2

Reliability and Quality Mathematics

2.1 Introduction

Since mathematics has played a pivotal role in the development of quality and reliability fields, it is essential to have a clear understanding of the mathematical concepts relevant to these two areas. Probability concepts are probably the most widely used mathematical concepts in both reliability and quality areas. The history of probability may be traced back to the sixteenth century, when a gambler's manual written by Girolamo Cardano (1501–1576) made reference to probability.

However, it was not until the seventeenth century when Pierre Fermat (1601–1665) and Blaise Pascal (1623–1662) solved the problem of dividing the winnings in a game of chance correctly and independently. Over the years many other people have contributed in the development of mathematical concepts used in the fields of reliability and quality. More detailed information on the history of mathematics and probability is available in [1, 2]. More specifically, both these documents are totally devoted to the historical developments in mathematics and probability. This chapter presents mathematical concepts considered useful to understand subsequent chapters of this book.

2.2 Arithmetic Mean, Mean Deviation, and Standard Deviation

These three measures are presented below, separately.

2.2.1 Arithmetic Mean

This is expressed by

$$m = \frac{\sum_{i=1}^{n} DV_i}{n} \tag{2.1}$$

where

n is the number of data values.
DVi is the data value i; for i = 1, 2, 3, …, n.
m is the mean value (*i. e.*, arithmetic mean).

Example 2.1

The quality control department of an automobile manufacturing company inspected a sample of 5 identical vehicles and discovered 15, 4, 11, 8, and 12 defects in each of these vehicles. Calculate the mean number of defects per vehicle (*i. e.*, arithmetic mean).

Using the above-specified data values in Equation (2.1) yields

$$m = \frac{15+4+11+8+12}{5}$$
$$= 10 \textit{ defects per vehicle}$$

Thus, the mean number of defects per vehicle or the arithmetic mean of the data set is 10.

2.2.2 Mean Deviation

This is one of the most widely used measures of dispersion. More specifically, it is used to indicate the degree to which a given set of data tend to spread about the mean. Mean deviation is expressed by

$$m_{\mathrm{d}} = \frac{\sum_{i=1}^{n} \left| DV_i - m \right|}{n} \tag{2.2}$$

where

n	is the number of data values.
DVi	is the data value i; for i = 1, 2, 3, …, n.
m_d	is the mean value deviation.
m	is the mean of the given data set.
$\lvert DV_i - m \rvert$	is the absolute value of the deviation of DVi from m.

Example 2.2

Find the mean deviation of the Example 2.1 data set.

In Example 2.1, the calculated mean value of the data set is 10 defects per vehicle. By substituting this calculated value and the given data into Equation (2.2), we get

$$m_d = \frac{\left[\lvert 15-10\rvert + \lvert 4-10\rvert + \lvert 11-10\rvert + \lvert 8-10\rvert + \lvert 12-10\rvert\right]}{5}$$
$$= \frac{[5+6+1+2+2]}{5}$$
$$= 3.2$$

Thus, the mean deviation of the Example 2.1 data set is 3.2.

2.2.3 Standard Deviation

This is expressed by

$$\sigma = \left[\frac{\sum_{i=1}^{n}\left(DV_i - m\right)^2}{n}\right]^{1/2} \tag{2.3}$$

where

σ is the standard deviation.

Standard deviation is a commonly used measure of data dispersion in a given data set about the mean, and its three properties pertaining to the normal distribution are as follows [3]:

- 68.27% of the all data values are within $m+\sigma$ and $m-\sigma$.
- 95.45% of the all data values are within $m-2\sigma$ and $m+2\sigma$.
- 99.73% of the all data values are within $m-3\sigma$ and $m+3\sigma$.

Example 2.3
Find the standard deviation of the data set given in Example 2.1.

Using the calculated mean value, *m*, of the given data set of Example 2.1 and the given data in Equation (2.3) yields

$$\sigma = \left[\frac{(15-10)^2+(4-10)^2+(11-10)^2+(8-10)^2+(12-10)^2}{5}\right]^{1/2}$$

$$= \left[\frac{5^2+(-6)^2+1^2+(-2)^2+2^2}{5}\right]^{1/2}$$

$$= 3.74$$

Thus, the standard deviation of the data set given in Example 2.1 is 3.74.

2.3 Some Useful Mathematical Definitions and Formulas

There are many mathematical definitions and formulas used in quality and reliability fields. This section presents some of the commonly used definitions and formulas in both these areas.

2.3.1 Laplace Transform

The Laplace transform is defined by [4] as

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t \tag{2.4}$$

where
t is time.
s is the Laplace transform variable.
$F(s)$ is the Laplace transform of function, $f(t)$.

Laplace transforms of four commonly occurring functions in reliability and quality work are presented in Table 2.1. Laplace transforms of other functions can be found in [4, 5].

Table 2.1. Laplace transforms of four commonly occurring functions

$f(t)$	$F(s)$
$e^{-\theta t}$	$\frac{1}{s+\theta}$
$\frac{d f(t)}{dt}$	$s F(s) - f(0)$
c (constant)	$\frac{c}{s}$
t	$\frac{1}{s^2}$

2.3.2 Laplace Transform: Initial-Value Theorem

If the following limits exist, then the initial-value theorem may be stated as

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s F(s) \tag{2.5}$$

2.3.3 Laplace Transform: Final-Value Theorem

Provided the following limits exist, then the final-value theorem may be stated as

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s) \tag{2.6}$$

2.3.4 Quadratic Equation

This is defined by

$$a y^2 + b y + c = 0 \quad for\ a \neq 0 \tag{2.7}$$

where
a, b, and c are constants.

Thus,

$$y = \frac{-b \pm \left(b^2 - 4ac\right)^{1/2}}{2a} \tag{2.8}$$

If a, b, and c are real and $M=b^2-4ac$ is the discriminant, then the roots of the equation are

- Real and equal if $M=0$
- Complex conjugate if $M<0$
- Real and unequal if $M>0$

If y_1 and y_2 are the roots of Equation (2.7), then we can write the following expressions:

$$y_1+y_2=\frac{-b}{a} \tag{2.9}$$

and

$$y_1\,y_2=\frac{c}{a} \tag{2.10}$$

Example 2.4
Solve the following quadratic equation:

$$y^2+13y+40=0 \tag{2.11}$$

Thus, in Equation (2.11), the values of a, b, and c are 1, 13, and 40, respectively. Using these values in Equation (2.8) yields

$$y=\frac{-13\pm\left[13^2-4(1)(40)\right]^{1/2}}{2(1)}$$

$$y=\frac{-13+3}{2}$$

Therefore,

$$y_1=\frac{-13+3}{2}$$
$$=-5$$

and

$$y_2=\frac{-13-3}{2}$$
$$=-8$$

Thus, the roots of Equation (2.11) are $y_1=-5$ and $y_2=-8$. More specifically, both these values of y satisfy Equation (2.11).

2.3.5 Newton Method

Newton's method is a widely used method to approximate the real roots of an equation that involves obtaining successive approximations. The method uses the following formula to approximate real roots of an equation [6, 7]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; for \; f'(x_n) \neq 0 \tag{2.12}$$

where
the prime ($'$) denotes differentiation with respect to x.
x_n is the value of the nth approximation.

The method is demonstrated through the following example.

Example 2.5
Approximate the real roots of the following equation by using the Newton's approach:

$$x^2 - 26 = 0 \tag{2.13}$$

As a first step, we write

$$f(x) = x^2 - 26 \tag{2.14}$$

By differentiating Equation (2.14) with respect to x, we get

$$\frac{\mathrm{d} f(x)}{\mathrm{d}x} = 2x \tag{2.15}$$

Inserting Equations (2.14) and (2.15) into Equation (2.12) yields

$$x_{n+1} = x_n - \frac{x_n^2 - 26}{2x_n} = \frac{x_n^2 + 26}{2x_n} \tag{2.16}$$

For $n = 1$ in Equation (2.16) we chose $x_1 = 5$ as the first approximation. Thus, Equation (2.16) yields

$$x_2 = \frac{x_1^2 + 26}{2x_1} = \frac{(5)^2 + 26}{2(5)} = 5.1$$

For $n = 2$, substituting the above-calculated value into Equation (2.16), we get

$$x_3 = \frac{x_2^2 + 26}{2x_2} = \frac{(5.1)^2 + 26}{2(5.1)} = 5.099$$

Similarly, for $n=3$, substituting the above-calculated value into Equation (2.16), we get

$$x_4 = \frac{x_3^2+26}{2x_3} = \frac{(5.099)^2+26}{2(5.099)} = 5.099$$

It is to be noted that the values x_3 and x_4 are the same, which simply means that the real root of Equation (2.13) is $x=5.099$. It can easily be verified by substituting this value into Equation (2.13).

2.4 Boolean Algebra Laws and Probability Properties

Boolean algebra is named after mathematician George Boole (1813–1864). Some of the Boolean algebra laws that can be useful in reliability and quality work are as follows [8, 9]:

$$A.B = B.A \tag{2.17}$$

where

A is an arbitrary event or set.

B is an arbitrary event or set.

Dot (.) between A and B or B and A denotes the intersection of events or sets. However, sometimes Equation (2.17) is written without the dot, but it still conveys the same meaning.

$$A+B = B+A \tag{2.18}$$

where

$+$ denotes the union of sets or events.

$$A(B+C) = AB+AC \tag{2.19}$$

where

C is an arbitrary set or event.

$$(A+B)+C = A+(B+C) \tag{2.20}$$

$$AA = A \tag{2.21}$$

$$A+A = A \tag{2.22}$$

$$A(A+B) = A \tag{2.23}$$

$$A+AB = A \tag{2.24}$$

$$(A+B)(A+C) = A+BC \tag{2.25}$$

As probability theory plays an important role in reliability and quality, some basic properties of probability are as follows [10–12]:

- The probability of occurrence of event, say X, is

$$0 \le P(X) \le 1 \tag{2.26}$$

- Probability of the sample space S is

$$P(S) = 1 \tag{2.27}$$

- Probability of the negation of the sample space is

$$P(\overline{S}) = 1 \tag{2.28}$$

where

$\overline{S}$ is the negation of the sample space S.

- The probability of occurrence and non-occurrence of an event, say X, is

$$P(X) + P(\overline{X}) = 1 \tag{2.29}$$

where

$P(X)$ is the probability of occurrence of event X.
$P(\overline{X})$ is the probability of non-occurrence of event X.

- The probability of an interaction of K independent events is

$$P(X_1 X_2 X_3 \ldots. X_K) = P(X_1)P(X_2)P(X_3)\ldots.P(X_K) \tag{2.30}$$

where

X_i is the ith event; for $i = 1, 2, 3, \ldots., K$.
$P(X_i)$ is the probability of occurrence of event X_i; for $i = 1, 2, 3, \ldots., K$.

- The probability of the union of K independent events is

$$P(X_1 + X_2 + X_3 + \ldots. + X_K) = 1 - \prod_{i=1}^{K} \left(1 - P\left(X_i\right)\right) \tag{2.31}$$

For $K = 2$, Equation (2.32) reduces to

$$P(X_1 + X_2) = P(X_1) + P(X_2) - P(X_1)\,P(X_2) \tag{2.32}$$

- The probability of the union of K mutually exclusive events is

$$\begin{aligned} &P(X_1 + X_2 + X_3 + \ldots. + X_K) = P(X_1) \\ &+P(X_2) + P(X_3) + \ldots + P(X_K) \end{aligned} \tag{2.33}$$

2.5 Probability-related Mathematical Definitions

There are various probability-related mathematical definitions used in performing reliability and quality analyses. Some of these are presented below [10–13]:

2.5.1 Definition of Probability

This is expressed by [11]

$$P(Y) = \lim_{m \to \infty} \left[\frac{M}{m} \right] \tag{2.34}$$

where

$P(Y)$ is the probability of occurrence of event Y.

M is the total number of times Y occurs in the m repeated experiments.

2.5.2 Cumulative Distribution Function

For a continuous random variable, this is expressed by

$$F(t) = \int_{-\infty}^{t} f(y)\,\mathrm{d}y \tag{2.35}$$

where

t is time (*i. e.*, a continuous random variable).

$F(t)$ is the cumulative distribution function.

$f(t)$ is the probability density function (in reliability work, it is known as the failure density function).

2.5.3 Probability Density Function

This is expressed by

$$f(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}\left[\int_{-\infty}^{t} f(y)\mathrm{d}y \right]}{\mathrm{d}t} \tag{2.36}$$

2.5.4 Expected Value

The expected value, $E(t)$, of a continuous random variable is expressed by

$$E(t) = M = \int_{-\infty}^{\infty} t\, f(t)\,\mathrm{d}t \tag{2.37}$$

where
$E(t)$ is the expected value of the continuous random variable t.
$f(t)$ is the probability density function.
M is the mean value.

2.5.5 Variance

This is defined by

$$\theta^2(t) = E(t^2) - [E(t)]^2 \tag{2.38}$$

or

$$\theta^2(t) = \int_{0}^{\infty} t^2 f(t)\mathrm{d}t - M^2 \tag{2.39}$$

where
$\sigma^2(t)$ is the variance of random variable t.

2.6 Statistical Distributions

In mathematical reliability and quality analyses, various types of probability or statistical distributions are used. Some of these distributions are presented below [13, 14].

2.6.1 Binomial Distribution

The binominal distribution is named after Jakob Bernoulli (1654–1705) and is used in situations where one is concerned with the probabilities of outcome such as the total number of occurrences (*e.g.*, failures) in a sequence of, say m, trials [1]. However, it should be noted that each trial has two possible outcomes (*e.g.*, success and failure), but the probability of each trial remains constant.

The distribution probability density function is defined by

$$f(x) = \binom{m}{x} p^x q^{m-x},\ for\ x = 0, 1, 2, ..., m \tag{2.40}$$

where
$f(x)$ is the binomial distribution probability density function.

$$\binom{m}{x} = \frac{m!}{x!(m-x)!}$$

x is the number occurrences (*e.g.*, failures) in m trials.
p is the single trial probability of success.
$q = 1 - p,$ is the single trial probability of failure.

The cumulative distribution function is given by

$$F(x) = \sum_{i=0}^{x} \frac{m!}{i!(m-i)!} p^i q^{m-i} \tag{2.41}$$

where
$F(x)$ is the cumulative distribution function or the probability of x or less failures in m trials.

The mean or the expected value of the distribution is [10]

$$E(x) = mp \tag{2.42}$$

2.6.2 Poisson Distribution

The Poisson distribution is named after Simeon Poisson (1781–1840), a French mathematician, and is used in situations where one is interested in the occurrence of a number of events that are of the same type. More specifically, this distribution is used when the number of possible events is large, but the occurrence probability over a specified time period is small. Waiting lines and the occurrence of defects are two examples of such a situation. The distribution probability density function is defined by

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ for } x = 0, 1, 2, \ldots \tag{2.43}$$

where
λ is the distribution parameter.

The cumulative Poisson distribution function is

$$F(x) = \sum_{i=0}^{x} \frac{\lambda^i e^{-\lambda}}{i!} \tag{2.44}$$

The mean or the expected value of the distribution is [10]

$$E(x) = \lambda \tag{2.45}$$

2.6.3 Normal Distribution

Although normal distribution was discovered by De Moivre in 1733, time to time it is also referred to as Gaussian distribution after German mathematician, Carl Friedrich Gauss (1777–1855). Nonetheless, it is one of the most widely used continuous random variable distributions, and its probability density function is defined by

$$f(t)=\frac{1}{\sigma\sqrt{2\Pi}}\exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right],-\infty\langle t\langle+\infty \tag{2.46}$$

where
t is the time variable.
μ and σ are the distribution parameters (*i. e.*, mean and standard deviation, respectively).

By substituting Equation (2.46) into Equation (2.35) we get the following equation for the cumulative distribution function:

$$F(t)=\frac{1}{\sigma\sqrt{2\Pi}}\int_{-\infty}^{t}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\mathrm{d}x \tag{2.47}$$

Using Equation (2.46) in Equation (2.37) yields the following expression for the distribution mean:

$$E(t)=\mu \tag{2.48}$$

2.6.4 Gamma Distribution

The Gamma Distribution is a two-parameter distribution, and in 1961 it was considered as a possible model in life test problems [15]. The distribution probability density function is defined by

$$f(t)=\frac{\lambda(\lambda t)^{K-1}}{\Gamma(K)}\exp(-\lambda t),\quad t\geq 0, K\rangle 0,\lambda\rangle 0 \tag{2.49}$$

where
t is time.
K is the shape parameter.
$\Gamma(K)$ is the gamma function.
$\lambda=\frac{1}{\theta},\theta$ is the scale parameter.

Using Equation (2.49) in Equation (2.35) yields the following equation for the cumulative distribution function:

$$F(t) = 1 - \frac{\Gamma(K, \lambda t)}{\Gamma(K)} \tag{2.50}$$

where
$\Gamma(K, \lambda t)$ is the incomplete gamma function.

Substituting Equation (2.49) into Equation (2.37) we get the following equation for the distribution mean:

$$E(t) = \frac{K}{\lambda} \tag{2.51}$$

Three special case distributions of the gamma distribution are the exponential distribution, the chi-square distribution, and the special case Erlangian distribution [16].

2.6.5 Exponential Distribution

This is probably the most widely used statistical distribution in reliability studies because it is easy to handle in performing reliability analysis and many engineering items exhibit constant failure rates during their useful life [17]. Its probability density function is defined by

$$f(t) = \lambda \mathrm{e}^{-\lambda t}, \qquad \lambda \rangle 0,\, t \geq 0 \tag{2.52}$$

where
t is time.
λ is the distribution parameter. In reliability work, it is called constant failure rate.

Substituting Equation (2.52) into Equation (2.35) we get the following equation for the cumulative distribution function:

$$F(t) = 1 - \mathrm{e}^{-\lambda t} \tag{2.53}$$

Using Equation (2.53) in Equation (2.37) yields the following equation for the distribution mean:

$$E(t) = \frac{1}{\lambda} \tag{2.54}$$

2.6.6 Rayleigh Distribution

The Rayleigh distribution is named after John Rayleigh (1842–1919) and is often used in the theory of sound and in reliability studies. The distribution probability density function is defined by

$$f(t) = \left(\frac{2}{\theta^2}\right) t \,\mathrm{e}^{-\left(\frac{t}{\theta}\right)^2}, \qquad \theta \rangle 0,\ t \geq 0 \tag{2.55}$$

where

t is time.

θ is the distribution parameter.

By substituting Equation (2.55) into Equation (2.35) we get the following equation for the cumulative distribution function:

$$F(t) = 1 - \mathrm{e}^{-\left(\frac{t}{\theta}\right)^2} \tag{2.56}$$

Inserting Equation (2.55) into Equation (2.37) yields the following equation for the distribution mean:

$$E(t) = \theta \Gamma\left(\frac{3}{2}\right) \tag{2.57}$$

where

$\Gamma(.)$ is the gamma function, which is expressed by

$$\Gamma(y) = \int_0^\infty t^{y-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \qquad y \rangle 0 \tag{2.58}$$

2.6.7 Weibull Distribution

The Weibull distribution is named after W. Weibull, a Swedish mechanical engineering professor who developed it in the early 1950s [17]. It is often used in reliability studies, and its probability density function is defined by

$$f(t) = \frac{\beta t^{\beta-1}}{\theta^\beta}\,\mathrm{e}^{-\left(\frac{t}{\theta}\right)^\beta}, \qquad \theta \rangle 0,\ \beta \rangle 0,\ t \geq 0 \tag{2.59}$$

where

t is time.

θ and β are the distribution scale and shape parameters, respectively.

Using Equation (2.59) in Equation (2.35) yields the following equation for the cumulative distribution function:

$$F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}} \tag{2.60}$$

By inserting Equation (2.59) into Equation (2.37) we get the following equation for the distribution mean:

$$E(t) = \theta \Gamma\left(1 + \frac{1}{\beta}\right) \tag{2.61}$$

It is to be noted that exponential and Rayleigh distributions are the special cases of this distribution for $\beta = 1$ and $\beta = 2$, respectively.

2.7 Problems

1. What is mean deviation?
2. Obtain the Laplace transform of the following function:

$$f(t) = \lambda e^{-\lambda t} \tag{2.62}$$

where
λ is a constant.
t is time.

3. Find roots of the following equation by using the quadratic formula:

$$x^2 + 15x + 50 = 0 \tag{2.63}$$

Approximate the real roots of the following equation by using the Newton method:

$$x^2 - 37 = 0 \tag{2.64}$$

4. Write down the five most important probability properties.
5. Prove that the total area under a continuous random variable probability density function curve is equal to unity.
6. Define the probability density function of a continuous random variable.
7. What are the special case distributions of the Weibull distribution?
8. Prove that the mean or the expected value of the gamma distribution is given by Equation (2.51).
9. Prove Equation (2.60).

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