2

Damage Models

Consider a standard cumulative damage model [11] for an operating unit: A unit is subjected to shocks and suffers some damage due to shocks. Let random variables X_j $(j = 1, 2, \dots)$ denote a sequence of interarrival times between successive shocks, and random variables W_j $(j = 1, 2, \dots)$ denote the damage produced by the jth shock, where $W_0 \equiv 0$. It is assumed that the sequence of $\{W_j\}$ is nonnegative, independently, and identically distributed, and furthermore, W_j is independent of X_i $(i \neq j)$. This is called a *jump process* [81] or *doubly stochastic process* [82].

Let $N(t)$ denote the random variable that is the total number of shocks up to time t $(t \geq 0)$. Then, define a random variable

$$Z(t) \equiv \sum_{j=0}^{N(t)} W_j \qquad (N(t) = 0, 1, 2, \dots), \tag{2.1}$$

where $Z(t)$ represents the total damage at time t. It is assumed that the unit fails when the total damage has exceeded a prespecified level K $(0 < K < \infty)$ for the first time (see Figure 2.1). Usually, a failure level K is statistically estimated and is already known. Of interest is a random variable $Y \equiv \min\{t; Z(t) > K\}$, *i.e.*, $\Pr\{Y \leq t\}$ represents the distribution of the failure time of the unit.

In this chapter, we consider two damage models: (1) the cumulative damage model where the total damage is additive, and (2) the independent damage model where the total damage is not additive, *i.e.*, it is independent of the previous damage level. For each model, we are interested in the following reliability quantities:

(i) $\Pr\{Z(t) \leq x\}$; the distribution of the total damage at time t.
(ii) $E\{Z(t)\}$; the total expected damage at time t.
(iii) $\Pr\{Y \leq t\}$; the first-passage time distribution to failure.
(iv) $E\{Y\}$; the mean time to failure (MTTF).

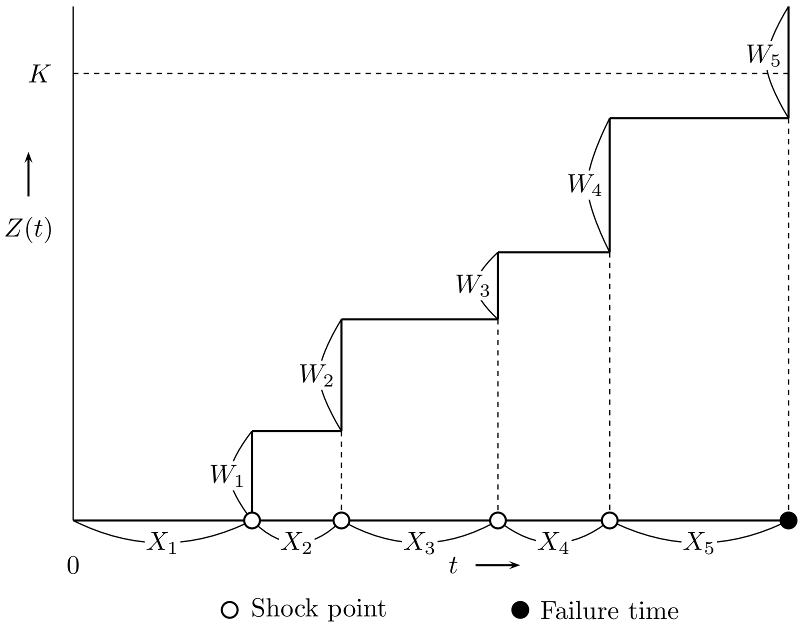

Fig. 2.1. Process for a standard cumulative damage model

(v) Failure rate or hazard rate $r(t)$; $r(t)\mathrm{d}t = \Pr\{t < Y \le t + \mathrm{d}t | Y > t\}$ is the probability that the unit surviving at time t will fail in $(t, t + \mathrm{d}t]$.
(vi) Probability function p_j; p_j is the probability that the unit fails at the jth shock.

Some reliability quantities have already been obtained [11, 33, 40]. This chapter summarizes only the known results that can be applied to maintenance policies discussed in later chapters and be useful in practical fields. A continuous wear process in which the total damage increases with time t is briefly introduced. Finally, five modified damage models are proposed. Several examples are presented. Some examples might appear to be theoretical and contrived, however, these would be useful for understanding the results easily.

2.1 Cumulative Damage Model

Consider a standard cumulative damage model: Successive shocks occur at time intervals X_j $(j = 1, 2, \dots)$ and each shock causes some damage to a unit in the amount W_j. The total damage due to shocks is additive.

It is assumed that $1/\lambda \equiv E\{X_j\} < \infty$, $1/\mu \equiv E\{W_j\} < \infty$, and $F(t) \equiv \Pr\{X_j \le t\}$, $G(x) \equiv \Pr\{W_j \le x\}$ for t, $x \ge 0$. Then, from (1.1) in Chapter 1, the probability that shocks occur exactly j times in $[0, t]$ is [11]

$$\Pr\{N(t) = j\} = F^{(j)}(t) - F^{(j+1)}(t) \qquad (j = 0, 1, 2, \ldots).$$

Thus,

$$\Pr\left\{\sum_{i=0}^{N(t)} W_i \leq x, N(t) = j\right\} = \Pr\left\{\sum_{i=0}^{N(t)} W_i \leq x \,\middle|\, N(t) = j\right\} \Pr\{N(t) = j\}$$
$$= G^{(j)}(x)[F^{(j)}(t) - F^{(j+1)}(t)] \qquad (j = 0, 1, 2, \ldots), \tag{2.2}$$

where $\varphi^{(j)}(t)$ denotes the j-fold Stieltjes convolution of any function $\varphi(t)$ with itself, and $\varphi^{(0)}(t) \equiv 1$ for $t \geq 0$.

Therefore, the distribution of $Z(t)$ defined in (2.1) is

$$\Pr\{Z(t) \leq x\} = \Pr\left\{\sum_{i=0}^{N(t)} W_i \leq x\right\}$$
$$= \sum_{j=0}^{\infty} \Pr\left\{\sum_{i=0}^{N(t)} W_i \leq x \,\middle|\, N(t) = j\right\} \Pr\{N(t) = j\}$$
$$= \sum_{j=0}^{\infty} G^{(j)}(x)[F^{(j)}(t) - F^{(j+1)}(t)], \tag{2.3}$$

and the survival probability is

$$\Pr\{Z(t) > x\} = \sum_{j=0}^{\infty} [G^{(j)}(x) - G^{(j+1)}(x)] F^{(j+1)}(t). \tag{2.4}$$

The total expected damage at time t is

$$E\{Z(t)\} = \int_0^{\infty} x \, \mathrm{d}\Pr\{Z(t) \leq x\}$$
$$= \frac{1}{\mu} \sum_{j=1}^{\infty} F^{(j)}(t) = \frac{M_F(t)}{\mu}, \tag{2.5}$$

where $M_F(t) \equiv \sum_{j=1}^{\infty} F^{(j)}(t)$ is called a renewal function of distribution $F(t)$ and represents the expected number of shocks in $[0, t]$. It can be intuitively known that $E\{Z(t)\}$ is given by the product of the average amount of damage suffered from shocks and the expected number of shocks in time t. This is useful for estimating the total expected damage at time t.

Furthermore, from Theorem 1.2, for the distribution F with finite rth moment μ_r and variance σ^2,

$$M(t) \equiv E\{N(t)\} = \frac{t}{\mu_1} + \left(\frac{\sigma^2}{2\mu_1^2} - \frac{1}{2}\right) + o(1),$$
$$V\{N(t)\} = \frac{\sigma^2 t}{\mu_1^3} + \left(\frac{5\sigma^4}{4\mu_1^4} + \frac{2\sigma^2}{\mu_1^2} + \frac{3}{4} - \frac{2\mu_3}{3\mu_1^3}\right) + o(1).$$

Thus, when F (G) has finite mean $1/\lambda$ $(1/\mu)$ and variance σ_F^2 (σ_G^2), approximately, for large t,

$$
\begin{aligned}
E\{Z(t)\} &= E\left\{E\left\{\sum_{j=1}^{N(t)} W_j \middle| N(t)\right\}\right\} = E\{N(t)\}E\{W_j\} \\
&\approx \frac{1}{\mu}\left(\lambda t + \frac{\lambda^2\sigma_F^2 - 1}{2}\right), \qquad (2.6)\\
V\{Z(t)\} &= E\{Z^2(t)\} - [E\{Z(t)\}]^2 \\
&= E\left\{\left\{\sum_{j=1}^{N(t)} W_j \sum_{i=1}^{N(t)} W_i \middle| N(t)\right\}\right\} - [E\{Z(t)\}]^2 \\
&= V\{N(t)\}[E\{W_j\}]^2 + E\{N(t)\}V\{W_j\} \\
&\approx \frac{1}{\mu}\left[\frac{\lambda t}{\mu}(\lambda^2\sigma_F^2 + \mu^2\sigma_G^2) + \frac{1}{\mu}\left(\frac{5\lambda^4\sigma_F^4}{4} + 2\lambda^2\sigma_F^2 + \frac{3}{4} - \frac{2\lambda^3\mu_3}{3}\right)\right] \\
&\quad + \frac{\sigma_G^2}{2}(\lambda^2\sigma_F^2 - 1). \qquad (2.7)
\end{aligned}
$$

Moreover, because

$$
\lim_{t\to\infty}\frac{E\{Z(t)\}}{t} = \frac{\lambda}{\mu}, \qquad \lim_{t\to\infty}\frac{V\{Z(t)\}}{t} = \frac{\lambda}{\mu^2}(\lambda^2\sigma_F^2 + \mu^2\sigma_G^2),
$$

by applying Takács theorem [83] (see Example 2.6 in [1]) to this model,

$$
\lim_{t\to\infty}\Pr\left\{\frac{Z(t) - \lambda t/\mu}{\sqrt{\lambda^3 t(\sigma_F^2/\mu^2 + \sigma_G^2/\lambda^2)}} \le x\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-u^2/2}\,\mathrm{d}u. \qquad (2.8)
$$

This was proved in [29] and generalized in [84–86].

Example 2.1. We wish to estimate the total damage when the probability that it is more than z in $t = 30$ days of operation is given by 0.90. The distributions of shock times and the amount of damage are unknown, but from sample data, the following estimations of means and variances are made:

$$
\begin{aligned}
1/\lambda &= 2\,\text{days}, & \sigma_F^2 &= 5\,(\text{days})^2, \\
1/\mu &= 1, & \sigma_G^2 &= 0.5.
\end{aligned}
$$

In this case, from (2.6), $E\{Z(30)\} \approx 15.125$. Then, from (2.8), when $t = 30$,

$$
\frac{Z(t) - \lambda t/\mu}{\sqrt{\lambda^3 t(\sigma_F^2/\mu^2 + \sigma_G^2/\lambda^2)}} = \frac{Z(30) - 15}{5.12}
$$

is approximately normally distributed with mean 0 and variance 1. Hence,

$$\Pr\{Z(t) > z\} = \Pr\left\{\frac{Z(30)-15}{5.12} > \frac{z-15}{5.12}\right\}$$
$$\approx \frac{1}{\sqrt{2\pi}}\int_{(z-15)/5.12}^{\infty} \mathrm{e}^{-u^2/2}\,\mathrm{d}u = 0.90.$$

Because $u_0 = -1.28$ such that $(1/\sqrt{2\pi})\int_{u_0}^{\infty}\mathrm{e}^{-u^2/2}\,\mathrm{d}u = 0.90$, $z = 15 - 5.12 \times 1.28 \approx 8.45$. Thus, the total damage is more than 8.45 in 30 days with probability 0.90.

Next, when a failure level is known as $K = 10$,

$$\Pr\{Z(t) > 10\} = \Pr\left\{\frac{Z(30)-10}{5.12} > \frac{10-15}{5.12}\right\}$$
$$\approx \frac{1}{\sqrt{2\pi}}\int_{-0.98}^{\infty} \mathrm{e}^{-u^2/2}\,\mathrm{d}u \approx 0.84.$$

Thus, the probability that the unit with a failure level $K = 10$ fails in 30 days is about 0.84. ∎

The first-passage time distribution to failure when the failure level is constant K, because the events of $\{Y \le t\}$ and $\{Z(t) > K\}$ are equivalent, is, from (2.4),

$$\begin{aligned}\Phi(t) &\equiv \Pr\{Y \le t\} = \Pr\{Z(t) > K\} \\ &= \sum_{j=0}^{\infty}[G^{(j)}(K) - G^{(j+1)}(K)]F^{(j+1)}(t),\end{aligned} \tag{2.9}$$

and its Laplace–Stieltjes (LS) transform is

$$\Phi^*(s) \equiv \int_0^{\infty}\mathrm{e}^{-st}\,\mathrm{d}\Phi(t) = \sum_{j=0}^{\infty}[G^{(j)}(K) - G^{(j+1)}(K)][F^*(s)]^{j+1}, \tag{2.10}$$

where $\varphi^*(s)$ denotes the LS transform of any function $\varphi(t)$, *i.e.*, $\varphi^*(s) \equiv \int_0^{\infty}\mathrm{e}^{-st}\mathrm{d}\varphi(t)$ for $s > 0$. Thus, the mean time to failure is

$$\begin{aligned}E\{Y\} &= \int_0^{\infty} t\,\mathrm{d}\Pr\{Y \le t\} = -\left.\frac{\mathrm{d}\Phi^*(s)}{\mathrm{d}s}\right|_{s=0} \\ &= \frac{1}{\lambda}\sum_{j=0}^{\infty}G^{(j)}(K) = \frac{1}{\lambda}[1 + M_G(K)],\end{aligned} \tag{2.11}$$

where $M_G(K) \equiv \sum_{j=1}^{\infty}G^{(j)}(K)$ represents the expected number of shocks before the total damage exceeds a failure level K.

Similarly, when G has finite mean $1/\mu$ and variance σ_G^2, approximately,

$$E\{Y\} \approx \frac{1}{\lambda}\left(\mu K + \frac{\mu^2\sigma_G^2 + 1}{2}\right). \tag{2.12}$$

In addition, when the distribution G has an IFR property, it has been shown that $\mu x - 1 < M_G(x) \leq \mu x$ from (1.20). Thus,

$$\frac{\mu K}{\lambda} < E\{Y\} \leq \frac{\mu K + 1}{\lambda}. \tag{2.13}$$

In Example 2.1, $E\{Y\}$ is approximately 21.5 days and $20 < E\{Y\} \leq 22$.

Finally, the failure rate is

$$\begin{aligned} r(t)\,\mathrm{d}t &= \frac{\Pr\{t < Y \leq t + \mathrm{d}t\}}{\Pr\{Y > t\}} \\ &= \frac{\sum_{j=0}^{\infty}[G^{(j)}(K) - G^{(j+1)}(K)]f^{(j+1)}(t)\,\mathrm{d}t}{\sum_{j=0}^{\infty} G^{(j)}(K)[F^{(j)}(t) - F^{(j+1)}(t)]}, \end{aligned} \tag{2.14}$$

where $f(t)$ is a density function of $F(t)$. Furthermore, because the probability that the unit fails at the $(j+1)$th shock is $p_{j+1} \equiv G^{(j)}(K) - G^{(j+1)}(K)$ $(j = 0, 1, 2, \dots)$, its survival distribution is

$$\overline{P}_j \equiv \sum_{i=j}^{\infty} p_{i+1} = G^{(j)}(K) \qquad (j = 0, 1, 2, \dots),$$

where $\overline{P}_0 \equiv 1$, *i.e.*, $\overline{P}_j$ represents the probability of surviving the first j shocks. Thus, the expected number of shocks until failure, including the shock at which the unit has failed, is

$$\sum_{j=1}^{\infty} j p_j = \sum_{j=0}^{\infty} G^{(j)}(K) = 1 + M_G(K).$$

$E\{Y\}$ in (2.11) is given by the product of the mean time between successive shocks and the expected number of shocks until the total damage has exceeded K. It is also approximately

$$\sum_{j=1}^{\infty} j p_j \approx \mu K + \frac{\mu^2 \sigma_G^2 + 1}{2}.$$

The discrete failure rate for a probability function $\{p_j\}_{j=1}^{\infty}$ is

$$r_{j+1} \equiv \frac{p_{j+1}}{\overline{P}_j} = \frac{G^{(j)}(K) - G^{(j+1)}(K)}{G^{(j)}(K)} \qquad (j = 0, 1, 2, \dots), \tag{2.15}$$

i.e., r_{j+1} represents the probability that the unit surviving at the jth shock will fail at the $(j+1)$th shock and is less than or equal to 1.

Next, suppose that shocks occur in a nonhomogeneous Poisson process with an intensity function $h(t)$ and a mean value function $H(t)$, *i.e.*, $H(t) \equiv \int_0^t h(u)\mathrm{d}u$ in **(2)** of Section 1.1. Then, from (1.1) and (1.26),

$$\Pr\{N(t) = j\} = \frac{[H(t)]^j}{j!}\mathrm{e}^{-H(t)} \qquad (j = 0, 1, 2, \dots). \tag{2.16}$$

Thus, by replacing $F^{(j)}(t)$ with $\sum_{i=j}^{\infty}\{[H(t)]^i/i!\}\mathrm{e}^{-H(t)}$ formally, we can rewrite all reliability quantities. For example,

$$\Pr\{Z(t) \le x\} = \sum_{j=0}^{\infty} G^{(j)}(x)\frac{[H(t)]^j}{j!}\mathrm{e}^{-H(t)}, \tag{2.17}$$

$$E\{Z(t)\} = \frac{H(t)}{\mu}, \tag{2.18}$$

$$E\{Y\} = \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^{\infty}\frac{[H(t)]^j}{j!}\mathrm{e}^{-H(t)}\,\mathrm{d}t. \tag{2.19}$$

If shocks occur at a constant time t_0 $(0 < t_0 < \infty)$, *i.e.*, $F(t)$ is the degenerate distribution placing unit mass at time t_0, and $F(t) \equiv 0$ for $t < t_0$, and 1 for $t \ge t_0$, then

$$\Pr\{Y \le t\} = 1 - G^{([t/t_0])}(K),$$

$$E\{Y\} = \int_0^{\infty} G^{([t/t_0])}(K)\,\mathrm{d}t,$$

where $[t/t_0]$ denotes the greatest integer less than or equal to t/t_0.

Finally, when $G(x) \equiv 0$ for $x < 1$ and 1 for $x \ge 1$, and $K = n$,

$$\Pr\{Y \le t\} = F^{(n+1)}(t), \qquad E\{Y\} = \frac{n+1}{\lambda},$$

that is, the unit fails certainly at the $(n+1)$th shock.

2.2 Independent Damage Model

Consider the independent damage model for an operating unit where the total damage is not additive, *i.e.*, any shock does no damage unless its amount has not exceeded a failure level K. If the damage due to some shock has exceeded for the first time a failure level K, then the unit fails (see Figure 2.2). The same assumptions as those of the previous model are made except that the total damage is additive. A typical example of this model is the fracture of brittle materials such as glasses [33], and semiconductor parts that have failed by some overcurrent or fault voltage. The generalized model with three types of shocks where shocks with a small level of damage are no damage to the unit, shocks with a large level of damage result in failure, and shocks with an intermediate level result in failure only with some probability, was considered [87].

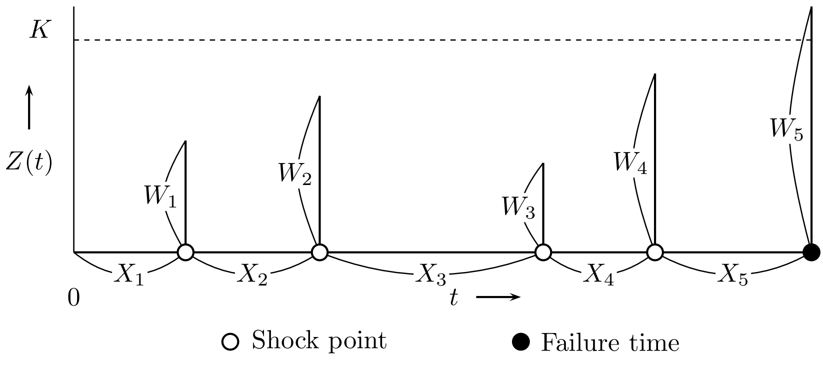

Fig. 2.2. Process for an independent damage model

In this case, the probability that the unit fails exactly at the $(j+1)$th shock $(j = 0, 1, 2, \dots)$ is $p_{j+1} = [G(K)]^j - [G(K)]^{j+1}$. Thus, the distribution of time to failure is

$$\Pr\{Y \leq t\} = \sum_{j=0}^{\infty} \{[G(K)]^j - [G(K)]^{j+1}\} F^{(j+1)}(t), \tag{2.20}$$

its LS transform is

$$\int_0^\infty \mathrm{e}^{-st}\, \mathrm{d}\Pr\{Y \leq t\} = \frac{[1 - G(K)]F^*(s)}{1 - G(K)F^*(s)}, \tag{2.21}$$

and the mean time to failure is

$$E\{Y\} = \frac{1}{\lambda[1 - G(K)]}. \tag{2.22}$$

Furthermore, the failure rates are

$$r(t) = \frac{\sum_{j=0}^\infty \{[G(K)]^j - [G(K)]^{j+1}\} f^{(j+1)}(t)}{\sum_{j=0}^\infty [G(K)]^j [F^{(j)}(t) - F^{(j+1)}(t)]}, \tag{2.23}$$

$$r_{j+1} = p_1 = 1 - G(K) \qquad (j = 0, 1, 2, \dots), \tag{2.24}$$

that is constant for any j.

If shocks occur in a nonhomogeneous Poisson process with a mean value function $H(t)$, then,

$$\Pr\{Y \leq t\} = \sum_{j=0}^{\infty} \left\{1 - [G(K)]^j\right\} \frac{[H(t)]^j}{j!} \mathrm{e}^{-H(t)} = 1 - \mathrm{e}^{-[1-G(K)]H(t)}, \tag{2.25}$$

and its mean time is

$$E\{Y\} = \int_0^\infty \mathrm{e}^{-[1-G(K)]H(t)}\,\mathrm{d}t. \tag{2.26}$$

The failure rate is

$$r(t) = [1 - G(K)]h(t), \tag{2.27}$$

that has the same property as that of an intensity function $h(t)$.

If shocks occur at a constant time t_0,

$$\Pr\{Y \le t\} = 1 - [G(K)]^{[t/t_0]},$$
$$E\{Y\} = \int_0^\infty [G(K)]^{[t/t_0]}\,\mathrm{d}t.$$

Example 2.2. Suppose that $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $G(x) = 1 - \mathrm{e}^{-\mu x}$, *i.e.*, shocks occur in a Poisson process with rate λ and each damage due to shocks is exponential with mean $1/\mu$. In this case, both a nonhomogeneous Poisson and renewal processes form the same Poisson process, *i.e.*,

$$F^{(j)}(t) = \sum_{i=j}^\infty \frac{[H(t)]^i}{i!}\mathrm{e}^{-H(t)} = \sum_{i=j}^\infty \frac{(\lambda t)^i}{i!}\mathrm{e}^{-\lambda t} \qquad (j = 0, 1, 2, \ldots).$$

In the cumulative damage model of Section 2.1, from (1.31),

$$\int_0^\infty \mathrm{e}^{-sx}\,\mathrm{d}\Pr\{Z(t) \le x\} = \mathrm{e}^{-\lambda[s/(s+\mu)t]}.$$

By inversion [65, p. 80],

$$\Pr\{Z(t) \le x\} = \mathrm{e}^{-\lambda t}\left[1 + \sqrt{\lambda\mu t}\int_0^x \mathrm{e}^{-\mu u}u^{-1/2}I_1\left(2\sqrt{\lambda\mu t u}\right)\mathrm{d}u\right],$$

where $I_i(x)$ is the Bessel function of order i for the imaginary argument defined by

$$I_i(x) \equiv \sum_{j=0}^\infty \left(\frac{x}{2}\right)^{2j+i}\frac{1}{j!(j+i)!}.$$

Thus, from (2.9), the distribution of time to failure is

$$\Pr\{Y \le t\} = 1 - \mathrm{e}^{-\lambda t}\left[1 + \sqrt{\lambda\mu t}\int_0^K e^{-\mu u}u^{-1/2}I_1\left(2\sqrt{\lambda\mu t u}\right)\mathrm{d}u\right].$$

Furthermore, from (2.5), (2.11), or (2.18), (2.19), and (2.7),

$$E\{Z(t)\} = \frac{\lambda t}{\mu}, \qquad V\{Z(t)\} = \frac{2\lambda t}{\mu^2},$$
$$E\{Y\} = \frac{1}{\lambda}\sum_{j=1}^\infty jp_j = \frac{\mu K + 1}{\lambda},$$

where note that $E\{Z(t)\}$ increases linearly with time t. Thus, we have the interesting result

$$\frac{E\{Z(t)\}}{K+1/\mu} = \frac{t}{E\{Y\}},$$

that represents that the ratio of the total expected damage at time t to a failure level plus one mean amount of damage is equal to that of the time t to the mean time to failure. If the mean time between shock times and their mean damage due to shocks are roughly estimated, the mean damage level and the mean time to failure are also estimated easily from these relations.

The failure rates are, from (2.14) and (2.15), respectively,

$$r(t) = \frac{\lambda \mathrm{e}^{-\lambda t-\mu K} I_0\left(2\sqrt{\lambda\mu t K}\right)}{1+\sqrt{\lambda\mu t}\int_0^K \mathrm{e}^{-\mu u}u^{-1/2} I_1\left(2\sqrt{\lambda\mu t u}\right)\mathrm{d}u},$$

$$r_{j+1} = \frac{(\mu K)^j/j!}{\sum_{i=j}^{\infty}[(\mu K)^i/i!]} \qquad (j=0,1,2,\dots),$$

that is strictly increasing in j from $\mathrm{e}^{-\mu K}$ to 1, because

$$\begin{aligned} r_{j+1}-r_j &= \frac{(\mu K)^j/j!}{\sum_{i=j}^{\infty}[(\mu K)^i/i!]} - \frac{(\mu K)^{j-1}/(j-1)!}{\sum_{i=j-1}^{\infty}[(\mu K)^i/i!]} \\ &= \frac{\sum_{i=j}^{\infty}[(\mu K)^{i+j-1}/(i!j!)](i-j)}{\sum_{i=j}^{\infty}[(\mu K)^i/i!]\sum_{i=j-1}^{\infty}[(\mu K)^i/i!]} > 0. \end{aligned}$$

In the independent damage model of Section 2.2, from (2.20) or (2.25),

$$\Pr\{Y \le t\} = 1-\exp(-\lambda t \mathrm{e}^{-\mu K}),$$

and from (2.22) or (2.26),

$$E\{Y\} = \frac{1}{r(t)} = \frac{1}{\lambda}\mathrm{e}^{\mu K},$$

that is, the first-passage time Y to failure has an exponential distribution with mean $\mathrm{e}^{\mu K}/\lambda$ and the failure rate is constant. ∎

2.3 Failure Rate

Investigate the reliability properties of the survival distribution $\overline{\Phi}(t) \equiv 1 - \Phi(t) = \Pr\{Y > t\}$ that the unit does not fail in $[0,t]$. Let $\overline{P}_j$ denote the probability of surviving the first j shocks ($j=0,1,2,\dots$), where $P_0 \equiv 0$, and $F_j(t)$ be the probability that j shocks occur in time t, where $F_0(t) \equiv 1$. Then, the survival distribution is written in the following general form:

$$\overline{\Phi}(t) = \sum_{j=0}^{\infty} \overline{P}_j \Pr\{N(t) = j\} = \sum_{j=0}^{\infty} \overline{P}_j [F_j(t) - F_{j+1}(t)]. \tag{2.28}$$

In particular, when shocks occur in a Poisson process with rate $\lambda > 0$, *i.e.*, $F(t) = 1 - \mathrm{e}^{-\lambda t}$ in Section 2.1,

$$\overline{\Phi}(t) = \sum_{j=0}^{\infty} \overline{P}_j \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t}. \tag{2.29}$$

The probabilistic properties of $\overline{\Phi}(t)$ were extensively investigated [34, 88]. We refer briefly only to these results that will be needed in the following chapters: The failure rate is, from (2.14),

$$r(t) = \lambda \left\{ 1 - \frac{\sum_{j=0}^{\infty} \overline{P}_{j+1} [(\lambda t)^j / j!]}{\sum_{j=0}^{\infty} \overline{P}_j [(\lambda t)^j / j!]} \right\} \leq \lambda. \tag{2.30}$$

When $\overline{P}_j = q^j$, *i.e.*, the total damage is not additive in Section 2.2, $\overline{\Phi}(t) = \mathrm{e}^{-\lambda(1-q)t}$ and $r(t) = \lambda(1-q)$ is constant.

Any distribution $F(t)$ is said to have the property of IFR (increasing failure rate) or IHR (increasing hazard rate) if and only if $[F(t+x) - F(t)]/\overline{F}(t)$ is increasing in t for $x > 0$ and $F(t) < 1$ [65], where $\overline{F}(t) \equiv 1 - F(t)$. Furthermore, it has been proved that $F(t)$ is IFR if and only if $r(t) \equiv f(t)/\overline{F}(t)$ is increasing in t. In this model, the following properties (i) and (ii) were proved [33]:

(i) The failure rate $r(t)$ in (2.30) is increasing if $(\overline{P}_j - \overline{P}_{j+1})/\overline{P}_j$ is increasing in j.

In addition, when the total damage is additive and shocks times are exponential, from (2.29),

$$\overline{\Phi}(t) = \sum_{j=0}^{\infty} G^{(j)}(K) \frac{(\lambda t)^j}{j!} \mathrm{e}^{-\lambda t}. \tag{2.31}$$

(ii) The failure rate average $\int_0^t r(u)\mathrm{d}u/t$ is increasing in t because $[G^{(j)}(x)]^{1/j}$ is decreasing in j. Note that if $r(t)$ is increasing, then $\int_0^t r(u)\mathrm{d}u/t$ is also increasing.

In particular, when $\overline{P}_j = G^{(j)}(K) = \sum_{i=j}^{\infty} [(\mu K)^i / i!] \mathrm{e}^{-\mu K}$, $\overline{P}_{j+1}/\overline{P}_j$ is strictly decreasing from Example 2.2, so that the failure rate $r(t)$ in (2.30) is strictly increasing from $\lambda \mathrm{e}^{-\mu K}$ to λ.

When shocks occur in a nonhomogeneous Poisson process with an intensity function $h(t)$ and a mean value function $H(t)$ [89], from (2.28),

$$\overline{\Phi}(t) = \sum_{j=0}^{\infty} \overline{P}_j \frac{[H(t)]^j}{j!} \mathrm{e}^{-H(t)}. \tag{2.32}$$

(iii) The failure rate $r(t)$ is increasing if $h(t)$ is increasing and $(\overline{P}_j - \overline{P}_{j+1})/\overline{P}_j$ is increasing.
(iv) The failure rate average $\int_0^t r(u)\mathrm{d}u/t$ is increasing if both $H(t)/t$ and $(\overline{P}_j - \overline{P}_{j+1})/\overline{P}_j$ are increasing.

When the total damage is additive, (2.32) is

$$\overline{\Phi}(t) = \sum_{j=0}^{\infty} G^{(j)}(K) \frac{[H(t)]^j}{j!} \mathrm{e}^{-H(t)}. \tag{2.33}$$

Then, properties (iii) and (iv) are rewritten as:

(v) The failure rate $r(t)$ is increasing if $h(t)$ is increasing and r_{j+1} in (2.15) is increasing.
(vi) The failure rate average $\int_0^t r(u)\mathrm{d}u/t$ is increasing if both $H(t)/t$ and r_{j+1} are increasing.

Such results were compactly summarized [90]. Moreover, when shocks occur in the birth process [68], in the counting process [72], and in the Lévy process [70], similar results were obtained.

After that, damage or shock models of this kind have been generalized and analyzed by many authors [91–107]. A general shock model, where the amount of damage due to shocks is correlated with their intervals, was analyzed [108–114]. Furthermore, bivariate and multivariate distributions derived from cumulative damage models were studied [115–123]. The failure rate was investigated for point, alternating, and diffused stresses [124].

2.4 Continuous Wear Processes

Let Y be the failure time of an operating unit. It is assumed that there exists a nonnegative function $h(t)$ such that

$$\Pr\{t < Y \le t + \Delta t\} = h(t)\Delta t + o(\Delta t) \tag{2.34}$$

for $\Delta t > 0$ and $t \ge 0$. Then, the probability of the unit surviving at time t is

$$R(t) = \Pr\{Y > t\} = \exp\left[-\int_0^t h(u)\,\mathrm{d}u\right] = \mathrm{e}^{-H(t)}, \tag{2.35}$$

that represents the reliability of the unit at time t and is given in (1.1) of [1]. In this case, the function $h(t)$ is called an *instantaneous wear* and $H(t) \equiv \int_0^t h(u)\,\mathrm{d}u$ is called an *accumulated wear* at time t [37]. In particular, when $H(t) = at/K$ for $a > 0$, $R(t) = \mathrm{e}^{-at/K}$ and $E\{Y\} = K/a$. Furthermore, when $H(t) = \lambda t^m$ $(m > 0)$, $R(t)$ becomes a Weibull distribution and $R(t) = \exp(-\lambda t^m)$.

On the other hand, assume that $h(t)$ is the realization of the stochastic process $\{W(t), t \geq 0\}$ with independent increments [35]. Then,

$$R(t) = E\left\{\exp\left[-\int_0^t W(u)\,\mathrm{d}u\right]\right\}. \tag{2.36}$$

If $Z(t)$ is simply the accumulated wear in a stochastic process with independent increments, then [34]

$$R(t) = E\{\mathrm{e}^{-Z(t)}\}. \tag{2.37}$$

The reliability function $R(t)$ was given by a gamma distribution [125] and some reliability functions were derived in more general assumptions [126].

The accumulated wear function $Z(t)$ usually increases with time t from 0, and the unit fails when $Z(t)$ has exceeded a failure level K. Next, suppose that $Z(t) = A_t t + B_t$ for $A_t \geq 0$. Then, the reliability at time t is

$$R(t) = \Pr\{Z(t) \leq K\} = \Pr\{A_t t + B_t \leq K\}. \tag{2.38}$$

(1) When $A_t \equiv a$ (constant), $K \equiv k$ (constant), and B_t is distributed normally with mean 0 and variance $\sigma^2 t$,

$$R(t) = \Pr\{B_t \leq k - at\} = \Phi\left(\frac{k - at}{\sigma\sqrt{t}}\right), \tag{2.39}$$

where $\Phi(x)$ is the standard normal distribution with mean 0 and variance 1, *i.e.*, $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} \mathrm{e}^{-u^2/2}\mathrm{d}u$.

(2) When $B_t \equiv 0$, $K \equiv k$, and A_t is distributed normally with mean a and variance σ^2/t,

$$R(t) = \Pr\{A_t \leq k/t\} = \Phi\left(\frac{k - at}{\sigma\sqrt{t}}\right), \tag{2.40}$$

that becomes equal to (2.39).

(3) When $A_t \equiv a$, $B_t \equiv 0$, and K is distributed normally with mean k and variance σ^2,

$$R(t) = \Pr\{at \leq K\} = \Phi\left(\frac{k - at}{\sigma}\right). \tag{2.41}$$

When K is distributed normally with mean k and variance $\sigma^2 t$, $R(t)$ is equal to (2.39) and (2.40).

Replacing $\alpha \equiv \sigma/\sqrt{ak}$ and $\beta \equiv k/a$ in (2.39) or (2.40),

$$R(t) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{t}} - \sqrt{\frac{t}{\beta}}\right)\right], \tag{2.42}$$

that is called the Birnbaum–Saunders distribution [36, 127]. This is widely applied to fatigue failure for material strength subject to stresses [128–130].

When $Z(t) = \mu t + \sigma B_t$ with positive drift μ and variance σ^2 where B_t is a standard Brownian motion, $Z(t)$ forms the Wiener process or Brownian motion process [62]. However, this has not been applied to actual damage models. When $Z(t) = A_t t + B_t$, if A_t, B_t and K are deterministic, *i.e.*, $A_t \equiv a$, $B_t \equiv b$, and $K \equiv k$, then the unit fails at time $t = (k - b)/a$. By fitting appropriate distributions to A_t, B_t, and K and estimating their parameters for practical systems, the function $Z(t)$ can be used as a continuous wear function in cumulative damage models. When $Z(t) = at$ and K is a random variable, the optimum policy where the unit is replaced at a planned time will be discussed in Section 5.2.

2.5 Modified Damage Models

Let us consider the following five damage models mainly based on our own work: (1) damage model with imperfect shock where some shock may produce no damage to a unit [40], (2) a failure level is a random variable with a general distribution $L(x)$ [131], (3) the total damage decreases exponentially with time [132], (4) the damage model of a system with n different units [133], and (5) the total damage increases with time [14, 134, 135]. Such damage models would be realistic in reliability models and be useful in practice. We derive the reliability quantities of each model and show simple examples when shock times are exponential.

(1) Imperfect Shock

It has been assumed that the damage due to a shock occurs and its amount is distributed with $G(x)$. However, it may be considered that some shocks do not produce any damage to a unit.

Suppose that the damage due to shocks occurs with probability p ($0 < p \leq 1$) and does not occur with probability $q \equiv 1 - p$. Other notations are the same as those of Sections 2.1 and 2.2. Then, substituting $F_1(t)$ in Example 1.1 in $F(t)$ in (2.3), (2.5), (2.9), (2.11), and (2.14), $\Pr\{Z(t) \leq x\}$, $E\{Z(t)\}$, $\Pr\{Y \leq t\}$, $E\{Y\}$, and $r(t)$ are given. In particular, from (2.10) and (2.11), respectively,

$$\int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}\Pr\{Y \leq t\} = \sum_{j=0}^{\infty}[G^{(j)}(K) - G^{(j+1)}(K)]\left[\frac{pF^*(s)}{1 - qF^*(s)}\right]^{j+1}, \quad (2.43)$$

$$E\{Y\} = \frac{1}{p\lambda}\sum_{j=0}^{\infty} G^{(j)}(K) = \frac{1}{p\lambda}[1 + M_G(K)]. \quad (2.44)$$

The corresponding results for the independent damage model are, from (2.21) and (2.22), respectively,

$$\int_0^\infty e^{-st}\,d\Pr\{Y \le t\} = \frac{p[1-G(K)]F^*(s)}{1-[q+pG(K)]F^*(s)}, \tag{2.45}$$

$$E\{Y\} = \frac{1}{p\lambda[1-G(K)]}. \tag{2.46}$$

(2) Random Failure Level and Time-Dependent Failure Level

Most units have individual variations in their ability to withstand shocks and are operating in a different environment. In such cases, a failure level K is not constant and would be random. Consider the case where a failure level K is a random variable with a general distribution $L(x)$ such that $L(0) = 0$ [33]. Then, for the cumulative damage model, the distribution of time to failure is

$$\Pr\{Y \le t\} = \sum_{j=0}^\infty F^{(j+1)}(t)\int_0^\infty [G^{(j)}(x) - G^{(j+1)}(x)]\,dL(x), \tag{2.47}$$

and its mean time is

$$E\{Y\} = \frac{1}{\lambda}\sum_{j=0}^\infty \int_0^\infty G^{(j)}(x)\,dL(x). \tag{2.48}$$

The failure rates are

$$r(t) = \frac{\sum_{j=0}^\infty f^{(j+1)}(t)\int_0^\infty [G^{(j)}(x) - G^{(j+1)}(x)]\,dL(x)}{\sum_{j=0}^\infty [F^{(j)}(t) - F^{(j+1)}(t)]\int_0^\infty G^{(j)}(x)\,dL(x)}, \tag{2.49}$$

$$r_{j+1} = \frac{\int_0^\infty [G^{(j)}(x) - G^{(j+1)}(x)]\,dL(x)}{\int_0^\infty G^{(j)}(x)\,dL(x)}. \tag{2.50}$$

For the independent damage model,

$$\Pr\{Y \le t\} = \sum_{j=0}^\infty F^{(j+1)}(t)\int_0^\infty \{[G(x)]^j - [G(x)]^{j+1}\}\,dL(x), \tag{2.51}$$

$$E\{Y\} = \frac{1}{\lambda}\sum_{j=0}^\infty \int_0^\infty [G(x)]^j\,dL(x). \tag{2.52}$$

For the cumulative model with imperfect shock,

$$\int_0^\infty e^{-st}\,d\Pr\{Y \le t\} = \sum_{j=0}^\infty \left[\frac{pF^*(s)}{1-qF^*(s)}\right]^{j+1}\int_0^\infty [G^{(j)}(x) - G^{(j+1)}(x)]\,dL(x). \tag{2.53}$$

Example 2.3. Suppose that all random variables are exponential, *i.e.*, $F(t) = 1 - e^{-\lambda t}$ and $G(x) = 1 - e^{-\mu x}$. Then, we obtain the explicit formulas for each model.

For imperfect shock, $F_1^*(s) = p\lambda/(s+p\lambda)$, *i.e.*, $F_1(t) = 1 - \mathrm{e}^{-p\lambda t}$ by inversion. Thus, substituting λ in $p\lambda$ in Example 2.2, we can obtain the corresponding results.

When a failure level $L(x)$ has also an exponential distribution $(1-\mathrm{e}^{-\theta x})$,

$$\int_0^\infty [G^{(j)}(x) - G^{(j+1)}(x)]\,\mathrm{d}L(x) = \frac{\theta\mu^j}{(\mu+\theta)^{j+1}}.$$

Thus, from (2.47),

$$\int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}\Pr\{Y \le t\} = \sum_{j=0}^\infty \left(\frac{\lambda}{s+\lambda}\right)^{j+1} \frac{\theta\mu^j}{(\mu+\theta)^{j+1}} = \frac{\lambda\theta}{s(\mu+\theta)+\lambda\theta}.$$

By inversion,

$$\Pr\{Y \le t\} = 1 - \exp\left(-\frac{\lambda\theta t}{\mu+\theta}\right),$$

$$E\{Y\} = \frac{1}{r(t)} = \frac{1}{\lambda}\sum_{j=1}^\infty jp_j = \frac{1}{\lambda}\left(\frac{\mu}{\theta}+1\right),$$

$$r_{j+1} = \frac{\theta}{\mu+\theta} = \frac{r(t)}{\lambda}.$$

It is of great interest that both failure rates are constant, and r_j corresponds to the ratio of (mean damage of one shock)/(mean failure level + mean damage of one shock).

For the independent damage model,

$$\begin{aligned}
\Pr\{Y > t\} &= \int_0^\infty \exp(-\lambda t\mathrm{e}^{-\mu x})\theta\mathrm{e}^{-\theta x}\,\mathrm{d}x = \sum_{j=0}^\infty \frac{(-\lambda t)^j}{j!}\int_0^\infty \theta\mathrm{e}^{-(\theta+j\mu)x}\,\mathrm{d}x \\
&= \sum_{j=0}^\infty \frac{(-\lambda t)^j}{j!}\frac{\theta}{\theta+j\mu}, \\
E\{Y\} &= \frac{1}{r(t)} = \frac{1}{\lambda}\sum_{j=1}^\infty jp_j \\
&= \frac{1}{\lambda}\int_0^\infty \mathrm{e}^{\mu x}\theta\mathrm{e}^{-\theta x}\,\mathrm{d}x = \begin{cases} \dfrac{\theta}{\lambda(\theta-\mu)} & (\theta > \mu), \\ \infty & (\theta \le \mu). \end{cases} \quad \blacksquare
\end{aligned}$$

Finally, suppose that the total damage due to shocks is investigated and is known statistically at the beginning. Then, if the unit with damage z_0 $(0 \le z_0 < K)$ begins to operate at time 0, we can obtain all reliability quantities by replacing K with $K - z_0$ [136].

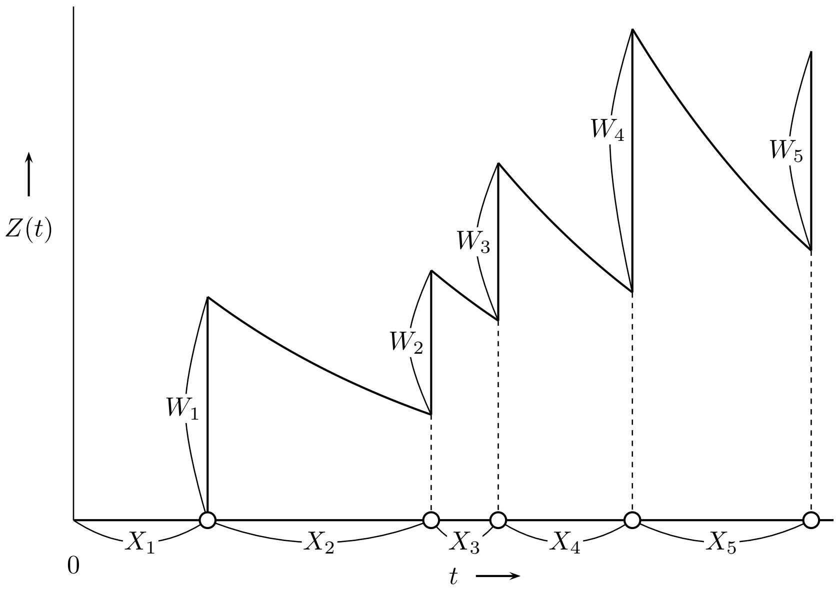

Fig. 2.3. Process for a cumulative damage model with annealing

(3) Damage with Annealing

The total damage in the usual reliability models is additive and does not decrease. In some materials, annealing, *i.e.*, lessening the damage, can take place such as rubber, fiber reinforced plastics, and polyurethane. We show two examples, using the results of [83].

Takács considered the following damage model: If a unit suffers damage W due to shock then its damage after time duration t is reduced to $W\mathrm{e}^{-\alpha t}$ $(0 < \alpha < \infty)$. Define

$$Z(t) \equiv \sum_{j=1}^{N(t)} W_j \exp[-\alpha(t - S_j)], \tag{2.54}$$

where $S_j \equiv \sum_{i=1}^{j} X_i$ $(j = 1, 2, \dots)$ (Figure 2.3). This also corresponds to the shot noise model in **(2)** of Section 10.1.

Suppose that shocks occur in a Poisson process with rate λ. Then, $\Phi(t,x) \equiv \Pr\{Z(t) \le x\}$ forms the following renewal equation [83, p. 105]:

$$\frac{\partial \Phi(t,x)}{\partial t} = -\lambda \left\{ \Phi(t,x) - \int_0^x G[(x-y)\mathrm{e}^{-\alpha t}]\,\mathrm{d}y \Phi(t,y) \right\}, \tag{2.55}$$

and its LS transform is

$$\frac{\partial \Phi^*(t,s)}{\partial t} = -\lambda [1 - G^*(s\mathrm{e}^{-\alpha t})] \Phi^*(t,s), \tag{2.56}$$

where $\Phi^*(t,s) \equiv \int_0^\infty \mathrm{e}^{-sx}\mathrm{d}\Phi(t,x)$ and $G^*(s) \equiv \int_0^\infty \mathrm{e}^{-sx}\,\mathrm{d}G(x)$. Solving this differential equation,

$$\Phi^*(t,s) = \exp\left\{-\lambda\int_0^t [1 - G^*(s\mathrm{e}^{-\alpha u})]\,\mathrm{d}u\right\}, \tag{2.57}$$

$$E\{Z(t)\} = -\left.\frac{\partial\Phi^*(t,s)}{\partial s}\right|_{s=0} = \frac{\lambda(1-\mathrm{e}^{-\alpha t})}{\alpha\mu}. \tag{2.58}$$

In addition, if $1/\mu = E\{W_j\} < \infty$, then $\lim_{t\to\infty}\Pr\{Z(t) \le x\}$ exists and its LS transform is

$$\Phi^*(\infty,s) = \exp\left[-\frac{\lambda}{\alpha}\int_0^1 \frac{1-G^*(su)}{u}\,\mathrm{d}u\right]. \tag{2.59}$$

Example 2.4.
(i) When $G(x) = 1 - \mathrm{e}^{-\mu x}$,

$$\Phi^*(t,s) = \left(\frac{s+\mu\mathrm{e}^{\alpha t}}{s+\mu}\right)^\nu \mathrm{e}^{-\lambda t},$$

where $\nu \equiv \lambda/\alpha$. Thus, by inversion,

$$\Pr\{Z(t) \le x\} = \mathrm{e}^{-\lambda t}\sum_{j=0}^\infty \binom{\nu+j-1}{j}(1-\mathrm{e}^{-\alpha t})^j \sum_{i=j}^\infty \frac{(\mu x\mathrm{e}^{\alpha t})^i}{i!}\exp(-\mu x\mathrm{e}^{\alpha t}).$$

In a similar way,

$$\Phi^*(\infty,s) = \left(\frac{\mu}{s+\mu}\right)^\nu,$$

$$\lim_{t\to\infty}\Pr\{Z(t) \le x\} = \int_0^x \frac{\mu(\mu u)^{\nu-1}}{\Gamma(\nu)}\mathrm{e}^{-\mu u}\,\mathrm{d}u,$$

that is a gamma distribution with mean ν/μ.
(ii) When $G(x) \equiv 0$ for $x < 1/\mu$ and 1 for $x \ge 1/\mu$, *i.e.*, the damage due to each shock is constant and its amount is $1/\mu$. From the results [83, p. 129],

$$\Phi^*(\infty,s) = \left(\frac{\mu}{s\gamma}\right)^\nu \exp\left(-\nu\int_{1/\mu}^\infty \frac{\mathrm{e}^{-su}}{u}\,\mathrm{d}u\right),$$

where $\gamma \equiv \mathrm{e}^c = 1.781072\cdots$ and $C \equiv 0.577215\cdots$ that is Euler's constant. By inversion,

$$\lim_{t\to\infty}\Pr\{Z(t) \le x\} = \frac{x^\nu + \sum_{j=1}^\infty[(-1)^j\nu^j/j!]\int_{j/\mu}^x (x-u)^\nu I^{(j)}(u)\,\mathrm{d}u}{(\gamma/\mu)^\nu\Gamma(1+\nu)},$$

where $I(y)$ is uniform over $[0,1/\mu]$. ∎

(4) n Different Units

Consider a system with n different units that are independent of each other. Successive shocks occur at time interval X_j with distribution $F(t) \equiv \Pr\{X_j \leq t\}$ $(j = 1, 2, \ldots)$. Each shock causes some damage to unit i $(i = 1, 2, \ldots, n)$ in the amount $W_{i;j}$ with distribution $G_i(x) \equiv \Pr\{W_{i;j} \leq x\}$ for all $j \geq 1$, where $W_{i;j}$ might be zero. Each unit fails when its total damage has exceeded its failure level K_i $(i = 1, 2, \ldots, n)$. A series system with n units subject to shocks was considered [137].

One typical example of this model would be the damage to railroad tracks, ties and pantographs. Such damage is mainly due to the number and sizes of running trains and depends on the weight and the speed of trains. In the case of $n = 3$, X_j is the time interval of trains, and $W_{i;j}$ $(i = 1, 2, 3)$ are the amounts of damage to the railroad tracks, ties, and pantographs, respectively, produced by one running train.

Letting $Z_i(t)$ denote the total damage to unit i $(i = 1, 2, \ldots, n)$ at time t, the joint distribution of $Z_i(t)$ is

$$\begin{aligned}&\Pr\{Z_i(t) \leq x_i \ (i = 1, 2, \ldots, n)\} \\ &\quad = \sum_{j=0}^{\infty} \Pr\{Z_i(t) \leq x_i \ (i = 1, 2, \ldots, n) | N(t) = j\} \Pr\{N(t) = j\}. \qquad (2.60)\end{aligned}$$

From the assumption that each amount of damage occurs independently,

$$\Pr\{Z_i(t) \leq x_i \ (i = 1, 2, \ldots, n) | N(t) = j\} = \prod_{i=1}^{n} G_i^{(j)}(x_i).$$

Thus, the joint distribution is

$$\Pr\{Z_i(t) \leq x_i \ (i = 1, 2, \ldots, n)\} = \sum_{j=0}^{\infty} \left[\prod_{i=1}^{n} G_i^{(j)}(x_i)\right] [F^{(j)}(t) - F^{(j+1)}(t)]. \qquad (2.61)$$

Suppose that a system fails when at least one of n units exceeds a failure level K_i, *i.e.*, the system is a n-unit series system. Then, the first-passage time distribution to system failure is

$$\begin{aligned}\Pr\{Y \leq t\} &= 1 - \Pr\{Z_i(t) \leq K_i \ (i = 1, 2, \ldots, n)\} \\ &= \sum_{j=0}^{\infty} \left[1 - \prod_{i=1}^{n} G_i^{(j)}(K_i)\right] [F^{(j)}(t) - F^{(j+1)}(t)], \qquad (2.62)\end{aligned}$$

and its mean time is

$$E\{Y\} = \frac{1}{\lambda} \sum_{j=0}^{\infty} \left[\prod_{i=1}^{n} G_i^{(j)}(K_i)\right]. \qquad (2.63)$$

Next, when a system fails if all of n units exceed a failure level K_i, *i.e.*, the system is an n-unit parallel system, the first-passage time distribution to system failure is

$$\Pr\{Y \le t\} = \sum_{j=0}^{\infty} \left\{ \prod_{i=1}^{n} [1 - G_i^{(j)}(K_i)] \right\} [F^{(j)}(t) - F^{(j+1)}(t)], \tag{2.64}$$

and its mean time is

$$E\{Y\} = \frac{1}{\lambda} \sum_{j=0}^{\infty} \left\{ 1 - \prod_{i=1}^{n} [1 - G_i^{(j)}(K_i)] \right\}. \tag{2.65}$$

When shocks occur in a nonhomogeneous Poisson process with a mean value function $H(t)$, the first-passage time distributions and their mean times are derived by replacing $F^{(j)}(t) - F^{(j+1)}(t)$ with $\{[H(t)]^j/j!\}\mathrm{e}^{-H(t)}$ formally.

Furthermore, suppose that a shock does no damage to unit i with probability $q_i \equiv 1 - p_i$, and otherwise, does some positive damage $W_{i;j}$ with distribution $G_i(x)$. In this case,

$$\Pr\{Z_i(t) \le x_i \ (i = 1, 2, \ldots, n) | N(t) = j\} = \prod_{i=1}^{n} \left[\sum_{m=0}^{j} \binom{j}{m} q_i^m p_i^{j-m} G_i^{(j-m)}(x_i) \right], \tag{2.66}$$

and hence, we can get the first-passage time distributions and their mean times from (2.62)–(2.65).

Example 2.5. Suppose that any amount of damage to unit i incurred from shocks is constant $1/\mu_i$, *i.e.*, $G_i(x) = 0$ for $x < 1/\mu_i$ and 1 for $x \ge 1/\mu_i$. Let $K_m \equiv \min\{\mu_1 K_1, \mu_2 K_2, \ldots, \mu_n K_n\}$ and $K_M \equiv \max\{\mu_1 K_1, \mu_2 K_2, \ldots, \mu_n K_n\}$. The first-passage time distribution and its mean time for a series system are, from (2.62) and (2.63),

$$\Pr\{Y \le t\} = F^{([K_m]+1)}(t), \qquad E\{Y\} = \frac{1}{\lambda}([K_m] + 1),$$

and for a parallel system are, from (2.64) and (2.65),

$$\Pr\{Y \le t\} = F^{([K_M]+1)}(t), \qquad E\{Y\} = \frac{1}{\lambda}([K_M] + 1),$$

where $[x]$ denotes the greatest integer contained in x.

Moreover, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $K_m \ge 1$, the failure rate is, for a series system,

$$r(t) = \frac{\lambda(\lambda t)^{[K_m]}/[K_m]!}{\sum_{j=0}^{[K_m]} (\lambda t)^j/j!},$$

and for a parallel system,

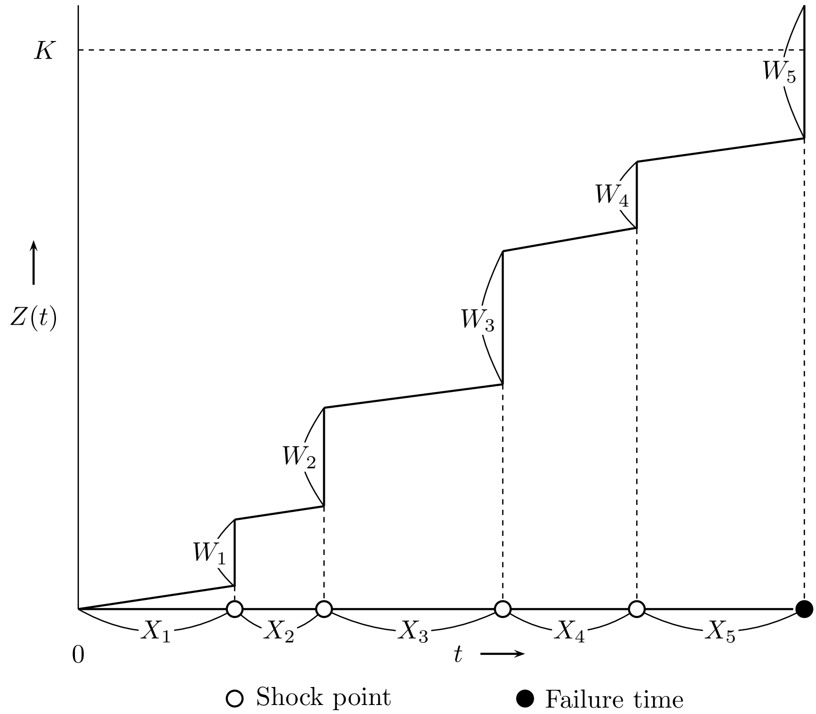

Fig. 2.4. Process for a cumulative damage model with two kinds of damages

$$r(t) = \frac{\lambda(\lambda t)^{[K_M]}/[K_M]!}{\sum_{j=0}^{[K_M]}(\lambda t)^j/j!},$$

both of which are $r(0) = 0$, and increase monotonically and become $r(\infty) = \lambda$ that is the constant failure rate of an exponential distribution $(1 - e^{-\lambda t})$. If $K_M < 1$, then $r(t) = \lambda$ for all $t \geq 0$. ▮

(5) Increasing Damage with Time

Consider the cumulative damage model with two kinds of damage (see Figure 2.4). One of them is caused by shock and is additive, and the other increases proportionately with time, that is, the total damage is accumulated subject to shocks and time at the rate of constant α ($\alpha > 0$), independent of shocks. A unit fails whether the total damage is exceeded with time or has exceeded a failure level K at some shock, and its failure is detected only at the time of shocks. Such a model would be the life of dry and storage batteries. A battery supplies electric power that is stored by chemical change according to its need. However, oxidation and deoxidation always occur irrespective of its use, that is, a battery always discharges a small quantity of electricity with time, and finally, it cannot be used.

Suppose that $S_j \equiv X_1 + X_2 + \cdots + X_j$, $Z_j \equiv W_1 + W_2 + \cdots + W_j$ $(j = 1, 2, \dots)$, and $S_0 \equiv Z_0 \equiv 0$. Because $\Pr\{S_j \le t\} = F^{(j)}(t)$ where $\Pr\{Z_j \le x\} = G^{(j)}(x)$ $(j = 0, 1, 2, \dots)$, the distribution of time to detect a failure at some shock is

$$\begin{aligned}\Pr\{Y \le t\} &= \sum_{j=0}^{\infty} \Pr\{Z_j + \alpha S_j < K \le Z_{j+1} + \alpha S_{j+1}, S_{j+1} \le t\} \\ &= \sum_{j=0}^{\infty} \int_0^t \left\{ \int_0^{t-u} [G^{(j)}(K - \alpha u) - G^{(j+1)}(K - \alpha(u + x))] \, \mathrm{d}F(x) \right\} \mathrm{d}F^{(j)}(u),\end{aligned} \tag{2.67}$$

where note that $G^{(j)}(x) \equiv 0$ for $x < 0$. Thus, the mean time to detect a failure at some shock is

$$\begin{aligned}&E\{Y\} = \\ &\sum_{j=0}^{\infty} \int_0^{\infty} \left\{ \int_0^{\infty} (t + x)[G^{(j)}(K - \alpha t) - G^{(j+1)}(K - \alpha(t + x))] \, \mathrm{d}F(x) \right\} \mathrm{d}F^{(j)}(t) \\ &\qquad = \frac{1}{\lambda} \sum_{j=0}^{\infty} \int_0^{K/\alpha} G^{(j)}(K - \alpha t) \, \mathrm{d}F^{(j)}(t).\end{aligned} \tag{2.68}$$

Similarly, the probability that the failure is detected at the $(j + 1)$th shock is

$$\begin{aligned}p_{j+1} &= \int_0^{\infty} \left\{ \int_0^{\infty} [G^{(j)}(K - \alpha t) - G^{(j+1)}(K - \alpha(t + x))] \, \mathrm{d}F(x) \right\} \mathrm{d}F^{(j)}(t) \\ &= \int_0^{K/\alpha} G^{(j)}(K - \alpha t) \, \mathrm{d}F^{(j)}(t) - \int_0^{K/\alpha} G^{(j+1)}(K - \alpha t) \, \mathrm{d}F^{(j+1)}(t) \\ &\qquad\qquad (j = 0, 1, 2, \dots),\end{aligned} \tag{2.69}$$

and the failure rate is

$$r_{j+1} = \frac{\int_0^{K/\alpha} G^{(j)}(K - \alpha t) \, \mathrm{d}F^{(j)}(t) - \int_0^{K/\alpha} G^{(j+1)}(K - \alpha t) \, \mathrm{d}F^{(j+1)}(t)}{\int_0^{K/\alpha} G^{(j)}(K - \alpha t) \, \mathrm{d}F^{(j)}(t)} \quad (j = 0, 1, 2, \dots). \tag{2.70}$$

This corresponds to the model where a failure level $K(t)$ at time t decreases with time t, *i.e.*, $K(t) = K - \alpha t$.

Example 2.6. It is intuitively estimated from (2.11) that because the average damage per unit of time is $\alpha + \lambda/\mu$, the mean time until the total damage has exceeded a failure level K is approximately

Table 2.1. Mean time to failure for two kinds of damage when $1/\lambda = 1$

$\alpha\mu$	$\mu K = 1$		$\mu K = 5$		$\mu K = 10$	
	λl	$\lambda E\{Y\}$	λl	$\lambda E\{Y\}$	λl	$\lambda E\{Y\}$
0.0	2.0	2.000	6.0	6.000	11.0	11.000
0.2	1.8	1.705	5.2	5.078	9.3	9.294
0.4	1.7	1.521	4.6	4.392	8.1	7.989
0.6	1.6	1.410	4.1	3.907	7.3	7.049
0.8	1.6	1.334	3.8	3.543	6.6	6.333
1.0	1.5	1.286	3.5	3.260	6.0	5.770
2.0	1.3	1.162	2.7	2.450	4.3	4.121
4.0	1.2	1.086	2.0	1.843	3.0	2.845

$$l = \frac{1}{\lambda}\left(\frac{K}{\alpha/\lambda + 1/\mu} + 1\right).$$

Table 2.1 presents $\lambda E\{Y\}$ and λl for $\alpha\mu$ and μK when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, $G(x) = 1 - \mathrm{e}^{-\mu x}$, and $1/\lambda = 1$. When $\alpha = 0$, this corresponds to the standard cumulative model given in Example 2.2. This table indicates that l shows a good upper bound for the mean time to failure. In actual models, l would be easily computed, and it would be used practically as one estimation of their mean failure times. ▮

Finally, if the total damage increases exponentially, *i.e.*,

$$Z(t) = \sum_{j=1}^{N(t)} W_j \exp\left[\alpha(t - S_j)\right], \tag{2.71}$$

then by arguments similar to those of **(3)**, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$,

$$\Phi^*(t,s) = \exp\left\{-\lambda \int_0^t [1 - G^*(s\mathrm{e}^{\alpha u})]\,\mathrm{d}u\right\}, \tag{2.72}$$

$$E\{Z(t)\} = \frac{\lambda(\mathrm{e}^{\alpha t} - 1)}{\alpha\mu}, \tag{2.73}$$

$$\Phi^*(\infty,s) = \exp\left[-\frac{\lambda}{\alpha}\int_1^\infty \frac{1 - G^*(su)}{u}\,\mathrm{d}u\right]. \tag{2.74}$$

This corresponds to the model where the total damage due to shocks is additive and also increases exponentially with time.

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