2

Positive Real Systems

The notion of *Positive Real* system may be seen as a generalization of the positive definiteness of a matrix to the case of a dynamical system with inputs and outputs. When the input-output relation (or mapping, or operator) is a constant matrix, testing its positive definiteness can be done by simply calculating the eigenvalues and checking that they are positive. When the input-output operator is more complex, testing positive realness becomes much more involved. This is the object of this chapter which is mainly devoted to positive real linear time-invariant systems. They are known as PR transfer functions.

The definition of Positive Real (PR) systems has been motivated by the study of linear electric circuits composed of resistors, inductors and capacitors. The driving point impedance from any point to any other point of such electric circuits is always PR. The result holds also in the sense that any PR transfer function can be realized with an electric circuit using only resistors, inductors and capacitors. The same result holds for any analogous mechanical or hydraulic systems. This idea can be extended to study analogous electric circuits with nonlinear passive components and even magnetic couplings as done by Arimoto [24] to study dissipative nonlinear systems. This leads us to the second interpretation of PR systems: they are systems which dissipate energy. As we shall see later in the book, the notion of *dissipative* systems, which applies to nonlinear systems, is closely linked to PR transfer functions.

This chapter reviews the main results available for PR linear systems. It starts with a short introduction to so-called *passive* systems. It happens that there has been a proliferation of notions and definitions of various kinds of PR or dissipative systems, since the early studies in the 1960s (to name a few: ISP, OSP, VSP, PR, SPR, WSPR, SSPR, MSPR, ESPR; see the index for the meaning of these acronyms). The study of their relationships (are they equivalent, which ones imply which other one?) is not so easy and we bring some elements of answers in this chapter and the next ones. This is why we introduce first in this chapter some basic definitions (passive systems, positive real systems, bounded real transfer functions), their relationships, and then we introduce other refined notions of PR systems. The reason why passive systems are briefly introduced before bounded real and positive real transfer functions, is that this allows one to make the link between an energy-related notion and the frequency domain notions, in a progressive way. This, however, is at the price of postponing a more rigorous and general exposition of passive systems until later in the book.

2.1 Dynamical System State-space Representation

In this book various kinds of evolution, or dynamical systems will be analyzed: linear, time invariant, nonlinear, finite-dimensional, infinite-dimensional, discrete time, non-smooth, "standard" differential inclusions, "unbounded" or "maximal monotone" differential inclusions *etc.* Whatever the system we shall be dealing with, it is of utmost importance to clearly define some basic ingredients:

- A state vector $x(\cdot)$ and a state space X
- A set of admissible inputs $\mathcal{U}$
- A set of outputs $\mathcal{Y}$
- An input/output mappping (or operator) $H : u \mapsto y$
- A state space representation which relates the derivative of $x(\cdot)$ to $x(\cdot)$ and $u(\cdot)$
- An output function which relates the output $y(\cdot)$ to the state $x(\cdot)$ and the input $u(\cdot)$

Such tools (or some of them) are necessary to write down the model, or system, that is under examination. When one works with pure input/output models, one doesn't need to define a state space X; however $\mathcal{U}$ and $\mathcal{Y}$ are crucial. In this book we will essentially deal with systems for which a state space representation has been defined. Then the notion of a (state) solution is central. Given some state space model under the form of an evolution problem (a differential equation or something looking like this), the first step is to provide informations on such solutions: the nature of the solutions (as time-functions, for instance), their uniqueness, their continuity with respect to the initial data and parameters, *etc.* This in turn is related to the set of admissible inputs $\mathcal{U}$. For instance, if the model takes the form of an ordinary differential equation (ODE) $\dot{x}(t) = f(x(t), u(t))$, the usual Carathéodory conditions will be in force to define $\mathcal{U}$ as a set of measurable functions, and $x(\cdot)$ will usually be an absolutely continuous function of time. In certain cases, one may want to extend $\mathcal{U}$ to measures, or even distributions. Then x may also be a measure or a distribution. Since it is difficult (actually impossible) to provide a general well-posedness result for all the systems that will be dealt with in the rest of the book, we will recall the well-posedness conditions progressively as new models are introduced. This will be the case especially for some classes of nonsmooth systems, where solutions may be absolutely continuous, or of local bounded variation.

From a more abstract point of view, one may define a general state-space deterministic model as follows [364, 510, 512]:

There exists a metric space X (the state space), a transition map $\psi : I\!R \times I\!R \times X \times \mathcal{U} \to X$, and a readout map $r : X \times I\!R^m \to I\!R^p$, such that:

- (i) The limit $x(t) = \lim_{t_0 \to -\infty} \psi(t_0, t, 0, u)$ is in X for all $t \in I\!R$ and all $u \in \mathcal{U}$ (then $x(t)$ is the state at time t)
- (ii) (Causality) $\psi(t_0, t_1, x, u_1) = \psi(t_0, t_1, x, u_2)$ for all $t_1 \geq t_0$, all $x \in X$, and all $u_1, u_2 \in \mathcal{U}$ such that $u_1(t) = u_2(t)$ in the interval $t_0 \leq t \leq t_1$
- (iii) (Initial state consistency) $\psi(t_0, t_0, x_0, u) = x_0$ for all $t_0 \in I\!R$, $u \in \mathcal{U}$, and all $x_0 \in X$
- (iv) (Semigroup property) $\psi(t_1, t_2, \psi(t_0, t_1, x_0, u), u) = \psi(t_0, t_2, x_0, u)$ for all $x_0 \in X$, $u \in \mathcal{U}$, whenever $t_0 \leq t_1 \leq t_2$
- (v) (Consistency with input-output relation) The input-output pairs (u, y) are precisely those described *via* $y(t) = r\left(\lim_{t_0 \to -\infty} \psi(t_0, t, 0, u), u(t)\right)$
- (vi) (Unbiasedness) $\psi(t_0, t, 0, 0) = 0$ whenever $t \geq t_0$ and $r(0, 0) = 0$
- (vii) (Time-invariance) $\psi(t_1 + T, t_2 + T, x_0, u_1) = \psi(t_1, t_2, x_0, u_2)$ for all $T \in I\!R$, all $t_2 \geq t_1$, and all $u_1, u_2 \in \mathcal{U}$ such that $u_2(t) = u_1(t + T)$

Clearly item (vii) will not apply to some classes of time-varying systems, and an extension is needed [512, §6]. There may be some items which do not apply well to differential inclusions where the solution may be replaced by a solution set (for instance the semigroup property may fail). The basic fact that X is a metric space will also require much care when dealing with some classes of systems whose state spaces are not spaces of functions (like descriptor variable systems that involve Schwarz' distributions). In the infinite-dimensional case X may be a Hilbert space (*i.e.* a space of functions) and one may need other definitions, see *e.g.* [39, 507]. An additional item in the above list could be the continuity of the transition map $\psi(\cdot)$ with respect to the initial data x_0. Some nonsmooth systems do not possess such a property, which may be quite useful in some stability results. A general exposition of the notion of a system can be found in [467, Chapter 2]. We now stop our investigations of what a system is since, as we said above, we shall give well-posedness results each time they are needed all through the book.

2.2 Definitions

In this section and the next one, we introduce input-output properties of a system, or operator $H : u \mapsto H(u) = y$. The system is assumed to be well-posed as an input-ouput system, i.e. we may assume that $H : \mathcal{L}_{2,e} \to \mathcal{L}_{2,e}$ [1].

[1] More details on $\mathcal{L}_p$ spaces can be found in Chapter 4.

Definition 2.1. *A system with input $u(\cdot)$ and output $y(\cdot)$ where $u(t), y(t) \in \mathbb{R}^m$ is passive if there is a constant β such that*

$$\int_0^t y^T(\tau)u(\tau)d\tau \geq \beta \tag{2.1}$$

for all functions $u(\cdot)$ and all $t \geq 0$. If, in addition, there are constants $\delta \geq 0$ and $\epsilon \geq 0$ such that

$$\int_0^t y^T(\tau)u(\tau)d\tau \geq \beta + \delta \int_0^t u^T(\tau)u(\tau)d\tau + \epsilon \int_0^t y^T(\tau)y(\tau)d\tau \tag{2.2}$$

for all functions $u(\cdot)$, and all $t \geq 0$, then the system is input strictly passive (ISP) if $\delta > 0$, output strictly passive (OSP) if $\epsilon > 0$, and very strictly passive (VSP) if $\delta > 0$ and $\epsilon > 0$. ■

Obviously $\beta \leq 0$ as the inequality (2.1) is to be valid for all functions $u(\cdot)$ and in particular the control $u(t) = 0$ for all $t \geq 0$, which gives $0 = \int_0^t y^T(s)u(s)ds \geq \beta$. Thus the definition could equivalently be stated with $\beta \leq 0$. The importance of the form of β in (2.1) will be illustrated in Examples 4.59 and 4.60; see also Section 4.4.2. Notice that $\int_0^t y^T(s)u(s)ds \leq \frac{1}{2}\int_0^t [y^T(s)y(s) + u^T(s)u(s)]ds$ is well defined since both $u(\cdot)$ and $y(\cdot)$ are in $\mathcal{L}_{2,e}$ by assumption.

Theorem 2.2. *Assume that there is a continuous function $V(\cdot) \geq 0$ such that*

$$V(t) - V(0) \leq \int_0^t y(s)^T u(s)ds \tag{2.3}$$

for all functions $u(\cdot)$, for all $t \geq 0$ and all $V(0)$. Then the system with input $u(\cdot)$ and output $y(\cdot)$ is passive. Assume, in addition, that there are constants $\delta \geq 0$ and $\epsilon \geq 0$ such that

$$V(t) - V(0) \leq \int_0^t y^T(s)u(s)ds - \delta \int_0^t u^T(s)u(s)ds - \epsilon \int_0^t y^T(s)y(s)ds \tag{2.4}$$

for all functions $u(\cdot)$, for all $t \geq 0$ and all $V(0)$. Then the system is input strictly passive if there is a $\delta > 0$, it is output strictly passive if there is an $\epsilon > 0$, and very strictly passive if there is a $\delta > 0$ and an $\epsilon > 0$ such that the inequality holds. ■

Proof: It follows from the assumption $V(t) \geq 0$ that

$$\int_0^t y^T(s)u(s)ds \geq -V(0)$$

for all functions $u(\cdot)$ and all $s \geq 0$, so that (2.1) is satisfied with $\beta := -V(0) \leq 0$. Input strict passivity, output strict passivity and very strict passivity are shown in the same way. ■

This indicates that the constant β is related to the initial conditions of the system; see also Example 4.59 for more informations on the role played by β. It is also worth looking at Corollary 3.3 to get more informations on the real nature of the function $V(\cdot)$: $V(\cdot)$ will usually be a function of the state of the system. The reader may have guessed such a fact by looking at the examples of Chapter 1.

Corollary 2.3. *Assume that there exists a continuously differentiable function $V(\cdot) \geq 0$ and a measurable function $d(\cdot)$ such that $\int_0^t d(s)ds \geq 0$ for all $t \geq 0$. Then*

1. *If*
$$\dot{V}(t) \leq y^T(t)u(t) - d(t) \tag{2.5}$$
for all $t \geq 0$ and all functions $u(\cdot)$, the system is passive.
2. *If there exists a $\delta > 0$ such that*
$$\dot{V}(t) \leq y^T(t)u(t) - \delta u^T(t)u(t) - d(t) \tag{2.6}$$
for all $t \geq 0$ and all functions $u(\cdot)$, the system is input strictly passive (ISP).
3. *If there exists a $\epsilon > 0$ such that*
$$\dot{V}(t) \leq y^T(t)u(t) - \epsilon y^T(t)y(t) - d(t) \tag{2.7}$$
for all $t \geq 0$ and all functions $u(\cdot)$, the system is output strictly passive (OSP).
4. *If there exists a $\delta > 0$ and a $\epsilon > 0$ such that*
$$\dot{V}(t) \leq y^T(t)u(t) - \delta u^T(t)u(t) - \epsilon y^T(t)y(t) - d(t) \tag{2.8}$$
for all $t \geq 0$ and all functions $u(\cdot)$, the system is very strictly passive (VSP). ■

If $V(\cdot)$ is the total energy of the system, then $\langle u, y\rangle = \int_0^t y^T(s)u(s)ds$ can be seen as the power supplied to the system from the control, while $d(t)$ can be seen as the power dissipated by the system. This means that the condition $\int_0^t d(s)ds \geq 0$ for all $t \geq 0$ means that the system is dissipating energy. The term $w(u, y) = u^T y$ is called the *supply rate* of the system.

Remark 2.4. All these notions will be examined in much more detail in Chapter 4; see especially Section 4.5.2. Actually the notion of passivity (or dissipativity) has been introduced in various ways in the literature. It is sometimes introduced as a pure input/output property of an operator (*i.e.* the constant β in (2.1) is not related to the state of the system) [125, 499, 500], and serves as a tool to prove some bounded input/bounded output stability results. Willems has, on the contrary, introduced dissipativity as a notion which involves the state space representation of a system, through so-called *storage functions* [510, 511]. We will come back to this subject in Chapter 4. Hill and Moylan started from an intermediate definition, where the constant β is assumed to depend on some initial state x_0 [206–209]. Then, under some controllability assumptions, the link with Willems' definition is made. In this chapter and the next one, we will essentially concentrate on linear time invariant dissipative systems, whose transfer functions are named *positive real* (PR). This is a very important side of passivity theory in Systems and Control theory.

2.3 Interconnections of Passive Systems

A useful result for passive systems is that parallel and feedback interconnections of passive systems are passive, and that certain strict passivity properties are inherited.

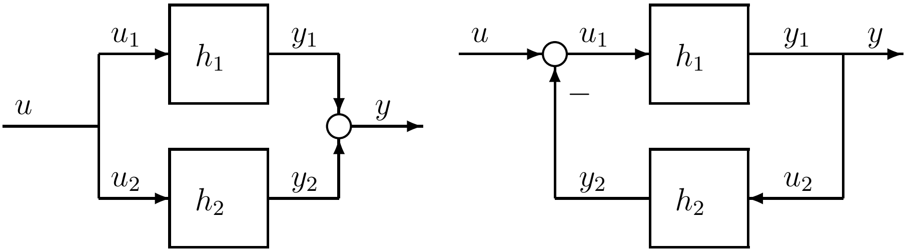

Fig. 2.1. Parallel and feedback interconnections.

To explore this we consider two passive systems with scalar inputs and outputs. Similar results are found for multivariable systems. System 1 has input u_1 and output y_1, and system 2 has input u_2 and output y_2. We make the following assumptions:

1. There are continuous differentiable functions $V_1(t) \geq 0$ and $V_2(t) \geq 0$.
2. There are functions $d_1(\cdot)$ and $d_2(\cdot)$ such that $\int_0^t d_1(s)ds \geq 0$ and $\int_0^t d_2(s)ds \geq 0$ for all $t \geq 0$.
3. There are constants $\delta_1 \geq 0$, $\delta_2 \geq 0$, $\epsilon_1 \geq 0$ and $\epsilon_2 \geq 0$ such that

$$\dot{V}_1(t) = y_1(t)u_1(t) - \delta_1 u_1^2(t) - \epsilon_1 y_1^2 - d_1(t) \tag{2.9}$$

$$\dot{V}_2(t) = y_2(t)u_2(t) - \delta_2 u_2^2(t) - \epsilon_2 y_2^2 - d_2(t) \tag{2.10}$$

Assumption 3 implies that both systems are passive, and that system i is strictly passive in some sense if any of the constants δ_i or ϵ_i are greater than zero. For the parallel interconnection we have $u_1 = u_2 = u$, $y = y_1 + y_2$, and

$$yu = (y_1 + y_2)u = y_1u + y_2u = y_1u_1 + y_2u_2 \tag{2.11}$$

By adding (2.9) (2.10) and (2.11), there exists a $V(\cdot) = V_1(\cdot) + V_2(\cdot) \geq 0$ and a $d_p = d_1 + d_2 + \epsilon_1 y_1^2 + \epsilon_2 y_2^2$ such that $\int_0^t d_p(t')dt' \geq 0$ for all $t \geq 0$, and

$$\dot{V}(t) = y(t)u(t) - \delta u^2(t) - d_p(t) \tag{2.12}$$

where $\delta = \delta_1 + \delta_2 \geq 0$. This means that the parallel interconnection system having input u and output y is passive and strictly passive if $\delta_1 > 0$ or $\delta_2 > 0$. For the feedback interconnection we have $y_1 = u_2 = y$, $u_1 = u - y_2$, and

$$yu = y_1(u_1 + y_2) = y_1u_1 + y_1y_2 = y_1u_1 + u_2y_2 \tag{2.13}$$

Again by adding (2.9) (2.10) and (2.11) we find that there is a $V(\cdot) = V_1(\cdot) + V_2(\cdot) \geq 0$ and a $d_{fb} = d_1 + d_2 + \delta_1 u_1^2$ such that $\int_0^t d_{fb}(s)ds \geq 0$ for all $t \geq 0$ and

$$\dot{V}(t) = y(t)u(t) - \epsilon y^2(t) - d_{fb}(t) \tag{2.14}$$

where $\epsilon = \epsilon_1 + \epsilon_2 + \delta_2$. This means that the feedback interconnection is passive, and in addition output strictly passive if $\epsilon_1 > 0$, $\epsilon_2 > 0$, or $\delta_2 > 0$. By induction it can be shown that any combination of passive systems in parallel or feedback interconnection is passive.

2.4 Linear Systems

Let us now deal with linear invariant systems, whose input-output relationships takes the form of a rational transfer function $H(s)$ (also denoted as $h(s)$), $s \in \mathbb{C}$, and $y(s) = H(s)u(s)$ where $u(s)$ and $y(s)$ are the Laplace transforms of the time-functions $u(\cdot)$ and $y(\cdot)$. Parseval's Theorem is very useful in the study of passive linear systems, as shown next. It is now recalled for the sake of completeness.

Theorem 2.5 (Parseval's Theorem). *Provided that the integrals exist, the following relation holds:*

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)y^*(j\omega)d\omega \tag{2.15}$$

where y^* denotes the complex conjugate of y and $x(j\omega)$ is the Fourier transform of $x(t)$, where $x(t)$ is a complex function of t, Lebesgue integrable. ■

Proof: The result is established as follows: the Fourier transform of the time function $x(t)$ is

$$x(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{2.16}$$

while the inverse Fourier transform is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)e^{j\omega t}d\omega \tag{2.17}$$

Insertion of (2.17) in (2.15) gives

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)e^{j\omega t}d\omega\right]y^*(t)dt \tag{2.18}$$

By changing the order of integration this becomes

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)\left[\int_{-\infty}^{\infty} y^*(t)e^{j\omega t}dt\right]d\omega \tag{2.19}$$

Here

$$\int_{-\infty}^{\infty} y^*(t)e^{j\omega t}dt = \left[\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt\right]^* = y^*(j\omega) \tag{2.20}$$

and the result follows. ■

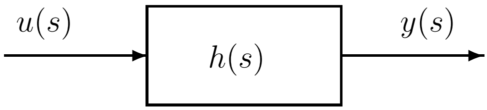

Fig. 2.2. Linear time-invariant system

We will now present important properties of a linear time-invariant passive system, which link the input-output passivity property to frequency-domain conditions, using Parseval's Theorem. These notions will be generalized later in the book, both in the case of LTI and nonlinear systems. Their usefulness will be illustrated through examples of stabilization.

Theorem 2.6. *Given a linear time-invariant linear system with rational transfer function $h(s)$, i.e.*

$$y(s) = h(s)u(s) \tag{2.21}$$

Assume that all the poles of $h(s)$ have real parts less than zero. Then the following assertions hold:

1. *The system is passive* $\Leftrightarrow$ $\mathbf{Re}[h(j\omega)] \geq 0$ *for all* $\omega \in [-\infty, +\infty]$.
2. *The system is input strictly passive (ISP)* $\Leftrightarrow$ *There exists a* $\delta > 0$ *such that* $\mathbf{Re}[h(j\omega)] \geq \delta > 0$ *for all* $\omega \in [-\infty, +\infty]$.
3. *The system is output strictly passive (OSP)* $\Leftrightarrow$ *There exists an* $\epsilon > 0$ *such that*

$$\mathbf{Re}[h(j\omega)] \geq \epsilon \left| h(j\omega) \right|^2$$
$$\Updownarrow$$
$$\left(\mathbf{Re}[h(j\omega)] - \tfrac{1}{2\epsilon}\right)^2 + \left(\mathbf{Im}[h(j\omega)]\right)^2 \leq \left(\tfrac{1}{2\epsilon}\right)^2$$

■

Remark 2.7. A crucial assumption in Theorem 2.6 is that all the poles have negative real parts. This assures that in Parseval's Theorem as stated in Theorem 2.5, the "integrals exist".

Proof: The proof is based on the use of Parseval's Theorem. In this Theorem the time integration is over $t \in [0, \infty)$. In the definition of passivity there is an integration over $t \in [0, T]$. To be able to use Parseval's Theorem in this proof we introduce the truncated function

$$u_t(\tau) = \begin{cases} u(\tau) & \text{when} \quad \tau \leq t \\ 0 & \text{when} \quad \tau > t \end{cases} \tag{2.22}$$

which is equal to $u(\tau)$ for all τ less than or equal to t, and zero for all τ greater than t. The Fourier transform of $u_T(t)$, which is denoted $u_T(j\omega)$, will be used in Parseval's Theorem. Without loss of generality we will assume that $y(t)$ and $u(t)$ are equal to zero for all $t \leq 0$. Then according to Parseval's Theorem

$$\int_0^t y(\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} y(\tau)u_t(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(j\omega)u_t^*(j\omega)d\omega \tag{2.23}$$

Insertion of $y(j\omega) = h(j\omega)u_T(j\omega)$ gives

$$\int_0^t y(\tau)u(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(j\omega)u_T(j\omega)u_t^*(j\omega)d\omega, \tag{2.24}$$

where

$$h(j\omega)u_t(j\omega)u_t^*(j\omega) = \{\mathbf{Re}[h(j\omega)] + j\mathbf{Im}[h(j\omega)]\}|u_t(j\omega)|^2 \tag{2.25}$$

The left hand side of (2.24) is real, and it follows that the imaginary part on the right hand side is zero. This implies that

$$\int_0^t u(\tau)y(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{Re}[h(j\omega)]|u_t(j\omega)|^2 d\omega \tag{2.26}$$

First, assume that $\mathbf{Re}[h(j\omega)] \geq \delta \geq 0$ for all ω. Then

$$\int_0^t u(\tau)y(\tau)d\tau \geq \frac{\delta}{2\pi}\int_{-\infty}^{\infty} |u_t(j\omega)|^2 d\omega = \delta\int_0^t u^2(\tau)d\tau \tag{2.27}$$

The equality is implied by Parseval's Theorem. It follows that the system is passive, and in addition input strictly passive if $\delta > 0$.

Then, assume that the system is passive. Thus there exists a $\delta \geq 0$ so that

$$\int_0^t y(s)u(s)dsz \geq \delta \int_0^t u^2(s)ds = \frac{\delta}{2\pi} \int_{-\infty}^{\infty} |u_t(j\omega)|^2 d\omega \tag{2.28}$$

for all $u(\cdot)$, where the initial conditions have been selected so that $\beta = 0$. Here $\delta = 0$ for a passive system, while $\delta > 0$ for a strictly passive system. Then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{Re}[h(j\omega)] |u_T(j\omega)|^2 d\omega \geq \frac{\delta}{2\pi} \int_{-\infty}^{\infty} |u_T(j\omega)|^2 d\omega \tag{2.29}$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathbf{Re}[h(j\omega)] - \delta) |u_T(j\omega)|^2 d\omega \geq 0 \tag{2.30}$$

If there exists a ω_0 so that $\mathbf{Re}[h(j\omega_0)] < \delta$, then inequality will not hold for all u because the integral on the left hand side can be made arbitrarily small if the control signal is selected to be $u(t) = U \cos \omega_0 t$. The results 1 and 2 follow.

To show result 3 we first assume that the system is output strictly passive, that is, there is an $\epsilon > 0$ such that

$$\int_0^t y(s)u(s)ds \geq \epsilon \int_0^t y^2(s)ds = \frac{\epsilon}{2\pi} \int_{-\infty}^{\infty} |h(j\omega)|^2 |u_t(j\omega)|^2 d\omega. \tag{2.31}$$

This gives the inequality (see (2.26))

$$\mathbf{Re}[h(j\omega)] \geq \epsilon \left| h(j\omega) \right|^2 \tag{2.32}$$

which is equivalent to

$$\epsilon \left[(\mathbf{Re}[h(j\omega)])^2 + (\mathbf{Im}[h(j\omega)])^2 \right] - \mathbf{Re}[h(j\omega)] \leq 0 \tag{2.33}$$

and the second inequality follows by straighforward algebra. The converse result is shown similarly as the result for input strict passivity. ■

Note that according to the theorem a passive system will have a transfer function which satisfies

$$|\angle h(j\omega)| \leq 90^\circ \quad \text{for all} \quad \omega \in [-\infty, +\infty] \tag{2.34}$$

In a Nyquist diagram the theorem states that $h(j\omega)$ is in the closed half plane $\mathbf{Re}\,[s] \geq 0$ for passive systems, $h(j\omega)$ is in $\mathbf{Re}\,[s] \geq \delta > 0$ for input strictly passive systems, and for output strictly passive systems $h(j\omega)$ is inside the circle with center in $s = 1/\,(2\epsilon)$ and radius $1/\,(2\epsilon)$. This is a circle that crosses the real axis in $s = 0$ and $s = 1/\epsilon$.

Remark 2.8. A transfer function $h(s)$ is rational if it is the fraction of two polynomials in the complex variable s, that is if it can be written in the form

$$h(s) = \frac{Q(s)}{R(s)} \tag{2.35}$$

where $Q(s)$ and $R(s)$ are polynomials in s. An example of a transfer function that is not rational is $h(s) = \tanh s$ which appears in connection with systems described by partial differential equations.

Example 2.9. Note the difference between the condition $\mathbf{Re}[h(j\omega)] > 0$ and the condition for input strict passivity in that there exists a $\delta > 0$ so that $\mathbf{Re}[h(j\omega_0)] \geq \delta > 0$ for all ω. An example of this is

$$h_1(s) = \frac{1}{1+Ts} \tag{2.36}$$

We find that $\mathbf{Re}[h_1(j\omega)] > 0$ for all ω because

$$h_1(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{1+(\omega T)^2} - j\frac{\omega T}{1+(\omega T)^2} \tag{2.37}$$

However there is no $\delta > 0$ that ensures $\mathbf{Re}[h(j\omega_0)] \geq \delta > 0$ for all $\omega \in [-\infty, +\infty]$. This is seen from the fact that for any $\delta > 0$ we have

$$\mathbf{Re}[h_1(j\omega)] = \frac{1}{1+(\omega T)^2} < \delta \quad \text{for all} \quad \omega > \sqrt{\frac{1-\delta}{\delta}}\frac{1}{T} \tag{2.38}$$

This implies that $h_1(s)$ is not input strictly passive. We note that for this system

$$|h_1(j\omega)|^2 = \frac{1}{1+(\omega T)^2} = \mathbf{Re}[h_1(j\omega)] \tag{2.39}$$

which means that the system is output strictly passive with $\epsilon = 1$.

Example 2.10. Consider a system with the transfer function

$$h_2(s) = \frac{s+c}{(s+a)(s+b)} \tag{2.40}$$

where a, b and c are positive constants. We find that

$$\begin{aligned} h_2(j\omega) &= \tfrac{j\omega+c}{(j\omega+a)(j\omega+b)} \\ &= \tfrac{(c+j\omega)(a-j\omega)(b-j\omega)}{(a^2+\omega^2)(b^2+\omega^2)} \\ &= \tfrac{abc+\omega^2(a+b-c)+j[\omega(ab-ac-bc)-\omega^3]}{(a^2+\omega^2)(b^2+\omega^2)}. \end{aligned}$$

From the above it is clear that

1. If $c \le a+b$, then $\mathbf{Re}[h_2(j\omega)] > 0$ for all $\omega \in I\!R$. As $\mathbf{Re}[h_2(j\omega)] \to 0$ when $\omega \to \infty$, *the* system is not input strictly passive.
2. If $c > a+b$, then $h_2(s)$ is not passive because $\mathbf{Re}[h_2(j\omega)] < 0$ for $\omega > \sqrt{abc/(c-a-b)}$.

Example 2.11. The systems with transfer functions

$$h_3(s) = 1 + Ts \tag{2.41}$$

$$h_4(s) = \frac{1+T_1 s}{1+T_2 s}, \quad T_1 < T_2 \tag{2.42}$$

are input strictly passive because

$$\mathbf{Re}[h_3(j\omega)] = 1 \tag{2.43}$$

and

$$\mathbf{Re}[h_4(j\omega)] = \frac{1+\omega^2 T_1 T_2}{1+(\omega T_2)^2} \in \left(\frac{T_1}{T_2}, 1\right] \tag{2.44}$$

Moreover $|h_4(j\omega)|^2 \le 1$, so that

$$\mathbf{Re}[h_4(j\omega)] \ge \frac{T_1}{T_2} \ge \frac{T_1}{T_2}\,|h_4(j\omega)|^2 \tag{2.45}$$

which shows that the system is output strictly passive with $\epsilon = T_1/T_2$. The reader may verify from a direct calculation of $|h_4(j\omega)|^2$ and some algebra that it is possible to have $\mathbf{Re}[h_4(j\omega)] \ge |h_4(j\omega)|^2$, that is, $\epsilon = 1$. This agrees with the Nyquist plot of $h_4(j\omega)$.

Example 2.12. A dynamic system describing an electrical one-port with resistors, inductors and capacitors is passive if the voltage over the port is input and the current into the port is output, or *vice versa*. In Figure 2.3 different passive one-ports are shown. We consider the voltage over the port to be the input and the current into the port as the output. The resulting transfer functions are admittances, which are the inverses of the impedances. Circuit 1 is a capacitor, circuit 2 is a resistor in parallel with a capacitor, circuit 3 is a resistor in series with a inductor and a capacitor, while circuit 4 is a resistor in series with a parallel connection of an inductor, a capacitor and a resistor. The transfer functions are

$$h_1(s) = Cs \tag{2.46}$$

$$h_2(s) = \frac{1}{R}(1 + RCs) \tag{2.47}$$

$$h_3(s) = \frac{Cs}{1 + RCs + LCs^2} \tag{2.48}$$

$$h_4(s) = \frac{1}{R_1}\frac{1 + \frac{L}{R}s + LCs^2}{1 + (\frac{L}{R_1} + \frac{L}{R})s + LCs^2} \tag{2.49}$$

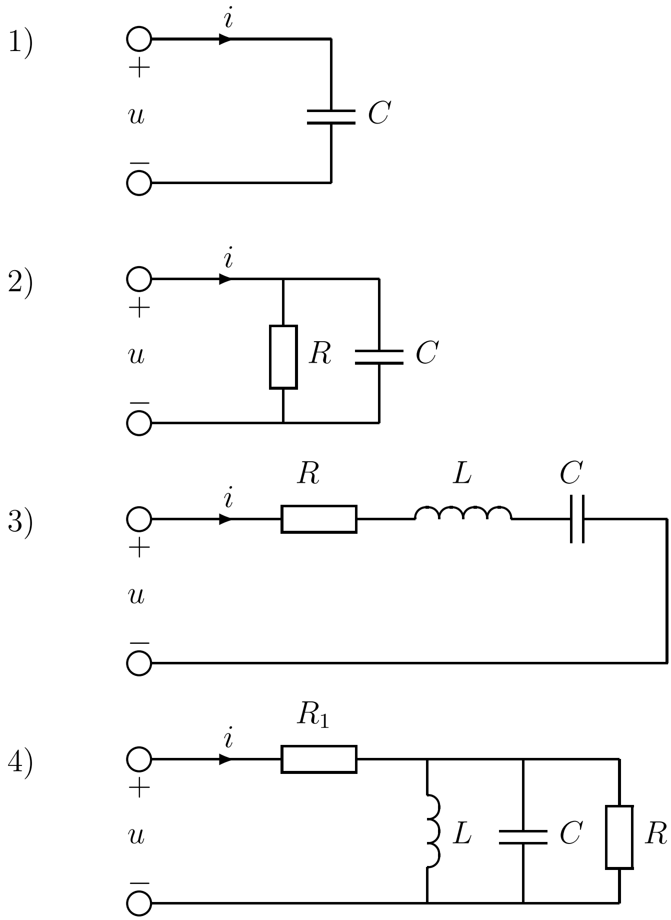

Fig. 2.3. Passive electrical one-ports

Systems 1, 2, 3 and 4 are all passive as the poles have real parts that are strictly less than zero, and in addition $\mathbf{Re}[h_i(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$ and $i \in \{1, 2, 3, 4\}$ (the fact that all the poles are in $\mathbf{Re}[s] < 0$ is important; see Theorem 2.14). It follows that the transfer functions have phases that satisfy $|\angle h_i(j\omega)| \leq 90°$. In addition system 2 is input strictly passive as $\mathbf{Re}[h_2(j\omega)] = 1/R > 0$ for all ω. For system 4 we find that

$$\mathbf{Re}[h_4(j\omega)] = \frac{1}{R_1} \frac{(1-\omega^2 LC)^2 + \omega^2 \frac{L^2}{R_1(R_1+R)}}{(1-\omega^2 LC)^2 + \omega^2 \frac{L^2}{(R_1+R)^2}} \geq \frac{1}{R_1+R} \tag{2.50}$$

which means that system 4 is input strictly passive. ■

So far we have only considered systems where the transfer functions $h(s)$ have poles with negative real parts. There are however passive systems that have transfer functions with poles on the imaginary axis. This is demonstrated in the following example:

Example 2.13. Consider the system $\dot{y}(t) = u(t)$ which is represented in transfer function description by $y(s) = h(s)u(s)$ where $h(s) = \frac{1}{s}$. This means that the transfer function has a pole at the origin, which is on the imaginary axis. For this system $\mathbf{Re}[h(j\omega)] = 0$ for all ω. However, we cannot establish passivity using Theorem 2.6 as this theorem only applies to systems where all the poles have negative real parts. Instead, consider

$$\int_0^t y(s)u(s)ds = \int_0^t y(s)\dot{y}(s)ds \tag{2.51}$$

A change of variables $\dot{y}(t)dt = dy$ gives

$$\int_0^t y(t')u(t')dt' = \int_{y(0)}^{y(t)} y(t')dy = \frac{1}{2}[y(t)^2 - y(0)^2] \geq -\frac{1}{2}y(0)^2 \tag{2.52}$$

and passivity is shown with $\beta = -\frac{1}{2}y(0)^2$. ■

It turns out to be relatively involved to find necessary and sufficient conditions on $h(j\omega)$ for the system to be passive when we allow for poles on the imaginary axis. The conditions are relatively simple and are given in the following Theorem.

Theorem 2.14. *Consider a linear time-invariant system with a rational transfer function $h(s)$. The system is passive if and only if*

1. *$h(s)$ has no poles in $\mathbf{Re}[s] > 0$.*
2. *$\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$ such that $j\omega$ is not a pole of $h(s)$.*
3. *If $j\omega_0$ is a pole of $h(s)$, then it is a simple pole, and the residual in $s = j\omega_0$ is real and greater than zero, that is, $Res_{s=j\omega_0} h(s) = \lim_{s\to j\omega_0}(s - j\omega_0)h(j\omega) > 0$.*

■

The above result is established in Section 2.12. Contrary to Theorem 2.6, poles on the imaginary axis are considered.

Corollary 2.15. *If a system with transfer function $h(s)$ is passive, then $h(s)$ has no poles in $\mathbf{Re}[s] > 0$.* ■

Proposition 2.16. *Consider a rational transfer function*

$$h(s) = \frac{(s+z_1)(s+z_2)\ldots}{s(s+p_1)(s+p_2)\ldots} \tag{2.53}$$

where $\mathbf{Re}[p_i] > 0$ and $\mathbf{Re}[z_i] > 0$ which means that $h(s)$ has one pole at the origin and the remaining poles in $\mathbf{Re}[s] < 0$, while all the zeros are in $\mathbf{Re}[s] < 0$. Then the system with transfer function $h(s)$ is passive if and only if $\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$. ■

Proof: The residual of the pole on the imaginary axis is

$$\mathrm{Res}_{s=0} h(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \tag{2.54}$$

Here the constants z_i and p_i are either real and positive, or they appear in complex conjugated pairs where the products $z_i z_i^* = |z_i|^2$ and $p_i p_i^* = |p_i|^2$ are real and positive. It is seen that the residual at the imaginary axis is real and positive. As $h(s)$ has no poles in $\mathbf{Re}\,[s] > 0$ by assumption, it follows that the system is passive if and only if $\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$. ■

Example 2.17. Consider two systems with transfer functions

$$h_1(s) = \frac{s^2 + a^2}{s(s^2 + \omega_0^2)}, \quad a \neq 0, \omega_0 \neq 0 \tag{2.55}$$

$$h_2(s) = \frac{s}{s^2 + \omega_0^2}, \quad \omega_0 \neq 0 \tag{2.56}$$

where all the poles are on the imaginary axis. Thus condition 1 in Theorem 2.14 is satisfied. Moreover,

$$h_1(j\omega) = -j \frac{a^2 - \omega^2}{\omega(\omega_0^2 - \omega^2)} \tag{2.57}$$

$$h_2(j\omega) = j \frac{\omega}{\omega_0^2 - \omega^2} \tag{2.58}$$

so that condition 2 also holds in view of $\mathbf{Re}[h_1(j\omega)] = \mathbf{Re}[h_2(j\omega)] = 0$ for all ω so that $j\omega$ is not a pole in $h\,(s)$. We now calculate the residual, and find that

$$\mathrm{Res}_{s=0} h_1(s) = \frac{a^2}{\omega_0^2} \tag{2.59}$$

$$\mathrm{Res}_{s=\pm j\omega_0} h_1(s) = \frac{\omega_0^2 - a^2}{2\omega_0^2} \tag{2.60}$$

$$\mathrm{Res}_{s=\pm j\omega_0} h_2(s) = \frac{1}{2} \tag{2.61}$$

We see that, according to Theorem 2.14, the system with transfer function $h_2(s)$ is passive, while $h_1(s)$ is passive whenever $a < \omega_0$.

Example 2.18. Consider a system with transfer function

$$h(s) = -\frac{1}{s} \tag{2.62}$$

The transfer function has no poles in $\mathbf{Re}\,[s] > 0$, and $\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \neq 0$. However, $Res_{s=0} h(s) = -1$, and Theorem 2.14 shows that the system is not passive This result agrees with the observation

$$\int_0^t y(s)u(s)ds = -\int_{y(0)}^{y(t)} y(s)dy = \frac{1}{2}[y(0)^2 - y(t)^2] \tag{2.63}$$

where the right hand side has no lower bound as $y(t)$ can be arbitrarily large.

2.5 Passivity of the PID Controllers

Proposition 2.19. *Assume that $0 \leq T_d < T_i$ and $0 \leq \alpha \leq 1$. Then the PID controller*

$$h_r(s) = K_p \frac{1 + T_i s}{T_i s} \frac{1 + T_d s}{1 + \alpha T_d s} \tag{2.64}$$

is passive. ■

This follows from Proposition 2.16.

Proposition 2.20. *Consider a PID controller with transfer function*

$$h_r(s) = K_p \beta \frac{1 + T_i s}{1 + \beta T_i s} \frac{1 + T_d s}{1 + \alpha T_d s} \tag{2.65}$$

where $0 \leq T_d < T_i$, $1 \leq \beta < \infty$ and $0 < \alpha \leq 1$. Then the controller is passive and, in addition, the transfer function gain has an upper bound $|h_r(j\omega)| \leq \frac{K_p \beta}{\alpha}$ and the real part of the transfer function is bounded away from zero according to $\mathbf{Re}\,[h_r(j\omega)] \geq K_p$ for all ω. ■

It follows from Bode diagram techniques that

$$|h_r(j\omega)| \leq K_p \beta \cdot 1 \cdot \frac{1}{\alpha} = \frac{K_p \beta}{\alpha} \tag{2.66}$$

The result on the $\mathbf{Re}\,[h_r(j\omega)]$ can be established using Nyquist diagram, or by direct calculation of $\mathbf{Re}\,[h_r(j\omega)]$. ■

2.6 Stability of a Passive Feedback Interconnection

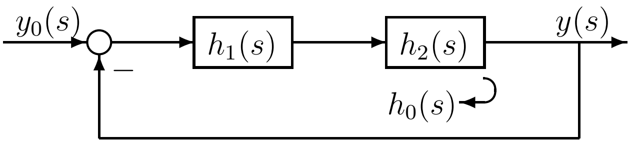

Fig. 2.4. Interconnection of a passive system $h_1(s)$ and a strictly passive system $h_2(s)$

Consider a feedback loop with loop transfer function $h_0(s) = h_1(s)h_2(s)$ as shown in Figure 2.4. If h_1 is passive and h_2 is strictly passive, then the phases of the transfer functions satisfy

$$|\angle h_1(j\omega)| \leq 90^\circ \quad \text{and} \quad |\angle h_2(j\omega)| < 90^\circ \tag{2.67}$$

It follows that the phase of the loop transfer function $h_0(s)$ is bounded by

$$|\angle h_0(j\omega)| < 180^\circ \tag{2.68}$$

As h_1 and h_2 are passive, it is clear that $h_0(s)$ has no poles in $\mathbf{Re}\,[s] > 0$. Then according to standard Bode-Nyquist stability theory the system is asymptotically stable and BIBO stable [2]. The same result is obtained if instead h_1 is strictly passive and h_2 is passive.

We note that, in view of Proposition 2.20, a PID controller with limited integral action is strictly stable. This implies that

- A passive linear system with a PID controller with limited integral action is BIBO stable.

For an important class of systems passivity or strict passivity is a structural property which is not dependent on the numerical values of the parameters of the system. Then passivity considerations may be used to establish stability even if there are large uncertainties or large variations in the system parameters. This is often referred to as robust stability. When it comes to performance it is possible to use any linear design technique to obtain high performance for the nominal parameters of the system. The resulting system will have high performance under nominal conditions, and in addition robust stability under large parameter variations.

2.7 Mechanical Analogs for PD Controllers

In this section we will study how PD controllers for position control can be represented by mechanical analogs when the input to the system is force and the output is position. Note that when force is input and position is output, then the physical system is not passive. We have a passive physical system if the force is the input and the velocity is the output, and then a PD controller from position corresponds to PI controller from velocity. For this reason we might have referred to the controllers in this section as PI controllers for velocity control.

We consider a mass m with position $x(\cdot)$ and velocity $v(\cdot) = \dot{x}(\cdot)$. The dynamics is given by $m\ddot{x}(t) = u(t)$ where the force u is the input. The desired position is $x_d(\cdot)$, while the desired velocity is $v_d(\cdot) = \dot{x}_d(\cdot)$. A PD controller $u = K_p(1 + T_d s)\,[x_d(s) - x(s)]$ is used. The control law can be written as

$$u(t) = K_p(x_d(t) - x(t)) + D(v_d(t) - v(t)) \tag{2.69}$$

where $D = K_p T_d$. The mechanical analog appears from the observation that this control force is the force that results if the mass m with position x is connected to the position x_d with a parallel interconnection of a spring with stiffness K_p and a damper with coefficient D as shown in Figure 2.5.

[2] Bounded Input-Bounded Output.

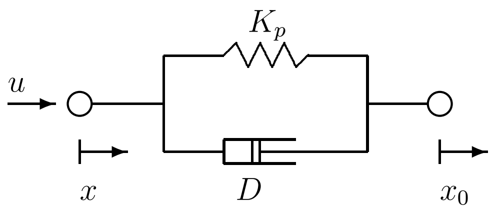

Fig. 2.5. Mechanical analog of PD controller with feedback from position

If the desired velocity is not available, and the desired position is not smooth a PD controller of the type

$$u(s) = K_p x_d(s) - K_p(1 + T_d s)x(s), \quad s \in \mathbb{C}$$

can be used. Then the control law is

$$u(t) = K_p(x_d(t) - x(t)) - Dv(t) \tag{2.70}$$

This is the force that results if the mass m is connected to the position x_d with a spring of stiffness K_p and a damper with coefficient D as shown in Figure 2.6.

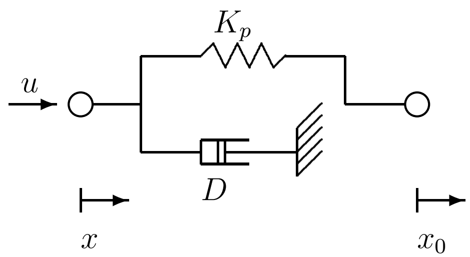

Fig. 2.6. Mechanical analog of a PD controller without desired velocity input

If the velocity is not measured the following PD controller can be used

$$u(s) = K_p \frac{1 + T_d s}{1 + \alpha T_d s}[x_d(s) - x(s)] \tag{2.71}$$

where $0 \leq \alpha \leq 1$ is the filter parameter. We will now demonstrate that this transfer function appears by connecting the mass m with position x to a spring with stiffness K_1 in series with a parallel interconnection of a spring with stiffness K and a damper with coefficient D as shown in Figure 2.7.

To find the expression for K_1 and K we let x_1 be the position of the connection point between the spring K_1 and the parallel interconnection. Then the force is $u = K_1(x_1 - x)$, which implies that $x_1(s) = x(s) + u(s)/K_1$. As

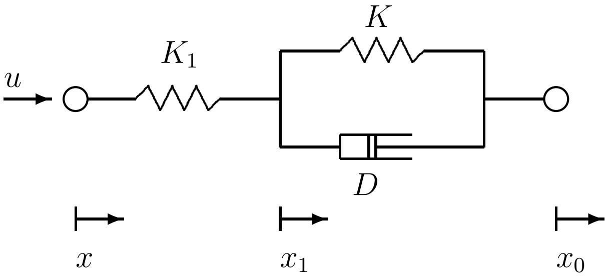

Fig. 2.7. Mechanical analog of a PD controller without velocity measurement

there is no mass in the point x_1 there must be a force of equal magnitude in the opposite direction from the parallel interconnection, so that

$$u(s) = K[x_d(s) - x_1(s)] + D[v_d(s) - v_1(s)] = (K + Ds)[x_d(s) - x_1(s)] \quad (2.72)$$

Insertion of $x_1(s)$ gives

$$u(s) = (K + Ds)[x_d(s) - x(s) - \frac{1}{K_1}u(s)] \quad (2.73)$$

We solve for $u(s)$ and the result is

$$\begin{aligned} u(s) &= K_1 \tfrac{K+Ds}{K_1+K+Ds}[x_d(s) - x(s)] \\ &= \tfrac{K_1 K}{K_1+K} \tfrac{1+\frac{D}{K}s}{1+\frac{K}{K_1+K}\frac{D}{K}s}[x_d(s) - x(s)] \end{aligned}$$

We see that this is a PD controller without velocity measurement where

$$\begin{cases} K_p &= \frac{K_1 K}{K_1+K} \\ T_d &= \frac{D}{K} \\ \alpha &= \frac{K}{K_1+K} \in [0, 1) \end{cases}$$

2.8 Multivariable Linear Systems

Theorem 2.21. *Consider a linear time-invariant system*

$$y(s) = H(s)u(s) \quad (2.74)$$

with a rational transfer function matrix $H(s) \in \mathbb{C}^{m \times m}$*, input* $u(t) \in \mathbb{R}^m$ *and input* $y(t) \in \mathbb{R}^m$*. Assume that all the poles of* $H(s)$ *are in* $\mathbf{Re}\,[s] < 0$*. Then,*

1. *The system is passive* $\Leftrightarrow \lambda_{min}[H(j\omega)+H^*(j\omega)] \geq 0$ *for all* $\omega \in [-\infty, +\infty]$*.*

2. *The system is input strictly passive* $\Leftrightarrow$ *There is a* $\delta > 0$ *so that* $\lambda_{min}[H(j\omega) + H^*(j\omega)] \geq \delta > 0$ *for all* $\omega \in [-\infty, +\infty]$.

■

Remark 2.22. Similarly to Theorem 2.6, a crucial assumption in Theorem 2.21 is that the poles have negative real parts, *i.e.* there is no pole on the imaginary axis.

Proof: Let $A \in \mathbb{C}^{m \times m}$ be some Hermitian matrix with eigenvalues $\lambda_i(A)$. Let $x \in \mathbb{C}^m$ be an arbitrary vector with complex entries. It is well-known from linear algebra that x^*Ax is real, and that $x^*Ax \geq \lambda_{min}(A)|x|^2$. From Parseval's Theorem we have

$$\begin{aligned} \int_0^\infty y^T(s)u_t(s)ds &= \sum_{i=1}^m \int_0^\infty y_i(s)(u_i)_t(s)ds \\ &= \sum_{i=1}^m \frac{1}{2\pi} \int_{-\infty}^\infty y_i^*(j\omega)(u_i)_t(j\omega)d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty y^*(j\omega)u_t(j\omega)d\omega \end{aligned}$$

where we recall that $u_t(\cdot)$ is a truncated function and that s in the integrand is a dumb integration variable (not to be confused with the Laplace transform!). This leads to

$$\begin{aligned} \int_0^t y^T(s)u(s)ds &= \int_0^\infty y^T(s)u_t(s)ds \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty y^*(j\omega)u_t(j\omega)d\omega \\ &= \frac{1}{4\pi} \int_{-\infty}^\infty [u_T^*(j\omega)y(j\omega) + y^*(j\omega)u_t(j\omega)]d\omega \\ &= \frac{1}{4\pi} \int_{-\infty}^\infty u_t^*(j\omega)[H(j\omega) + H^*(j\omega)]u_t(j\omega)d\omega \end{aligned}$$

Because $H(j\omega) + H^*(j\omega)$ is Hermitian we find that

$$\int_0^t y^T(\tau)u(\tau)d\tau \geq \frac{1}{4\pi} \int_{-\infty}^\infty \lambda_{\min}[H(j\omega) + H^*(j\omega)]|u_t(j\omega)|^2 d\omega \tag{2.75}$$

The result can be established along the lines of Theorem 2.6. ■

2.9 The Scattering Formulation

By a change of variables an alternative description can be established where passivity corresponds to small gain. We will introduce this idea with an example from linear circuit theory. Consider a linear time-invariant system describing an electrical one-port with voltage e, current i and impedance $z(s)$ so that

$$e(s) = z(s)i(s) \tag{2.76}$$

Define the wave variables

$$a = e + z_0 i \quad \text{and} \quad b = e - z_0 i \tag{2.77}$$

where z_0 is a positive constant. The Laplace transform is

$$\begin{aligned} a(s) &= [z(s) + z_0]i(s) \\ b(s) &= [z(s) - z_0]i(s) \end{aligned}$$

Combining the two equations we get

$$b(s) = g(s)a(s) \tag{2.78}$$

where

$$g(s) = \frac{z(s) - z_0}{z_0 + z(s)} = \frac{\frac{z(s)}{z_0} - 1}{1 + \frac{z(s)}{z_0}} \tag{2.79}$$

is the *scattering function* of the system. The terms wave variable and scattering function originate from the description of transmission lines where a can be seen as the incident wave and b can be seen as the reflected wave.

If the electrical circuit has only passive elements, that is, if the circuit is an interconnection of resistors, capacitors and inductors, the passivity inequality satisfies

$$\int_0^t e(\tau)i(\tau)d\tau \geq 0 \tag{2.80}$$

where it is assumed that the initial energy stored in the circuit is zero. We note that

$$a^2 - b^2 = (e + z_0 i)^2 - (e - z_0 i)^2 = 4z_0 ei \tag{2.81}$$

which implies

$$\int_0^t b^2(\tau)d\tau = \int_0^t a^2(\tau)d\tau - 4z_0 \int_0^t e(\tau)i(\tau)d\tau \tag{2.82}$$

From this it is seen that passivity of the system with input i and output e corresponds to small gain for the system with input a and output b in the sense that

$$\int_0^t b^2(\tau)d\tau \leq \int_0^t a^2(\tau)d\tau \tag{2.83}$$

This small gain condition can be interpreted loosely in the sense that the energy content b^2 of the reflected wave is smaller than the energy a^2 of the incident wave. For the general linear time-invariant system

$$y(s) = h(s)u(s) \tag{2.84}$$

introduce the wave variables

$$a = y + u \quad \text{and} \quad b = y - u \tag{2.85}$$

where, as above, a is the incident wave and b is the reflected wave. As for electrical circuits it will usually be necessary to include a constant z_0 so that $a = y + z_0 u$ $b = y - z_0 u$ so that the physical units agree. We tacitly suppose that this is done by letting $z_0 = 1$ with the appropriate physical unit. The scattering function is defined by

$$g(s) \triangleq \frac{b}{a}(s) = \frac{y-u}{y+u}(s) = \frac{h(s)-1}{1+h(s)} \tag{2.86}$$

Theorem 2.23. *Consider a system with rational transfer function $h(s)$ with no poles in $\mathbf{Re}[s] \geq 0$, and scattering function $g(s)$ given by (2.86). Then*

1. *The system is passive if and only if $|g(j\omega)| \leq 1$ for all $\omega \in [-\infty, +\infty]$.*
2. *The system is input strictly passive, and there is a γ so that $|h(j\omega)| \leq \gamma$ for all $\omega \in [-\infty, +\infty]$ if and only if there is a $\gamma' \in (0,1)$ so that $|g(j\omega)|^2 \leq 1 - \gamma'$.* ■

Proof: Consider the following computation

$$\begin{aligned} |g(j\omega)|^2 &= \tfrac{|h(j\omega)-1|^2}{|h(j\omega)+1|^2} \\ &= \tfrac{|h(j\omega)|^2 - 2\mathbf{Re}[h(j\omega)]+1}{|h(j\omega)|^2+2\mathbf{Re}[h(j\omega)]+1} \\ &= 1 - \tfrac{4\mathbf{Re}[h(j\omega)]}{|h(j\omega)+1|^2} \end{aligned} \tag{2.87}$$

It is seen that $|g(j\omega)| \leq 1$ if and only if $\mathbf{Re}[h(j\omega)] \geq 0$. Result 1 then follows as the necessary and sufficient condition for the system to be passive is that $\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$. Concerning the second result, we show the "if" part. Assume that there is a δ so that $\mathbf{Re}[h(j\omega)] \geq \delta > 0$ and a γ so that $|h(j\omega)| \leq \gamma$ for all $\omega \in [-\infty, +\infty]$. Then

$$|g(j\omega)|^2 \geq 1 - \frac{4\delta}{(\gamma+1)^2} \tag{2.88}$$

and the result follows with $0 < \gamma' < \min\left(1, \frac{4\delta}{(\gamma+1)^2}\right)$. Next assume that $g(j\omega)|^2 \leq 1 - \gamma'$ for all ω. Then

$$4\mathbf{Re}\,[h(j\omega)] \geq \gamma' \left(|h(j\omega)|^2 + 2\mathbf{Re}[h(j\omega)] + 1\right) \tag{2.89}$$

and strict passivity follows from

$$\mathbf{Re}\,[h(j\omega)] \geq \frac{\gamma'}{4-2\gamma'} > 0 \tag{2.90}$$

Finite gain of $h(j\omega)$ follows from

$$\gamma'|h(j\omega)|^2 - (4-2\gamma')\,\mathbf{Re}[h(j\omega)] + \gamma' \leq 0 \tag{2.91}$$

which in view of the general result $|h(j\omega)| > \mathbf{Re}[h(j\omega)]$ gives the inequality

$$|h(j\omega)|^2 - \frac{(4-2\gamma')}{\gamma'}\,|h(j\omega)| + 1 \leq 0 \tag{2.92}$$

This implies that

$$|h\,(j\omega)| \leq \frac{(4-2\gamma')}{\gamma'} \tag{2.93}$$

■

We shall come back on the relationships between passivity and bounded realness in the framework of dissipative systems and H_∞ theory; see Section 5.9. A comment on the input-output change in (2.85): the association of the new system with transfer function $g(s)$ merely corresponds to writing down $uy = \frac{1}{4}(a+b)(a-b) = \frac{1}{4}(a^2-b^2)$. Thus if $\int_0^t u(s)y(s)ds \geq 0$ one gets $\int_0^t a^2(s)ds \geq \int_0^t b^2(s)ds$: the $\mathcal{L}_2$-norm of the new output $b(t)$ is bounded by the $\mathcal{L}_2$-norm of the new intput $a(t)$.

2.10 Impedance Matching

In this section we will briefly review the concept of impedance matching. Again an electrical one-port is studied. The one-port has a voltage source e, serial impedance z_0,output voltage v and current i. The circuit is coupled to the load which is a passive one-port with driving point impedance $z_L(s)$ as shown in Figure 2.8.

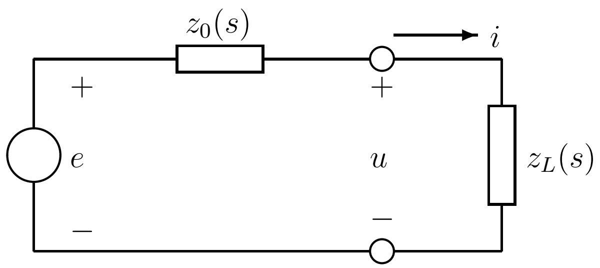

Fig. 2.8. Impedance matching

The following problem will be addressed: suppose $z_0\,(s)$ is given and that $e\,(t) = E\sin\omega_e t$. Select $z_L\,(s)$ so that the power dissipated in z_L is maximized.

The current is given by

$$i(s) = \frac{e(s)}{z_0(s) + z_L(s)} \tag{2.94}$$

while the voltage over z_L is

$$u(s) = z_L(s)i(s) \tag{2.95}$$

The power dissipated in z_L is therefore

$$\begin{aligned} P(\omega_e) &= \tfrac{1}{2}\mathbf{Re}[u_L(j\omega_e)i^*(j\omega_e)] \\ &= \tfrac{1}{2}\mathbf{Re}[z_L(j\omega_e)]i(j\omega_e)i^*(j\omega_e) \\ &= \tfrac{1}{2}\tfrac{\mathbf{Re}[z_L(j\omega_e)]}{[z_0(j\omega_e)+z_L(j\omega_e)]^*[z_0(j\omega_e)+z_L(j\omega_e)]}E^2 \end{aligned}$$

where $(\cdot)^*$ denotes the complex conjugate. Denote

$$z_0(j\omega_e) = \alpha_0 + j\beta_0 \quad \text{or} \quad z_L(j\omega_e) = \alpha_L + j\beta_L \tag{2.96}$$

This gives

$$P = \frac{1}{2}\frac{\alpha_L E^2}{(\alpha_0+\alpha_L)^2 + (\beta_0+\beta_L)^2} \tag{2.97}$$

We see that if $\alpha_L = 0$, then $P = 0$, whereas for nonzero α_L then $|\beta_L| \to \infty$, gives $P \to 0$. A maximum for P would be expected somewhere between these extremes. Differentiation with respect to β_L gives

$$\frac{\partial P}{\partial \beta_L} = \frac{E^2}{2}\frac{-2\alpha_L(\beta_0+\beta_L)}{[(\alpha_0+\alpha_L)^2 + (\beta_0+\beta_L)^2]^2} \tag{2.98}$$

which implies that the maximum *of* P appears for $\beta_L = -\beta_0$. Differentiation with respect to α_L with $\beta_L = -\beta_0$ gives

$$\frac{\partial P}{\partial a} = \frac{E^2}{2}\frac{\alpha_0^2 - {\alpha_L}^2}{[(\alpha_0+\alpha_L)^2 + (\beta_0+\beta_L)^2]^2} \tag{2.99}$$

and it is seen that the maximum is found for $\alpha_L = \alpha_0$. This means that the maximum power dissipation in z_L is achieved with

$$z_L(j\omega_e) = z_0^*(j\omega_e) \tag{2.100}$$

This particular selection of $z_L(j\omega_e)$ is called impedance matching. If the voltage source $e(t)$ is not simply a sinusoid but a signal with a arbitrary spectrum, then it is not possible to find a passive impedance $z_L(s)$ which satisfies the impedance matching condition or a general series impedance $z_0(j\omega)$. This is because the two impedances are required to have the same absolute values, while the phase have opposite signs. This cannot be achieved for one particular $z_L(s)$ for all frequencies.

However, if $z_0(j\omega) = z_0$ is a real constant, then impedance matching at all frequencies is achieved with $z_L = z_0$. We now assume that z_0 is a real constant, and define the wave variables to be

$$a = u + z_0 i \quad \text{or} \quad b = u - z_0 i \tag{2.101}$$

Then it follows that

$$a = e \tag{2.102}$$

for the system in Figure 2.8. A physical interpretation of the incident wave a is as follows: let u be the input voltage to the one-port and let i be the current into the port. Consider the extended one-port where a serial impedance z_0 is connected in to the one-port as shown in Figure 2.9. Then a is the input voltage of the extended one-port.

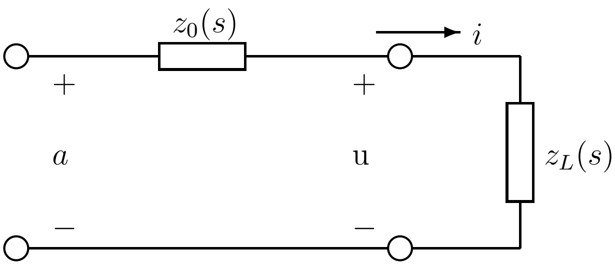

Fig. 2.9. Extended one-port with a serial impedance z_0

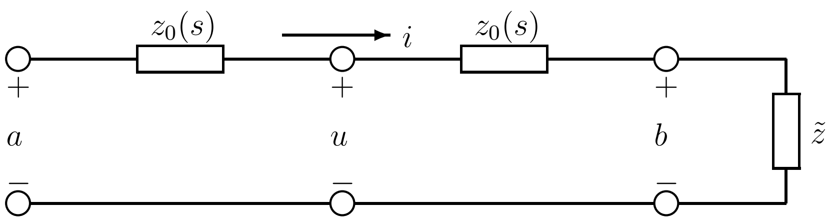

Fig. 2.10. Physical interpretation of the reflected wave b where $\tilde{z} = z_L(s) - z_0(s)$

The physical interpretation of the reflected wave b is shown in figure 2.10. We clearly see that if $z_L = z_0$, then

$$u = z_0 i \quad \Rightarrow \quad b = 0 \tag{2.103}$$

This shows that if impedance matching is used with z_0 being constant, then the scattering function is

$$g(s) = \frac{b(s)}{a(s)} = 0 \tag{2.104}$$

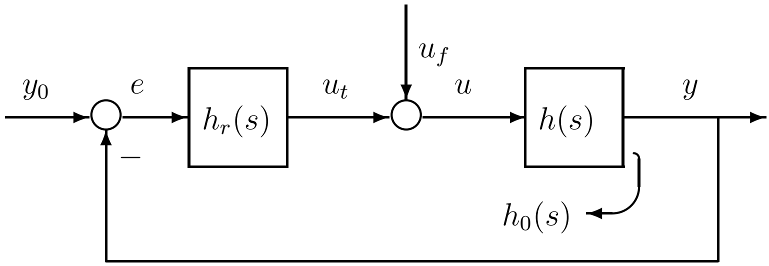

Fig. 2.11. Feedback interconnection of two passive systems

2.11 Feedback Loop

A feedback interconnection of two passive linear time-invariant systems is shown in Figure 2.11 where signals are given by

$$y(s) = h(s)u(s), \qquad u_t(s) = h_r(s)e(s) \tag{2.105}$$

$$u(t) = u_f(t) + u_t(t), \qquad e(t) = y_0(t) - y(t) \tag{2.106}$$

We can think of $h(s)$ as describing the plant to be controlled, and $h_r(s)$ as describing the feedback controller. Here u_t is the feedback control and u_f is the feedforward control. We assume that the plant $h(s)$ and that the feedback controller $h_r(s)$ are strictly passive with finite gain. Then, as shown in Section 2.6 we have $\angle|h_0(j\omega)| < 180°$ where $h_0(s) := h(s)h_r(s)$ is the loop transfer function, and the system is BIBO stable.

A change of variables in now introduced to bring the system into a scattering formulation. The new variables are

$$a \triangleq y + u \quad \text{and} \quad b \triangleq y - u$$

for the plant and

$$a_r \triangleq u_t + e \quad \text{and} \quad b_r \triangleq u_t - e$$

for the feedback controller. In addition input variables

$$a_0 \triangleq y_0 + u_f \quad \text{and} \quad b_0 \triangleq y_0 - u_f$$

are defined. We find that

$$a_r = u_t + y_0 - y = u - u_f + y_0 - y = b_0 - b \tag{2.107}$$

and

$$b_r = u_t - y_0 + y = u - u_f - y_0 + y = a - a_0 \tag{2.108}$$

The associated scattering functions are

$$g(s) \triangleq \frac{h(s) - 1}{1 + h(s)} \quad \text{and} \quad g_r(s) \triangleq \frac{h_r(s) - 1}{1 + h_r(s)}$$

Now, $h(s)$ and $h_r(s)$ are passive by assumption, and as a consequence they cannot have poles in $\mathbf{Re}[s] > 0$. Then it follows that $g(s)$ and $g_r(s)$ cannot have poles in $\mathbf{Re}[s] > 0$ because $1 + h(s)$ is the characteristic equations for $h(s)$ with a unity negative feedback, which obviously is a stable system. Similar arguments apply for $1 + h(s)$. The system can then be represented as in Figure 2.12 where

$$b(s) = g(s)a(s), \qquad b_r(s) = g_r(s)a_r(s) \tag{2.109}$$

$$a(t) = b_r(t) + a_0(t), \qquad a_r(t) = b_0(t) - b(t) \tag{2.110}$$

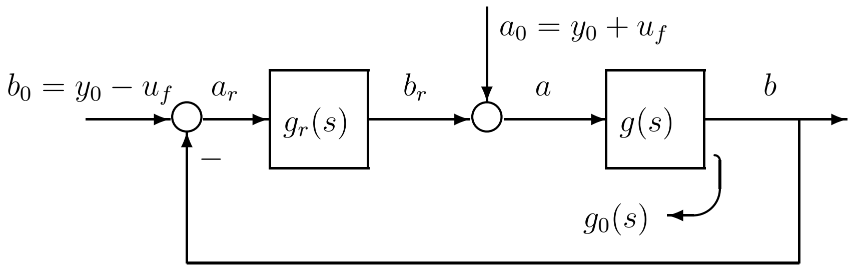

Fig. 2.12. Equivalent system

In the passivity setting, stability was ensured when two passive systems were interconnected in a feedback structure because the loop transfer function $h_0(j\omega)$ had a phase limitation so that $\angle h_0(j\omega) > -180°$. We would now like to check if there is an interpretation for the scattering formulation that is equally simple. This indeed turns out to be the case. We introduce the loop transfer function

$$g_0(s) \triangleq g(s)g_r(s) \tag{2.111}$$

of the scattering formulation. The function $g_0(s)$ cannot have poles in $\mathbf{Re}[s] > 0$ as $g(s)$ and $g_r(s)$ have no poles in $\mathbf{Re}[s] > 0$ by assumption. Then we have from Theorem 2.23:

1. $|g(j\omega)| \leq 1$ for all $\omega \in [-\infty, +\infty]$ because $h(s)$ is passive.
2. $|g_r(j\omega)| < 1$ for all $\omega \in [-\infty, +\infty]$ because $h_r(s)$ is strictly passive with finite gain.

As a consequence of this,

$$|g_0(j\omega)| < 1 \tag{2.112}$$

for all $\omega \in [-\infty, +\infty]$, and according to the Nyquist stability criterion the system is BIBO stable.

2.12 Bounded Real and Positive Real Transfer Functions

Bounded real and positive real are two important properties of transfer functions related to passive systems that are linear and time-invariant. We will in this section show that a linear time-invariant system is passive if and only if the transfer function of the system is positive real. To do this we first show that a linear time-invariant system is passive if and only if the scattering function, which is the transfer function of the wave variables, is bounded real. Then we show that the scattering function is bounded real if and only if the transfer function of the system is positive real. We will also discuss different aspects of these results for rational and irrational transfer functions.

We consider a linear time-invariant system $y(s) = h(s)u(s)$ with input u and output y. The incident wave is denoted $a \stackrel{\Delta}{=} y + u$, and the reflected wave is denoted $b \stackrel{\Delta}{=} y - u$. The scattering function $g(s)$ is given by

$$g(s) = \frac{h(s) - 1}{1 + h(s)} \tag{2.113}$$

and satisfies $b(s) = g(s)a(s)$. We note that

$$u(t)y(t) = \frac{1}{4}[a^2(t) - b^2(t)] \tag{2.114}$$

For linear time-invariant systems the properties of the system do not depend on the initial conditions, as opposed to nonlinear systems. We therefore assume for simplicity that the initial conditions are selected so that the energy function $V(t)$ is zero for initial time, that is $V(0) = 0$. The passivity inequality is then

$$0 \leq V(t) = \int_0^t u(s)y(s)ds = \frac{1}{4}\int_0^t [a^2(s) - b^2(s)]ds \tag{2.115}$$

The properties bounded real and positive real will be defined for functions that are analytic in the open right half plane $\mathbf{Re}[s] > 0$. We recall that a function $f(s)$ is *analytic* in a domain only if it is defined and infinitely differentiable for all points in the domain. A point where $f(s)$ ceases to be analytic is called a *singular point*, and we say that $f(s)$ has a *singularity* at this point. If $f(s)$ is rational, then $f(s)$ has a finite number of singularities, and the singularities are called *poles*. The poles are the roots of the denominator polynomial $R(s)$ if $f(s) = Q(s)/R(s)$, and a pole is said to be simple pole if it is not a multiple root in $R(s)$.

Definition 2.24. *A function $g(s)$ is said to be bounded real if*

1. *$g(s)$ is analytic in $\mathbf{Re}[s] > 0$.*
2. *$g(s)$ is real for real and positive s.*
3. *$|g(s)| \leq 1$ for all $\mathbf{Re}[s] > 0$.*

This definition extends to matrix functions $G(s)$ as follows:

Definition 2.25. *A transfer matrix $G(s) \in \mathbb{C}^{m\times m}$ is bounded real if all elements of $G(s)$ are analytic for $\mathbf{Re}[s] \geq 0$ and the $H_\infty-$norm satisfies $||G(s)||_\infty \leq 1$ where we recall that $||G(s)||_\infty = \sup_{\omega\in I\!R} \sigma_{\max}(G(j\omega))$. Equivalently the second condition can be replaced by: $I_m - G^T(-j\omega)G(j\omega) \geq 0$ for all $\omega \in I\!R$. Strict Bounded Realness holds when the ≥ 0 inequalities are replaced by > 0 inequalities.*

■

Theorem 2.26. *Consider a linear time-invariant system described by $y(s) = h(s)u(s)$, and the associated scattering function $a = y + u$, $b = y - u$ and $b(s) = g(s)a(s)$ where*

$$g(s) = \frac{h(s)-1}{1+h(s)} \tag{2.116}$$

which satisfies $b(s) = g(s)a(s)$ $a = y + u$ and $b = y - u$. Then the system $y(s) = h(s)u(s)$ is passive if and only if $g(s)$ is bounded real. ■

Proof: Assume that $y(s) = h(s)u(s)$ is passive. Then (2.115) implies that

$$\int_0^t a^2(\tau)d\tau \geq \int_0^t b^2(\tau)d\tau \tag{2.117}$$

for all $t \geq 0$. It follows that $g(s)$ cannot have any singularities in $\mathbf{Re}[s] > 0$ as this would result in exponential growth in $b(t)$ for any small input $a(t)$. Thus, $g(s)$ must satisfy condition 1 in the definition of bounded real.

Let σ_0 be an arbitrary real and positive constant, and let $a(t) = e^{\sigma_0 t}\mathbf{1}(t)$ where $\mathbf{1}(t)$ is the unit step function. Then the Laplace transform of $a(t)$ is $a(s) = \frac{1}{s-\sigma_0}$, while $b(s) = \frac{g(s)}{s-\sigma_0}$. Suppose that the system is not initially excited so that the inverse Laplace transform for rational $g(s)$ gives

$$b(t) = \sum_{i=1}^{n}\left(\mathrm{Res}_{s=s_i}\frac{g(s)}{s-\sigma_0}\right)e^{s_i t} + \left(\mathrm{Res}_{s=\sigma_0}\frac{g(s)}{s-\sigma_0}\right)e^{\sigma_0 t}$$

where s_i are the poles of $g(s)$ that satisfy $\mathbf{Re}\,[s_i] < 0$, and $\mathrm{Res}_{s=\sigma_0}\frac{g(s)}{s-\sigma_0} = g(\sigma_0)$. When $t \to \infty$ the term including $e^{\sigma_0 t}$ will dominate the terms including

$e^{s_i t}$, and $b(t)$ will tend to $g(\sigma_0)e^{\sigma_0 t}$. The same limit for $b(t)$ will also be found for irrational $g(s)$. As $a(t)$ is real, it follows that $g(\sigma_0)$ is real, and it follows that $g(s)$ must satisfy condition 2 in the definition of bounded realness.

Let $s_0 = \sigma_0 + j\omega_0$ be an arbitrary point in $\mathbf{Re}[s] > 0$, and let the input be $a(t) = \mathbf{Re}[e^{s_0 t}\mathbf{1}(t)]$. Then $b(t) \to \mathbf{Re}[g(s_0)e^{s_0 t}]$ as $t \to \infty$ and the power

$$P(t) := \frac{1}{4}[a^2(t) - b^2(t)] \tag{2.118}$$

will tend to

$$P(t) = \frac{1}{4}[e^{2\sigma_0 t}\cos^2\omega_0 t - |g(s_0)|^2 e^{2\sigma_0 t}\cos^2(\omega_0 t + \phi)]$$

where $\phi = \arg[g(s_0)]$. This can be rewritten using $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$, and the result is

$$\begin{aligned} 8P(t) &= (1 + \cos 2\omega_0 t)e^{2\sigma_0 t} - |g(s_0)|^2[1 + \cos(2\omega_0 t + 2\phi)]e^{2\sigma_0 t} \\ &= [1 - |g(s_0)|^2]e^{2\sigma_0 t} + \mathbf{Re}[\left(1 - g(s_0)^2\right) e^{2s_0 t}] \end{aligned}$$

In this expression s_0 and σ_0 are constants, and we can integrate $P(t)$ to get the energy function $V(T)$:

$$\begin{aligned} V(t) &= \textstyle\int_{-\infty}^{t} P(s)ds \\ &= \tfrac{1}{16\sigma_0}[1 - |g(s_0)|^2]e^{2\sigma_0 t} + \tfrac{1}{16}\mathbf{Re}\{\tfrac{1}{s_0}[1 - g(s_0)^2]e^{2s_0 t}\} \end{aligned}$$

First it is assumed that $\omega_0 \neq 0$. Then $\mathbf{Re}\{\frac{1}{s_0}[1 - g(s_0)^2]e^{2s_0 t}\}$ will be a sinusoidal function which becomes zero for certain values of t. For such values of t the condition $V(t) \geq 0$ implies that

$$\frac{1}{16\sigma_0}[1 - |g(s_0)|^2]e^{2\sigma_0 t} \geq 0$$

which implies that

$$1 - |g(s_0)|^2 \geq 0$$

Next it is assumed that $\omega_0 = 0$ such that $s_0 = \sigma_0$ is real. Then $g(s_0)$ will be real, and the two terms in $V(t)$ become equal. This gives

$$0 \leq V(t) = \frac{1}{8\sigma_0}[1 - g^2(s_0)]e^{2\sigma_0 t}$$

and with this it is established that for all s_0 in $\mathbf{Re}[s] > 0$ we have

$$1 - |g(s_0)|^2 \geq 0 \Rightarrow |g(s_0)| \leq 1$$

To show the converse we assume that $g(s)$ is bounded real and consider

$$g(j\omega) = \lim_{\substack{\sigma \to 0 \\ \sigma > 0}} g(\sigma + j\omega) \tag{2.119}$$

Because $g(s)$ is bounded and analytic for all $\mathbf{Re}\,[s] > 0$ it follows that this limit exists for all ω, and moreover

$$|g(j\omega)| \le 1$$

Then it follows from Parseval's Theorem that with a_T being the truncated version of a we have

$$\begin{aligned} 0 &\le \tfrac{1}{8\pi} \textstyle\int_{-\infty}^{\infty} |a_t(j\omega)|^2 \left(1 - |g(j\omega)|^2\right) d\omega \\ &= \tfrac{1}{4} \textstyle\int_0^t [a^2(s) - b^2(s)] ds \\ &= \textstyle\int_0^t u(s) y(s) ds \end{aligned}$$

which shows that the system must be passive. ■

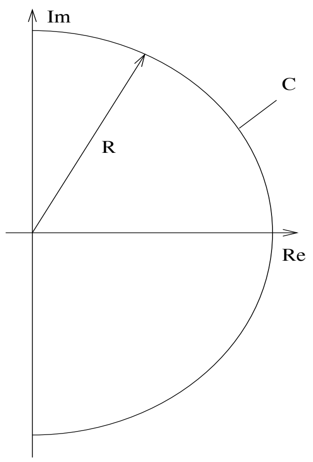

Fig. 2.13. Contour in the right half plane.

Define the contour C which encloses the right half plane as shown in Figure 2.13. The maximum modulus theorem is as follows. Let $f(s)$ be a function that is analytic inside the contour C. Let M be the upper bound on $|f(s)|$ on C. Then $|f(s)| \le M$ inside the contour, and equality is achieved at some point inside C if and only if $f(s)$ is a constant. This means that if $g(s)$ is bounded real, and $|g(s)| = 1$ for some point in $\mathbf{Re}[s] > 0$, then $|g(s)|$ achieves its maximum inside the contour C, and it follows that $g(s)$ is a constant in $\mathbf{Re}[s] \ge 0$. Because $g(s)$ is real for real $s > 0$, this means that $g(s) = 1$ for all

s in $\mathbf{Re}[s] \geq 0$. In view of this $[1 - g(s)]^{-1}$ has singularities in $\mathbf{Re}[s] > 0$ if and only if $g(s) = 1$ for all s in $\mathbf{Re}[s] \geq 0$.

If $g(s)$ is assumed to be a rational function the maximum modulus theorem can be used to reformulate the condition on $|g(s)|$ to be a condition on $|g(j\omega)|$. The reason for this is that a rational transfer function satisfying $|g(j\omega)| \leq 1$ for all ω will also satisfy

$$\lim_{\omega \to \infty} |g(j\omega)| = \lim_{|s| \to \infty} |g(s)| \tag{2.120}$$

Therefore, for a sufficiently large contour C we have that $|g(j\omega)| \leq 1$ implies $|g(s)| \leq 1$ for all $\mathbf{Re}[s] > 0$ whenever $g(s)$ is rational. This leads to the following result:

Theorem 2.27. *A rational function $g(s)$ is bounded real if and only if*

1. *$g(s)$ has no poles in $\mathbf{Re}[s] \geq 0$.*
2. *$|g(j\omega)| \leq 1$ for all $\omega \in [-\infty, +\infty]$.*

■

Let us now state a new definition.

Definition 2.28. *A transfer function $h(s)$ is said to be positive real (PR) if*

1. *$h(s)$ is analytic in $\mathbf{Re}[s] > 0$*
2. *$h(s)$ is real for positive real s*
3. *$\mathbf{Re}[h(s)] \geq 0$ for all $\mathbf{Re}[s] > 0$*

The last condition above is illustrated in Figure 2.14 where the Nyquist plot of a PR transfer function $H(s)$ is shown. The notion of positive realness extends to multivariable systems:

Definition 2.29. *The transfer matrix $H(s) \in \mathbb{C}^{m \times m}$ is positive real if:*

- *$H(s)$ has no pole in $\mathbf{Re}[s] > 0$*
- *$H(s)$ is real for all positive real s*
- *$H(s) + H^\star(s) \geq 0$ for all $\mathbf{Re}[s] > 0$* ■

An interesting characterization of multivariable PR transfer functions is as follows:

Theorem 2.30. *Let the transfer matrix $H(s) = C(sI_n - A)^{-1} + D \in \mathbb{C}^{m \times m}$, where the matrices A, B, C, and D are real, and every eigenvalue of A has a negative real part. Then $H(s)$ is positive real if and only if $y^\star[H^\star(j\omega) + H(j\omega)]y = y^\star \Pi(j\omega) y \geq 0$ for all $\omega \in \mathbb{R}$ and all $y \in \mathbb{C}^m$.* ■

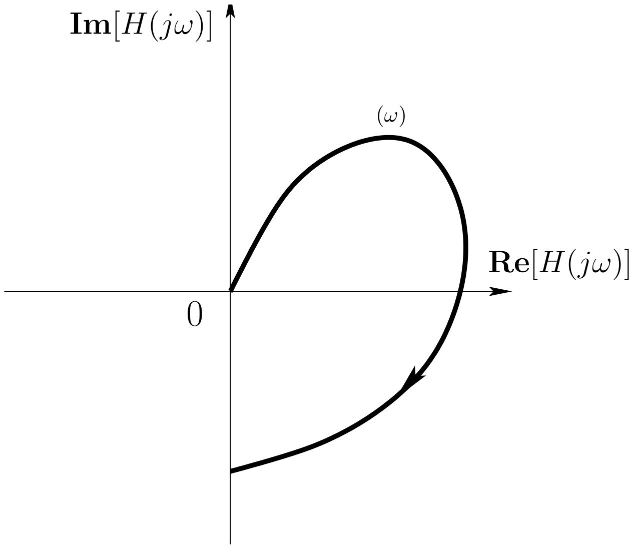

Fig. 2.14. Positive real transfer function

This result was proved in [8, p.53]. The rational matrix $\Pi(s) = c(sI_n - A)^{-1}B - B^T(sI_n + A^T)^{-1}C^T + D + D^T$ is known as the Popov function of the system. It is a rational spectral function, since it satisfies $\Pi(s) = \Pi^T(-s)$. The introduction of the spectral function $\Pi(s)$ allows us to state a result on which we shall come back in Section 3.3. Let $\Lambda : \mathcal{L}_{2,e} \to \mathcal{L}_{2,e}$ be a rational input-output operator $u(\cdot) \mapsto y(\cdot) = \Lambda(u(\cdot))$. Assume that the kernel of Λ has a minimal realization (A, B, C, D). In other words, the operator is represented in the Laplace transform space by a transfer matrix $H(s) = C(sI_n - A)^{-1}B + D$, where (A, B) is controllable and (A, C) is observable. The rational matrix $\Pi(s)$ is the spectral function associated to Λ.

Proposition 2.31. *The rational operator Λ is non-negative, i.e.*

$$\int_0^t u(\tau)\Lambda(u(\tau))d\tau \geq 0$$

for all $u \in \mathcal{L}_{2,e}$, if and only if its associated spectral function $\Pi(s)$ is non-negative. ■

Proof: We assume that $u(t) = 0$ for all $t < 0$ and that the system is causal. Let the output $y(\cdot)$ be given as

$$y(t) = Du(t) + \int_0^t Ce^{A(t-\tau}Bu(\tau)d\tau \tag{2.121}$$

Let $U(s)$ and $Y(s)$ denote the Laplace transforms of $u(\cdot)$ and $y(\cdot)$, respectively. Let us assume that $\Pi(s)$ has no pole on the imaginary axis. From Parseval's Theorem one has

$$\int_{-\infty}^{+\infty} [y^T(t)u(t) + u^T(t)y(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [Y^\star(j\omega)U(j\omega) + U^\star(j\omega)Y(j\omega)]d\omega \tag{2.122}$$

One also has $Y(s) = \big(D + C(sI_n - A)^{-1}B\big)\,U(s)$. Therefore

$$\int_{-\infty}^{+\infty} [y^T(t)u(t) + u^T(t)y(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U^\star(j\omega)\Pi(j\omega)U(j\omega)d\omega. \tag{2.123}$$

It follows that:

- $\Pi(j\omega) \geq 0$ for all $\omega \in I\!R$ implies that $\int_{-\infty}^{+\infty}[y^T(t)u(t) + u^T(t)y(t)]dt \geq 0$ for all admissible $u(\cdot)$.
- Reciprocally, given a couple (ω_0, U_0) that satisfies $U_0^T\Pi(j\omega_0)U_0 < 0$, there exists by continuity an interval Ω_0 such that $U_0^T\Pi(j\omega)U_0 < 0$ for all $\omega \in \Omega_0$. Consequently the inverse Fourier transform $v_0(\cdot)$ of the function

$$U(j\omega) = \begin{cases} U_0 \text{ if } \omega \in \Omega_0 \\ \\ 0 \quad \text{if } \omega \notin \Omega_0 \end{cases} \tag{2.124}$$

makes the quadratic form $\frac{1}{2\pi}\int_{\Omega_0} U_0^T\Pi(j\omega)U_0 d\omega < 0$. Therefore positivity of Λ and of its spectral function are equivalent properties.

If $\Pi(s)$ has poles on the imaginary axis, then Parseval's Theorem can be used under the form

$$\int_{-\infty}^{+\infty} e^{-2at}[y^T(t)u(t)+u^T(t)y(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U^\star(a+j\omega)S(a+j\omega)U(a+j\omega)d\omega \tag{2.125}$$

which is satisfied for all real a, provided the line $a + j\omega$ does not contain any pole of $\Pi(s)$. ■

Remark 2.32. It is implicit in the proof of Proposition 2.31 that the initial data on $y(\cdot)$ and $u(\cdot)$ and their derivatives, up to the required orders, are zero. Consequently, the positivity of the operator $\Lambda(\cdot)$, when associated to a state space representation (A, B, C, D), is characterized with the initial state $x(0) = 0$. Later on in Chapter 4, we shall give a definition of dissipativity, which generalizes that of positivity for a rational operator such as $\Lambda(\cdot)$, and which precisely applies with $x(0) = 0$; see Definition 4.22.

It is sometimes taken as a definition that a spectral function $\Pi(s)$ is non-negative if there exists a PR function $H(s)$ such that $\Pi(s) = H(s) + H^T(-s)$

[145, Definition 6.2]. We shall make use of Proposition 2.31 in Section 5.10 on hyperstability. Notice that Proposition 2.31 does not imply the stability of the above mentioned operator (provided one has associated a state space realization to this operator). The stability is in fact obtained if one makes further assumptions like the observability and controllability. We shall come back on these points in the next chapters on dissipative systems and their stability, via the Kalman-Yakubovich-Popov Lemma; see Remark 3.32.

The next theorem links bounded realness with positive realness.

Theorem 2.33. *Consider the linear time-invariant system $y(s) = h(s)u(s)$, and the scattering formulation $a = y+u$, $b = y-u$ and $b(s) = g(s)a(s)$ where*

$$g(s) = \frac{h(s)-1}{1+h(s)} \tag{2.126}$$

Assume that $g(s) \neq 1$ for all $\mathbf{Re}[s] > 0$. Then $h(s)$ is positive real if and only if $g(s)$ is bounded real. ■

Proof: Assume that $g(s)$ is bounded real and that $g(s) \neq 1$ for all $\mathbf{Re}[s] > 0$. Then $[1-g(s)]^{-1}$ exists for all s in $\mathbf{Re}[s] > 0$. From (2.126) we find that

$$h(s) = \frac{1+g(s)}{1-g(s)} \tag{2.127}$$

where $h(s)$ is analytic in $\mathbf{Re}[s] > 0$ as $g(s)$ is analytic in $\mathbf{Re}[s] > 0$, and $[1-g(s)]^{-1}$ is nonsingular by assumption in $\mathbf{Re}[s] > 0$. To show that $\mathbf{Re}[h(s)] \geq 0$ for all $\mathbf{Re}[s] > 0$ the following computation is used:

$$\begin{aligned} 2\mathbf{Re}[h(s)] &= h^*(s) + h(s) \\ &= \tfrac{1+g^*(s)}{1-g^*(s)} + \tfrac{1+g(s)}{1-g(s)} \\ &= 2\tfrac{1-g^*(s)g(s)}{[1-g^*(s)][1-g(s)]} \end{aligned} \tag{2.128}$$

We see that $\mathbf{Re}[h(s)] \geq 0$ for all $\mathbf{Re}[s] > 0$ whenever $g(s)$ is bounded real.

Next assume that $h(s)$ is positive real. Then $h(s)$ is analytic in $\mathbf{Re}[s] > 0$, and $[1+h(s)]$ is nonsingular in $\mathbf{Re}[s] > 0$ as $\mathbf{Re}[h(s)] \geq 0$ in $\mathbf{Re}[s] > 0$. It follows that $g(s)$ is analytic in $\mathbf{Re}[s] > 0$. From (2.128) it is seen that $|g(s)| \leq 1$ in $\mathbf{Re}[s] > 0$; it follows that $g(s)$ is bounded real. ■

It is noteworthy that Theorem 2.33 extends to multivariable systems:

Theorem 2.34. *Let $H(s) \in \mathbb{C}^{m\times m}$ be a square transfer function, with $det(H(s) + H(-s)) \neq 0$ for $\mathbf{Re}[s] \geq 0$, and $H(j\infty) + H(^T(j\infty) \geq 0$. Then the bounded realness of $G(s) = (G(s) - I_m)(G(s) + I_m)^{-1}$ implies that $H(s)$ is positive real.* ■

From Theorem 2.26 and Theorem 2.33 it follows that:

Corollary 2.35. *A system with transfer function $h(s)$ is passive if and only if the transfer function $h(s)$ is positive real.*

Example 2.36. A fundamental result in electrical circuit theory is that if the transfer function $h(s)$ is rational and positive real, then there exists an electrical one-port built from resistors, capacitors and inductors so that $h(s)$ is the impedance of the one-port [126, p. 815]. If e is the voltage over the one-port and i is the current entering the one-port, then $e(s) = h(s)i(s)$. The system with input i and output e must be passive because the total stored energy of the circuit must satisfy

$$\dot{V}(t) = e(t)i(t) - g(t) \tag{2.129}$$

where $g(t)$ is the dissipated energy.

Example 2.37. The transfer function

$$h(s) = \frac{1}{\tanh s} \tag{2.130}$$

is irrational, and positive realness of this transfer function cannot be established from conditions on the frequency response $h(j\omega)$. We note that $\tanh s = \sinh s / \cosh s$, where $\sinh s = \frac{1}{2}(e^s - e^{-s})$ and $\cosh s = \frac{1}{2}(e^s + e^{-s})$. First we investigate if $h(s)$ is analytic in the right half plane. The singularities are given by

$$\sinh s = 0 \Rightarrow e^s - e^{-s} = 0 \Rightarrow e^s(1 - e^{-2s}) = 0$$

Here $|e^s| \geq 1$ for $\mathbf{Re}[s] > 0$, while

$$e^s(1 - e^{-2s}) = 0 \Rightarrow e^{-2s} = 1$$

Therefore the singularities are found to be

$$s_k = jk\pi, \quad k \in \{0, \pm 1, \pm 2 \ldots\} \tag{2.131}$$

which are on the imaginary axis. This means that $h(s)$ is analytic in $\mathbf{Re}[s] > 0$. Obviously, $h(s)$ is real for real $s > 0$. Finally we check if $\mathbf{Re}\,[h(s)]$ is positive in $\mathbf{Re}[s] > 0$. Let $s = \sigma + j\omega$. Then

$$\begin{aligned}\cosh s &= \tfrac{1}{2}[e^\sigma(\cos\omega + j\sin\omega) + e^{-\sigma}(\cos\omega - j\sin\omega)] \\ &= \cosh\sigma\cos\omega + j\sinh\sigma\sin\omega\end{aligned}$$

while

$$\sinh s = \sinh\sigma\cos\omega + j\cosh\sigma\sin\omega \tag{2.132}$$

This gives

$$\mathbf{Re}[h(s)] = \frac{\cosh\sigma\sinh\sigma}{|\sinh s|^2} > 0, \quad \mathbf{Re}\,[s] > 0 \tag{2.133}$$

where it is used that $\sigma = \mathbf{Re}\,[s]$, and the positive realness of $h(s)$ has been established. ∎

Consider a linear system represented by a rational function $H(s)$ of the complex variable $s = \sigma + j\omega$:

$$H(s) = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \tag{2.134}$$

where $a_i, b_i \in I\!R$ are the system parameters n is the order of the system and $r = n - m$ is the relative degree. For rational transfer functions it is possible to find conditions on the frequency response $h(j\omega)$ for the transfer function to be positive real. The result is presented in the following theorem:

Theorem 2.38. *A rational function $h(s)$ is positive real if and only if*

1. *$h(s)$ has no poles in $\mathbf{Re}[s] > 0$.*
2. *$\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$ such that $j\omega$ is not a pole in $h(s)$.*
3. *If $s = j\omega_0$ is a pole in $h(s)$, then it is a simple pole, and if ω_0 is finite, then the residual*

$$\mathrm{Res}_{s=j\omega_0} h(s) = \lim_{s\to j\omega_0}(s - j\omega_0)h(s)$$

is real and positive. If ω_0 is infinite, then the limit

$$R_\infty := \lim_{\omega\to\infty}\frac{h(j\omega)}{j\omega}$$

is real and positive. ∎

Proof: The proof can be established by showing that conditions 2 and 3 in this Theorem are equivalent to the condition

$$\mathbf{Re}[h(s)] \geq 0 \tag{2.135}$$

for all $\mathbf{Re}[s] > 0$ for $h(s)$ with no poles in $\mathbf{Re}[s] > 0$.

First assume that conditions 2 and 3 hold. We use a contour C as shown in Figure 2.15 which goes from $-j\Omega$ to $j\Omega$ along the $j\omega$ axis with small semicircular indentations into the right half plane around points $j\omega_0$ that are poles of $h(s)$. The contour C is closed with a semicircle into the right half plane. On the part of C that is on the imaginary axis $\mathbf{Re}[h(s)] \geq 0$ by assumption. On the small indentations

$$h(s) \approx \frac{\mathrm{Res}_{s=j\omega_0} h(s)}{s - j\omega_0} \tag{2.136}$$

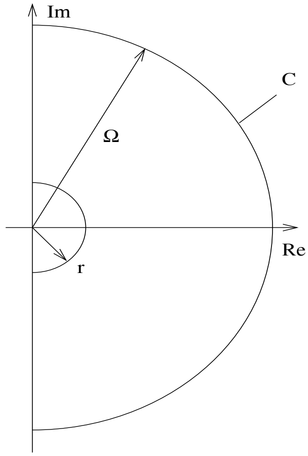

Fig. 2.15. Contour C of $h(s)$ in the right half plane.

As $\mathbf{Re}[s] \geq 0$ on the small semi-circles and $\mathrm{Res}_{s=j\omega_0} h(s)$ is real and positive according to condition 3, it follows that $\mathbf{Re}[h(s)] \geq 0$ on these semi-circles. On the large semi-circle into the right half plane with radius Ω we also have $\mathbf{Re}[h(s)] \geq 0$ and the value is a constant equal to $\lim_{\omega\to\infty} \mathbf{Re}[h(j\omega)]$, unless $h(s)$ has a pole at infinity at the $j\omega$ axis, in which case $h(s) \approx sR_\infty$ on the large semi-circle. Thus we may conclude that $\mathbf{Re}[h(s)] \geq 0$ on C. Define the function

$$f(s) = e^{-\mathbf{Re}[h(s)]}$$

Then $|f(s)| \leq 1$ on C, and in view of the maximum modulus theorem, $|f(s)| \leq 1$ for all $s \in \mathbf{Re}[s] > 0$. It follows that $\mathbf{Re}[h(s)] \geq 0$ in $\mathbf{Re}[s] > 0$, and the result is shown.

Next assume that $\mathbf{Re}[h(s)] \geq 0$ for all $\mathbf{Re}[s] > 0$. Then condition 2 follows because

$$h(j\omega) = \lim_{\substack{\sigma\to 0\\ \sigma>0}} h(\sigma + j\omega)$$

exists for all ω such that $j\omega$ is not a pole in $h(s)$. To show condition 3 we assume that ω_0 is a pole of multiplicity m for $h(s)$. On the small indentation with radius r into the right half plane we have $s - j\omega_0 = re^{j\theta}$ where $-\pi/2 \leq \theta \leq \pi/2$. Then

$$h(s) \approx \frac{\mathrm{Res}_{s=j\omega_0} h(s)}{r^m e^{jm\theta}} = \frac{\mathrm{Res}_{s=j\omega_0} h(s)}{r^m} e^{-jm\theta} \tag{2.137}$$

Clearly, here it is necessary that $m = 1$ to achieve $\mathbf{Re}[h(s)] \geq 0$ because the term $e^{-jm\theta}$ gives an angle from $-m\pi/2$ to $m\pi/2$ in the complex plane. Moreover, it is necessary that $\mathrm{Res}_{s=j\omega_0} h(s)$ is positive and real because $e^{-jm\theta}$ gives an angle from $-\pi/2$ to $\pi/2$ when $m = 1$. The result follows. ■

The foregoing theorem extends to multivariable systems:

Theorem 2.39. *The rational function $H(s) \in \mathbb{C}^{m \times m}$ is positive real if and only if:*

- *$H(s)$ has no poles in $\mathbf{Re}[s] > 0$*
- *$H(j\omega) + H^{\star}(j\omega) \geq 0$ for all positive real ω such that $j\omega$ is not a pole of $H(\cdot)$*
- *If $i\omega_0$, finite or infinite, is a pole of $H(\cdot)$, it is a simple pole and the corresponding residual K_0 is a semi positive definite Hermitian matrix.*

■

2.13 Examples

2.13.1 Mechanical Resonances

Motor and Load with Elastic Transmission

An interesting and important type of system is a motor that is connected to a load with an elastic transmission. The motor has moment of inertia J_m, the load has moment of inertia J_L, while the transmission has spring constant K and damper coefficient D. The dynamics of the motor is given by

$$J_m \ddot{\theta}_m(t) = T_m(t) - T_L(t) \tag{2.138}$$

where $\theta_m(\cdot)$ is the motor angle, $T_m(\cdot)$ is the motor torque, which is considered to be the control variable, and $T_L(\cdot)$ is the torque from the transmission. The dynamics of the load is

$$J_L \ddot{\theta}_L(t) = T_L(t) \tag{2.139}$$

The transmission torque is given by

$$T_L = -D\left(\dot{\theta}_L - \dot{\theta}_m\right) - K\left(\theta_L - \theta_m\right) \tag{2.140}$$

The load dynamics can then be written in Laplace transform form as

$$\left(J_L s^2 + Ds + K\right)\theta_L(s) = (Ds + K)\,\theta_m(s) \tag{2.141}$$

which gives

$$\frac{\theta_L}{\theta_m}(s) = \frac{1 + 2Z\frac{s}{\Omega_1}}{1 + 2Z\frac{s}{\Omega_1} + \frac{s^2}{\Omega_1^2}} \tag{2.142}$$

where

$$\Omega_1^2 = \frac{K}{J_L} \tag{2.143}$$

and

$$\frac{2Z}{\Omega_1} = \frac{D}{K} \tag{2.144}$$

By adding the dynamics of the motor and the load we get

$$J_m \ddot{\theta}_m(t) + J_L \ddot{\theta}_L(t) = T_m(t) \tag{2.145}$$

which leads to

$$J_m s^2 \theta_m(s) + J_L s^2 \frac{1 + 2Z\frac{s}{\Omega_1}}{1 + 2Z\frac{s}{\Omega_1} + \frac{s^2}{\Omega_1^2}} \theta_m(s) = T_m(s) \tag{2.146}$$

and from this

$$\frac{\theta_m}{T_m}(s) = \frac{1 + 2Z\frac{s}{\Omega_1} + \frac{s^2}{\Omega_1^2}}{Js^2(1 + 2\zeta\frac{s}{\omega_1} + \frac{s^2}{\omega_1^2})} \tag{2.147}$$

where

$$J = J_m + J_L \tag{2.148}$$

is the total inertia of motor and load, and the resonant frequency ω_1 is given by

$$\omega_1^2 = \frac{1}{1 - \frac{J_L}{J}} \Omega_1^2 = \frac{J}{J_m} \Omega_1^2 \tag{2.149}$$

while the relative damping is given by

$$\zeta = \sqrt{\frac{J}{J_m}} Z \tag{2.150}$$

We note that the parameters ω_1 and ζ depend on both motor and load parameters, while the parameters Ω_1 and Z depend only on the load.

The main observation in this development is the fact that $\Omega_1 < \omega_1$. This means that the transfer function $\theta_m(s)/T_m(s)$ has a complex conjugated pair of zeros with resonant frequency Ω_1, and a pair of poles at the somewhat higher resonant frequency ω_1. The frequency response is shown in Figure 2.16 when $K = 20$, $J_m = 20$, $J_L = 15$ and $D = 0.5$. Note that the elasticity does not give any negative phase contribution.

By multiplying the transfer functions $\theta_L(s)/\theta_m(s)$ and $\theta_m(s)/T_m(s)$ the transfer function

$$\frac{\theta_L}{T_m}(s) = \frac{1 + 2Z\frac{s}{\Omega_1}}{Js^2(1 + 2\zeta\frac{s}{\omega_1} + \frac{s^2}{\omega_1^2})} \tag{2.151}$$

is found from the motor torque to the load angle.

The resulting frequency response is shown in Figure 2.17. In this case the elasticity results in a negative phase contribution for frequencies above ω_1.

Example 2.40. Typically the gear is selected so that $J_m = J_L$. This gives

$$\Omega_1 = \frac{1}{\sqrt{2}} \omega_1 = 0.707\omega_1 \tag{2.152}$$

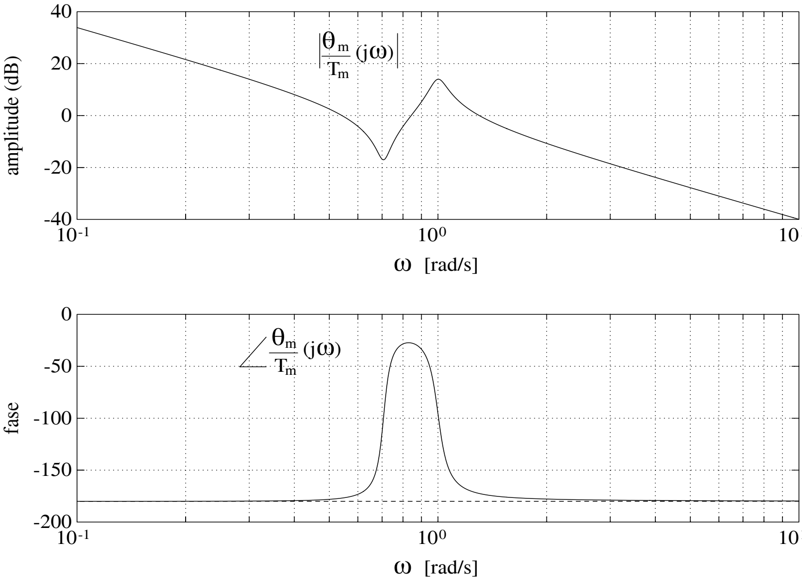

Fig. 2.16. Frequency response of $\theta_m(s)/T_m(s)$

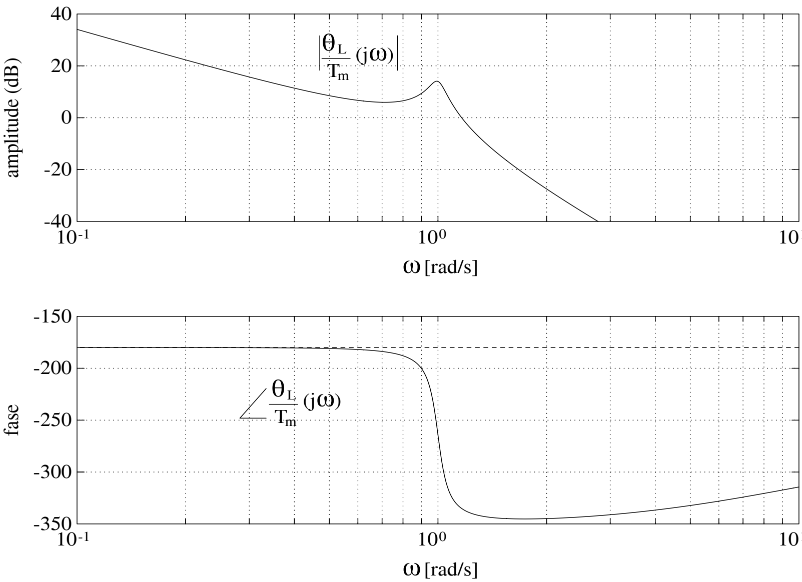

Fig. 2.17. Frequency response of $\theta_L(s)/\theta_m(s)$

Example 2.41. Let $Z = 0.1$ and $J_m = J_L$. In this case

$$\frac{\theta_L}{T_m}(s) = \frac{1 + \frac{s}{3.535\omega_1}}{Js^2(1 + 2\zeta\frac{s}{\omega_1} + \frac{s^2}{\omega_1^2})} \tag{2.153}$$

Passivity Inequality

The total energy of motor and load is given by

$$V(\omega_m, \omega_L, \theta_L, \theta_m) = \frac{1}{2}J_m\omega_m^2 + \frac{1}{2}J_L\omega_L^2 + \frac{1}{2}K[\theta_L - \theta_m]^2 \tag{2.154}$$

where $\omega_m(t) = \dot{\theta}_m(t)$ and $\omega_L(t) = \dot{\theta}_L(t)$. The rate of change of the total energy is equal to the power supplied from the control torque $T_m(t)$ minus the power dissipated in the system. This is written

$$\dot{V}(t) = \omega_m(t)T_m(t) - D[\omega_L(t) - \omega_m(t)]^2 \tag{2.155}$$

We see that the power dissipated in the system is $D[\omega_L(t) - \omega_m(t)]^2$ which is the power loss in the damper. Clearly the energy function $V(t) \geq 0$ and the power loss satisfies $D[\Delta\omega(t)]^2 \geq 0$,. It follows that

$$\int_0^t \omega_m(s)T_m(s)ds = V(t) - V(0) + \int_0^t D[\Delta\omega(s)]^2 ds \geq -V(0) \tag{2.156}$$

which implies that the system with input $T_m(\cdot)$ and output $\omega_m(\cdot)$ is passive. It follows that

$$\mathbf{Re}[h_m(j\omega)] \geq 0 \tag{2.157}$$

for all $\omega \in [-\infty, +\infty]$. From energy arguments we have been able to show that

$$-180^\circ \leq \angle\frac{\theta_m}{T_m}(j\omega) \leq 0^\circ. \tag{2.158}$$

2.13.2 Systems with Several Resonances

Passivity

Consider a motor driving n inertias in a serial connection with springs and dampers. Denote the motor torque by T_m and the angular velocity of the motor shaft by ω_m. The energy in the system is

$$
\begin{aligned}
V(\omega_m, \theta_m, \theta_{Li}) = \tfrac{1}{2} J_m \omega_m^2 &+ \tfrac{1}{2} K_{01}(\theta_m - \theta_{L1})^2 \\
&+ \tfrac{1}{2} J_{L1} \omega_{L1}^2 + \tfrac{1}{2} K_{12}(\theta_{L1} - \theta_{L2})^2 + \ldots \\
&+ \tfrac{1}{2} J_{L,n-1} \omega_{L,n-1}^2 + \tfrac{1}{2} K_{n-1,n}(\theta_{L,n-1} - \theta_{Ln})^2 \\
&+ \tfrac{1}{2} J_{Ln} \omega_{Ln}^2
\end{aligned}
$$

Clearly, $V(\cdot) \geq 0$. Here J_m is the motor inertia, ω_{Li} is the velocity of inertia J_{Li}, while $K_{i-1,i}$ is the spring connecting inertia $i-1$ and i and $D_{i-1,i}$ is the coefficient of the damper in parallel with $K_{i-1,i}$. The index runs over $i = 1, 2, \ldots, n$. The system therefore satisfies the equation

$$\dot{V}(t) = T_m(t)\omega_m(t) - d(t) \tag{2.159}$$

where

$$d(t) = D_{12}(\omega_{L1}(t) - \omega_{L2}(t))^2 + \ldots + D_{n-1,n}(\omega_{L,n-1}(t) - \omega_{Ln}(t))^2 \geq 0 \tag{2.160}$$

represents the power that is dissipated in the dampers, and it follows that the system with input T_m and output ω_m is passive. If the system is linear, then the passivity implies that the transfer function

$$h_m(s) = \frac{\omega_m}{T_m}(s) \tag{2.161}$$

has the phase constraint

$$|\angle h_m(j\omega)| \leq 90^\circ \tag{2.162}$$

for all $\omega \in [-\infty, +\infty]$. It is quite interesting to note that the only information that is used to find this phase constraint on the transfer function is that the system is linear, and that the load is made up from passive mechanical components. It is not even necessary to know the order of the system dynamics, as the result holds for an arbitrary n.

2.13.3 Two Motors Driving an Elastic Load

In this section we will see how passivity considerations can be used as a guideline for how to control two motors that actuate on the same load through elastic interconnections consisting of inertias, springs and dampers as shown in Figure 2.18.

The motors have inertias J_{mi}, angle q_{mi} and motor torque T_{mi} where $i \in \{1, 2\}$. Motor 1 is connected to the inertia J_{L1} with a spring with stiffness K_{11} and a damper D_{11}. Motor 2 is connected to the inertia J_{L2} with a spring with stiffness K_{22} and a damper D_{22}. Inertia J_{Li} has angle q_{Li}. The two inertias are connected with a spring with stiffness K_{12} and a damper D_{12}.

The total energy of the system is

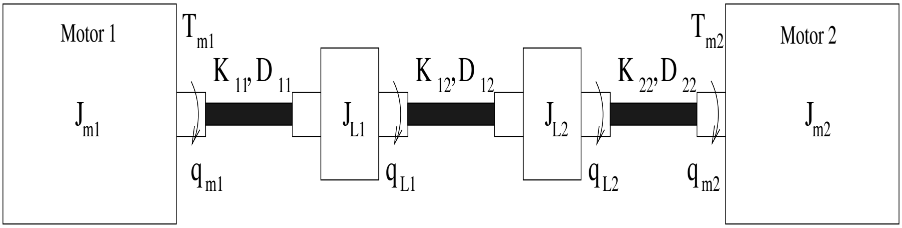

Fig. 2.18. Two motors actuating on one load

$$V(q_{m1}, q_{m2}, q_{Li}) = \frac{1}{2}[J_{m1}q_{m1}^2 + J_{m2}q_{m2}^2 + J_{L1}q_{L1}^2 + J_{L2}q_{L2}^2$$
$$+K_{11}(q_{m1} - q_{L1})^2 + K_{22}(q_{m2} - q_{L2})^2 + K_{12}(q_{L1} - q_{L2})^2]$$

and the time derivative of the energy when the system evolves is

$$\dot{V}(t) = T_{m1}\dot{q}_{m1}(t) + T_{m2}\dot{q}_{m2}(t) - D_{11}(\dot{q}_{m1}(t) - \dot{q}_{L1}(t))^2$$
$$+D_{22}(\dot{q}_{m2}(t) - \dot{q}_{L2}(t))^2 \quad +D_{12}(\dot{q}_{L1}(t) - \dot{q}_{L2}(t))^2$$

It is seen that the system is passive from $(T_{m1}, T_{m2})^T$ to $(\dot{q}_{m1}, \dot{q}_{m2})^T$. The system is multivariable, with controls T_{m1} and T_{m2} and outputs q_{m1} and q_{m2}. A controller can be designed using multivariable control theory, and passivity might be a useful tool in this connection. However, here we will close one control loop at a time to demonstrate that independent control loops can be constructed using passivity arguments. The desired outputs are assumed to be $q_{m1} = q_{m2} = 0$. Consider the PD controller

$$T_{m2} = -K_{p2}q_{m2} - K_{v2}\dot{q}_{m2} \tag{2.163}$$

for motor 2 which is passive from $\dot{q}_{m2}$ to $-T_{m2}$. The mechanical analog of this controller is a spring with stiffness K_{p2} and a damper K_{v2} which is connected between the inertia J_{m2} and a fixed point. The total energy of the system with this mechanical analog is

$$V(q_{m1}, q_{m2}, q_{L1}, q_{L2}) = \frac{1}{2}[J_{m1}q_{m1}^2 + J_{m2}q_{m2}^2 + J_{L1}q_{L1}^2 + J_{L2}q_{L2}^2$$
$$+K_{11}(q_{m1} - q_{L1})^2 + K_{22}(q_{m2} - q_{L2})^2$$
$$+K_{12}(q_{L1} - q_{L2})^2 + K_{p2}q_2^2]$$

and the time derivative is

$$\dot{V}(t) = T_{m1}(t)\dot{q}_{m1}(t) - D_{11}(\dot{q}_{m1}(t) - \dot{q}_{L1}(t))^2 + D_{22}(\dot{q}_{m2}(t) - \dot{q}_{L2}(t))^2$$
$$+D_{12}(\dot{q}_{L1}(t) - \dot{q}_{L2}(t))^2 - K_{v2}\dot{q}_2^2(t)$$

It follows that the system with input T_{m1} and output $\dot{q}_{m1}$ is passive when the PD controller is used to generate the control T_{m2}. The following controller can then be used:

$$T_1(s) = K_{v1}\beta\frac{1+T_is}{1+\beta T_is}\dot{q}_1(s) = K_{v1}[1+(\beta-1)\frac{1}{1+\beta T_is}]sq_1(s) \tag{2.164}$$

This is a PI controller with limited integral action if $\dot{q}_1$ is considered as the output of the system. The resulting closed loop system will be BIBO stable independently from system and controller parameters, although in practice, unmodelled dynamics and motor torque saturation dictate some limitations on the controller parameters. As the system is linear, stability is still ensured even if the phase of the loop transfer function becomes less that $-180°$ for certain frequency ranges. Integral effect from the position can therefore be included for one of the motors, say motor 1. The resulting controller is

$$T_1(s) = K_{p1}\frac{1+T_is}{T_is}q_1(s) + K_{v1}sq_1 \tag{2.165}$$

In this case the integral time constant T_i must be selected *e.g.* by Bode diagram techniques so that stability is ensured.

2.14 Strictly Positive Real (SPR) Systems

Consider again the definition of Positive Real transfer function in Definition 2.28. The following is the standard definition of Strictly Positive Real (SPR) transfer functions.

Definition 2.42 (Strictly Positive Real). *A rational transfer function $H(s) \in \mathbb{C}^{m\times m}$ that is not identically zero for all s, is strictly positive real (SPR) if $H(s-\epsilon)$ is PR for some $\epsilon > 0$.*

Let us now consider two simple examples:

Example 2.43. The transfer function of an asymptotically stable first order system is given by

$$H(s) = \frac{1}{s+\lambda} \tag{2.166}$$

where $\lambda > 0$. Replacing s by $\sigma + j\omega$ we get

$$H(s) = \frac{1}{(\sigma+\lambda)+j\omega} = \frac{\sigma+\lambda-j\omega}{(\sigma+\lambda)^2+\omega^2} \tag{2.167}$$

Note that $\forall$ $\mathbf{Re}[s] = \sigma > 0$ we have $\mathbf{Re}[H(s)] \geq 0$. Therefore $H(s)$ is PR. Furthermore $H(s-\epsilon)$ for $\epsilon = \frac{\lambda}{2}$ is also PR and thus $H(s)$ is also SPR.

Example 2.44. Consider now a simple integrator (*i.e.* take $\lambda = 0$ in the previous example)

$$H(s) = \frac{1}{s} = \frac{1}{\sigma + j\omega} = \frac{\sigma - j\omega}{\sigma^2 + \omega^2}. \tag{2.168}$$

It can be seen that $H(s) = \frac{1}{s}$ is PR but not SPR.

In view of Theorem 2.6, one may wonder whether an SPR transfer function is ISP, OSP. See Examples 4.62, 4.64, 4.65.

2.14.1 Frequency Domain Conditions for a Transfer Function to be SPR

The definition of SPR transfer functions given above is in terms of conditions in the s complex plane. Such conditions become relatively difficult to be verified as the order of the system increases. The following theorem establishes conditions in the frequency domain ω for a transfer function to be SPR.

Theorem 2.45 (Strictly Positive Real). *[226] A rational transfer function $h(s)$ is SPR if*

1. *$h(s)$ is analytic in $\mathbf{Re}[s] \geq 0$, i.e. the system is asymptotically stable*
2. *$\mathbf{Re}[h(j\omega)] > 0$, for all $\omega \in (-\infty, \infty)$ and*
3. *a) $\lim_{\omega^2 \to \infty} \omega^2 \mathbf{Re}[h(j\omega)] > 0$ when $r = 1$,*
   b) $\lim_{\omega^2 \to \infty} \mathbf{Re}[h(j\omega)] > 0$, $\lim_{|\omega| \to \infty} \frac{h(j\omega)}{j\omega} > 0$ when $r = -1$,

where r is the relative degree of the system. ■

Proof: *Necessity:* If $h(s)$ is SPR, then from Definition 2.42, $h(s - \epsilon)$ is PR for some $\epsilon > 0$. Hence, there exists an $\epsilon^* > 0$ such that for each $\epsilon \in [\,0, \epsilon^*)$, $h(s - \epsilon)$ is analytic in $\mathrm{Re}[s] < 0$. Therefore, there exists a real rational function $W(s)$ such that [8]

$$h(s - \epsilon) + h(-s + \epsilon) = W(s - \epsilon)W(-s + \epsilon) \tag{2.169}$$

where $W(s)$ is analytic and nonzero for all s in Re $[s] > -\epsilon$. Let $s = \epsilon + j\omega$; then from (2.169) we have

$$2 \text{ Re } [h(j\omega)] = |W(j\omega)|^2 > 0, \qquad \forall \omega \in (-\infty, \infty) \tag{2.170}$$

Now $h(s)$ can be expressed as

$$h(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{2.171}$$

If $m = n - 1$, *i.e.*, $r = 1$, $b_{n-1} \neq 0$, then from (2.171) it follows that $b_{n-1} > 0$ and $a_{n-1} b_{n-1} - b_{n-2} - \epsilon b_{n-1} > 0$ for $h(s - \epsilon)$ to be PR, and

$$\lim_{\omega^2 \to \infty} \omega^2 \text{ Re } [h(j\omega)] = a_{n-1}b_{n-1} - b_{n-2} \geq \epsilon b_{n-1} > 0 \tag{2.172}$$

If $m = n + 1$, *i.e.*, $r = -1$, $b_{n+1} \neq 0$, then

$$\text{Re } [h(j\omega - \epsilon)] = \frac{1}{|a(j\omega - \epsilon)|^2} \left[(b_n - b_{n+1}a_{n-1} - \epsilon b_{n+1})\, \omega^{2n} + \cdots \right] \tag{2.173}$$

Since Re $[h(j\omega - \epsilon)] \geq 0 \ \forall \ \omega \in (-\infty, \infty)$ and

$$\lim_{|\omega| \to \infty} \frac{h(j\omega - \epsilon)}{j\omega} = b_{n-1} \geq 0,$$

then $b_{n+1} > 0$, $b_n - b_{n+1}a_{n-1} \geq \epsilon b_{n+1} > 0$, and therefore 3.b) follows directly.

Sufficiency; Let (A, b, c, d, f) be a minimal state representation of $h(s)$, *i.e.*,

$$h(s) = c(sI - A)^{-1}b + d + fs \tag{2.174}$$

From (2.174) we can write

$$h(s - \epsilon) = c(sI - A)^{-1}b + d + fs + \epsilon \left[c(sI - A - \epsilon I)^{-1}(sI - A)^{-1}b - f \right] \tag{2.175}$$

Hence,

$$\mathbf{Re}\,[h(j\omega - \epsilon)] = \mathbf{Re}\,[h(j\omega)] + \epsilon \mathbf{Re}\,[g(j\omega - \epsilon)] \tag{2.176}$$

where $g(j\omega - \epsilon) = c(j\omega I_n - A - \epsilon I)^{-1}(j\omega I_n - A)^{-1}b - f$. There exists an $\epsilon^* > 0$ such that for all $\epsilon \in [0, \epsilon^*)$ and $\omega \in (-\infty, \infty)$, $(j\omega I_n - A - \epsilon I)^{-1}$ is analytic. Therefore for each $\epsilon \in [0, \epsilon^*)$, $|\mathbf{Re}\ [g(j\omega - \epsilon)]| < k_1 < \infty$ for all $\omega \in (-\infty, \infty)$ and some $k_1 > 0$. If $r = 0$, then Re $[h(j\omega)] > k_2 > 0$ for all ω and some $k_2 > 0$. Therefore

$$\mathbf{Re}\,[h(j\omega - \epsilon)] = \mathbf{Re}\,[h(j\omega)] + \epsilon \mathbf{Re}\,[g(j\omega - \epsilon)] > k_2 - \epsilon k_1 > 0 \tag{2.177}$$

for all $\omega \in (-\infty, \infty)$ and $0 < \epsilon < \min\{\epsilon^*, k_2/k_1\}$. Hence, $h(s - \epsilon)$ is PR and therefore $h(s)$ is SPR.

If $r = 1$, then Re $[h(j\omega)] > k_3 > 0$ for all $|\omega| < \omega_0$ and ω^2 Re $[h(j\omega)] > k_4 > 0$ for all $|\omega| \geq \omega_0$, where ω_0, k_3, k_4 are finite positive constants. Similarly, $\left|\omega^2 \text{ Re } [g(j\omega - \epsilon)]\right| < k_5$ and $|\text{Re } [g(j\omega - \epsilon)]| < k_6$ for all $\omega \in (-\infty, \infty)$ and some finite positive constants k_5, k_6. Therefore, $\mathbf{Re}\ [h(j\omega - \epsilon)] > k_3 - \epsilon k_6$ for all $|\omega| < \omega_0$ and $\omega^2 \mathbf{Re}\ [h(j\omega - \epsilon)] > k_4 - \epsilon k_5$ for all $|\omega| \geq \omega_0$. Then, for $0 < \epsilon < \min\{k_3/k_6, \epsilon^*, k_4/k_5\}$ and $\forall \omega \in (-\infty, \infty)$, $\mathbf{Re}\ [h(j\omega - \epsilon)] > 0$. Hence, $h(s - \epsilon)$ is PR and therefore $h(s)$ is SPR.

If $r = -1$, then $d > 0$ and therefore

$$\mathbf{Re}\,[h(j\omega-\epsilon)] > d - \epsilon k_1 \tag{2.178}$$

Hence, for each ϵ in the interval $[0, \min\{\epsilon^*, d/k_1\})$, $\mathbf{Re}\ [h(j\omega-\epsilon)] > 0$ for all $\omega \in (-\infty, \infty)$. Since

$$\lim_{\omega\to\infty} \frac{h(j\omega)}{j\omega} = f > 0$$

then

$$\lim_{\omega\to\infty} \frac{h(j\omega-\epsilon)}{j\omega} = f > 0$$

and therefore, all the conditions of Definition 2.28 and Theorem 2.38 are satisfied by $h(s-\epsilon)$; hence $h(s-\epsilon)$ is PR, *i.e.*, $h(s)$ is SPR and the sufficiency proof is complete. ■

Remark 2.46. It should be noted that when $r = 0$, conditions 1 and 2 of the Theorem, or 1 and $\mathbf{Re}[h(j\omega)] > \delta > 0$ for all $\omega \in [-\infty, +\infty]$, are both necessary and sufficient for $h(s)$ to be SPR.

Notice that $H(s)$ in (2.166) satisfies condition 3.a), but $H(s)$ in (2.168) does not. Let us now give a multivariable version of Theorem 2.45, whose proof is given in [256] and is based on [226, 508].

Theorem 2.47. *Let $H(s) \in \mathbb{C}^{m\times m}$ be a proper rational transfer matrix, and suppose that det$(H(s) + H^T(s))$ is not identically zero. Then $H(s)$ is SPR if and only if*

- *$H(s)$ has all its poles with negative real parts*
- *$H(j\omega) + H^T(-j\omega) > 0$ for all $\omega \in \mathbb{R}$*
  and one of the following three conditions is satisfied:
  - *$H(\infty) + H^T(\infty) > 0$*
  - *$H(\infty) + H^T(\infty) = 0$ and $\lim_{\omega\to\infty} \omega^2[H(j\omega) + H^T(-j\omega)] > 0$*
  - *$H(\infty) + H^T(\infty) \geq 0$ (but not zero nor nonsingular)and there exist positive constants σ and δ such that*

$$\omega^2 \sigma_{\min}[H(j\omega) + H^T(-j\omega)] \geq \sigma, \quad \forall|\omega| \geq \delta \tag{2.179}$$

■

2.14.2 Necessary Conditions for $H(s)$ to be PR (SPR)

In general, before checking all the conditions for a specific transfer function to be PR or SPR, it is useful to check first that it satisfies a set of necessary conditions. The following are necessary conditions for a system to be PR (SPR)

- $H(s)$ is (asymptotically) stable.
- The Nyquist plot of $H(j\omega)$ lies entirely in the (closed) right half complex plane.
- The relative degree of $H(s)$ is either $r = 0$ or $r = \pm 1$.
- $H(s)$ is (strictly) minimum-phase, *i.e.* the zeros of $H(s)$ lie in $\mathbf{Re}[s] \leq 0$ $(\mathbf{Re}[s] < 0)$.

Remark 2.48. In view of the above necessary conditions it is clear that unstable systems or nonminimum phase systems are not positive real. Furthermore proper transfer functions can be PR only if their relative degree is 0 or 1. This means for instance that a double integrator, *i.e.* $H(s) = \frac{1}{s^2}$ is not PR. This remark will turn out to be important when dealing with passivity of nonlinear systems. In particular for a robot manipulator we will be able to prove passivity from the torque control input to the velocity of the generalized coordinates but not to the position of the generalized coordinates.

2.14.3 Tests for SPRness

Stating necessary and sufficient conditions for a transfer function to be PR or SPR is a first fundamental step. A second step consists in usable criteria which allow one to determine if a given rational function is SPR or not. Work in this direction may be found in [31,132,146,177,205,341,396,455,504,528,536]. We can for instance quote a result from [455].

Theorem 2.49. *[455] Consider $H(s) = C(sI_n - A)^{-1}B \in \mathbb{C}$. $H(s)$ is SPR if and only if 1) $CAB < 0$, 2) $CA^{-1}B < 0$, 3) A is stable, 4) $A(I_n - \frac{ABC}{CAB})A$ has no eigenvalue on the open negative real axis $(-\infty, 0)$. Consider now $H(s) = C(sI_n - A)^{-1}B + D \in \mathbb{C}$, $D > 0$. $H(s)$ is SPR if and only if 1) A is stable, 2) the matrix $(A - \frac{BC}{D})A$ has no eigenvalue on the closed negative real axis $(-\infty, +\infty]$.* ∎

Stability means here that all the eigenvalues are in the open left-half of the complex plane $\mathbf{Re}[s] < 0$, and may be called strict stability. An interpretation of SPRness is that (A, B, C, D) with $D \neq 0$ is SPR if and only if the matrix pencil $A^{-1} + \lambda(A - \frac{BC}{D})$ is nonsingular for all $\lambda > 0$ [455].

2.14.4 Interconnection of Positive Real Systems

One of the important properties of positive real systems is that the inverse of a PR system is also PR. In addition the interconnection of PR systems in parallel or in negative feedback (see Figure 2.19) inherit the PR property. More specifically we have the following properties (see [226]):

- $H(s)$ is PR (SPR) $\Leftrightarrow \frac{1}{H(s)}$ is PR (SPR).
- If $H_1(s)$ and $H_2(s)$ are SPR so is $H(s) = \alpha_1 H_1(s) + \alpha_2 H_2(s)$ for $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_1 + \alpha_2 > 0$.

- If $H_1(s)$ and $H_2(s)$ are SPR, so is $H(s) = \frac{H_1(s)}{1+H_1(s)H_2(s)}$.

Remark 2.50. Note that a transfer function $H(s)$ need not be proper to be PR or SPR. For instance, the non-proper transfer function s is PR.

Remark 2.51. Let us recall that if (A, B, C, D) is a realization of the transfer function $H(s) \in \mathbb{C}$, *i.e.* $C(sI_n - A)^{-1}B + D = H(s)$, and if $D \neq 0$, then $(A - \frac{BC}{D}, \frac{B}{D}, -\frac{C}{D}, \frac{1}{D})$ is a realization of a system with transfer function $\frac{1}{H(s)}$ (see for instance [246, p.76]).

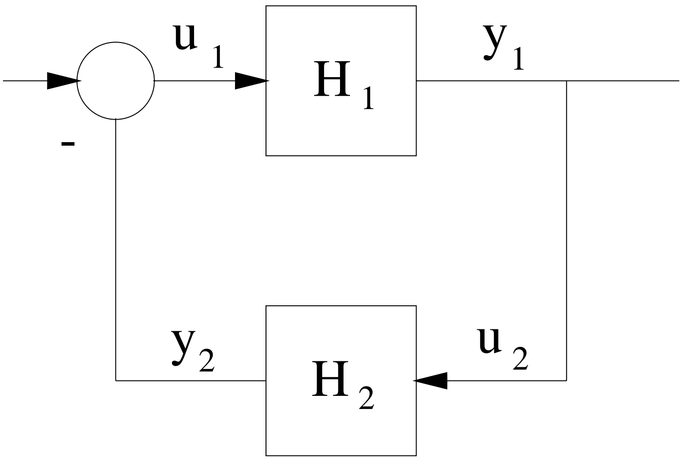

Fig. 2.19. Negative feedback interconnection of H_1 and H_2

2.14.5 Special Cases of Positive Real Systems

We will now introduce two additional definitions of classes of systems. Both of them are PR systems but one of them is weaker than SPR systems and the other is stronger. Weak SPR (WSPR) are important because they allow the extension of the KYP Lemma presented in Chapter 3 for systems other than PR. They are also important because they allow to relax the conditions for stability of the negative feedback interconnection of a PR system and an SPR system. We will actually show that the negative feedback interconnection between a PR system and a WSPR produces an asymptotically stable system. Both properties will be seen later.

Remark 2.52. Consider again an electric circuit composed of an inductor in parallel with a capacitor. Such a circuit will exhibit sustained oscillatory behavior. If we have instead a lossy capacitor in parallel with a lossy inductor, it is clear that the energy stored in the system will be dissipated. However, it is sufficient that at least one of the two is a lossy element (either a lossy capacitor or a lossy inductor) to guarantee that the oscillatory behavior will asymptotically converge to zero. This example motivates the notion of weakly SPR transfer function.

Definition 2.53. (Weakly SPR) *A rational function $H(s) \in \mathbb{C}$ is weakly SPR (WSPR) if*

1. $H(s)$ *is analytic in* $\mathbf{Re}[s] \geq 0$.
2. $\mathbf{Re}[H(j\omega)] > 0$, *for all* $\omega \in (-\infty, \infty)$. ■

In the multivariable case one replaces the second condition by $H(j\omega) + H^T(-j\omega) > 0$ for all $\omega \in I\!R$. It is noteworthy that a transfer function may be WSPR but not be SPR; see an example below. WSPRness may be seen as an intermediate notion between PR and SPR. See Section 5.3 for more analysis on WSPR systems, which shows in particular and in view of Examples 4.62 and 4.64 that WSPR is not SPR.

Definition 2.54. (Strong SPR) *A rational function $H(s) \in \mathbb{C}$ is strongly SPR (SSPR) if*

1. $H(s)$ *is analytic in* $\mathbf{Re}[s] \geq 0$.
2. $\mathbf{Re}[H(j\omega)] \geq \delta > 0$, *for all* $\omega \in [-\infty, \infty]$ *and some* $\delta \in I\!R$. ■

In the multivariable case the second condition for SSPRness becomes $H(j\omega) + H^T(-j\omega) > 0$ for all $\omega \in I\!R$ and $H(\infty) + H^T(\infty) > 0$, or as $H(j\omega) + H^T(-j\omega) > \delta I_m$ for all $\omega \in [-\infty, \infty]$. From Theorem 2.6, it can be seen that a SSPR transfer function is ISP. If the system has a minimal state space realization (A, B, C, D) then $H(s) + H^T(-s) = C(sI_n - A)^{-1}B - B^T(sI_n + A^T)^{-1}C^T + D + D^T$ so that the second condition implies $D + D^T > 0 \Rightarrow D > 0$. This may also be deduced from the fact that $C(sI_n - A)^{-1}B + D = \sum_{i=1}^{+\infty} CA^{i-1}Bs^{-i} + D$. The next result may be useful to characterize SSPR functions.

Lemma 2.55. *[146] A proper reational matrix $H(s) \in \mathbb{C}^{m\times m}$ is SSPR if and only if its principal minors $H_i(s) \in \mathbb{C}^{i\times i}$ are proper rational SSPR matrices, respectively, for $i = 1, ..., m-1$, and det$(H(j\omega)+H^T(-j\omega)) > 0$ for all $\omega \in I\!R$.*

The next lemma is close to Theorem 2.34.

Lemma 2.56. *Let $G(s) \in \mathbb{C}^{m\times m}$ be a proper rational matrix satisfying det$(I_m + G(s)) \neq 0$ for $\mathbf{Re}[s] \geq 0$. Then the proper rational matrix $H(s) = (I_m + G(s))^{-1}(I_m - G(s)) \in \mathbb{C}^{m\times m}$ is SSPR if and only if $G(s)$ is strictly bounded real.*

Let us now illustrate the various definitions of PR, SPR and WSPR functions on examples.

Example 2.57. Consider again an asymptotically stable first order system

$$H(s) = \frac{1}{s+\lambda}, \quad \text{with} \quad \lambda > 0 \tag{2.180}$$

Let us check the conditions for $H(s)$ to be SPR.

1. $H(s)$ has only poles in $\mathbf{Re}[s] < 0$
2. $H(j\omega)$ is given by

$$H(j\omega) = \frac{1}{\lambda + j\omega} = \frac{\lambda - j\omega}{\lambda^2 + \omega^2} \tag{2.181}$$

Therefore,

$$\mathbf{Re}[H(j\omega)] = \frac{\lambda}{\lambda^2 + \omega^2} > 0 \;\; \forall \omega \in (-\infty, \infty) \tag{2.182}$$

- $\lim\limits_{\omega^2 \to \infty} \omega^2 \mathbf{Re}[H(j\omega)] = \lim\limits_{\omega^2 \to \infty} \frac{\omega^2 \lambda}{\lambda^2 + \omega^2} = \lambda > 0$

Therefore $\frac{1}{s+\lambda}$ is SPR. However $\frac{1}{s+\lambda}$ is not SSPR because there does not exist a $\delta > 0$ such that $\mathbf{Re}[H(j\omega)] > \delta, \quad$ for all $\omega \in [-\infty, \infty]$ since $\lim\limits_{\omega^2 \to \infty} \frac{\lambda}{\lambda^2 + \omega^2} = 0$.

Example 2.58. Similarly it can be proved that $H(s) = \frac{1}{s}$ and $H(s) = \frac{s}{s^2+\omega^2}$ are PR but they are not WSPR. $H(s) = 1$ and $H(s) = \frac{s+a^2}{s+b^2}$ are both SSPR.

The following is an example of a system that is WSPR but is not SPR.

Example 2.59. Consider the second order system

$$H(s) = \frac{s+\alpha+\beta}{(s+\alpha)(s+\beta)} \; ; \; \alpha, \;\; \beta > 0 \tag{2.183}$$

Let us verify the conditions for $H(s)$ to be WSPR. $H(j\omega)$ is given by

$$\begin{aligned} H(j\omega) &= \tfrac{j\omega+\alpha+\beta}{(j\omega+\alpha)(j\omega+\beta)} \\ &= \tfrac{(j\omega+\alpha+\beta)(\alpha-j\omega)(\beta-j\omega)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} \\ &= \tfrac{(j\omega+\alpha+\beta)(\alpha\beta-j\omega(\alpha+\beta)-\omega^2)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} \end{aligned} \tag{2.184}$$

Thus

$$\begin{aligned}\mathbf{Re}[H(j\omega)] &= \tfrac{\omega^2(\alpha+\beta)+(\alpha+\beta)(\alpha\beta-\omega^2)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} \\ &= \tfrac{\alpha\beta(\alpha+\beta)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} > 0, \text{ for all } \omega \in (-\infty,\infty)\end{aligned} \tag{2.185}$$

so $H(s)$ is weakly SPR. However $H(s)$ is not SPR since

$$\lim_{\omega^2\to\infty} \frac{\omega^2\alpha\beta(\alpha+\beta)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} = 0 \tag{2.186}$$

Example 2.60. [213] The transfer function $\frac{s+\alpha}{(s+1)(s+2)}$ is

- PR if $0 \leq \alpha \leq 3$
- WSPR if $0 < \alpha \leq 3$
- SPR if $0 < \alpha < 3$

Let us point out that other definitions exist for positive real transfer functions, like the following one:

Definition 2.61. *[430] [γ-**PR**] Let $0 < \gamma < 1$. The transfer function $H(s) \in \mathbb{C}^{m\times m}$ is said to be γ-positive real if it is analytic in $\mathbf{Re}[s] \geq 0$ and satisfies*

$$(\gamma^2-1)H^\star(s)H(s) + (\gamma^2+1)(H^\star(s)+H(s)) + (\gamma^2-1)I_m \geq 0 \tag{2.187}$$

for all $s \in \mathbf{Re}[s] \geq 0$. ■

Then the following holds:

Proposition 2.62. *[430] If a system is $\gamma-$positive real, then it is SSPR. Conversely, if a system is SSPR, then it is $\gamma-$positive real for some $0 < \gamma < 1$.*

■

For single input-single output systems ($m = 1$) the index γ can be used to measure the maximal phase difference of transfer functions. The transfer function $H(s) \in \mathbb{C}$ is $\gamma-$PR if and only if the Nyquist plot of $H(s)$ is in the circle centered at $\frac{1+\gamma^2}{1-\gamma^2}$ and radius $\frac{2\gamma}{1-\gamma^2}$.

Lemma 2.63. *[430] Let $m = 1$. If the system (A,B,C,D) with transfer function $H(s) = C(sI_n - A)^{-1}B + D$ is $\gamma-$PR, then*

$$|arg(H(s))| \leq arctan\left(\frac{2\gamma}{1-\gamma^2}\right) \text{ for all } \mathbf{Re}[s] \geq 0 \tag{2.188}$$

■

Other classes of PR systems exist which may slightly differ from the above ones; see *e.g.* [149,245]. In particular a system is said to be extended SPR if it is SPR and if $H(j\infty)+H^T(-j\infty) > 0$. From the series expansion of a rational transfer matrix one deduces that $H(j\omega) = \sum_{i=1}^{+\infty} CA^{i-1}B(j\omega)^{-i} + D$ which implies that $D + D^T > 0$. The definition of SSPRness in [245, Definition 3] and Definition 2.54 are not the same, as they impose that $H(\infty)+H^T(\infty) \geq 0$ only, with $\lim_{\omega\to\infty} \omega^2[H(j\omega) + H^T(-j\omega)] > 0$ if $H(\infty) + H^T(\infty)$ is singular. The notion of marginally SPR (MSPR) transfer functions is introduced in [245]. MSPR functions satisfy inequality 2 of Definition 2.53, however they are allowed to possess poles on the imaginary axis.

2.15 Applications

2.15.1 SPR and Adaptive Control

The concept of SPR transfer functions is very useful in the design of some type of adaptive control schemes. This will be shown next for the control of an unknown plant in a state space representation and it is due to Parks [394] (see also [240]). Consider a linear time-invariant system in the following state space representation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2.189}$$

with state $x(t) \in I\!R^n$, input $u(t) \in I\!R$ and output $y(t) \in I\!R$. Let us assume that there exists a control input

$$u = -L^T x + r(t) \tag{2.190}$$

where $r(t)$ is a reference input and $L \in I\!R^n$, such that the closed loop system behaves as the reference model

$$\begin{cases} \dot{x}_r(t) = (A - BL^T)x_r(t) + Br(t) \\ \\ y_r(t) = Cx_r(t) \end{cases} \tag{2.191}$$

We also assume that the above reference model has an SPR transfer function. From the Kalman-Yakubovich-Popov Lemma, which will be presented in detail in the next chapter, this means that there exists a matrix $P > 0$, a matrix L', and a positive constant ε such that

$$\begin{cases} A_{cl}^T P + PA_{cl} &= -L'L'^T - \varepsilon P \\ \\ PB &= C^T \end{cases} \tag{2.192}$$

where

$$A_{cl} = A - BL^T$$

Since the system parameters are unknown, let us consider the following adaptive control law:

$$\begin{aligned} u &= -\hat{L}^T x + r(t) \\ &= -L^T x + r(t) - \tilde{L}^T x \end{aligned} \tag{2.193}$$

where $\hat{L}$ is the estimate of L and $\tilde{L}$ is the parametric error

$$\tilde{L}(t) = \hat{L}(t) - L$$

Introducing the above control law into the system (2.189) we obtain

$$\dot{x}(t) = (A - BL^T)x(t) + B(r(t) - \tilde{L}^T(t)x(t)) \tag{2.194}$$

Define the state error $\tilde{x} = x - x_r$ and the output error $e = y - y_r$. From the above we obtain

$$\begin{cases} \frac{d\tilde{x}}{dt}(t) = A_{cl}\tilde{x}(t) - B\tilde{L}^T(t)x(t) \\ e(t) \quad = C\tilde{x}(t) \end{cases} \tag{2.195}$$

Consider the following Lyapunov function candidate

$$V(\tilde{x}, \tilde{L}) = \tilde{x}^T P \tilde{x} + \tilde{L}^T P_L \tilde{L} \tag{2.196}$$

where $P > 0$ and $P_L > 0$. Therefore

$$\dot{V}(\tilde{x}, \tilde{L}) = \tilde{x}^T(A_{cl}^T P + PA_{cl})\tilde{x} - 2\tilde{x}^T PB\tilde{L}^T x + 2\tilde{L}^T P_L \frac{d\tilde{L}}{dt}$$

Choosing the following parameter adaptation law

$$\frac{d\hat{L}}{dt}(t) = P_L^{-1} x(t)e(t) = P_L^{-1} x(t)C\tilde{x}(t)$$

we obtain

$$\dot{V}(\tilde{x}, \tilde{L}) = \tilde{x}^T(A_{cl}^T P + PA_{cl})\tilde{x} - 2\tilde{x}^T(PB - C^T)\tilde{L}^T x$$

Introducing (2.192) in the above we get

$$\dot{V}(\tilde{x}) = -\tilde{x}^T(L'L'^T + \varepsilon P)\tilde{x} \leq 0 \tag{2.197}$$

It follows that $\tilde{x}, x$ and $\tilde{L}$ are bounded. Integrating the above we get

$$\int_0^t \tilde{x}^T(s)(L'L'^T + \varepsilon P)\tilde{x}(s)ds \leq V(\tilde{x}(0), \tilde{L}(0))$$

Thus $\tilde{x} \in \mathcal{L}_2$. From (2.195) it follows that $\frac{d\tilde{x}}{dt}(\cdot)$ is bounded and we conclude that $\tilde{x}(\cdot)$ converges to zero.

2.15.2 Adaptive Output Feedback

In the previous section we presented an adaptive control based on the assumption that there exists a state feedback control law such that the resulting closed-loop system is SPR. In this section we present a similar approach but this time we only require output feedback. In the next section we will present the conditions under which there exists an output feedback that renders the closed loop SPR. The material in this section and the next have been presented in [219]. Consider again the system (2.189) in the MIMO (multiple-input multiple-output) case, *i.e.*, with state $x(t) \in I\!R^n$, input $u(t) \in I\!R^m$ and output $y(t) \in I\!R^p$. Assume that there exists a constant output feedback control law

$$u(t) = -Ky(t) + r(t) \tag{2.198}$$

such that the closed loop system

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) + Br(t) \\ \\ y(t) = Cx(t) \end{cases} \tag{2.199}$$

with

$$\bar{A} = A - BKC$$

is SPR, *i.e.* there exists a matrix $P > 0$, a matrix L', and a positive constant ε such that [3]

$$\begin{cases} \bar{A}^T P + P\bar{A} = -L'L'^T - \varepsilon P \\ \\ PB \qquad\qquad = C^T \end{cases} \tag{2.200}$$

Since the plant parameters are unknown, consider the following adaptive controller for $r(t) = 0$:

$$u(t) = -\hat{K}(t)y(t)$$

where $\hat{K}(t)$ is the estimate of K at time t. The closed loop system can be written as

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) - B(\hat{K}(t) - K)y(t) \\ \\ y(t) = Cx(t) \end{cases}$$

Define

$$\tilde{K}(t) = \hat{K}(t) - K$$

[3] Similarly to in the foregoing section, this is a consequence of the Kalman-Yakubovich-Popov Lemma for SPR systems.

and consider the following Lyapunov function candidate

$$V(x, \tilde{K}) = x^T P x + \mathrm{tr}\left(\tilde{K}^T \Gamma^{-1} \tilde{K}\right)$$

where $\Gamma > 0$ is an arbitrary positive definite matrix. The time derivative of $V(\cdot)$ along the system trajectories is given by

$$\dot{V}(x, \tilde{K}) = x^T(\bar{A}^T P + P\bar{A})x - 2x^T PB\tilde{K}y + 2\mathrm{tr}\left(\tilde{K}^T \Gamma^{-1} \frac{d}{dt}\left(\tilde{K}\right)\right)$$

Introducing (2.189) and (2.200) we obtain

$$\dot{V}(x, \tilde{K}) = x^T(\bar{A}^T P + P\bar{A})x - 2\mathrm{tr}\left(\tilde{K}yy^T - \tilde{K}^T \Gamma^{-1} \frac{d}{dt}\left(\tilde{K}\right)\right)$$

Choosing the parameter adaptation law

$$\frac{d}{dt}\left(\hat{K}\right)(t) = \Gamma y(t) y^T(t)$$

and introducing (2.192) we obtain

$$\dot{V}(x) = -x^T(L' L'^T + \varepsilon P)x \leq 0$$

Therefore $V(\cdot)$ is a Lyapunov function and thus $x(\cdot)$ and $\hat{K}(\cdot)$ are both bounded. Integrating the above equation it follows that $x \in \mathcal{L}_2$. Since $\dot{x}(\cdot)$ is also bounded we conclude that $x(t) \to 0$ as $t \to 0$.

Hence the proposed adaptive control law stabilizes the system as long as the assumption of the existence of a constant output feedback that makes the closed-loop transfer matrix SPR is satisfied. The conditions for the existence of such control law are established in the next section. Further work on this topic may be found in [42] who relax the symmetry of the Markov parameter CB.

2.15.3 Design of SPR Systems

The adaptive control scheme presented in the previous section motivates the study of constant output feedback control designs such that the resulting closed-loop is SPR. The positive real synthesis problem is important in its own right and has been investigated by [179, 428, 480, 505]. This problem is quite close to the so-called *passification* or *passivation* by output feedback [153, 156, 280]. Necessary and sufficient conditions have been obtained in [219] for a linear system to become SPR under constant output feedback. Furthermore, they show that if no constant feedback can lead to an SPR closed-loop system, then no dynamic feedback with proper feedback transfer matrix can do it neither. Hence, there exists an output feedback such that the closed-loop system is SPR if and only if there exists a constant output feedback rendering the closed-loop system SPR.

Consider again the system (2.189) in the MIMO case, *i.e.*, with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$ and output $y(t) \in \mathbb{R}^p$ and the constant output feedback in (2.198). The closed loop is represented in Figure 2.20 where $G(s)$ is the transfer function of the system (2.189). The equation of the closed-loop $T(s)$ of Figure 2.20 is given in (2.199).

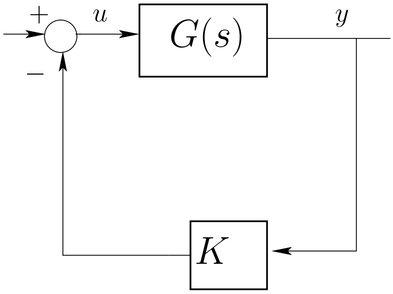

Fig. 2.20. Closed-loop system $T(s)$ using constant output feedback

Theorem 2.64. *[41] Any strictly proper strictly minimum-phase system (A, B, C) with the $m \times m$ transfer function $G(s) = C(sI_n - A)^{-1}B$ and with $CB > 0$ and symmetric, can be made SPR via constant output feedback.* ■

The fact that the zeroes of the system have to satisfy $\mathbf{Re}[s] < 0$ is crucial. Consider $G(s) = \frac{s^2+1}{(s+1)(s+2)(s+5)}$. There does not exist any static output feedback $u = ky + w$ which renders the closed-loop transfer function PR. Indeed if $\omega^2 = \frac{9-k}{8-k}$ then $\mathbf{Re}[T(j\omega)] < 0$ for all $k < 0$. Therefore the strict minimum phase assumption is necessary. Recall that a static state feedback does not change the zeroes of a linear time invariant system. We now state the following result where we assume that B and C are full rank.

Theorem 2.65 (SPR synthesis [219]). *There exists a constant matrix K such that the closed-loop transfer function matrix $T(s)$ in Figure 2.20 is SPR if and only if*

$$B^T C = C^T B > 0$$

and there exists a positive definite matrix X such that

$$C_\perp^T herm\{B_\perp X B_\perp^T A\} C_\perp < 0$$

When the above conditions hold, K is given by

$$K = C^{\dagger} Z (I - C_{\perp}(C_{\perp}^T Z C_{\perp})^{-1} C_{\perp}^T Z) C^{\dagger T} + S$$

where $Z = herm\{PA\}$ *and* $P = C(B^T C)^{-1} C^T + B_{\perp} X B_{\perp}^T$*, and* S *is an arbitrary positive definite matrix.* ■

The notation used above is $\mathrm{herm}\{X\} \stackrel{\Delta}{=} \frac{1}{2}(X + X^*)$, and $X_{\perp}$ is defined as $X_{\perp}^T X = 0$ and $X_{\perp}^T X_{\perp} = I_n, X \in I\!R^{n\times n}$.

In the single-input single-output case, the necessary condition $B^T C > 0$ implies the relative degree of $G(s)$ is one. It is noteworthy that the above two results apply to systems with no feedthrough term, *i.e.* $D = 0$. An answer is provided in [480, Theorem 4.1], where this time one considers a dynamic output feedback. The system (A, B, C, D) is partitioned as $B = [B_1 \quad B_2]$, $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{pmatrix}$. It is assumed that (A, B_2) is stabilizable and that (A, C_2) is detectable. The closed-loop system is said *internally stable* if the matrix $\begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ & \\ B_K C_2 & A_K \end{pmatrix}$ is stable (has eigenvalues with strictly negative real parts), where (A_K, B_K, C_K, D_K) is the dynamic feedback controller.

Theorem 2.66. *[480] There exists a strictly proper (dynamic) state feedback such that the closed-loop system is internally stable and extended SPR if and only if there exists two matrices* F *and* L *such that*

- $D_{11} + D_{11}^T > 0$
- *The algebraic Riccati inequality*

$$\begin{array}{l}(A + B_2 F)^T P + P(A + B_2 F) + (C_1 + D_{12} F - B_1^T P)^T (D_{11} + D_{11}^T)^{-1}. \\ \\ .(C_1 + D_{12} F - B_1^T P) < 0\end{array} \tag{2.201}$$

  has a positive definite solution P_f
- *The algebraic Riccati inequality*

$$\begin{array}{l}(A + L C_2)^T G + G(A + L C_2) + (B_1 + L D_{12} - G C_1^T)^T (D_{11} + D_{11}^T)^{-1}. \\ \\ .(B_1 + L D_{12} - G C_1^T) < 0\end{array} \tag{2.202}$$

  has a positive definite solution G_f,
- *The spectral radius* $\rho(G_f P_f) < 1$. ■

The conditions such that a system can be rendered SPR via static state feedback are relaxed when an observer is used in the control loop. However this creates additional difficulty in the analysis because the closed-loop system loses its controllability. See Section 3.4 for more information. Other works related with the material exposed in this section, may be found in [49, 50, 177, 205, 330, 448, 465, 497, 515, 516]. Despite there being no close relationship with the material of this section, let us mention [19] where model reduction which preserves passivity is considered. Spectral conditions for a single-input single-output system to be SPR, are provided in [455]. The SPRness is also used in identification of LTI systems [12]. Robust stabilisation when a PR uncretainty is considered is studied in [180].

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