial^2 q(p)}{\partial p^2} < 0,$$

that is, the centralized objective function (2.6) is strictly concave in price for $p \in [c, P]$. This implies that equation (2.7) has a unique solution which maximizes (2.6).

Game Analysis

We consider now a decentralized supply chain characterized by non-cooperative or competing firms and assume first that both players make their decisions simultaneously. The supplier chooses the wholesale price w and the retailer selects his price, p, or equivalently his margin, m, and hence buys $q(p)$ products. The supplier then delivers the products. Since this pricing game is deterministic, all products that the retailer buys will be sold.

Using the first-order optimality conditions for the retailer's problem, we find that the retailer's best response is determined by the following expression

$$\frac{\partial J_r(m,w)}{\partial m} = q(w+m) + m\frac{\partial q(p)}{\partial p} = 0\,. \tag{2.8}$$

It is easy to verify that the retailer's objective function is strictly concave in m and, thus, (2.8) has a unique solution, or, in other words, the retailer's best response function is unique. Comparing (2.8) and (2.7) and taking into account that $w>c$ (otherwise the supplier has no profit), we conclude with the following result:

Proposition 2.1. *In vertical competition of the pricing game, if the supplier makes a profit, i.e., $w>c$, the retail price will be greater and the retailer's order less than the system-wide optimal (centralized) price and order quantity respectively.*

Proof: Substituting $p = w+m$ into (28) we have

$$q(p) + (p-w)\frac{\partial q(p)}{\partial p} = 0\,. \tag{2.9}$$

Comparing (2.7) and (2.9) we observe that

$$q(p) + (p-w)\frac{\partial q(p)}{\partial p} = q(p^*) + (p^* - c)\frac{\partial q(p^*)}{\partial p} = 0, \tag{2.10}$$

while taking into account that $w>c$ and $\frac{\partial q}{\partial p} < 0$,

$$q(p^*)+(p^*-w)\frac{\partial q(p^*)}{\partial p} > q(p^*)+(p^*-c)\frac{\partial q(p^*)}{\partial p}=0. \quad (2.11)$$

Next, by denoting $f(p)=q(p)+(p-w)\frac{\partial q(p)}{\partial p}$, and recalling $\frac{\partial q}{\partial p}<0$ and $\frac{\partial^2 q(p)}{\partial p^2}\leq 0$, we find that

$$\frac{\partial f(p)}{\partial p}=\frac{\partial q(p)}{\partial p}+\frac{\partial q(p)}{\partial p}+(p-w)\frac{\partial^2 q(p)}{\partial p^2}<0$$

Thus, to have (2.10) we need $f(p)<f(p^*)$, which, with respect to the last inequality, requires, $p>p^*$ and, hence, $q(p)<q(p^*)$, as stated in Proposition 1.

Note, that our conclusion that vertical pricing competition (2.1)-(2.5) increases retail price and decreases the retailer's order quantity does not depend on whether both players make a simultaneous decision or whether the supplier first sets the wholesale price and plays the role of the Stackelberg leader, as is often the case in practice. In either of the two cases, the overall efficiency of the supply chain deteriorates under vertical competition.

Equilibrium

To determine the Nash pricing equilibrium, which corresponds to simultaneous moves of the supplier and retailer, we next consider the optimality conditions for the supplier's objective function,

$$\frac{\partial J_s(m,w)}{\partial w}=q(w+m)+(w-c)\frac{\partial q(w+m)}{\partial p}=0\,. \quad (2.12)$$

One can readily verify that the supplier's objective function is strictly concave in w, $\frac{\partial^2 J_s(m,w)}{\partial w^2}<0$ and, thus, the supplier's best response (2.12) is unique as well. As a result, the Nash equilibrium, (w^n,m^n) is found by solving simultaneously the following system of equations

$$q(w+m)+m\frac{\partial q(w+m)}{\partial p}=0\,, \quad (2.13)$$

$$q(w+m)+(w-c)\frac{\partial q(w+m)}{\partial p}=0\,. \quad (2.14)$$

Solving (2.13) and (2.14) results in

$$w\text{-}c\text{-}m=0 \text{ and } q(c+2m)+m\frac{\partial q(c+2m)}{\partial p}=0\,.$$

Assuming that the solution $w+m=P$, $q(P)=0$ cannot be optimal since it leads to zero profit for all supply chain members, we conclude with the following result.

Proposition 2.2. *The pair (w^n,m^n), where m^n satisfies the following equation*

$$q(c+2m^n)+m^n\frac{\partial q(c+2m^n)}{\partial p}=0. \tag{2.15}$$

and $w^n=m^n+c$ constitutes a unique Nash equilibrium of the pricing game with $0<m^n<(P-c)/2$.

Proof: To see that a solution of equation (2.15) always exists and that it is unique, assume $m^n=0$. Then, since $P>c$ and $q(P)=0$, $q(c+2m^n)>0$, while the second term in (2.15) is zero. Thus, $f(m^n)=q(m^n)+m^n\frac{\partial q(m^n)}{\partial p}>0$ when $m^n=0$. On the other hand, let $c+2m^n=P$, since $q(P)=0$, while the second term in (2.15) is strictly negative as $m^n=(P-c)/2>0$, we have $f(m^n)=q(m^n)+m^n\frac{\partial q(m^n)}{\partial p}<0$. Finally, taking into account that $\frac{\partial f(m^n)}{\partial m^n}<0$, we conclude that the solution of $f(m^n)=0$ is unique and $0<m^n<(P-c)/2$.

Next, we assume that the supplier makes the first move by setting the wholesale price. The retailer then decides on what price to set and, hence, the quantity to order. To find the Stackelberg equilibrium, we need to maximize the supplier's objective with m subject to the best retailer's response $m=m^R(w)$ determined by (2.8),

$$J_s(m,w)=(w-c)q(w+m^R(w)).$$

Differentiating the supplier's objective function we have

$$\frac{\partial J_s(m,w)}{\partial w}=q(w+m^R(w))+(w-c)\frac{\partial q(w+m)}{\partial p}\frac{\partial m^R(w)}{\partial w}=0,$$

where $\frac{\partial m^R(w)}{\partial w}$ is determined by differentiating (2.8) with m set equal to $m^R(w)$.

$$\frac{\partial q(w+m)}{\partial p}(1+\frac{\partial m^R(w)}{\partial w})+\frac{\partial m^R(w)}{\partial w}\frac{\partial q(p)}{\partial p}+m\frac{\partial^2 q(p)}{\partial p^2}(1+\frac{\partial m^R(w)}{\partial w})=0.$$

Thus

$$\frac{\partial m^R(w)}{\partial w}=-\left(\frac{\partial q(w+m)}{\partial p}+m\frac{\partial^2 q(w+m)}{\partial p^2}\right)\Bigg/\left(\frac{\partial q(w+m)}{\partial p}+\frac{\partial q(w+m)}{\partial p}+m\frac{\partial^2 q(w+m)}{\partial p^2}\right). \tag{2.16}$$

Equation (2.16) naturally implies
the greater the supplier's wholesale price w, the lower the retailer's margin m.

Based on (2.16) and (2.8) we conclude that a pair (w^s,m^s) constitutes a Stackelberg equilibrium of the pricing game if there exists a joint solution in w and m of the following equations

$$q(w+m)+(w-c)\frac{\partial q(w+m)}{\partial p}\frac{\partial m}{\partial w}=0\,,$$

$$q(w+m)+m\frac{\partial q(w+m)}{\partial p}=0\,,$$

where

$$\frac{\partial m}{\partial w}=-\left(\frac{\partial q(w+m)}{\partial p}+m\frac{\partial^2 q(w+m)}{\partial p^2}\right)\Bigg/\left(\frac{\partial q(w+m)}{\partial p}+\frac{\partial q(w+m)}{\partial p}+m\frac{\partial^2 q(w+m)}{\partial p^2}\right)$$

We do not study here the existence and uniqueness of the Stackelberg solution. Instead we revisit Examples 2.1 and 2.2, which determine both Stackelberg and Nash solutions for a special case of the pricing game.

Example 2.4

Let the demand be linear in price, $q(p)=a-bp$ and the supplier's cost negligible, $c=0$. Thus we obtain the problem solved in Example 2.1. Note that the demand requirements, $\frac{\partial q}{\partial p}=-b<0$ and $\frac{\partial^2 q}{\partial p^2}\le 0$ are met for the selected function. Using Proposition 2.2. we solve (2.15),

$$q(2m^n)+m^n\frac{\partial q(2m^n)}{\partial p}=a-b2m^n+m^n(-b)=0,\ w^n=m^n$$

to find Nash equilibrium $w^n=m^n=\frac{a}{3b}$, hence, $p^n=w^n+m^n=\frac{2a}{3b}$ and $q(p^n)=\frac{a}{3}$, as is also the case in Example 2.1. The payoff for the equilibrium is identical for both players, $J_r(m^n,w^n)=J_s(m^n,w^n)=\frac{a^2}{9b}$. Similarly, one can verify that the Stackelberg solution is the same as in Example 2.2,

$$w^s=\frac{a}{2b},\ m^s=\frac{a}{4b},\ p^s=w^s+m^s=\frac{3a}{4b},\ q(p^s)=\frac{a}{4},$$

$$J_s(m^s,w^s)=\frac{a^2}{8b}\ \text{and}\ J_r(m^s,w^s)=\frac{a^2}{16b}.$$

Finally, the centralized solution (2.7) (see also Example 2.3) is

$$q(p^*) + (p^* - c)\frac{\partial q(p^*)}{\partial p} = a - bp^* + p^*(-b) = 0,$$

that is,

$$m^* + w^* = p^* = \frac{a}{2b},\ q(p^*) = \frac{a}{2} \text{ and } J(p^*) = \frac{a^2}{4b}.$$

Comparing these results we find that the system-wide optimal order is greater than that of the Nash or Stackelberg strategy

$$q(p^s) = \frac{a}{4} < q(p^n) = \frac{a}{3} < q(p^*) = \frac{a}{2},$$

which agrees with Proposition 2.1. Correspondingly, the retail prices increase under vertical competition

$$p^s = \frac{3a}{4b} > p^n = \frac{2a}{3b} > p^* = \frac{a}{2b}.$$

and the overall chain payoff deteriorates

$$J_s(m^s,w^s) + J_r(m^s,w^s) = \frac{3a^2}{16b} < J_r(m^n,w^n) + J_s(m^n,w^n) = \frac{2a^2}{9b} < J(p^*) = \frac{a^2}{4b}.$$

Example 2.5

The goal of this example is twofold. First of all, it is rarely possible to find an equilibrium analytically. This example illustrates how to conduct the analysis numerically with Maple. Secondly, the condition imposed on the second derivative of demand is sufficient for the equilibrium to be unique, but it is not necessary, as the example demonstrates.

Let the demand be non-liner in price, $q(p)=a-bp^{\alpha}$. Assuming that $0<\alpha<1$, we observe that the demand requirements with respect to the first derivative are met, $\frac{\partial q}{\partial p} = -b\alpha p^{\alpha-1} < 0$, while with respect to the second $\frac{\partial^2 q}{\partial p^2} = b\alpha(1-\alpha)p^{\alpha-2} > 0$ is not. Using Proposition 2.2., we employ (2.13) and (2.14) to obtain numerically the retailer's and supplier's best response respectively, $m=m^R(w)$ and $w=w^R(m)$. Specifically, we first set the left-hand side of (2.13) as L1

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>L1:=a-b*(w+m)^alpha-m*alpha*(w+m)^(alpha-1);
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$$L1 := a - b\,(w+m)^{\alpha} - m\,\alpha\,(w+m)^{(\alpha-1)}$$

and the left-hand side of (2.14) as L2.

```
> L2:=a-b*(w+m)^alpha-(w-c)*alpha*(w+m)^(alpha-1);
```

$$L2 := a - b\,(w+m)^{\alpha} - (w-c)\,\alpha\,(w+m)^{(\alpha-1)}$$

Next we substitute specific parameters of the example α=`0.5`, a=`15`, b=`2`, c=`1` to have numeric left-hand sides L11 and L12 respectively

```
>L11:=subs(alpha=0.5, a=15, b=2, c=1, L1);
```

$$L11 := 15 - 2\,(w+m)^{0.5} - \frac{0.5\,m}{(w+m)^{0.5}}$$

```
> L12:=subs(alpha=0.5, a=15, b=2, c=1, L2);
```

$$L12 := 15 - 2\,(w+m)^{0.5} - \frac{0.5\,(w-1)}{(w+m)^{0.5}}\,.$$

Next we find the equilibrium by solving the system of equations L11=0 and L12=0

```
>solve({L11=0, L12=0}, {m,w});
```

$$\{m = 21.83319513,\, w = 22.83319513\}$$

To verify that the equilibrium is unique, we find the best retailer's response $m^R(w)$ numerically as mR

```
> mR:=solve(L11=0,m);
```

$$mR := 18. + 1.200000000\,\sqrt{225. + 5.\,w} - 0.8000000000\,w,$$
$$18. - 1.200000000\,\sqrt{225. + 5.\,w} - 0.8000000000\,w$$

and the inverse function $mRinv$ of the best supplier's response $w^R(m)$

```
>mRinv:=solve(L12=0,m);
```

$$mRinv := 28.37500000 + 1.875000000\,\sqrt{229. - 4.\,w} - 1.250000000\,w,$$
$$28.37500000 - 1.875000000\,\sqrt{229. - 4.\,w} - 1.250000000\,w$$

Both responses have two solutions, positive and negative. Since the margin is non-negative, we select only positive solutions $mR[1]$ and $mRinv[2]$ and plot them on the same graph.

```
>plot([mR[1],mRinv[1]],w=1..45,legend=["Retailer",
"Supplier"]);
```

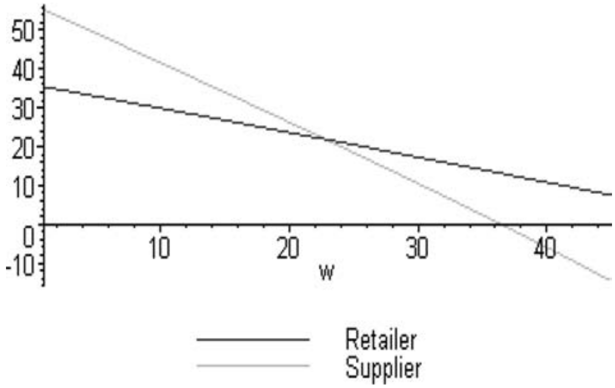

Figure 2.2. The pricing equilibrium

From Figure 2.2 we observe that there is only one point where the responses intersect. This is the Nash equilibrium point which we found numerically as m^n =21.833 and w^n =22.833.

The centralized solution (2.7) is found similarly with Maple

```
> L:=a-b*p^alpha-(p-c)*alpha*p^(alpha-1);
```

$$L := a - b\,p^{\alpha} - (p-c)\,\alpha\,p^{(\alpha-1)}$$

```
> L11:=subs(alpha=0.5, a=15, b=2, c=1, L);
```

$$L11 := 15 - 2\,p^{0.5} - \frac{0.5\,(p-1)}{p^{0.5}}$$

```
> popt:=solve(L11=0,p);
```

$$popt := 36.39890107$$

Comparing the system-wide optimal price with the equilibrium Nash price, we find that $p^*=36.398<p^n=m^n+w^n=21.833+22.833=44.666$.

Coordination

According to Proposition 2.1, vertical competition has a negative effect on the supply chain. The retailer orders less, the retail price goes up and profits shrink. Moreover, although the supplier's leadership allows the supplier to increase his profit, in the specific case of linear price demand (see Example 2.4), the leadership is also destructive as it further reduces the total profit in the supply chain. The negative effect of the vertical competition is due to the well-known double marginalization effect. This effect takes place if the retailer ignores the supplier's profit margin, w-c, when ordering as shown in Proposition 2.1. Specifically, when recalling that $p=w+m$, the retailer's best response (2.9)

$$q(p)+(p-w)\frac{\partial q(p)}{\partial p}=0,$$

can be written as

$$q(p)+m\frac{\partial q(p)}{\partial p}=0,$$

which implies that though the demand depends on price $p=w+m$, the retailer accounts only for his margin m instead of ordering as indicated by the centralized approach (2.7)

$$q(p)+(p-c)\frac{\partial q(p)}{\partial p}=q(p)+(w-c+m)\frac{\partial q(p)}{\partial p}=0$$

and thus adding the supplier's margin, w-c, to m. Equivalently, from equation (2.14)

$$q(p)+(w-c)\frac{\partial q(p)}{\partial p}=0$$

we observe that the supplier ignores the retailer's margin m when setting the wholesale price. The remaining question is how to induce the retailer to order more, or the supplier to reduce the wholesale price, i.e., how to coordinate the supply chain and thus increase its total profit. Of course, the supplier may set the wholesale price at his marginal cost, $w=c$, or the retailer may set his margin at zero. Equation (2.7) then becomes identical to (2.9) and the supply chain is perfectly coordinated. However, the supply chain member who gives up his margin gets no profit at all. The most popular way of dealing with such a problem is by discounting or by collaboration for profit sharing.

One approach to discounting is a simple two-part tariff. If the supplier is the leader, he can set $w=c$, but charge the retailer a fixed fee. In this way, the supplier can regulate his share in the total supply chain profit without a special contract. Moreover, if the supplier sets the fixed fee very close to the centralized supply chain profit, $J(p^*)$, then the retailer gets almost no profit and still orders the system-wide optimal quantity $q(p^*)$ as well as sets system-wide optimal price p^*.

Regardless of whether there is a leader or not, signing a profit-sharing contract is an alternative way to mitigate the double marginalization. In such a contact, the parties would explicitly set their shares of the total supply chain profit, $J(p^*)$ with η, $0\leq \eta \leq 1$, so that the retailer gets $\eta J(p^*)$ and the supplier $(1-\eta)J(p^*)$. This, however, is already cooperative rather than competitive behavior. To illustrate one possibility for coordination with cooperation, we briefly consider an example of bargaining over the wholesale price and retailer's margin in terms of the Nash bargain, which solves

$$\max_{m,w} [J_r(w,m)-j_r][J_s(w,m)-j_s],$$

where j_r and j_s represent the outside options to each party. Employing the demand function of this section and assuming that all outside options are normalized to zero, i.e., $j_r=0$ and $j_s=0$, we have the following bargaining problem:

$$\max_{m,w} J^B(m,w)= \max_{m,w} mw[q(w+m)]^2.$$

If $q(w+m)$ is such that $J^B(m,w)$ is concave, then applying the first-order optimality conditions we obtain the following two equations

$$q(m+w)+2m\frac{\partial q(m+w)}{\partial p}=0\,,$$

$$q(m+w)+2(w-c)\frac{\partial q(m+w)}{\partial p}=0\,.$$

From these equations we immediately find that $m=w-c$ and thereby the two equations result in a single condition:

$$q(m+w)+(m+w-c)\frac{\partial q(m+w)}{\partial p}=0\,.$$

Taking into account that $p=m+w$, we observe that the derived condition is identical to the system-wide optimality condition (2.7). Thus, if $J^B(m,w)$ is concave, the Nash bargain perfectly coordinates the supply chain for the case of the pricing game. The only difference is that the system-wide optimal solution specifies only the optimal price p^* (since the transfer costs are not important for a centralized system), while the Nash bargain solution of the pricing problem results in equal margins, $m=w-c$, and shares, $J_r(w,m)=J_s(w,m)$, for both parties.

The multi-echelon effect

It is intuitively clear that the greater the number of the upstream suppliers involved, the more margins are added to the supply chain and thereby the greater the deterioration of the expected system performance. Specifically, let an upstream distributor have a marginal cost c_d per product and let him sell his products to the supplier at a price w_d. Then the retail price would be $p= w+m$, $w\geq c+w_d$ and the resulting problems of the three-echelon supply chain are defined as follows.

The distributor's problem

$$\max_{w_d} J_d(w_d,w,m)= \max_{w_d} (w_d-c_d)q(w+m)$$

s.t.

$$w_d \geq c_d.$$

The supplier's problem

$$\max_{w} J_s(w_d, w, m) = \max_{w} (w\text{-}c\text{-}w_d) q(w+m)$$

s.t.

$$w \geq c + w_d.$$

The retailer's problem

$$\max_{m} J_r(w_d, w, m) = \max_{m} m q(w+m)$$

s.t.

$$m \geq 0,\ q(w+m) \geq 0.$$

The centralized problem

$$\max_{m,w} J(m,w) = \max_{m,w} (\ m + w - c - c_d) q(w+m)$$

s.t.

$$m \geq 0,\ q(w+m) \geq 0,\ w \geq c + w_d.$$

Consequently the system-wide optimal retail margin is determined by

$$\frac{\partial J(m,w)}{\partial m} = q(p) + (m + w - c - c_d)\frac{\partial q(p)}{\partial p} = 0,$$

while the equation for an optimal margin when the parties are non-cooperative remains the same

$$\frac{\partial J_r(m,w)}{\partial m} = q(p) + m\frac{\partial q(p)}{\partial p} = 0\ .$$

We thus observe that the retailer when ordering, accounts for his margin m and ignores both the supplier's margin $w\text{-}c\text{-}w_d$ and the distributor's margin $w_d\text{-}c_d$, which is, $w\text{-}c\text{-}c_d$ in total. Again, by employing the two-part tariff, the supply chain becomes perfectly coordinated. This is accomplished if the distributor and the supplier set the wholesale prices equal to their marginal costs, i.e., $w_d = c_d$ and $w = c + c_d$, respectively and charge a fixed cost per transaction.

2.2.2 THE PRODUCTION GAME

Previously we were concerned with vertical competition. Now we shall study the effect of horizontal production competition (see Figure 2.3). Consider two manufacturers producing the same or substitutable types of product over a period of time and thus competing horizontally for the same customers, possibly for the same retailer. Accordingly, the manufacturers are suppliers with ample capacity and the order period is longer than the suppliers' lead-time. This means that both suppliers are able to deliver on time any quantity q_1 and q_2 to the retailer. The retailer, on the other hand, adopts the so-called vendor managed inventory (VMI) policy, in which the suppliers decide on the quantities to deliver while the retailer simply charges a fixed percentage from sales. Since the retailer has no part in the competition, he does not affect the system-wide optimal solution, equilibrium order quantities, or prices.

Further, in the previous section we assumed that the retailer demand is a function of product price which is referred to as Bertrand's model of competition pricing. In this section we assume that the retail price is a function of customer demand which is referred to as Cournot's model of production competition. Specifically, the product is characterized by an endogenous price function of total demand $Q=q_1+q_2$, $p=p(Q)$, which, since the products are fully substitutable, is symmetric in q_1 and q_2. We assume that this symmetric function is down-sloping (concave) in the total quantity of the products, i.e., $\frac{\partial p}{\partial q_1}=\frac{\partial p}{\partial q_2}<0$ and concave, $\frac{\partial^2 p}{\partial Q^2}\le 0$, i.e., $\frac{\partial^2 p}{\partial {q_1}^2}=\frac{\partial^2 p}{\partial {q_2}^2}=\frac{\partial^2 p}{\partial q_1 \partial q_2}\le 0$. The suppliers incur identical unit production cost c, $c<p(0)$, and seek to maximize profits, i.e., they maximize their margins, $p(Q)-c$, times the demand, q_1 or q_2.

The problem of supplier 1

$$\max_{q_1} J_1(q_1,q_2)=\max_{q_1}\ q_1[p(q_1+q_2)-c] \tag{2.17}$$

s.t.

$$q_1\ge 0,\ p(q_1+q_2)\ge c.$$

The problem of Supplier 2

$$\max_{q_2} J_2(q_1,q_2)=\max_{q_2}\ q_2[p(q_1+q_2)-c] \tag{2.18}$$

s.t.

$$q_2\ge 0,\ p(q_1+q_2)\ge c,$$

where $p(Q)$ is the price at which the retailer can sell Q product units; q_1 and q_2 are the quantities produced by suppliers (manufacturers) 1 and 2 respectively and sold by the retailer; $Q=q_1+q_2$ is the total quantity sold by the retailer; and c is the unit production cost for both suppliers.

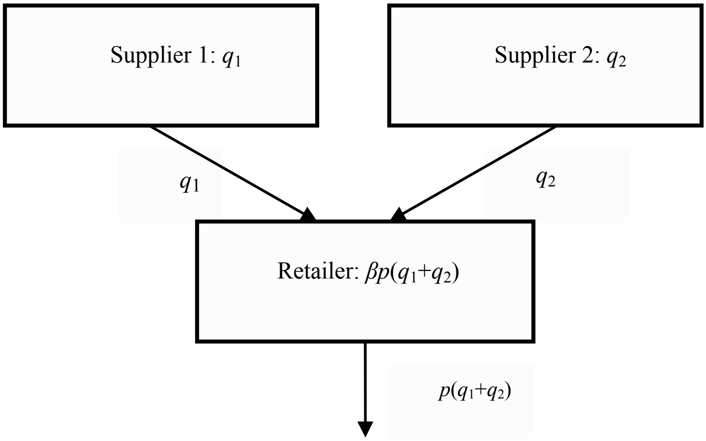

Figure 2.3. Horizontal competition for the same retailer

Exactly, (2.17) and (2.18) can be presented as

$$\max_{q_1} J_1(q_1,q_2)=\max_{q_1} \beta q_1[p(q_1+q_2)-c];$$

$$\max_{q_2} J_2(q_1,q_2)=\max_{q_2} \beta q_2[p(q_1+q_2)-c],$$

where β is percentage paid to the retailer by each manufacturer. Since coefficient β does not affect the optimality conditions, it is omitted. Moreover, since the retailer's profit is

$$J_r(q_1,q_2)= (1-\beta)q_1[p(q_1+q_2)-c] + (1-\beta)q_2[p(q_1+q_2)-c],$$

the centralized objective function does not involve β at all since it represents internal supply chain transfers. Thus, if the supply chain is horizontally integrated, that is, if a single decision maker is in charge, then we have the following single problem as a benchmark.

The centralized problem

$$\max_{q_1,q_2} J(q_1,q_2) = \max_{q_1,q_2} [J_1(q_1,q_2)+J_2(q_1,q_2)]=$$

$$\max_{q_1,q_2} \; q_1[p(q_1+q_2)-c]+ q_2[p(q_1+q_2)-c] \tag{2.19}$$

s.t.

$$q_1 \geq 0,\; q_2 \geq 0,\; p(q_1+q_2) \geq c.$$

System-wide optimal solution

We first study the centralized problem by employing the first-order optimality conditions

$$\frac{\partial J(q_1,q_2)}{\partial q_1} = p(q_1+q_2)-c+q_1\frac{\partial p(Q)}{\partial Q}\frac{\partial Q}{\partial q_1}+q_2\frac{\partial p(Q)}{\partial Q}\frac{\partial Q}{\partial q_1}=0,$$
$$\frac{\partial J(q_1,q_2)}{\partial q_2} = p(q_1+q_2)-c+q_2\frac{\partial p(Q)}{\partial Q}\frac{\partial Q}{\partial q_2}+q_1\frac{\partial p(Q)}{\partial Q}\frac{\partial Q}{\partial q_2}=0.$$

Since the two problems are symmetric, $Q=q_1+q_2$, $\frac{\partial p}{\partial q_1}=\frac{\partial p}{\partial q_2}=\frac{\partial p}{\partial Q}$, only total order Q matters in terms of optimality. Considering the symmetric solution to the above system of equations as well, $q^*= q_1^*=q_2^*$, we obtain the following equation

$$p(2q^*)-c+2q^*\frac{\partial p(2q^*)}{\partial Q}=0. \tag{2.20}$$

Define Q' so that $p(Q')=c$. Then it is easy to verify that,

$$\frac{\partial^2 J}{\partial q_1{}^2}=\frac{\partial^2 J}{\partial q_2{}^2}=\frac{\partial^2 J}{\partial q_1\partial q_2}=2\frac{\partial p}{\partial Q}+q_1\frac{\partial^2 p}{\partial Q^2}+q_2\frac{\partial p^2}{\partial Q^2}<0.$$

This implies that the Hessian of $J(q_1,q_2)$ is semi-definite negative and thus the function $J(q_1,q_2)$ is jointly concave in production quantities q_1 and q_2 for $q_1+q_2\in[0,Q']$. Though this does not ensure the uniqueness of the optimal solution, by differentiating the left-hand side of equation (2.20) in $q=q^*$ we obtain for the symmetric solution

$$\frac{\partial^2 J}{\partial q^2}=4\frac{\partial p}{\partial Q}+4q\frac{\partial^2 p}{\partial Q^2}<0,$$

that is, the left-hand side of (2.20) is strictly monotone in q. Thus, equation (2.20) has a unique solution as formalized in the following proposition.

Proposition 2.3. *The pair (q_1^*,q_2^*), where $q_1^*=q_2^*=q^*$ satisfy equation (2.20) constitutes a unique symmetric system-wide optimal order with $0<q^*<Q'/2$.*

Proof: Since the left-hand side of equation (2.20) is strictly decreasing in q, if there is a feasible solution to (2.20), it is unique. To see that a solution of (2.20) always exists, assume $q=0$, then, since $p(0)>c$, the left-hand side of (2.20) is positive. On the other hand, if $2q=Q'$, since $p(Q')=c$, while the last term of (2.20) is strictly negative as $q=Q/2>0$, we find that the left-hand side of (2.20) is negative. Thus a feasible solution always exists and $0<q<Q'/2$.

Game analysis

Consider now a decentralized supply chain characterized by non-cooperative firms and assume that both players simultaneously decide how many products to produce and supply to the retailer. Using the first-order optimality conditions for the suppliers' problems we find

$$\frac{\partial J(q_1,q_2)}{\partial q_1} = p(q_1+q_2) - c + q_1\frac{\partial p(q_1+q_2)}{\partial q_1}=0,$$

$$\frac{\partial J(q_1,q_2)}{\partial q_2} = p(q_1+q_2) - c + q_2\frac{\partial p(q_1+q_2)}{\partial q_2}=0.$$

Again, since the two problems are symmetric, the competition is symmetric. That is, the solution to this system of equations is $q= q_1=q_2$, which satisfies the following equation

$$p(2q) - c + q\frac{\partial p(2q)}{\partial Q}=0. \qquad (2.21)$$

Comparing (2.21) and (2.20), we conclude with the result highlighting the differences between the centralized and (Nash) game solution.

Proposition 2.4. *In horizontal competition of the production game with equal power players, the retail price will be lower and the quantities produced by the manufacturers higher than the system-wide optimal price and production quantity respectively.*

Proof: Comparing (2.21) and (2.20) we observe that if $q=q^*$, then

$$p(2q) - c + q\frac{\partial p(2q)}{\partial Q} > p(2q^*) - c + 2q^*\frac{\partial p(2q^*)}{\partial Q}=0,$$

while the derivative of the left-hand side of this inequality with respect to q is negative. Thus, $q>q^*$, which, in regard to the down-sloping price function $p(2q)$, means that $p(2q)<p(2q^*)$.

Nash solution

Since it is easy to verify that the suppliers' objective functions are strictly concave in their production quantities, each supplier has a unique, best-response function. In addition, since the derivative of the left-hand side of (2.21) is strictly negative, (2.21) has a unique solution.

Proposition 2.5. *The pair (q_1^n,q_2^n), which satisfies $q_1^n=q_2^n= q^n$ and*

$$p(2q^n) - c + q^n\frac{\partial p(2q^n)}{\partial Q} = 0 \qquad (2.22)$$

constitutes a unique Nash equilibrium of the production game with $0<q^n< Q'/2$.

Proof: The proof is identical to that for proposition (2.3).
The uniqueness of the Nash solution implies that both parties will tend to attain the equilibrium when pursuing their own profits.

The effect of partial product substitutability

Let the product that the second supplier produces partially substitute for the brand of the first supplier. This is expressed by the ratio $0 \le \lambda \le 1$, so that $p=p(Q)=p(q_1+\lambda q_2)$. Then, the Nash optimality conditions take the following form

$$\frac{\partial J(q_1,q_2)}{\partial q_1} = p(Q) - c + q_1 \frac{\partial p(Q)}{\partial Q} = 0,$$

$$\frac{\partial J(q_1,q_2)}{\partial q_2} = p(Q) - c + q_2 \frac{\partial p(Q)}{\partial Q}\lambda = 0.$$

Though these conditions are no longer symmetric, subtracting one equation from the other we find

$$q_1^n=\lambda q_2^n.$$

Thus, $Q=q_1^n+\lambda q_2^n=2\lambda q_2^n$ and q_2^n is determined by

$$p(2\lambda q_2^n) - c + q_2^n \frac{\partial p(2\lambda q_2^n)}{\partial Q}\lambda = 0.$$

In other words, the equilibrium exists, but the production quantities are now proportional rather than identical.

Stackelberg solution

Next we assume that one of the suppliers is the leader, say supplier-one. To find the Stackelberg equilibrium, we need to maximize supplier-one's objective with q_1, subject to the best supplier-two's response $q_2= q_2^R(q_1)$. Let $q_2= q_2^R(q_1)$ satisfy the following equation

$$p(q_1+q_2) - c + q_2 \frac{\partial p(q_1+q_2)}{\partial q_2} = 0. \qquad (2.23)$$

The Stackelberg equilibrium is determined by maximizing the following function

$$\max_{q_1} J_1(q_1)= \max_{q_1} \; q_1[p(q_1+ q_2^R(q_1))-c].$$

Differentiating this function we find

$$\frac{\partial J_1(q_1)}{\partial q_1} = p(q_1+q_2^R(q_1)) - c + q_1 \frac{\partial p(q_1+q_2^R(q_1))}{\partial Q}\left(1+\frac{\partial q_2^R(q_1)}{\partial q_1}\right) = 0, \qquad (2.24)$$

where $\frac{\partial q_2{}^R(q_1)}{\partial q_1}$ is determined by differentiating (2.23) with q_2 set equal to $q_2^R(q_1)$

$$\frac{\partial p(Q)}{\partial Q}(1+\frac{\partial q_2{}^R(q_1)}{\partial q_1})+\frac{\partial q_2{}^R(q_1)}{\partial q_1}\frac{\partial p(Q)}{\partial Q}+q_2{}^R(q_1)\frac{\partial p^2(Q)}{\partial Q^2}(1+\frac{\partial q_2{}^R(q_1)}{\partial q_1})=0.$$

Thus

$$\frac{\partial q_2{}^R(q_1)}{\partial q_1}=-\left(\frac{\partial p(Q)}{\partial Q}+q_2{}^R(q_1)\frac{\partial p^2(Q)}{\partial Q^2}\right)\Bigg/\left(2\frac{\partial p(Q)}{\partial Q}+q_2{}^R(q_1)\frac{\partial p^2(Q)}{\partial Q^2}\right). \quad (2.25)$$

Equation (2.25) implies, $\frac{\partial q_2{}^R(q_1)}{\partial q_1}<0$,

the greater the production of the first supplier, q_1, the lower the production of the second supplier, $q_2^R(q_1)$.

Based on (2.23), (2.24) and (2.25) we conclude that the pair (q_1^s, q_2^s) constitutes the Stackelberg equilibrium of the production game if there exists a joint solution in q_1 and q_2 of the following equations:

$$p(q_1+q_2)-c+q_1\frac{\partial p(q_1+q_2)}{\partial Q}(1+\frac{\partial q_2}{\partial q_1})=0,$$

$$p(q_1+q_2)-c+q_2\frac{\partial p(q_1+q_2)}{\partial Q}=0,$$

where

$$\frac{\partial q_2}{\partial q_1}=-\left(\frac{\partial p(Q)}{\partial Q}+q_2\frac{\partial p^2(Q)}{\partial Q^2}\right)\Bigg/\left(2\frac{\partial p(Q)}{\partial Q}+q_2\frac{\partial p^2(Q)}{\partial Q^2}\right),\ Q=q_1+q_2.$$

We illustrate this with the following example:

Example 2.6

Let the price be linear in production quantity, $p=a-bQ$, $Q=q_1+q_2$, $p(0)=a>c$. Note that the price requirements, $\frac{\partial p}{\partial q_1}=\frac{\partial p}{\partial q_2}=-b<0$ and $\frac{\partial^2 p}{\partial q_1{}^2}=\frac{\partial^2 p}{\partial q_2{}^2}=\frac{\partial^2 p}{\partial q_1\partial q_2}=0$ are met for the selected function. Using Proposition 2.5 we solve (2.22),

$$p(2q^n)-c+q^n\frac{\partial p(2q^n)}{\partial Q}=a-2bq^n-c+q^n(-b)=0$$

and find that $q_1^n = q_2^n = \frac{a-c}{3b}$, hence, $p^n = \frac{1}{3}a + \frac{2}{3}c$. The payoffs for the equilibrium are thus identical for both players, $J_1(q_1^n, q_2^n) = J_2(q_1^n, q_2^n) = \frac{(a-c)^2}{9b}$.

Based on (2.23) we can identify the best response function of the second supplier

$$p(q_1 + q_2) - c + q_2 \frac{\partial p(q_1 + q_2)}{\partial q_2} = a - b(q_1 + q_2) - c + q_2(-b) = 0,$$

and thus

$$q_2 = q_2^r(q_1) = \frac{a - bq_1 - c}{2b}.$$

This response is then employed in (2.24) and (2.25) to find the Stackelberg equilibrium. Equivalently, by substituting this response into the first supplier objective function

$$\max_{q_1} q_1[p(q_1 + q_2^R(q_1)) - c] = \max_{q_1} q_1[\frac{a}{2} - \frac{bq_1}{2} - \frac{c}{2}].$$

and using the first-order optimality conditions, we obtain an explicit resolution of equation (2.24) for our example,

$$\frac{\partial J_1}{\partial q_1} = [\frac{a}{2} - \frac{bq_1}{2} - \frac{c}{2}] + q_1[-\frac{b}{2}] = 0.$$

Accordingly, $q_1^s = \frac{a-c}{2b}$, $q_2^s = \frac{a-c}{4b}$, $p^s = \frac{a+3c}{4}$, $J_1(q_1^s, q_2^s) = \frac{(a-c)^2}{8b}$ and $J_2(q_1^s, q_2^s) = \frac{(a-c)^2}{16b}$. Note that instead of equal payoff under a simultaneous Nash strategy, the first supplier, who is the leader, gains a profit which is twice as much as the follower's profit under a sequential Stackelberg strategy.

Finally, the centralized solution (2.20) is

$$p(2q^*) - c + 2q^* \frac{\partial p(2q^*)}{\partial Q} = a - 2bq^* - c + 2q^*(-b) = 0.$$

Or, $q_1^* = q_2^* = \frac{a-c}{4b}$, hence, $p^* = \frac{1}{2}a + \frac{1}{2}c$ and the system-wide optimal supply chain profit is $J(q_1^*, q_2^*) = \frac{(a-c)^2}{4b}$.

Comparing these results, we find for the first supplier, that his production quantity under the centralized approach is smaller than both that of the Nash strategy and that obtained when the supplier is the Stackelberg leader

$$q_1^s = \frac{a-c}{2b} > q_1^n = \frac{a-c}{3b} > q_1^* = \frac{a-c}{4b}.$$

For the second supplier, the production level is the same under the Stackelberg follower strategy and the system-wide policy, but higher for the Nash strategy.

$$q_2^n = \frac{a-c}{3b} > q_2^s = q_2^* = \frac{a-c}{4b}.$$

Both results agree with Proposition 2.4 which compares Nash and system-wide strategies. Correspondingly, given $p(0)=a>c$, the retail prices decrease

$$p^s = \frac{a+3c}{4} < p^n = \frac{1}{3}a + \frac{2}{3}c < p^* = \frac{1}{2}a + \frac{1}{2}c$$

and the overall supply chain payoff deteriorates under horizontal competition,

$$J_1(q_1^s, q_2^s) + J_2(q_1^s, q_2^s) = \frac{3(a-c)^2}{16b} < J_1(q_1^n, q_2^n) + J_2(q_1^n, q_2^n) =$$

$$\frac{2(a-c)^2}{9b} < J(q_1^*, q_2^*) = \frac{(a-c)^2}{4b}.$$

Example 2.7

This example illustrates how the equilibrium can be analyzed numerically. Let the price be exponential in the production quantity, $p=ae^{-bQ}$, $Q=q_1+q_2$, $p(0)=a>c$. Note that, $\frac{\partial p}{\partial q_1} = \frac{\partial p}{\partial q_2} = -abe^{-bQ} < 0$, while for the second order condition $\frac{\partial^2 p}{\partial q_1^2} = \frac{\partial^2 p}{\partial q_2^2} = \frac{\partial^2 p}{\partial q_1 \partial q_2} = ab^2 e^{-bQ} > 0$ implying that the equilibrium is not necessarily unique. The Nash equilibrium is determined by (2.22)

$$ae^{-b2q^n} - c - q^n abe^{-b2q^n} = 0.$$

Setting the left-hand side of this equation as L in Maple

```
>L:=a*exp(-b*2*q)-c-q*a*b*exp(-b*2*q);
```

$$L := a\,\mathbf{e}^{(-2\,b\,q)} - c - q\,a\,b\,\mathbf{e}^{(-2\,b\,q)}$$

and substituting specific parameters of the problem `a=15,b=0.1,c=1`, we have

```
> L1:=subs(a=15, b=0.1, c=1, L);
```

$$L1 := 15\,\mathbf{e}^{(-0.2\,q)} - 1 - 1.5\,q\,\mathbf{e}^{(-0.2\,q)}$$

The solution to this transcendental equation is found with Maple's SOLVE

```
> solve(L1=0, q);
```

$$7.191168444$$

To verify that the Nash equilibrium is unique, we construct a plot of the left-hand side Y=$L1$

```
> plot(L1, q=0..10);
```

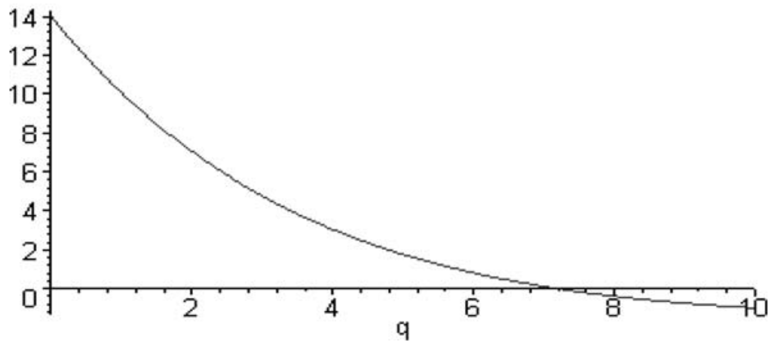

Figure 2.4. The Nash equilibrium

From this plot (see Figure 2.4) we observe that for feasible orders $q_1{}^n = q_2{}^n \geq 0$, there is only one intersection of $Y=L1$ with line, $Y=0$, which is the Nash equilibrium, $q_1{}^n = q_2{}^n = 7.191168444$.

Similarly, employing equation (2.20) to find the system-wide optimal solution with Maple:

```
> LL:=a*exp(-b*2*q)-c-2*q*a*b*exp(-b*2*q);
```

$$LL := a\,\mathbf{e}^{(-2\,b\,q)} - c - 2\,q\,a\,b\,\mathbf{e}^{(-2\,b\,q)}$$

```
> LL1:=subs(a=15, b=0.1, c=1, LL);
```

$$LL1 := 15\,\mathbf{e}^{(-0.2\,q)} - 1 - 3.0\,q\,\mathbf{e}^{(-0.2\,q)}$$

```
> solve(LL1=0, q);
```

$$4.224140740$$

Comparing the system wide optimal production quantity with the Nash quantity we find $q^*=4.224 < q_1{}^n = q_2{}^n = 7.191$.

Coordination

According to Proposition 2.4, although retailers and consumers may benefit from non-cooperating suppliers leading to a fall in retail prices and an increase in production as well as consumption of products, the horizontal competition has a negative effect on the supply chain's profits. Thus, just as with the double marginalization effect, the deterioration in the supply chain performance arises because each manufacturer, when deciding on the quantity to produce, ignores the quantity which the other manufacturer is producing. This can be termed a "double quantification". Indeed, in vertical competition the supplier sells the retailer products which are then resold to the customers. Two margins are being imposed then on the same product quantity. On the other hand, in horizontal competition, each supplier produces a number of products, but sells them at the same price. The price is due to the two quantities being produced. Ignoring one of the quantities, such as ignoring one of the margins, yields results that are different from the system-wide optimal solution.

The essential means to coordinate horizontal competition is thus to cooperate. By simply agreeing to simultaneously set the production quantities equal to the system-wide optimal quantity, rather than to the non-cooperative equilibrium quantities, the suppliers, will be perfectly coordinating the supply chain and increasing their profits equally without any internal supply chain transfers.

The multi-echelon effect

Recalling the effect of vertical competition on the supply chain discussed in the previous section, it is apparent that the more upstream suppliers that are involved, the more margins are added to the supply chain. This results in a decrease in the quantity produced and an increase in prices. This is to say, double marginalization may coordinate the supply chain if its effect is not stronger than that of the horizontal competition. Specifically, let an upstream distributor who has a marginal cost c_d per product play a supply part or sell products to both suppliers at price w_d. (Of course, if the suppliers are not symmetric, then the wholesale price that they can get from the distributor may be different). The corresponding problems of the three-echelon supply chain with two horizontally competing suppliers are as follows (as aforementioned in this section, we consider the case when the retailer does not compete and therefore his problem is not accounted for):

The problem of supplier 1

$$\max_{q_1} J_1(q_1,q_2)=\max_{q_1}\ q_1[p(q_1+q_2)-c-w_d]$$

s.t.

$$q_1\geq 0,\ p(q_1+q_2)\ \geq\ c+w_d.$$

The problem of supplier 2

$$\max_{q_2} J_2(q_1,q_2) = \max_{q_2}\ q_2[p(q_1+q_2)-c-w_d]$$

s.t.

$$q_2 \geq 0,\ p(q_1+q_2) \geq c+w_d$$

The distributor's problem

$$\max_{w_d} J_d(w_d,w,m) = \max_{w_d}\ (w_d-c_d)(q_1+q_1)$$

s.t.

$$w_d \geq c_d.$$

The centralized problem

$$\max_{q_1,q_2} J(q_1,q_2) = \max_{q_1,q_2}\ q_1[p(q_1+q_2)-c-c_d]+q_2[p(q_1+q_2)-c-c_d]$$

s.t.

$$q_1 \geq 0,\ q_2 \geq 0,\ p(q_1+q_2) \geq c+c_d.$$

Assuming that the suppliers are at a Nash equilibrium, the equation for an optimal order quantity $q=q_1=q_2$ for the symmetric suppliers is similar to (2.21). The only difference could be that w_d is subtracted

$$p(2q)-c-w_d+q\frac{\partial p(2q)}{\partial Q}=0.$$

A system-wide optimal solution, on the other hand, is similar to (2.20) but corrected by c_d,

$$p(2q^*)-c-c_d+2q^*\frac{\partial p(2q^*)}{\partial Q}=0\,.$$

Comparing these two equations, we find that both suppliers account for their margins, $p(2q)-c-w_d$, and ignore the distributor's margin w_d-c_d, which, if added, as in the centralized solution, results in a total of $p(2q)-c-c_d$. Since $w_d > c_d$ and the derivatives of the left hand sides of these equations are negative, the Nash production quantity q decreases compared to the system-wide optimal solution. On the other hand, when the quantity which the other party produces is ignored (as discussed in this section), the (Nash) production quantity q decreases compared to the system-wide optimal solution. Thus, if for $q=q^*$ the following holds

$$p(2q)-c-w_d+q\frac{\partial p(2q)}{\partial Q} > p(2q^*)-c-c_d+2q^*\frac{\partial p(2q^*)}{\partial Q}\,,$$

or, equivalently,

$$-q*\frac{\partial p(2q*)}{\partial Q} > w_d - c_d,$$

Then, the effect of horizontal competition between the two suppliers is stronger than that of the vertical competition between the suppliers and additional upstream parties coordinate the supply chain. More precisely, the quantity produced and sold by the three-echelon supply chain will be lower than that of the corresponding two-echelon chain which does not involve an additional upstream distributor.

Finally, it is worth noting that horizontal competition in multi-echelon supply chains opens up a whole spectrum of collaboration activities. For example, horizontally competing producers may coordinate the quantities they order from an upstream supplier to bargain lower wholesale prices. Interested readers are referred to Davidson (1988), Horn and Wolinsky (1988) and Viehoff (1987) who have addressed the benefits of various bargaining schemes.

2.3 STOCKING COMPETITION WITH RANDOM DEMAND

In contrast to the previous section, we now assume that the retailer demand is random and proceed to adapt two classic newsvendor models into two stocking/pricing games. In one game the supplier sets the wholesale price to sell some of his stock while the retailer decides on the quantity to purchase in order to replenish his stock. The retailer incurs no fixed order cost. We refer to this game as the stocking game.

The other game is related to a manufacturer who pays a setup cost for each production order. To avoid this irreversible cost, the manufacturer has the alternative of outsourcing current in-house production to a supplier. Similar to the stocking game, the supplier decides on the wholesale price and does not charge a fixed order cost. Unlike the stocking game, the manufacturer determines first whether to outsource the production at this wholesale price or to produce in-house and then determining the proper quantity to order. We refer to this game as the outsourcing game.

2.3.1 THE STOCKING GAME

The classical, single-period, newsboy or newsvendor problem formulation assumes random exogenous demand, *d*, in contrast to previously discussed pricing and production problems with deterministic but endogenous demands. The selling season is short and there is no time for additional orders so if the retailer orders less than the demand at the end of period, then shortage h^- cost per unit of unsatisfied demand is incurred. The shortage cost normally includes lost sales and a loss of customer goodwill. On the other hand, if the retailer orders more than he is able to sell, unit inventory cost h^+ (mitigated by salvage cost) is incurred for units left over at the end of period. The fixed-order cost is assumed to be negligible. The retailer's goal is to find order quantity, q, to maximize expected overall profits. The described newsvendor problem assumes that the product purchasing cost is fixed and given. However, if we take into account a supplier who independently maximizes his profit and thus impacts the retailer's optimal solution by choosing a wholesale price, w, the newsvendor problem is reduced to a game.

Let retailer's margin, m, be fixed, $f(D)$ and $F(a)=\int_0^a f(D)dD$ be the demand probability density and cumulative distribution functions respectively. Then, the retailer's problem is formulated as follows.

The retailer's problem

$$\max_q J_r(q,w)= \max_q \{E[ym - h^+x^+ - h^-x^-]-wq\}, \tag{2.26}$$

s.t.

$$x=q-d, \tag{2.27}$$

$$q\geq 0, \tag{2.28}$$

where $x^+=\max\{0, x\}$ and $x^- =\max\{0, -x\}$ are inventory surplus and shortage at the end of selling season respectively, and $y=\min\{q,d\}$ is the number of products sold.

Applying conditional expectation to (2.26), the objective function transforms into the following form

$$\max_q J_r(q,w)= \max_q \{$$

$$\int_0^q mDf(D)dD+\int_q^\infty mqf(D)dD-\int_0^q h^+(q-D)f(D)dD-\int_q^\infty h^-(D-q)f(D)dDwq\}. \tag{2.29}$$

The first term in the objective function $E[ym]=\int_0^q mDf(D)dD+\int_q^\infty mqf(D)dD$ represents income from selling y product units; the second and the third terms, $E[h^+x^+]=\int_0^q h^+(q-D)f(D)dD$, $E[h^-x^-]=\int_q^\infty h^-(D-q)f(D)dD$ represent losses due the inventory surplus and shortage respectively; and the last term, wq, is the amount paid to the supplier.

Note, that the retailer orders products from the supplier if he expects non-negative profit. In other words, there is a maximum wholesale price, w^M, that the supplier can charge. Taking this into account, as well as the unit production cost, c, of the supplier, we formulate the supplier's problem.

The supplier's problem

$$\max_{w} J_s(q,w) = (w-c)q \tag{2.30}$$

s.t.

$$c \le w \le w^M. \tag{2.31}$$

The corresponding centralized problem is based on the sum of two objective functions (2.30) and (2.26), which results in a function independent of the wholesale price, w, representing a transfer within the supply chain.

The centralized problem

$$\max_{q} J(q) = \max_{q} \{E[ym - h^+x^+ - h^-x^-] - cq\} \tag{2.32}$$

s.t.

$$x = q - d,\ q \ge 0.$$

System-wide optimal solution

We first study the centralized problem. Similar to (2.29), by determining the expectation of (2.32), we obtain

$$\max_{q} J(q) = \max_{q} \{$$

$$\int_0^q mDf(D)dD + \int_q^\infty mqf(D)dD - \int_0^q h^+(q-D)f(D)dD - \int_q^\infty h^-(D-q)f(D)dD - cq\}.$$

By employing the first-order optimality condition to this function, we have

$$\frac{\partial J(q)}{\partial q} = mqf(q) - mqf(q) + \int_q^\infty mf(D)dD - \int_0^q h^+ f(D)dD + \int_q^\infty h^- f(D)dD - c = 0,$$

which, after simple manipulations, results in

$$m(1-F(q)) - h^+F(q) + h^-(1-F(q)) - c = 0\,.$$

Thus we find that the traditional newsvendor expression for the optimal order quantity q^*, which is feasible if $m+h^->c$,

$$F(q^*)=\frac{m+h^- -c}{m+h^- +h^+}. \tag{2.33}$$

We can also verify the sufficient condition, i.e., that the objective function (2.30) is concave,

$$\frac{\partial^2 J(q)}{\partial q^2}=-(m+h^+ +h^-)f(q)\le 0. \tag{2.34}$$

Let $f(D)>0$ for $d^{\min}\le D\le d^{\max}$. Then, since ordering less than the minimum demand, $d^{\min}$, as well as more than the maximum demand, $d^{\max}$, does not make any sense, the centralized objective function is strictly concave and thus we find a unique solution.

The effect of initial inventory

Note that if the retailer has an initial inventory, x^0, that is, $x=x^0+q-d$, then by using the same arguments we observe that the only change in (2.33) is in the argument of $F(.)$:

$$F(x^0+q)=\frac{m+h^- -c}{m+h^- +h^+}. \tag{2.35}$$

Let s satisfy the equation,

$$F(s)=\frac{m+h^- -c}{m+h^- +h^+}, \tag{2.36}$$

then s is the base stock, and the optimal order quantity is interpreted as the well-known order-up-to policy,

$$q^*=\begin{cases} s-x^0, \text{if } s>x^0\\ 0, \text{otherwise.}\end{cases}$$

Service level

For the risk of shortage, we have the probability $P[x<0]=1-\alpha$, where α is referred to as the service level. From (2.32) it follows that the service level in the centralized supply chain is $P[x\ge 0]=F(q^*)$, or, equivalently,

$$\alpha=\frac{m+h^- -c}{m+h^- +h^+}. \tag{2.37}$$

When $x^0>s$, the service level is higher than the specified level α.

Game analysis

We consider now a decentralized supply chain characterized by non-cooperative firms and assume first that both players make their decisions simultaneously. The supplier chooses the wholesale price w and the retailer selects the order quantity, q. The supplier then produces q units at unit cost c and delivers them to the retailer.

Using the first-order optimality conditions for the retailer's problem, we have

$$\frac{\partial J(q,w)}{\partial q} = mqf(q) - mqf(q) + \int_q^\infty mf(D)dD - \int_0^q h^+ f(D)dD + \int_q^\infty h^- f(D)dD - w = 0$$

Thus, we find that the maximum wholesale price, $w^M = m + h^-$, so that if $w \le w^M$, the best retailer's response is determined by

$$F(q) = \frac{m + h^- - w}{m + h^- + h^+}. \tag{2.38}$$

From (2.38) we observe, that if $w = w^M$, the retailer does not order at all, while if $w < w^M$, then comparing (2.33) and (2.38) and taking into account $w \ge c$ and $\frac{\partial F(q)}{\partial q} > 0$, we conclude with results similar to those found for the pricing game with endogenous demand.

Proposition 2.6. *In vertical competition of the stocking game, if the supplier makes a profit, i.e., w>c, the retailer's order quantity and the customer service level are lower than the system-wide optimal order quantity and service level.*

Note that if the retailer would account for the supplier's margin, w-c, by including it into the numerator of (2.38), equation (2.38) would transform into (3.33). We thus find the double marginalization effect discussed in the pricing game. In addition, this effect decreases the customer service level unless the supplier does not want to profit from the sale and sets $w=c$. On the other hand, since the supplier's objective function (2.30) is linear in w, we conclude that the supplier would set the wholesale price as high as possible, i.e., $w=w^M$ under the Nash strategy. In such a case, the retailer makes no profit and orders nothing. As a result of the Nash strategy, there is neither business nor customer service between the supplier and the retailer.

Similar to the pricing game of the previous section, the statement of Proposition 2.6 that vertical competition causes the supply chain performance to deteriorate does not depend on whether the players make a simultaneous decision or if the supplier first sets wholesale price, as is often the case in practice. In what follows, we show that under the supplier's leadership, the Stackelberg equilibrium's wholesale price does not equal the maximum purchasing price w^M.

Equilibrium

Assume that the supplier is a leader in the Stackelberg game. The supplier's objective function with q subject to the optimal retailer's response $q=q^R(w)$ is determined by (2.38),

$$J_s(q,w)= (w-c)\, q^R(w).$$

Differentiating the supplier's objective function, we have

$$\frac{\partial J_s(q,w)}{\partial w}=q^R(w)+(w-c)\frac{\partial q^R(w)}{\partial w}=0\,. \qquad (2.39)$$

The value of $\frac{\partial q^R(w)}{\partial w}$ is determined by differentiating (2.38) with q set equal to $q^R(w)$,

$$f(q^R(w))\frac{\partial q^R(w)}{\partial w}=-\frac{1}{m+h^-+h^+}\,.$$

As a result:
The greater the wholesale price, the lower the quantity that the retailer orders and by substituting

$$\frac{\partial q^R(w)}{\partial w}=-\frac{1}{\left(m+h^-+h^+\right)f(q^R(w))}$$

into (2.38), we have

$$\frac{\partial J_s(q,w)}{\partial w}=q^R(w)-\frac{w-c}{(m+h^-+h^+)f(q^R(w))}=0\,, \qquad (2.40)$$

where

$$F(q^R(w))=\frac{m+h^- -w}{m+h^-+h^+}\,. \qquad (2.41)$$

We conclude with the following proposition.

Proposition 2.7. *Let $f(D)>0$ for $D\geq 0$, otherwise $f(D)=0$. The pair (w^s,q^s), where w^s and $q^s= q^R(w^s)$ satisfy*

$$q^R(w^s)-\frac{w^s-c}{(m+h^-+h^+)f(q^R(w^s))}=0\,,\ F(q^R(w^s))=\frac{m+h^- -w^s}{m+h^-+h^+}\,,$$

constitutes a Stackelberg equilibrium of the stocking game with $c<w^s< m+h^-=w^M$.

Proof: First we consider equation (2.40) and verify that

$$\frac{\partial J_s(q,c)}{\partial w}=q^R(c)>0\,,\ \frac{\partial J_s(w^M)}{\partial w}=-\frac{w^M-c}{(m+h^-+h^+)f(0)}<0\ .$$

Since $f(D)>0$ *for* $D\geq 0$ we observe that

$$\frac{\partial J_s(q,w)}{\partial w}=q^R(w)-\frac{w-c}{(m+h^-+h^+)f(q^R(w))}$$

is a continuous function for $c \le w \le w^M$. We conclude that there is at least one root, $\frac{\partial J_s(q, w^s)}{\partial w} = 0$, $c<w^s<w^M$, as stated in Proposition 2.7.

To have a unique Stackelberg wholesale price, however, we require that the supplier's objective function be strictly concave, $\frac{\partial^2 J_s(q,w)}{\partial w^2}$<0, that is,

$$\frac{\partial q^R(w)}{\partial w}-\frac{1}{(m+h^-+h^+)f(q^R(w))}+\frac{w-c}{(m+h^-+h^+)[f(q^R(w))]^2}\frac{\partial f(q^R)}{\partial q^R}\frac{\partial q^R(w)}{\partial w}<0, \quad (2.42)$$

which apparently does not hold for every distribution.

Example 2.8

Let the demand be characterized by the uniform distribution,

$$f(D)=\begin{cases}\frac{1}{A}, & \text{for } 0\le D\le A;\\ 0, \text{otherwise}\end{cases} \text{ and } F(a)=\frac{a}{A}, 0\le a\le A.$$

Then the supplier objective function is strictly concave, as (2.42) holds. Using (2.40) - (2.41) we find

$$q^R(w^s)-\frac{w^s-c}{(m+h^-+h^+)}A=0 \text{ and } F(q^R(w^s))=\frac{q^R(w^s)}{A}=\frac{m+h^--w^s}{m+h^-+h^+}.$$

Thus,

$$\frac{m+h^--w^s}{m+h^-+h^+}A-\frac{w^s-c}{(m+h^-+h^+)}A=0,$$

which results in

$$w^s=\frac{m+h^-+c}{2},\ q^s=q^R(w^s)=\frac{m+h^--c}{m+h^-+h^+}\frac{A}{2}, \quad (2.43)$$

while the system-wide optimal order quantity is twice as large,

$$q^*=\frac{m+h^--c}{m+h^-+h^+}A. \quad (2.44)$$

Recalling our assumption that $w^M=m+h^->c$, we observe that $c<w^s<w^M$ and $0<q^s<A/2$. Thus, this problem has always a unique Stackelberg equilibrium.

Example 2.9

Let the demand be characterized by an exponential distribution, i.e.,

$$f(D) = \begin{cases} \lambda e^{-\lambda D}, & \text{for } D \ge 0; \\ 0, \text{otherwise} \end{cases} \quad \text{and } F(a) = 1 - e^{-\lambda a}, \, a \ge 0.$$

Then according to (2.40), we have the equation for the Stackelberg wholesale price

$$q^R(w) - \frac{w-c}{(m+h^- + h^+)\lambda e^{-\lambda q^R(w)}} = 0,$$

where according to (2.41)

$$1 - e^{-\lambda q^R(w)} = \frac{m+h^- - w}{m+h^- + h^+}$$

and thus

$$q^R(w) = \frac{1}{\lambda} \ln \frac{m+h^- + h^+}{w+h^+}.$$

Substituting this into the equation of the Stackelberg wholesale price, we obtain the following expression

$$\frac{1}{\lambda} \ln \frac{m+h^- + h^+}{w+h^+} - \frac{w-c}{(w+h^+)\lambda} = 0.$$

We solve this equation with Maple by first setting the left hand side as L

```
>L:=ln((m+hplus+hminus)/(w+hplus)-(w-c)/(w+hplus);
```

$$L := \ln\left(\frac{m + hplus + hminus}{w + hplus}\right) - \frac{w - c}{w + hplus}$$

Then substituting specific values for `m=15, hplus=1, hminus=10, c=2`

```
>L1:=subs(m=15, hplus=1, hminus=10, c=2, L);
```

$$L1 := \ln\left(\frac{26}{w+1}\right) - \frac{w-2}{w+1}$$

we verify with a plot Y= L1 that it crosses line Y=0 only once and thus the Stackelberg wholesale price is unique.

```
>plot(L1, w=2..15);
```

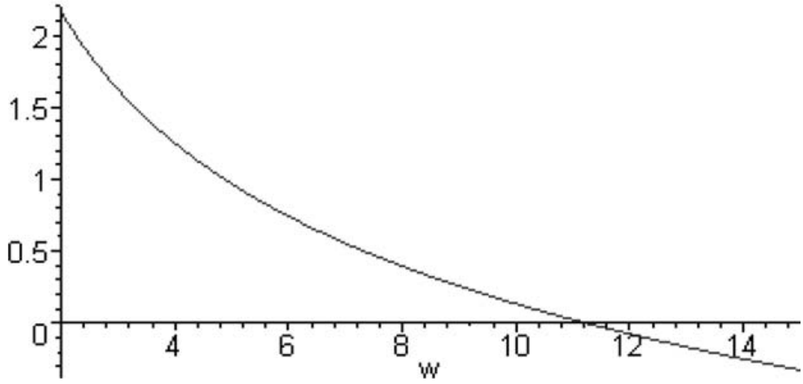

Figure 2.5. The Stackelberg wholesale price

Next we solve equation L1=0 in a general form

```
> ws:=solve(L1=0, w);
```

$$ws := -\frac{\text{LambertW}\left(\frac{3}{26}\,\mathbf{e}\right) - 3}{\text{LambertW}\left(\frac{3}{26}\,\mathbf{e}\right)}$$

and evaluate the result numerically

```
> evalf(ws);
```

$$11.22512050$$

Finally we calculate the equilibrium order quantity by using the best retailer's response function $q^R(w)=\frac{1}{\lambda}\ln\frac{m+h^- + h^+}{w+h^+}$.

```
> q:=1/lambda*ln((m+hplus+hminus)/(w+hplus));
```

$$q := \frac{\ln\left(\frac{m + hplus + hminus}{w + hplus}\right)}{\lambda}$$

and substituting the specific parameters of the problem

```
>qR:=subs(m=15, hplus=1, hminus=10, lambda=0.1,
w=evalf(ws), c=2, q);
```

$$qR := 10.\ \ln(2.126768403)\ .$$

Evaluating numerically the last result leads to

```
> qs=evalf(qR);
```

$$qs = 7.546036459\ .$$

Thus w^s=11.225 and q^s= 7.546. The system-wide optimal order quantity is determined by (2.33)

$$q^* = \frac{1}{\lambda}\ln\frac{m+h^- + h^+}{c+h^+},$$

which with Maple results in

```
> qopt:=1/lambda*ln((m+hplus+hminus)/(c+hplus));
```

$$qopt := \frac{\ln\left(\frac{m + hplus + hminus}{c + hplus}\right)}{\lambda}$$

```
>qswopt:=subs(m=15, hplus=1, hminus=10, lamb-
da=0.1, c=2, qopt);
```

$$qswopt := 10.\ \ln\left(\frac{26}{3}\right)$$

```
>evalf(qswopt);
```

$$21.59484249$$

Comparing the system-wide optimal solution with the equilibrium solution we find that the system-wide optimal order is almost three-times as large.

$$q^s = 7.546 < q^* = 21.594.$$

Coordination

According to Proposition 2.6, vertical competition under exogenous random demand has a negative effect on the supply chain: the retailer orders less and the service level decreases. This is similar to the pricing competition considered in the previous section and again the negative effect is due to the double marginalization. As opposed to the pricing game, there is no Nash equilibrium in the stocking game while the supplier's leadership has a positive effect on the chain. More precisely, there is an equilibrium if the supplier assumes leadership.

Due to the same double marginalization effect, the coordination in this game is similar to that discussed for the pricing game: discounting and profit sharing. We present here a straightforward approach for developing a coordinating quantity discounting scheme.

First we generalize the supplier's objective function $J_s(q,w)=(w-c)q$ to make the wholesale price dependent on the order quantity, q,

$$J_s(q,w)= w(q)-cq.$$

Then the retailer's best-response (2.38) takes the following form

$$F(q)= \frac{m+h^- - \partial w/\partial q}{m+h^- + h^+}. \qquad (2.45)$$

We do not specify any specific requirement for wholesale price $w(q)$ but impose conditions on the rate of change of $w(q)$

$$\frac{\partial w(q)}{\partial q} < c,\ \frac{\partial^2 w(q)}{\partial q^2} \geq 0,\ \text{if } q<q^* \text{ and } \frac{\partial w(q)}{\partial q} \geq c,\ \text{if } q>q^*.$$

These conditions imply that the function $w(q)$ may have various discounting schemes for $0 \le q \le q^*$. Next we show that if the conditions are met, the supplier can select any value for $w(q)$, $w(q^*) < w^M$, and still have the retailer ordering the system-wide optimal quantity.

Proposition 2.8. *Let $w(q^*) < w^M$, and the discounting scheme be such that if $w(q)$ is a continuous function of q, $\frac{\partial w(q)}{\partial q} < c$ and $\frac{\partial^2 w(q)}{\partial q^2} \le 0$ for $q < q^*$, and $\frac{\partial w(q)}{\partial q} \ge c$ for $q > q^*$, then the supplier orders the system-wide optimal quantity q^*.*

Proof: Since $w(q)$ is continuous, $\frac{\partial^2 w(q)}{\partial q^2} \ge 0$ for $q < q^*$ and $\frac{\partial w(q)}{\partial q} \ge c > 0$ for $q > q^*$, the wholesale price $w(q)$ is a convex function, a solution which satisfies (2.45). Note that derivative of $w(q)$ at $q = q^*$ is not required to exist. We thus represent it by the sub-gradient, $\frac{\partial w(q^*)}{\partial q} = e, a \le e \le c$ where $a = \lim_{q \to q^*, q < q^*} \frac{w(q) - w(q^*)}{q - q^*} < c$. There can be three possible solutions to (2.45). Assume there exists an optimal solution q', $q' < q^*$, such that $\frac{\partial w(q)}{\partial q} \le a < c$ and (2.45) is met. Recalling that $F(q^*) = \frac{m + h^- - c}{m + h^- + h^+}$, we find that if (2.45) is met and $\frac{\partial w(q)}{\partial q} < c$, then $q' > q^*$, which contradicts our initial assumption. Similarly, we observe that another solution, say q'', $q'' > q^*$ and thus $\frac{\partial w(q)}{\partial q} \ge c$ contradicts (2.45). The only solution left is $q''' = q^*$, $\frac{\partial w(q^*)}{\partial q} = e$. Substituting this into (2.45) we find

$$F(q''') = \frac{m + h^- - e}{m + h^- + h^+},$$

which is satisfied for $e = c$ as $a \le e \le c$ and $q'' = q^*$.

A trivial example of linear discounting that satisfies Proposition 2.8 is

$$w(q) = \begin{cases} A - aq, 0 \le q \le q^*; \\ A - aq^* + c(q - q^*), \text{otherwise}, \end{cases}$$

where $A - aq^* < w^M$.

2.3.2 THE OUTSOURCING GAME

In this section, the classical, single-period newsvendor model with a setup cost is turned into an outsourcing game. We consider a single manufacturer with two potential situations. He either incurs a fixed cost per each production order or the product produced is characterized by frequently changing characteristics and/or technology. These changes may be due to new product features and/or technological developments so that each change induces a non-negligible fixed cost. The basic assumptions remain unchanged: the demand is random with known density, $f(D)$ and cumulative $F(a)$ distribution function. In addition we assume a short selling season. If the manufacturer's production or supply order is less than the demand realized at the end of period, then a shortage cost h^- per unit of unsatisfied demand is incurred and there is no time for additional orders. Otherwise, if there is a surplus, the unit inventory cost h^+ is incurred at the end of period.

Accordingly, the manufacturer has two options. One is to order the production in-house, which incurs an irreversible fixed cost C as well as variable cost c_m per unit product. This is in contrast to the newsvendor model considered in the previous section, where the retailer's fixed-order cost was assumed to be negligible. The other option involves outsourcing the production to a single supplier. Then the manufacturer incurs only the variable purchasing cost w per product unit and the supplier incurs a unit production cost c. We assume that $c>c_m$, no initial inventory, and a profitable in-house production (at least when there is no initial inventory at the manufacturer's plant). Otherwise outsourcing is always advantageous. Both the manufacturer and the supplier are profit maximizers.

The manufacturer's problem

$$\max_q J_m(q,w)=$$

$$\max\{\max_q \{E[ym - h^+x^+ - h^-x^-]-wq\}, \max_q \{E[ym - h^+x^+ - h^-x^-]-c_mq-C\}\}, \quad (2.46)$$

s.t.

$$x=q-d, \quad (2.47)$$

$$q\geq 0, \quad (2.48)$$

where $x^+=\max\{0, x\}$ and $x^- =\max\{0, -x\}$ are respectively inventory surplus and shortage at the end of a period, and $y=\min\{q,d\}$ is the number of products sold.

The manufacturer's objective function (2.46) consists of two parts. The first part $\max_q \{E[ym - h^+x^+ - h^-x^-]-wq\}$ represents the profit which the manufacturer can gain if he decides to outsource the production. The other part is the profit from in-house production (assuming that the production is profitable). Since the first part is identical to that studied in the previous section, application of conditional expectation to the first part of (2.46) results into (2.29). Thus, the optimal manufacturer's outsourcing order q' for (2.29) is given by (2.38),

$$F(q')=\frac{m+h^- - w}{m+h^- + h^+}.$$

If we assume that C=0, then the second part of (2.46) differs from the first part by c_m only, replaced with w. Consequently, if C=0, then the optimal response for the second part of (2.46), q'', is

$$F(q'')=\frac{m+h^- - c_m}{m+h^- + h^+}.$$

Introduce a cost function, $\pi(q)$, such that

$$\pi(q)=E[ym - h^+x^+ - h^-x^-]. \tag{2.49}$$

Then,

$$\pi(q')-wq'=$$

$$\int_0^{q'} mDf(D)dD+\int_{q'}^{\infty} mq'f(D)dD-\int_0^{q'} h^+(q'-D)f(D)dD-\int_{q'}^{\infty} h^-(D-q')f(D)dD\, wq'$$

is the maximum profit if outsourcing is selected (the first part of (2.46)). The maximum profit when in-house production is selected (the second part of (2.46)) is

$$\pi(q'')-c_m q''-C=$$

$$\int_0^{q''} mDf(D)dD+\int_{q''}^{\infty} mq''f(D)dD-\int_0^{q''} h^+(q''-D)f(D)dD-\int_{q''}^{\infty} h^-(D-q'')f(D)dD-c_m q''-C.$$

Thus, the optimal manufacturer's choice for a given wholesale price is summarized by

$$q=\begin{cases} q', \text{if } \pi(q')-wq'\geq \pi(q'')-c_m q''-C \\ q'', \text{otherwise,} \end{cases} \tag{2.50}$$

where q' is the outsourcing order, while q'' is the in-house production (according to our assumption that in-house production is at least worth-while, $\pi(q'')-c_m q''-C>0$). Furthermore, condition (2.50) assumes that out-sourcing is a dominating strategy when profits from in-house production and outsourcing are identical.

Let outsourcing at supplier's marginal cost be advantageous compared to in-house production profit,

$$\pi(q'')\text{-}c_m q''\text{-}C \le \pi(q')\text{-}cq',\ F(q')=\frac{m+h^- -c}{m+h^- +h^+}.$$

This, along with (2.50) and the fact that outsourcing profit decreases when the wholesale price increases, implies that the maximum purchase price $w^o \ge c$ always exists such that

$$\pi(q'')\text{-}c_m q''\text{-}C=\pi(q') - w^o q',\ F(q')= \frac{m+h^- -w^o}{m+h^- +h^+}.$$

Using (2.49), w^o is the smallest root of the expression below

$$\int_0^{q''} mDf(D)dD+\int_{q''}^{\infty} mq''f(D)dD-\int_0^{q''} h^+(q''-D)f(D)dD-\int_{q''}^{\infty} h^-(D-q'')f(D)dD\text{-}c_m q''\text{-}C=$$

$$\int_0^{q'} mDf(D)dD+\int_{q'}^{\infty} mq'f(D)dD-\int_0^{q'} h^+(q'-D)f(D)dD-\int_{q'}^{\infty} h^-(D-q')f(D)dD\text{-}$$

$$w^o q', \tag{2.51}$$

where $F(q'')= \dfrac{m+h^- -c_m}{m+h^- +h^+}$ and $F(q')= \dfrac{m+h^- -w^o}{m+h^- +h^+}$.

On the other hand, if outsourcing is not advantageous, then $\pi(q'')\text{-}c_m q''\text{-}C>\pi(q')\text{-}cq'$, $F(q')= \dfrac{m+h^- -c}{m+h^- +h^+}$ and $c_m<w^o<c$. Thus condition (2.50) can be reformulated as follows

$$q=\begin{cases} q', \text{if } c\le w\le w^0, \\ q'', \text{if } w^0<c, \end{cases} \tag{2.52}$$

where $F(q'')= \dfrac{m+h^- -c_m}{m+h^- +h^+}$ and $F(q')= \dfrac{m+h^- -w^o}{m+h^- +h^+}$.

The interpretation of (2.52) is straightforward. If purchasing at the marginal cost of the supplier is not beneficial compared to the in-house production, then there is no wholesale price, $w>c$, to encourage outsourcing.

The supplier's problem is similar to that of the previous section.

The supplier's problem

$$\max_w J_s(q,w)= (w\text{-}c)q \tag{2.53}$$

s.t.

$$c\le w\le w^0. \tag{2.54}$$

Note that if $\pi(q'')\text{-}c_m q''\text{-}C\le\pi(q')\text{-}cq'$, then the supplier's problem has a feasible solution. Otherwise, $c_m<w^o<c$, and the supplier's problem has no

feasible solution since, in order to compete with in-house production, the supplier has to set the wholesale price below his marginal cost, $w<c$.

Correspondingly, the centralized problem is split into two cases. If $\pi(q'')-c_m q''-C \leq \pi(q')-cq'$, or equivalently, $w^o \geq c$, the centralized problem is reduced to that considered in the previous section. Indeed, if the supply chain is integrated, then wholesale-related costs represent a transfer within the chain which does not affect the system-wide optimal solution. Then the supplier will deliver products at his marginal cost c and no fixed irreversible cost will be paid since in-house production is not implemented.

The centralized problem

$$\max_q J(q) = \max_q \{E[ym - h^+x^+ - h^-x^-]-cq\} \tag{2.55}$$

s.t.

$$x=q-d,\ q \geq 0.$$

If $w^o<c$, then the centralized objective function is identical to the second part of (2.46), which is the classical newsvendor problem with a setup cost

$$\max_q \{E[ym - h^+x^+ - h^-x^-]-c_m q-C\ . \tag{2.56}$$

In other words, the manufacturer's problem and the centralized problem become identical in such a case.

System-wide optimal solution

The centralized problem (2.55) was studied in the previous section. If outsourcing is selected, i.e., $w^o>c$, the system-wide optimal order quantity $q^{*\prime}$ is unique and defined by (2.33).

$$F(q^{*\prime}) = \frac{m+h^- - c}{m+h^- + h^+}.$$

Note that if the supply chain is centralized, then it simply has two options to produce the product (at the manufacturer and at the supplier). Therefore, it is the production at the supplier option (if chosen) rather than outsourcing.

Similarly, if production at the manufacturer is selected, $w^o<c$, the optimal solution is the newsvendor solution

$$F(q^{*\prime\prime}) = \frac{m+h^- - c_m}{m+h^- + h^+}. \tag{2.57}$$

The effect of initial inventory

Since the supplier does not impose any fixed-order cost, the effect of initial inventories on outsourcing is identical to that for the centralized system as discussed in the previous section,

$$F(x^0+q^{*\prime})=\frac{m+h^- -c}{m+h^- +h^+}.$$

To study the effect of initial inventories on production at the manufacturer's plant, let $x^0<S$, (otherwise it is not optimal to produce at all) and $x= x^0+q-d$. Then the profit from not ordering anything is

$$\pi(x^0)=$$

$$\int_0^{x^0} mDf(D)dD+\int_{x^0}^{\infty} mx^0 f(D)dD-\int_0^{x^0} h^+(x^0-D)f(D)dD-\int_{x^0}^{\infty} h^-(D-x^0)f(D)dD.$$

On the other hand, if the manufacturer produces $q>0$ products, the profit is

$$\pi(q+x^0)\text{-}c_m q\text{-}C.$$

The optimal solution for this objective function is determined by (2.57)

$$F(q^{*\prime\prime}+x^0)=\frac{m+h^- -c_m}{m+h^- +h^+}.$$

Denote $S= q^{*\prime\prime}+x^0$, then the optimal in-house profit for a given x^0 is

$$\pi^0(S)\text{-}c_m(S\text{-}\ x^0)\text{-}C.$$

Note that if $x^0=0$, then assuming that in-house production is profitable under conditions of no initial inventory, we have, $\pi(S)\text{-}c_m(S\text{-}\ x^0)\text{-}C>0$, while $\pi(x^0)<0$ since we do not sell anything when $x^0=0$. That is,

$$\pi(S)\text{-}c_m(S\text{-}\ x^0)\ \text{-}C> \pi(x^0),$$

or equivalently,

$$\pi(S)\text{-}c_mS\ \text{-}C> \pi(x^0)\text{-}c_mx^0,$$

which implies that it is optimal to produce in-house when $x^0=0$. When initial inventories increase $x^0>0$, then the left-hand part of the inequality remains unchanged while the right-hand part increases towards its maximum which is attained at $x^0=S$. Thus, when $x^0=S,\ C>0$, we have

$$\pi(S)\text{-}c_mS\ \text{-}\ C< \pi(x^0)\text{-}c_mx^0,$$

which implies that it is optimal not to produce when $x^0=S$. The right-hand side of the inequality represents the traditional newsvendor objective function, $\pi(x^0)\text{-}c_mx^0$, which monotonically increases when x^0 increases towards S. We conclude that there exists $x^0=s<S,$ such that,

$$\pi(S)\text{-}c_mS\ \text{-}\ C= \pi(s)\text{-}c_ms.$$

Thus, if $x^0<s$, then $\pi(S)\text{-}c_mS\ \text{-}C> \pi(x^0)\text{-}c_mx^0$ and it is profitable to produce so that $S= q^{*\prime\prime}+x^0$. On the other hand, if $x^0>s$, then $\pi(S)\text{-}c_mS\ \text{-}\ C< \pi(x^0)\text{-}c_mx^0$ and it is not profitable to produce. Consequently, in contrast to the optimal order-up-to policy when no fixed order cost is incurred, we obtain the so-called security stock (s, S) policy which is widely used in industry as well,

$$q^{*\prime\prime} = \begin{cases} S - x^0, \text{if } x^0 < s \\ 0, \text{otherwise,} \end{cases}$$

where s is the smallest value that satisfies $\pi(S)$-$c_m S$ -C= $\pi(s)$-$c_m s$.

Game analysis

To simplify the presentation, we assume x^0=0 and consider now a decentralized supply chain characterized by non-cooperating firms. Let the supplier first set the wholesale price. If w^o<c, then regardless of the wholesale price, an in-house production for q” is chosen. Otherwise, the manufacturer decides to outsource and issues an order, q', which the supplier delivers.

Since in-house (2.57) and the centralized in-house solutions are identical, we further focus on outsourcing, i.e., $w^o \geq c$. Let us first assume that w^o=c, then the supplier has zero profit by setting w=c, and simply sustains himself since the manufacturer’s dominating policy is to outsource (2.50) when the profit from in-house production is equal to the outsourcing profit.

Let w^o>c. Using the results from the previous section, the optimal order is determined by (2.38)

$$F(q')= \frac{m + h^- - w}{m + h^- + h^+}.$$

This, similar to Proposition 2.6, implies the double marginalization effect.

Proposition 2.9. In the outsourcing game, *if w^o>c and the supplier makes a profit*, *i.e., w>c, the manufacturer’s order quantity and the customer service level are lower than the system-wide centralized order quantity and service level.*

Again, similar to the observation from the previous section, since the supplier’s objective function is linear in w, the supplier would want to set the wholesale price as high as possible, i.e., w=w^o under the Nash strategy. This causes supply chain performance to deteriorate. In contrast to the inventory game of the previous section, if the manufacturer’s dominating policy is to outsource when the profit from in-house production is equal to the profit from outsourcing, then the manufacturer will still outsource at w=w^o.

Equilibrium

Given $w^o>c$, Proposition 2.7 proves that there is a Stackelberg equilibrium price $c<w^s<m+h^-$. However, since $q'>0$ and $\pi(q')-w^o q'=\pi(q'')-c_m(q'')-C>0$, then $w^o<w^M=m+h^-$. This implies that the Stackelberg wholesale price found with respect to Proposition 2.7 may be greater than w^o. In such a case it is set to $w^s=w^o$.

Based on Proposition 2.7 and the manufacturer's optimal response (2.52), we summarize our results.

If $w^o<c$, then produce q'' products in-house, where

$$F(q'')=\frac{m+h^- -c_m}{m+h^- +h^+}.$$

If $w^o=c$, then outsource; the equilibrium wholesale price is $w^s=c$, and the outsourcing quantity q' is such that

$$F(q')=\frac{m+h^- -c}{m+h^- +h^+}.$$

If $w^o>c$, then outsource; find w' and $q'=q^R(w')$ (according to Proposition 2.7), i.e.,

$$q^R(w')-\frac{w'-c}{(m+h^- +h^+)f(q^R(w'))}=0,\ F(q^R(w'))=\frac{m+h^- -w'}{m+h^- +h^+}.$$

If $w'<w^o$, then the equilibrium wholesale price is $w^s=w'$ and the outsourcing order is q', otherwise *$w^s=w^o$ and the outsourcing order q' is such that* $F(q')=\dfrac{m+h^- -w^0}{m+h^- +h^+}$.

Example 2.10

Let the demand be characterized by the uniform distribution,

$$f(D)=\begin{cases}\dfrac{1}{A}, & \text{for } 0\le D\le A;\\ 0, \text{otherwise}\end{cases} \quad\text{and}\quad F(a)=\frac{a}{A},\ 0\le a\le A.$$

Then using the results of Example 2.8, we have a unique solution for each case.

If $w^o<c$, then produce $q''=\dfrac{m+h^- -c_m}{m+h^- +h^+}A$ products in-house, which is equivalent to the system-wide optimal solution.

If w^o=c, then we outsource; the equilibrium wholesale price is w^s=c and the outsourcing quantity is $q^s = \frac{m+h^- - c}{m+h^- + h^+} A$ *products, which is equivalent to the system-wide optimal order.*

If $\frac{m+h^- + c}{2} \le w^o$ *(and thus w^o>c), then we outsource; the equilibrium wholesale price is* $w^s = \frac{m+h^- + c}{2}$ *and the outsourcing order is* $q^s = q' = \frac{m+h^- - c}{m+h^- + h^+} \frac{A}{2}$.

If $\frac{m+h^- + c}{2} > w^o > c$*, then we outsource; the equilibrium wholesale price is* $w^s = w^o$ *and outsourcing order quantity is* $q^s = q' = \frac{m+h^- - w^0}{m+h^- + h^+} \frac{A}{2}$ *products,*

where w^o satisfies the expression

$$\int_0^{q''} m\frac{D}{A} dD + \int_{q''}^{\infty} m\frac{q''}{A} dD - \int_0^{q''} \frac{h^+}{A}(q''-D)dD - \int_{q''}^{\infty} \frac{h^-}{A}(D-q'')dD - c_m q'' - C\}$$

$$= \int_0^{q'} m\frac{D}{A} dD + \int_{q'}^{\infty} m\frac{q'}{A} dD - \int_0^{q'} \frac{h^+}{A}(q'-D)dD - \int_{q'}^{\infty} \frac{h^-}{A}(D-q')dD -$$

$$w^o q'\}, \ q'' = \frac{m+h^- - c_m}{m+h^- + h^+} A \text{ and } q' = \frac{m+h^- - w^o}{m+h^- + h^+} A .$$

Example 2.11

Let the demand be characterized by an exponential distribution, i.e.,

$$f(D) = \begin{cases} \lambda e^{-\lambda D}, & \text{for } D \ge 0; \\ 0, \text{otherwise} \end{cases} \quad \text{and } F(a) = 1 - e^{-\lambda a}, \ a \ge 0.$$

We first formalize equation (2.51) for w^o which, for the exponential distribution yields,

$$\int_0^{q''} [mD - h^+(q''-D)]\lambda e^{-\lambda D} dD + \int_{q''}^{\infty} [mq'' - h^-(D-q'')]\lambda e^{-\lambda D} dD - c_m q'' - C =$$

$$=\int_0^{q'}[mD-h^+(q'-D)]\lambda e^{-\lambda D}dD+\int_{q'}^{\infty}[mq'-h^-(D-q')]\lambda e^{-\lambda D}dD\text{-}w^o q',$$

where $q''=\dfrac{1}{\lambda}\ln\dfrac{m+h^-+h^+}{h^++c_m}$ and $q'=\dfrac{1}{\lambda}\ln\dfrac{m+h^-+h^+}{h^++w^0}$.

We calculate this expression with Maple. Specifically, we set the order quantities q'' and q' as q2 and q1 respectively,

```
> q2:=1/lambda*ln((m+hplus+hminus)/(cm+hplus));
```

$$q2 := \frac{\ln\left(\frac{m + hplus + hminus}{cm + hplus}\right)}{\lambda}$$

```
> q1:=1/lambda*ln((m+hplus+hminus)/(w0+hplus));
```

$$q1 := \frac{\ln\left(\frac{m + hplus + hminus}{w0 + hplus}\right)}{\lambda}$$

Next we define the left-hand side and right-hand side of (2.51) as LHS and RHS

```
>LHS:=int((m*D-hplus*(q2-D))*lambda*exp(-lambda*
D),D=0..q2)+int((m*q2-hminus*(D-q2))*lambda*exp(-
lambda*D), D=q2..infinity)-cm*q2-C:
>RHS:=int((m*D-hplus*(q1-D))*lambda*exp(-lambda*
D),D=0..q1)+int((m*q1-hminus*(D-q1))*lambda*exp(-
lambda*D), D=q1..infinity)-w0*q1:
```

Then to see how fixed cost, C, effects the solution, specific values are substituted for the parameters of the problem except for C.

```
> LHSC:=subs(m=15,hplus=1,hminus=10,cm=2,lambda=0.1,
LHS);
> RHS1:=subs(m=15,hplus=1,hminus=10,cm=2,lambda=0.1,
RHS);
```

After evaluating the left-hand side and the right-hand side

```
> LHSCe:=evalf(LHSC);
```

$$LHSCe := 65.2154725 - 1.\,C$$

```
> RHSe:=evalf(RHS1);
```

$$RHSe := -15.76923077\ \ln\left(\frac{26.}{w0+1.}\right) + 168.7967107 + 8.796710786\ w0 - 15.76923077\ \ln\left(\frac{26.}{w0+1.}\right) w0 + 5.769230769\ \ln\left(\frac{1}{w0+1.}\right) + 5.769230769\ w0\ln\left(\frac{1}{w0+1.}\right)$$

we solve (2.51) in w^0

```
> solutionw0:=solve(LHSCe=RHSe, w0);
```
and plot the solution as a function of the fixed cost
```
>plot(solutionw0, C=0..200);
```

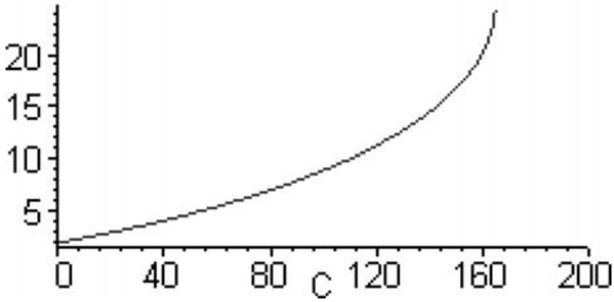

Figure 2.6. The effect of the fixed cost C on the maximum wholesale price w_0

The plot (Figure 2.6) implies that the higher the fixed cost, C, the greater w^0 and thus the smaller the chance that in-house production is beneficial compared to the outsourcing. For example, if C=120
```
> LHSes:=subs(C=120, LHSCe);
```
$$LHSes := -54.7845275$$

then
```
>solve(LHSes=RHSe, w0);
```
$$11.26258264$$

w^0=11.2625 and thus if supplier's cost c>11.2625, the in-house production is advantageous (and is system-wide optimal) at quantity q''*=q2opt=21.594
```
>q2opt:=evalf(subs(m=15, hplus=1, hminus=10, cm=2,
lambda=0.1, q2));
```
$$q2opt := 21.59484249$$

Otherwise, if $c \le 11.2625$, then outsourcing is advantageous and the Stackelberg equilibrium wholesale price w' and order quantity q' are calculated as described in the previous section. Note that in case of $w'>w^o$, the Stackelberg wholesale price equals w^o and the order quantity is computed correspondingly.

Coordination

If $w^0>c$, then outsourcing has a negative impact compared to the corresponding centralized supply chain, the manufacturer orders less and the service level decreases. This is similar to the vertical inventory game without a setup cost considered in the previous section. In contrast to that game, this effect is reduced when $c \leq w^o < w^s$, where w^s is calculated under an assumption of no constraints, i.e., according to Proposition (2.7). In addition, there can be a special case when $w^o = c$, and thus the supplier is forced to set the wholesale price equal to its marginal cost, $w=c$. This eliminates double marginalization, the manufacturer outsources the system-wide optimal quantity and the supply chain becomes perfectly coordinated regardless of whether the supplier is leader in a Stackelberg game or the firms make decisions simultaneously using a Nash strategy. On the other hand, since the case when the manufacturer prefers in-house production is identical to the corresponding centralized problem, no coordination is needed. Consequently, the case which requires coordination is when $w^0 > c$. This case coincides with that derived for the inventory game with no setup cost. Thus, the coordinating measures discussed in the previous section are readily applied to an outsourcing-based supply chain.

An alternative way of improving the supply chain performance is to develop a risk-sharing contract which would make it possible to coordinate the chain in an efficient manner as discussed in the following section.

2.4 INVENTORY COMPETITION WITH RISK SHARING

In competitive conditions discussed so far, the retailer incurs the overall risk associated with uncertain demands. The fact that expected profit is the criterion for decision-making implies that the retailer does not have an assured profit. The supplier, on the other hand, profits by the quantity he sells. If the supplier is sensitive to the retailer's service level, he may agree to mitigate demand uncertainty by buying back left-over products at the end of selling season or offer an option for additional urgent deliveries to cover cases of higher than expected demand. These well-known types of risk-sharing contracts make it possible to improve the service level as well as to coordinate the supply chain as discussed in the following sections. (See also Ritchken and Tapiero 1986).

2.4.1 THE INVENTORY GAME WITH A BUYBACK OPTION

A modification of the traditional newsvendor problem considered here arises when the supplier agrees to buy back leftovers at the end of selling season at a price, $b(w)$, $\dfrac{\partial b(w)}{\partial w} \geq 0$ and $\dfrac{\partial^2 b(w)}{\partial w^2} \geq 0$. This means that the uncertainty associated with random demand may result in inventory associated costs, $b(w)x^+$ at the supplier's site while at the retailer's site it is an income $b(w)x^+$ rather than a cost. Thus the supplier mitigates the retailer's risk associated with demand overestimation or, in other words, the supplier shares costs associated with demand uncertainty. The other parameters of the problem remain the same as those of the stocking game.

The retailer's problem

$$\max_q J_r(q,w)= \max_q \{E[ym + b(w)x^+ - h^-x^-]-wq\}, \tag{2.58}$$

s.t.

$$x=q-d,$$
$$q \geq 0,$$

where $x^+=\max\{0, x\}$, $x^- =\max\{0, -x\}$ and $y=\min\{q,d\}$.

Applying conditional expectation to (2.58) the objective function transforms into

$$\max_q J_r(q,w)= \max_q \{$$

$$\int_0^q mDf(D)dD+\int_q^\infty mqf(D)dD+\int_0^q b(w)(q-D)f(D)dD-\int_q^\infty h^-(D-q)f(D)dD-wq\}. \tag{2.59}$$

The first term in the objective function, $E[ym]= \int_0^q mDf(D)dD + \int_q^\infty mqf(D)dD$, represents income from selling y product units; the second, $E[b(w)x^+]= \int_0^q b(w)(q-D)f(D)dD$, represents income from selling leftover goods at the end of the period; the third, $E[h^-x^-]= \int_q^\infty h^-(D-q)f(D)dD$, represents losses due to an inventory shortage; while the last term, wq, is the amount paid to the supplier for purchasing q units of product. As discussed earlier, there is a maximum wholesale price, w^M, that the supplier can charge so that the retailer will still continue to buy products. Taking this into account we formulate the supplier's problem.

The supplier's problem

$$\max_w J_s(q,w)= \max_w (w-c)q-E[b(w)x^+] \tag{2.60}$$

s.t.

$$c \le w \le w^M.$$

The first term $(w\text{-}c)q$ in (2.60) represents the supplier's income from selling q products at margin $w\text{-}c$, while the second, $E[b(w)x^+]$ is the payment for the returned leftovers to the supplier. To simplify the problem, we here assume that leftovers are salvaged at a negligible price rather than stored at the supplier's site. The centralized problem is then based on the sum of two objective functions (2.59) and (2.60) which results in a function independent of the wholesale price, w.

The centralized problem

$$\max_q J(q) = \max_q \{E[ym - h^- x^-] - cq\} \tag{2.61}$$

s.t.

$$x = q - d,\ q \ge 0.$$

Note that since w and b represent transfers within the supply chain, system-wide profit does not depend on them.

System-wide optimal solution

Applying conditional expectation to (2.61) and the first-order optimality condition, we find that

$$\frac{\partial J(q)}{\partial q} = mqf(q) - mqf(q) + \int_q^\infty mf(D)dD - \int_q^\infty h^- f(D)dD - c = 0,$$

which results in

$$F(q^*) = \frac{m + h^- - c}{m + h^-}. \tag{2.62}$$

Since this result differs from (2.33) by only h^+ set at zero, the objective function in (2.61) is strictly concave under the same assumptions. Similarly, the service level in the centralized supply chain with a buyback contract is

$$\alpha = \frac{m + h^- - c}{m + h^-}, \tag{2.63}$$

This is different from $\alpha = \dfrac{m + h^- - c}{m + h^- + h^+}$ of the traditional newsvendor problem only because of our assumption that surplus products are salvaged at a negligible price rather than stored at the supplier's site.

Game analysis

Consider now a decentralized supply chain characterized by non-cooperative firms and assume that both players make their decisions simultaneously. The supplier chooses the wholesale price w and thereby buyback $b(w)$ price while the retailer selects the order quantity, q. The supplier then delivers the products and buys back leftovers.

Using the first-order optimality conditions for the retailer's problem, we find $w^M=m+h^-$, so that if $w \le w^M$, then

$$F(q)=\frac{m+h^- - w}{m+h^- - b(w)}. \tag{2.64}$$

Since the retailer's objective function is strictly concave, we conclude from (2.64), the following result.

Proposition 2.10. *In vertical competition, if the supplier makes a profit, i.e., $w>c$, a buyback contract induces increased retail orders and an improved customer service level compared to that obtained in the corresponding stocking game.*

Proof: To prove this proposition, compare the optimal orders with the non-cooperative buyback option

$$F(q)=\frac{m+h^- - w}{m+h^- - b(w)},$$

and without the buyback option

$$F(q)=\frac{m+h^- - w}{m+h^- + h^+}.$$

From Proposition 2.10 we conclude that the buyback contract has a coordinating effect on the supply chain. Moreover, comparing (2.62) and (2.64), we observe that in contrast to the stocking game, with buyback contracts, i.e., $b(c)>0$, when setting $w=c$, the retailer orders even more than the system-wide optimal quantity since there is less risk of overestimating demands. In such a case, the supplier has only losses due to buying back leftover products. Thus, the supplier can select $w>c$ so that the retailer's non-cooperative order will be equal to the system-wide optimal order quantity. This coordinating choice will be discussed below after analyzing possible equilibria.

Equilibrium

Let us first consider the case of $\frac{\partial b(w)}{\partial w} > 0$, $\frac{\partial^2 b(w)}{\partial w^2} > 0$ and assume that $b(w)$ is chosen such that $\lim_{w} J_s(q,w) = -\infty$, i.e., the solution set is compact. Then the Nash equilibrium can be found by differentiating the supplier's objective function $J_s(q,w)=(w-c)q-E[b(w)x^+]=(w-c)q-\int_0^q b(w)(q-D)f(D)dD$,

$$\frac{\partial J_s(q,w)}{\partial w} = q - \frac{\partial b(w)}{\partial w}\int_0^q (q-D)f(D)dD = 0. \qquad (2.65)$$

Verifying the second-order optimality condition, we also find

$$\frac{\partial^2 J_s(q,w)}{\partial w^2} = -\frac{\partial^2 b(w)}{\partial w^2}\int_0^q (q-D)f(D)dD < 0. \qquad (2.66)$$

Since the functions of both supplier and retailer are strictly concave and the solution space is compact, we readily conclude that a Nash equilibrium exists (see, for example, Basar and Olsder 1999).

Proposition 2.11. *The pair (w^n,q^n), such that*

$$q^n - \frac{\partial b(w^n)}{\partial w}\int_0^{q^n} (q^n - D)f(D)dD = 0,\ F(q^n) = \frac{m+h^- - w^n}{m+h^- - b(w^n)}$$

constitutes a Nash equilibrium of the inventory game under a buyback option.

An interesting case arises when $b(w)$ is a linear function of w. In such a case, similar to the traditional stocking game, $J_s(q,w)$ depends linearly on w, i.e., the supplier would set the wholesale price as high as possible. Unlike the stocking game, this situation does not lead to no-business under a buyback contract. Indeed, by setting w close to but less than w^M, the supplier may still be able to induce the retailer to order the desired quantity by properly choosing a function $b^*=b^*(w)$. In fact, this strategy leads to perfect coordination regardless of the fact whether the supplier is the Stackelberg leader or the decision is made simultaneously. This is because under any wholesale price w, $b^*=b^*(w)$ would ensure the same response from the retailer by increasing w the supplier increases his profit. Thus, this time we find *the greater the wholesale price, the greater the supplier's profit while the order quantity remains the same*.

Example 2.12

Let $\frac{\partial b(w)}{\partial w} > 0$, $\frac{\partial^2 b(w)}{\partial w^2} > 0$ and the demand be characterized by the uniform distribution,

$$f(D) = \begin{cases} \frac{1}{A}, & \text{for } 0 \le D \le A; \\ 0, \text{otherwise} \end{cases} \quad \text{and } F(a) = \frac{a}{A},\ 0 \le a \le A.$$

Then using (2.64), we find

$$q^n = \frac{m + h^- - w^n}{(m + h^- - b(w^n))} A.$$

Substituting into (2.65) we have

$$\frac{m + h^- - w^n}{m + h^- - b(w^n)} A - \frac{\partial b(w^n)}{\partial w} \left(\frac{m + h^- - w^n}{m + h^- - b(w^n)} \right)^2 \frac{A}{2} = 0.$$

Rearranging this last equation we obtain

$$\frac{m + h^- - w^n}{m + h^- - b(w^n)} A \left(1 - \frac{\partial b(w^n)}{\partial w} \frac{m + h^- - w^n}{m + h^- - b(w^n)} \frac{1}{2} \right) = 0.$$

Since $w^n = w^M = m + h^-$ results in no order at all, the Nash equilibrium is found by

$$1 - \frac{\partial b(w^n)}{\partial w} \frac{m + h^- - w^n}{m + h^- - b(w^n)} \frac{1}{2} = 0.$$

If for example, $b(w) = \alpha + \beta w^2$, and the buyback price does not exceed the maximum price, $\alpha + \beta [w^M]^2 < m + h^-$, then we have a unique Nash equilibrium

$$w^n = \frac{1}{\beta}\left(1 - \frac{\alpha}{m + h^-}\right),\quad q^n = \frac{m + h^- - w^n}{m + h^- - \alpha - \beta [w^n]^2} A.$$

On the other hand, the system-wide optimal order is

$$q^* = \frac{m + h^- - c}{m + h^-} A.$$

Coordination

As discussed in previous sections, discounting, for example, a two-part tariff is one tool which provides coordination by inducing a non-cooperative solution to tend to the system-wide optimum.

In this section we show that buyback contacts provide an efficient means for coordinating vertically competing supply chain participants. Specifically, when $b(w)$ is a linear function of w, the supplier's objective function depends linearly on w. This implies that it is optimal for the supplier to set the wholesale price as high as possible. However, unlike the traditional stocking game, this situation does not lead to no orders if the supplier chooses $b^*=b^*(w)$ as described below.

Let the best retailer's response q defined by (2.64) be identical to the system-wide optimal solution q^* defined by (2.62),

$$\frac{m+h^- - c}{m+h^-} = \frac{m+h^- - w}{m+h^- - b^*(w)}. \tag{2.67}$$

From (2.67) we conclude that if

$$b^*(w) = \left(m+h^-\right)\frac{w-c}{m+h^- - c}, \tag{2.68}$$

then $q=q^*$ for any $w<w^M$. Thus, if $b^*(w)$ is set according to (2.68), the supplier can maximize his profit by choosing w very close to w^M. This would leave the retailer still ordering a system-wide optimal quantity which would perfectly coordinate the supply chain. This result is independent of the fact whether the supplier first sets w and $b^*(w)$ (as Stackelberg leader) or whether decisions on w and q are made simultaneously (Nash strategy) if function $b^*(w)$ is known to the retailer.

Example 2.13

Let the demand be characterized by an exponential distribution, i.e.,

$$f(D) = \begin{cases} \lambda e^{-\lambda D}, & \text{for } D \ge 0; \\ 0, \text{otherwise} \end{cases} \quad \text{and } F(a) = 1 - e^{-\lambda a}, \; a \ge 0$$

and $b^*=b^*(w)$ be chosen by the supplier so that the best retailer's response q is identical to the system-wide optimal solution q^*, that is, $b^*(w)$ is determined by (2.68). Then the equilibrium wholesale and buyback prices are

$$w=w^M-\varepsilon=m+h^- -\varepsilon \text{ and } b^*(w) = \left(m+h^-\right)\left(1-\frac{\varepsilon}{m+h^- - c}\right),$$

where ε is a small number and the equilibrium order quantity is

$$q= \frac{1}{\lambda}\ln\frac{m+h^-}{c}.$$

Note that the smaller the ε, the greater the supplier's share of the risk associated with uncertain demands and the greater the share of the overall supply chain profit that the supplier gains on account of the retailer. When ε is very small, the retailer returns all unsold products at almost the same wholesale price he purchased them. He therefore has no risk at all in case the demand realization will be lower than the quantity stocked.

2.4.2 THE INVENTORY GAME WITH A PURCHASING OPTION

Similar to the buyback option, this modification of the stocking game arises when the supplier is willing to mitigate the risk the retailer incurs with respect to the uncertainty of customer demands. Specifically, similar to a buyback contract, the supplier may agree to have an inventory surplus at the end of the selling season. In contrast to the buyback contract, this surplus is due to an option which is offered to the retailer. The option allows the retailer to issue an urgent or fast order, to be shipped immediately, at a predetermined option price, $m>u(w)>w$, $\frac{\partial u(w)}{\partial w} \geq 0$, close to the end of the selling season. The retailer will exercise this option only if customer demand exceeds his inventories. It is this difference between the retailer's backorder and the supplier's inventory level which the retailer's option purchase covers. If the supplier is unable to satisfy such a backorder, he will compensate the retailer for his loss. Thus, under this type of contract, the supplier assumes the customer service level at the retailer's site by mitigating the retailer's backlog costs. We assume that the system parameters are such that the supplier's order q_s exceeds the retailer's order q_r, $q_r<q_s$ (an exact requirement for this to hold is stated in Proposition 2.13) which ensures an inventory game between the retailer and supplier. Furthermore, we assume that the wholesale price and the retailer's margin are fixed and the supplier cost is negligible unless it is an urgent order. This enables us to focus solely on the inventory game where the supplier and retailer have to choose a quantity to order. To draw an analogy with our previous analysis, we allow the wholesales price to change when coordination aspects are discussed.

The retailer's problem

$$\max_{q_r} J_r(q_r,q_s)= \max_{q_r} \{E[my+(m- u(w))x_r^- - h_r^+x_r^+ - h_r^-x_s^-] - wq_r\}, \quad (2.69)$$

s.t.

$$x_r=q_r\text{-}d,$$
$$x_s=q_s - q_r- x_r^-,$$
$$q_r \geq 0,$$

where $x_r^+=\max\{0, x_r\}$, $x_r^- =\max\{0, -x_r\}$ and $y=\min\{d, q_r\}$,

In this single-period formulation, x_r is the retailer's inventory level by the end of a period prior to an urgent order when realization, D, of random demand d is already known; x_r^+ is the retailer's inventory surplus at the end of the period; x_r^- is the retailer's inventory shortage prior to an urgent order; the urgent quantity ordered by the retailer for immediate shipment,

h_r^+, h_r are the retailer's inventory holding and shortage costs respectively; and q_r is the quantity ordered by the retailer at the beginning of the period and shipped by the end of the period. If the supplier does not have enough products to ship, then a purchase option implies that the supplier covers the difference between the retailer's margin and the option price m-$u(w)$ for unsold product.

Applying conditional expectation to (2.69), the objective function transforms into

$$\max_{q_r} J_r(q_r,q_s) = \max_{q_r} \{$$

$$\int_0^{q_r} mDf(D)dD + \int_{q_r}^{\infty} mq_r f(D)dD + \int_{q_r}^{\infty} (m-u(w))(D-q_r)f(D)dD - \int_0^{q_r} h_r^+(q_r-D)f(D)dD$$

$$-\int_{q_s}^{\infty} h_r^-(D-q_s)f(D)dD - wq_r \}. \tag{2.70}$$

The first term in the objective function, $E[ym] = \int_0^{q_r} mDf(D)dD + \int_{q_r}^{\infty} mq_r f(D)dD$, represents the income from selling $y=\min\{d,q_r\}$ product units; the second, $E[(m\text{-}u(w))x_r^-] = \int_{q_r}^{\infty}(m-u(w))(D-q_r)f(D)dD$, represents the income from backlog at the end of the period; the third and the fourth, $E[h_r^+x_r^+] = \int_0^{q_r} h_r^+(q_r-D)f(D)dD$, $E[h_r x_s] = \int_{q_s}^{\infty} h_r^-(D-q_s)f(D)dD$, are the surplus and shortage costs; and the last term, wq_r, is the amount paid to the supplier for a regular order.

The supplier's problem

$$\max_{q_s} J_s(q_r,q_s) =$$

$$\max_{q_s} \{wq_r + E[(u(w)-c)(x_r^- - x_s^-) - (m - u(w))\, x_s^- - h_s^+ x_s^+]\}, \tag{2.71}$$

s.t.

$$x_s = q_s - q_r - x_r,$$
$$x_r = q_r - d,$$
$$q_s \geq 0,$$
$$x_s^+ = \max\{0, x_s\},\ x_s = \max\{0, -x_s\},$$

where x_s is the supplier's inventory level by the end of period after an urgent order; q_s is the quantity ordered by the supplier at the beginning of the period and shipped in time for reshipment from the supplier to the retailer by the end of the period; $u(w)$ is the option price; h_s^+ is the supplier's inventory holding cost; and c is the cost of processing the urgent order.

After simple manipulations with (2.71)

$$J_s(q_r,q_s)= wq_r +E[(u(w)-c)x_r - (m-c)x_s^- - h_s^+x_s^+]$$

and determining expectation, we have

$$J_s(q_r,q_s)=wq_r+\int_{q_r}^{\infty}(u(w)-c)(D-q_r)f(D)dD-\int_{q_s}^{\infty}(m-c)(D-q_s)f(D)dD- \\ \int_{q_r}^{q_s}h_s^+(q_s-D)f(D)dD-\int_{0}^{q_r}h_s^+(q_s-q_r)f(D)dD\}. \tag{2.72}$$

The first term in the objective function, wq_r, is the income from selling q_r products; the second, $E[(u(w)-c)x_r]=\int_{q_r}^{\infty}(u(w)-c)(D-q_r)f(D)dD$, represents income from the optional order; the third, $E[(m-c)x_s] =\int_{q_s}^{\infty}(m-c)(D-q_s)f(D)dD$, represents the compensation paid by the supplier for the part of the optional order which the supplier is unable to deliver (i.e., this is the supplier's shortage cost); and the last term, $E[h_s^+x_s^+]=\int_{q_r}^{q_s}h_s^+(q_s-D)f(D)dD$ $+\int_{0}^{q_r}h_s^+(q_s-q_r)f(D)dD$, is the inventory surplus cost incurred by the supplier.

The centralized problem is based on the sum of two of the objective functions (2.69) and (2.71).

The centralized problem

$$\max_{q_r,q_s} J(q_r,q_s)= \max_{q_r,q_s} \{E[my+(m-c)(x_r^- - x_s^-) - h_r^+x_r^+ - h_s^+x_s^+ - h_r^-x_s^-]\} \tag{2.73}$$

s.t.

$$x_s=q_s - q_r - x_r,$$
$$x_r=q_r - d,$$
$$q_r\geq 0,\ q_s\geq 0.$$

Note that since w, $u(w)$ and $(m-c)x_s^-$ represent transfers within the supply chain, the system-wide profit does not depend on w, $u(w)$ and is reduced by $(m\text{-}c)x_s^-$ to account only for the satisfied part $(x_r^- - x_s^-)$ of the optional (urgent) order. Applying conditional expectation to (2.73) we have explicitly,

$$J(q_r,q_s)=\int_0^{q_r} mDf(D)dD+\int_{q_r}^{\infty} mq_r f(D)dD+\int_{q_r}^{\infty}(m-c)(D-q_r)f(D)dD-$$

$$\int_{q_s}^{\infty}(m-c)(D-q_s)f(D)dD-\int_{q_s}^{\infty}h_r^-(D-q_s)f(D)dD-\int_0^{q_r}h_r^+(q_r-D)f(D)dD$$

$$-\int_{q_r}^{q_s}h_s^+(q_s-D)f(D)dD-\int_0^{q_r}h_s^+(q_s-q_r)f(D)dD=mE[D]-\int_{q_r}^{\infty}c(D-q_r)f(D)dD$$

$$-\int_{q_s}^{\infty}(m-c)(D-q_s)f(D)dD\ -\int_{q_s}^{\infty}h_r^-(D-q_s)f(D)dD$$

$$-\int_0^{q_r}h_r^+(q_r-D)f(D)dD\ -\int_{q_r}^{q_s}h_s^+(q_s-D)f(D)dD-\int_0^{q_r}h_s^+(q_s-q_r)f(D)dD.$$

System-wide optimal solution

The first-order optimality condition with respect to q_r results in

$$\frac{\partial J(q_r,q_s)}{\partial q_r}=\int_{q_r}^{\infty}cf(D)dD\ -\int_0^{q_r}h_r^+f(D)dD+h_s^+(q_s-q_r)f(q_r)$$

$$-h_s^+(q_s-q_r)f(q_r)=0.$$

Thus, the system-wide unique optimal order quantity of the supplier is

$$F(q_r^*)=\frac{c}{c+h_r^+}. \tag{2.74}$$

Similarly, the first-order optimality condition with respect to q_s yields,

$$\frac{\partial J(q_r,q_s)}{\partial q_s}=(m-c)(1-F(q_s))-h_s^+(F(q_s)-F(q_r))-h_s^+F(q_r)+$$

$$h_r^-(1-F(q_s))=0.$$

Thus, the system-wide unique optimal supplier's order is

$$F(q_s^*)=\frac{m-c+h_r^-}{m-c+h_s^+ +h_r^-}. \tag{2.75}$$

Furthermore, since the first derivative in one of the variables is independent of the other variable, the corresponding Hessian is negative definite and this newsvendor type of the objective function is strictly concave in both decision variables.

Game analysis

Consider now a decentralized supply chain characterized by non-cooperative firms and assume that both players make their decisions simultaneously. After the retailer and supplier choose their orders q_r and q_s , the supplier delivers q_r units as a regular order and (x_r^- - x_s^-) as an urgent order as well as covers the retailer for losses if the urgent order does saturate the demand, x_s^-.

Applying the first-order optimality condition to the retailer's objective function (2.70) we find

$$\frac{\partial J(q_r,q_s)}{\partial q_r}=$$

$$mq_r f(q_r)-mq_r f(q_r)+\int_{q_r}^{\infty} mf(D)dD-\int_{q_r}^{\infty}(m-u(w))f(D)dD-\int_{0}^{q_r} h_r^+ f(D)dD\text{-}w=$$

$$=m(1-F(q_r))-(m-u(w))(1-F(q_r))-h_r^+F(q_r)-w=0,$$

that is,

$$F(q_r)=\frac{u(w)-w}{u(w)+h_r^+}. \tag{2.76}$$

Equation (2.76) represents a unique, newsvendor-type, optimal solution. As long as our assumption $u(w)<m$ holds, the regular order is independent of the retailer's margin. Shortage cost h_r^- is not a part of this equation since the purchasing option causes a shortage which depends on the supplier's order quantity rather than on the retailer's decision.

To determine the Nash equilibrium, we next differentiate the supplier's objective function (2.72),

$$\frac{\partial J(q_r,q_s)}{\partial q_s}=\int_{q_s}^{\infty}(m-c)f(D)dD-\int_{q_r}^{q_s}h_s^+ f(D)dD-\int_{0}^{q_r}h_s^+ f(D)dD=$$

$$=(m-c)(1-F(q_s))-h_s^+(F(q_s)-F(q_r))-h_s^+F(q_r)=0$$

that is,

$$F(q_s)=\frac{m-c}{m-c+h_s^+}. \tag{2.77}$$

This solution is unique and identical to (2.75) if h_r^-=0, that is, *the supplier's equilibrium order is system-wide optimal if h_r^- is negligible*.

However, if h_r^->0, then $q_s^*>q_s$.

Equilibrium

It is easy to verify that the second derivative with respect to the supplier's order quantity is negative and the supplier's objective function is also strictly concave. Thus, imposing our assumption, $q_r \le q_s$, we readily conclude with the following statement.

Proposition 2.12. *Let* $\frac{m-c}{m-c+h_s^+} \ge \frac{u(w)-w}{u(w)+h_r^+}$. *The pair* (q_r^n, q_s^n), *such that*

$$F(q_r^n) = \frac{u(w)-w}{u(w)+h_r^+} \text{ and } F(q_s^n) = \frac{m-c}{m-c+h_s^+}$$

constitutes a unique Nash equilibrium of the inventory game under a purchasing option.

Since $c<u(w)<m$, then we can assume that $u(w)$-$w \le c$. If this condition holds, then $F(q_r *) = \frac{c}{c+h_r^+} > F(q_r^n) = \frac{u(w)-w}{u(w)+h_r^+}$ which, of course, is not a new discovery. In contrast to previous results, the total order also includes urgent order, x_r^--x_s^-, while the supplier's inventory level, $F(q_s) = \frac{m-c}{m-c+h_s^+}$ determines the service level in the supply chain with a purchasing option. We thus conclude with the following property:

Proposition 2.13. *Let* $\frac{m-c}{m-c+h_s^+} \ge \frac{u(w)-w}{u(w)+h_r^+}$. *In vertical competition, if* $u(w)$-$w \le c$, *a contract with a purchasing option induces lower order quantities from the retailer and supplier as well as a lower service level than the system-wide optimal solution.*

Next, comparing the retailer's order with (q_r^n) and without (q_r) purchasing option (see the stocking game in Section 2.3.2), we conclude that

$$F(q_r^n) = \frac{u(w)-w}{u(w)+h_r^+} < F(q_r) = \frac{m+h_r^- - w}{m+h_r^- + h_r^+},$$

as $u(w)<m$.

Proposition 2.14. *Let* $\frac{m-c}{m-c+h_s^+} \ge \frac{u(w)-w}{u(w)+h_r^+}$. *In vertical competition, a contract with a purchasing option induces a lower regular order quantity by the retailer compared to the contract without a purchasing option, while the service level depends on* h_s^+.

From Proposition 2.14, it follows that unless the supplier's inventory holding cost is too high, a contract with a purchasing option improves the service level, but the regular order quantity decreases. This is expected, since, given the possibility of an urgent order, it is beneficial for the retailer to reduce the regular order and wait for demand to realize and only then increase profit by an urgent purchase if the demand exceeds the regular order stock. Note that since the urgent order is random, $x_r^- - x_s^-$, and always non-negative, it means that

$$E[x_r^- - x_s^-] = \int_{q_r}^{\infty}(D-q_r)f(D)dD - \int_{q_s}^{\infty}(D-q_s)f(D)dD, \tag{2.78}$$

is not zero and thus the overall quantity ordered by the retailer is greater than that of a regular order. Moreover, the regular order quantity can be increased since a contract with a purchasing option allows efficient coordination by the proper choice of the option price, $u(w)$. These results are demonstrated in the following example.

Example 2.14

Let the demand be characterized by the uniform distribution,

$$f(D) = \begin{cases} \dfrac{1}{A}, & \text{for } 0 \le D \le A; \\ 0, \text{otherwise} \end{cases} \quad \text{and } F(a) = \frac{a}{A}, 0 \le a \le A.$$

Then using Proposition 2.12, we find the Nash equilibrium

$$q_r^{\ n} = \frac{u(w)-w}{u(w)+h_r^+}A \text{ and } q_s^{\ n} = \frac{m-c}{m-c+h_s^+}A.$$

The centralized solution is

$$q_r{}^* = \frac{c}{c+h_r^+}A \text{ and } q_s{}^* = \frac{m-c+h_r^-}{m-c+h_s^+ + h_r^-}A \ .$$

The average urgent order is thus,

$$E[x_r^- - x_s^-] = \int_{q_r}^{\infty}(D-q_r)f(D)dD -$$

$$\int_{q_s}^{\infty}(D-q_s)f(D)dD = (q_s^{\ n} - q_r^{\ n})[1 - \frac{1}{2A}(q_s^{\ n} + q_r^{\ n})] > 0,$$

while the total average retailer's order is

$$q_r^n + (q_s^{\ n} - q_r^{\ n})[1 - \frac{1}{2A}(q_s^{\ n} + q_r^{\ n})].$$

Coordination

Coordination under a purchasing option is similar to buyback contacts where a proper choice of the buyback price, $b(w)$, induces the retailer to choose a system-wide optimal order quantity. Specifically, if the supplier chooses the option price $u(w)$ as a linear function of w, $u^*(w)$, so that

$$\frac{u^*(w)-w}{u^*(w)+h_r^+}=\frac{c}{c+h_r^+},$$

and thus

$$u^*(w)=\frac{w+\dfrac{ch_r^+}{c+h_r^+}}{1-\dfrac{c}{c+h_r^+}}, \tag{2.79}$$

then $q_r^n=q_r^*$. Moreover, since $u^*(w)$ is chosen as a linear function of w, the supplier, as is the case with the buyback contacts, can increase the wholesale price very close to its maximum level and thus gain most of the supply chain profit while still having the retailer order the system-wide optimal quantity. The overall game will, however, become perfectly coordinated only if the retailer's shortage cost is negligible. If it is not negligible, sharing inventory-related costs may have a positive effect on the supply chain's performance.

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