Solutions

Problems of Chapter 1

1.1 A real square matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if its determinant is non zero. The determinant of A is a polynomial in the entries a_{ij} of A, whose set of zeros in $\mathbb{R}^{n\times n}$ is closed. Therefore $det(A)$ is generically different from zero and A is generically invertible.

1.2 A square, symmetric, real matrix A of order n, that can be seen as a point in $\mathbb{R}^{(n^2+n)/2}$, is positive definite if and only if $x^t\ A\ x > 0$ for every $x \in \mathbb{R}^n$. The trivial case $n = 1$ is obvious: positiveness is not a generic property for real numbers. The general case can be dealt with by recalling that A is positive definite if and only if the determinants of all its upper-left submatrices A_i, $i = 1, \ldots, n$, are positive. Then, consider the function $D : \mathbb{R}^{(n^2+n)/2} \to \mathbb{R}^n$ defined, for any square, symmetric real matrix $A \in \mathbb{R}^{(n^2+n)/2}$, by

$$D(A) = (det(A_1), \ldots, det(A_n))^T$$

and take e.g. the point $P = (1, \ldots, 1, -1)^T \in \mathbb{R}^n$. Since the matrix

$$A = \begin{bmatrix} 1 \ldots & 0 & 0 \\ \ldots\ldots\ldots & \ldots & \\ 0 \ldots & 1 & 0 \\ 0 \ldots & 0 & -1 \end{bmatrix},$$

belongs to $D^{-1}(P)$ and D is a continuous function in the entries of A, the conterimage $D^{-1}(V_\epsilon(P))$ of any spherical neighborhood $V_\epsilon(P)$ of P of radius $\epsilon > 0$ is a neighborhood of A. Taking $0 < \epsilon \leq 1$, $D^{-1}(V_\epsilon(P))$ consists of non positive definite matrices, then the conclusion follows.

1.3 The controllability matrix

$$[BABA^2B...A^{n-1}B]$$

has *rank* equal to n when at least one of its $n \times n$ minors has non zero determinant. The same argument as in (1.1) proves that this is a generic property.

1.4 It can be shown that the function $f(x) = \begin{cases} e^{-1/x^2}, & \text{if} x < 0 \\ 0, & \text{if} x \geq 0 \end{cases}$ admits continuous derivatives $f^{(m)}(x)$ of any order m in all points of $I\!R$ and $f^{(m)}(0) = 0$ for all m. The proof, by induction, is mainly based on the fact that for any integer m

$$lim_{x \to 0^-} \ x^{-m} \ e^{-1/x^2} \ = 0$$

According to what precedes, the Taylor series of f at 0 is

$$\sum_{n=0}^{\infty} \frac{0}{n!} \ x^n = 0 \neq f(x)$$

unless $x \geq 0$. Consequently f is not analytic at 0, since it does not admit a Taylor expansion in a neighborhood of $x = 0$.

1.5 Integration of one-forms

(a) The differential of the one-form $v = (1 + cos(x + y))dx + cos(x + y)dy$, is $\mathrm{d}v = (-sin(x+y)) \ dx \wedge dy + (-sin(x+y)) \ dy \wedge dx = (-sin(x+y)) \ dx \wedge dy - (-sin(x + y)) \ dx \wedge dy = 0$.
Then, v is closed and a function φ that, locally, satisfies the relation $v = \mathrm{d}\varphi$ is $\varphi = x + sin(x + y)$.

(b) The differential of the one-form $v = \dfrac{x + 2y}{x^3 y} dx + \dfrac{1}{xy^2} dy$
is $\mathrm{d}v = \dfrac{-x^4}{x^6 y^2} \ dx \wedge dy + \dfrac{-y^2}{x^2 y} \ dy \wedge dx = \dfrac{-1}{x^2 y} \ dx \wedge dy - \dfrac{-1}{x^2 y} \ dx \wedge dy = 0$. Then, v is a closed one-form and a function φ that, locally, satisfies the relation $v = \mathrm{d}\varphi$ is $\varphi = x + sin(x + y)$ that, locally, satisfies the relation $\varphi = -\dfrac{x + y}{x^2 y}$.

(c) The differential of the one-form $v = \dfrac{x}{x^2 + y^2} dx + \dfrac{y}{x^2 + y^2} dy$ is
$\mathrm{d}v = \dfrac{-2xy}{(x^2 + y^2)^2} \ dx \wedge dy + \dfrac{-2xy}{(x^2 + y^2)^2} \ dy \wedge dx = \dfrac{-2xy + 2xy}{(x^2 + y^2)^2} \ dx \wedge dy = 0$.
Then, v is closed a function φ that, locally, satisfies the relation $v = \mathrm{d}\varphi$ is $\varphi = \dfrac{1}{2} \ log(x^2 + y^2)$.

(d) The differential of the one-form $v = -\dfrac{y}{x^2 + y^2} \ dx + \dfrac{x}{x^2 + y^2} \ dy$ is

$$\mathrm{d}v = \frac{-x^2 + y^2}{(x^2 + y^2)^2} \ dx \wedge dy + \frac{-x^2 + y^2}{(x^2 + y^2)^2} \ dy \wedge dx =$$
$$= \frac{-x^2 + y^2 - (-x^2 + y^2)}{(x^2 + y^2)^2} \ dx \wedge dy = 0.$$

Then, v is closed and a function φ that, locally, satisfies the relation $v = \mathrm{d}\varphi$ can be found, in this case, computing $\displaystyle\int\left(-\frac{y}{x^2+y^2}\right)dx$. We find $\varphi = -\arctan(\frac{x}{y})$.

(e) The differential of the one-form $v = \dfrac{x}{x^2+y^2+z^2}dx + \dfrac{y}{x^2+y^2+z^2}dy + \dfrac{z}{x^2+y^2+z^2}dz$ is

$$\begin{aligned}
\mathrm{d}v = {} & \frac{-2xy}{(x^2+y^2+z^2)^2}\, dx\wedge dy + \frac{-2xz}{(x^2+y^2+z^2)^2} dx\wedge dz + \frac{-2xy}{(x^2+y^2+z^2)^2}\, dy\wedge dx + \\
& + \frac{-2yz}{(x^2+y^2+z^2)^2} dy\wedge dz + \frac{-2xz}{(x^2+y^2+z^2)^2}\, dz\wedge dx + \frac{-2yz}{(x^2+y^2+z^2)^2} dz\wedge dy = \\
& = \frac{-2xy+2xy}{(x^2+y^2+z^2)^2}\, dx\wedge dy + \frac{-2xz+2xz}{(x^2+y^2+z^2)^2} dx\wedge dz + \frac{-2yz+2yz}{(x^2+y^2+z^2)^2} dy\wedge dz = 0
\end{aligned}$$

Then, v is closed and a function φ that, locally, satisfies the relation $v = \mathrm{d}\varphi$ can be found, in this case, computing $\displaystyle\int\left(\frac{x}{x^2+y^2+z^2}\right)dx$. We find $\varphi = \dfrac{1}{2}\, log(x^2+y^2+z^2)$.

(f) The differential of the one-form $v = -\dfrac{y+z}{(x-y-z)^2}dx + \dfrac{x}{(x-y-z)^2}dy + \dfrac{x}{(x-y-z)^2}dz$ is

$$\begin{aligned}
\mathrm{d}v = {} & \frac{-x-y-z}{(x-y-z)^3}\, dx\wedge dy + \frac{-x-y-z}{(x-y-z)^3} dx\wedge dz + \frac{-x-y-z}{(x-y-z)^3}\, dy\wedge dx + \\
& + \frac{2x}{(x-y-z)^3} dy\wedge dz + \frac{-x-y-z}{(x-y-z)^3}\, dz\wedge dx + \frac{2x}{(x-y-z)^3} dz\wedge dy = \\
= {} & \frac{-x-y-z-(-x-y-z)}{(x-y-z)^3}\, dx\wedge dy + \frac{-x-y-z-(-x-y-z)}{(x-y-z)^3} dx\wedge dz + \\
& + \frac{2x-2x}{(x-y-z)^3} dy\wedge dz = 0
\end{aligned}$$

Then, v is closed and a function φ that, locally, satisfies the relation $v = \mathrm{d}\varphi$ can be found, in this case, computing $\displaystyle\int\left(-\frac{y+z}{(x-y-z)^2}\right)dx$. We find $\varphi = \dfrac{y+z}{x-y-z}$.

1.6 The one-form $\omega = (-x^3cos(y))dx + (xsin(y))dy$ is not closed, since we have

$$\mathrm{d}\omega = (x^3sin(y))\ dx\wedge dy + sin(y)\ dy\wedge dx \neq 0$$

However,

$$\begin{aligned}\mathrm{d}\omega \wedge \omega = (-x^6\ sin(y)\ cos(y))\ dx \wedge dy \wedge dx + x^4\ sin(y)^2\ dy \wedge dx \wedge dy + \\ -x^3\ sin(y)\ cos(y)\ dy \wedge dx \wedge dx + x\ sin(y)^2\ dy \wedge dx \wedge dy = 0\end{aligned}$$

Then, by theorem 1.15, ω is colinear to an exact form, *i.e.* there exist λ and φ in $\mathcal{K}$ such that $\lambda\omega = \mathrm{d}\varphi$.
The *integrating factor* λ must satisfy the relation $\mathrm{d}(\lambda\omega) = 0$. Since

$$\begin{aligned}\mathrm{d}(\lambda\omega) &= \left[-x^3\ cos(y)\ \frac{\partial\lambda}{\partial\ y} + \lambda x^3\ sin(y)\right] dx \wedge dy + \left[\frac{\partial\lambda}{\partial\ x} xsin(y) + \lambda sin(y)\right] dy \wedge dx = \\ &= \left[-x^3 cos(y)\frac{\partial\lambda}{\partial\ y} + \lambda\ x^3 sin(y) - xsin(y)\ \frac{\partial\lambda}{\partial\ x} - \lambda\ sin(y)\right]\ dx \wedge dy \\ &= \left[-x^3 cos(y)\frac{\partial\lambda}{\partial\ y} + sin(y)\left(\frac{\mathrm{d}\lambda}{\mathrm{d}x}\ x + (1-x^3)\ \lambda\right)\right]\ dx \wedge dy = 0\end{aligned} \tag{12.50}$$

Assume that λ depend only on x, i.e. $\dfrac{\partial\lambda}{\partial\ y} = 0$, then equation (12.50) is equivalent to

$$\frac{\mathrm{d}\lambda}{\mathrm{d}x}\ x + (1-x^3)\ \lambda = 0 \tag{12.51}$$

$\lambda = -\dfrac{1}{x}\ exp\ (x^3/3)$ solves equation (12.51) and

$$\lambda\omega = (-x^2\ exp\ (x^3/3)\ cos(y))dx + (exp\ (x^3/3) sin(y))dy$$

is, by construction, a closed one-form. A function φ that, locally, satisfies the relation $\mathrm{d}\varphi = \lambda\omega$ can be found computing $\left(x^2 \cos(y)\ e^{x^3/3}\right) dx$. We find $\varphi = -\ e^{x^3/3} cos(y)$.

1.7 Exterior differentiation
$\mathrm{d}\omega = (cos(x+y)\ dx \wedge dy + (2x\ dy \wedge dx = [cos(x+y) - 2x]\ \ dx \wedge dy.$
(b) $\mathrm{d}\omega = -sin(z)\ dx \wedge dz$
(c) $\omega = (x^2)\ dx \wedge dy$, $\mathrm{d}\omega = 0$.
(d) $\mathrm{d}\omega = dy \wedge dz \wedge dx = -dy \wedge dx \wedge dz = dx \wedge dy \wedge dz$
(e) $\mathrm{d}\omega = 0$.

1.8 Since we have that

$\mathrm{d}\xi_{i_0} \wedge \ldots \wedge \mathrm{d}\xi_{i_j} \wedge \ldots \wedge \mathrm{d}\xi_{i_k} \wedge \ldots \wedge \mathrm{d}\xi_{i_s} = -\ \mathrm{d}\xi_{i_0} \wedge \ldots \wedge \mathrm{d}\xi_{i_k} \wedge \ldots \wedge \mathrm{d}\xi_{i_j} \wedge \ldots \wedge \mathrm{d}\xi_{i_s}$,
if $\mathrm{d}\xi_{i_j} = \mathrm{d}\xi_{i_k}$ for some index j and k, the conclusion follows.

1.9 Exterior product
(a)
$$\begin{aligned}& dx \bigwedge (sin(y)dy \bigwedge (x\ dx + (y^2)dy) = \\ &= (sin(y)\ dx \wedge dy) \bigwedge \left(x\ dx + y^2 dy\right) = \\ &= xsin(y)\ dx \wedge dy \wedge dx + sin(y)\ y^2\ dx \wedge dy \wedge dy = 0 \quad \text{by 1.8.}\end{aligned}$$

(b) $\big(cos(xy)dx + (y^3)dy\big) \bigwedge (z\ dx + y\ dz) =$
$= z\ cos(xy)\ dx \wedge dx + y\ cos(xy)\ dx \wedge dz + y^3\ z\ dy \wedge dx + y^4\ dy \wedge dz =$
$= -z\ y^3\ dx \wedge dy + y\ cos(xy)\ dx \wedge dz + y^4\ dy \wedge dz$

(c) $(2x\ dx + (x+y)^2 dy + (1-z)dz) \bigwedge (ydx - xdz) =$
$= -2x^2\ dx \wedge dz - y(x+y)^2\ dx \wedge dy - x(x+y)^2\ dy \wedge dz -\ y(1-z)\ dx \wedge dz = .$
$= -y(x+y^2)\ dx \wedge dy + (-2x^2 - y + yz)\ dx \wedge dz - x(x+y)^2\ dy \wedge dz$

(d) $(e^x)dx \wedge dy + x\ dy \wedge dz$

(e) $(dx \wedge dy) \bigwedge (cos(x+y)dy \wedge dz) = cos(x+y)dx \wedge dy \wedge dy \wedge dz = 0$ by 1.8.

Problems of Chapter 2

2.1 The ball and beam

1. Set $x_1 = r$, $x_2 = \dot{r}$, $u = \alpha$, and $y = r$. Then,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \left(\frac{J}{R^2} + m\right)^{-1} \left[m x_1 \dot{u}^2 - mg \sin u\right] \\ y &= x_1 \end{aligned}$$

is the generalized realization which is sought.

2. Introduce the notation $x_3 = u$, and $x_4 = \dot{u}$. Write the extended system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \left(\frac{J}{R^2} + m\right)^{-1} \left[m x_1 x_4^2 - mg \sin x_3\right] \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \ddot{u} \\ y &= x_1 \end{aligned}$$

Compute

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4\} \\ \mathcal{H}_2 &= \mathrm{span}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\} \\ \mathcal{H}_3 &= \mathrm{span}\{\mathrm{d}x_1, \omega\} \end{aligned}$$

where $\omega = \left(\frac{J}{R^2} + m\right) \mathrm{d}x_2 - 2m x_1 x_4 \mathrm{d}x_3$. Since $\mathcal{H}_3$ is not fully integrable, there does not exist any classical state-space realization.

2.2 The actuated ball and beam

Set $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \alpha$, $x_4 = \dot{\alpha}$, $u = \ddot{\alpha}$, and $y = r$. Then, a classical realization is obviously obtained as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \left(\dfrac{J}{R^2} + m\right)^{-1} \left[m x_1 x_4^2 - mg \sin x_3\right] \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = u \\ \ \ y = x_1 \end{cases}$$

2.2 The pendulum on a cart

Theorem 2.16 can not be applied directly as the input-output equation is not given. To cope with the auxiliary variable r , it is necessary to derive the so-called inverse dynamic model, *i.e.* to write explicitly $\ddot{r}$ and $\ddot{\theta}$. From the direct dynamic model

$$\begin{aligned} (M+m)\ddot{r} + b\dot{r} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2 \sin\theta &= F \\ (I+ml^2)\ddot{\theta} + mgl\sin\theta &= -ml\ddot{r}\cos\theta \end{aligned}$$

the elimination of $\ddot{\theta}$ is obtained by multiplying the first equation by $(I+ml^2)$ and the second equation by $(-ml\cos\theta)$. The resulting summation yields

$$A\ddot{r} + b(I+ml^2)\dot{r} - ml(I+ml^2)\dot{\theta}^2\sin\theta - m^2gl^2\sin\theta\cos\theta = (I+ml^2)F$$

where $A = (I+ml^2)(M+m) - m^2l^2\cos^2\theta$. Thus,

$$\ddot{r} = \frac{-b(I+ml^2)\dot{r} + ml(I+ml^2)\dot{\theta}^2\sin\theta + m^2gl^2\sin\theta\cos\theta + (I+ml^2)F}{(I+ml^2)(M+m) - m^2l^2\cos^2\theta}$$

The elimination of $\ddot{r}$ in the direct dynamic model is obtained in a similar vein by multiplying the first equation by $ml\cos\theta$ and the second equation by $(M+m)$:

$$B\ddot{\theta} + bml\dot{r}\cos\theta - m^2l^2\sin\theta\cos\theta = (M+m)mgl\sin\theta + ml\cos\theta F$$

where $B = m^2l^2\cos^2\theta - (M+m)(I+ml^2)$. Thus,

$$\ddot{\theta} = \frac{-bml\dot{r}\cos\theta + m^2l^2\sin\theta\cos\theta + (M+m)mgl\sin\theta + ml\cos\theta F}{m^2l^2\cos^2\theta - (M+m)(I+ml^2)}$$

Set $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, and $x_4 = \dot{\theta}$, then direct inspection of the inverse dynamic model yields the following classical realization:

$$\begin{cases} \dot{x} = \begin{pmatrix} x_2 \\ \frac{-b(I+ml^2)x_2 + ml(I+ml^2)x_4^2\sin x_3 + m^2gl^2\sin x_3\cos x_3}{(I+ml^2)(M+m) - m^2l^2\cos^2 x_3} \\ x_4 \\ \frac{-bmlx_2\cos x_3 + m^2l^2\sin x_3\cos x_3 + (M+m)mgl\sin x_3}{m^2l^2\cos^2 x_3 - (M+m)(I+ml^2)} \end{pmatrix} \\ \qquad + \begin{pmatrix} 0 \\ \frac{(I+ml^2)}{(I+ml^2)(M+m) - m^2l^2\cos^2 x_3} \\ 0 \\ \frac{ml\cos x_3}{m^2l^2\cos^2 x_3 - (M+m)(I+ml^2)} \end{pmatrix} F \\ y = x_1 \end{cases}$$

Problems of Chapter 3

3.1 The actuated ball and beam
The dynamics were obtained as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \left(\dfrac{J}{R^2} + m\right)^{-1} \left[m x_1 x_4^2 - mg \sin x_3\right] \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = u \end{cases}$$

Compute

$$\mathcal{H}_2 = \operatorname{span}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}$$

Obviously, $\mathrm{d}x_1 \in \mathcal{H}_3$ since its time-derivative is in $\mathcal{H}_2$. Let ω be a general vector of $\mathcal{H}_3$, thus $\omega \in \mathcal{H}_2$ and

$$\omega = \alpha_1 \mathrm{d}x_1 + \alpha_2 \mathrm{d}x_2 + \alpha_3 \mathrm{d}x_3$$

where $\alpha_i \in \mathcal{K}$, for $i = 1, 2, 3$. The problem consists in finding all solutions $(\alpha_1, \alpha_2, \alpha_3)$ such that $\dot{\omega} \in \mathcal{H}_2$. Compute:

$$\dot{\omega} = \sum_i \dot{\alpha}_i \mathrm{d}x_i + \sum_i \alpha_i \mathrm{d}\dot{x}_i$$

Since $\sum_i \dot{\alpha}_i \mathrm{d}x_i \in \mathcal{H}_2$ for any α_i, write

$$\begin{aligned} \dot{\omega} &= * + \textstyle\sum_i \alpha_i \mathrm{d}\dot{x}_i \\ &= * + \alpha_1 \mathrm{d}x_2 + \alpha_2 m \left(\dfrac{J}{R^2} + m\right)^{-1} \mathrm{d}\left[x_1 x_4^2 - g \sin x_3\right] + \alpha_3 \mathrm{d}x_4 \\ &= * + [2\alpha_2 m \left(\dfrac{J}{R^2} + m\right)^{-1} x_1 x_4 + \alpha_3] \mathrm{d}x_4 \end{aligned}$$

Thus, $\dot{\omega} \in \mathcal{H}_2$ if and only if $2\alpha_2 m \left(\dfrac{J}{R^2} + m\right)^{-1} x_1 x_4 + \alpha_3 = 0$. A nonzero solution of the latter is

$$\begin{aligned} \alpha_2 &= 1 \\ \alpha_3 &= -2m \left(\dfrac{J}{R^2} + m\right)^{-1} x_1 x_4 \end{aligned}$$

and

$$\mathcal{H}_3 = \operatorname{span}\{\mathrm{d}x_1, \mathrm{d}x_2 - 2m \left(\dfrac{J}{R^2} + m\right)^{-1} x_1 x_4 \mathrm{d}x_3\}$$

To compute $\mathcal{H}_4$, denote again ω as a general vector of $\mathcal{H}_3$:

$$\omega = \alpha_1 \mathrm{d}x_1 + \alpha_2 \left[\mathrm{d}x_2 - 2m \left(\dfrac{J}{R^2} + m\right)^{-1} x_1 x_4 \mathrm{d}x_3\right]$$

Compute $\dot{\omega}$, and denote terms which belong to $\mathcal{H}_3$ by $*$

$$\begin{aligned}
\dot{\omega} = & * + \alpha_1 \mathrm{d}x_2 + \alpha_2 \left[m \left(\frac{J}{R^2} + m \right)^{-1} \mathrm{d} \left[x_1 x_4^2 - g \sin x_3 \right] \right. \\
& \left. -2m \left(\frac{J}{R^2} + m \right)^{-1} (x_2 x_4 + x_1 u) \mathrm{d}x_3 - 2m \left(\frac{J}{R^2} + m \right)^{-1} x_1 x_4 \mathrm{d}x_4 \right] \\
= & * + \alpha_1 \mathrm{d}x_2 + \alpha_2 \left[-mg \left(\frac{J}{R^2} + m \right)^{-1} \cos x_3 - 2m \left(\frac{J}{R^2} + m \right)^{-1} (x_2 x_4 + x_1 u) \right] \mathrm{d}x_3 \\
& + \alpha_2 \left[2m \left(\frac{J}{R^2} + m \right)^{-1} x_1 x_4 - 2m \left(\frac{J}{R^2} + m \right)^{-1} x_1 x_4 \right] \mathrm{d}x_4 \\
= & * + \alpha_1 \mathrm{d}x_2 + \alpha_2 \left[-mg \left(\frac{J}{R^2} + m \right)^{-1} \cos x_3 - 2m \left(\frac{J}{R^2} + m \right)^{-1} (x_2 x_4 + x_1 u) \right] \mathrm{d}x_3
\end{aligned}$$

$\dot{\omega} \in \mathcal{H}_3$ if, for instance,

$$\begin{aligned}
\alpha_1 &= -mg \left(\frac{J}{R^2} + m \right)^{-1} \cos x_3 - 2m \left(\frac{J}{R^2} + m \right)^{-1} (x_2 x_4 + x_1 u) \\
\alpha_2 &= -2m \left(\frac{J}{R^2} + m \right)^{-1} x_1 x_4
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{H}_4 = \mathrm{span}\, \{ & \left[-mg \left(\frac{J}{R^2} + m \right)^{-1} \cos x_3 - 2m \left(\frac{J}{R^2} + m \right)^{-1} (x_2 x_4 + x_1 u) \right] \mathrm{d}x_1 \\
& + \left[-2m \left(\frac{J}{R^2} + m \right)^{-1} x_1 x_4 \right] \left[\mathrm{d}x_2 - 2m \left(\frac{J}{R^2} + m \right)^{-1} x_1 x_4 \mathrm{d}x_3 \right] \}
\end{aligned}$$

Since $\dim \mathcal{H}_4 = 1$, either $\mathcal{H}_5 = \mathcal{H}_4$ or $\mathcal{H}_5 = 0$. By time derivation of the one-form in $\mathcal{H}_4$, one checks that $\mathcal{H}_5 = 0$ and the system is thus accessible.

3.2 The hopping robot

The dynamics (3.19) of the hopping robot are

$$\dot{x} = \begin{pmatrix} x_2 \\ x_1 x_6^2 \\ x_4 \\ 0 \\ x_6 \\ -2\frac{x_2 x_6}{x_1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1/m \\ 0 & 0 \\ 1/J & 0 \\ 0 & 0 \\ -\frac{1}{m x_1^2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Its controllability indices are derived from the dimensions of the $\mathcal{H}_k$ spaces. Compute

$$\mathcal{H}_2 = \mathrm{span}\{\mathrm{d}x_1, \mathrm{d}x_3, \mathrm{d}x_5, J\mathrm{d}x_4 + m x_1^2 \mathrm{d}x_6\}$$

To compute $\mathcal{H}_3$, consider a general vector $\omega \in \mathcal{H}_2$:

$$\omega = \alpha_1 \mathrm{d}x_1 + \alpha_2 \mathrm{d}x_3 + \alpha_3 \mathrm{d}x_5 + \alpha_4 (J\mathrm{d}x_4 + mx_1^2 \mathrm{d}x_6)$$

Compute $\dot{\omega}$, and denote terms which belong to $\mathcal{H}_2$ by $*$

$$\begin{aligned}\dot{\omega} &= * + \alpha_1 \mathrm{d}x_2 + \alpha_2 \mathrm{d}x_4 + \alpha_3 \mathrm{d}x_6 + \alpha_4 \left[2mx_1x_2\mathrm{d}x_6 - 2mx_1^2 \mathrm{d}(\tfrac{x_2x_6}{x_1})\right] \\ &= * + \alpha_1 \mathrm{d}x_2 + \alpha_2 \mathrm{d}x_4 + \alpha_3 \mathrm{d}x_6 + \alpha_4 \left[2mx_1x_2\mathrm{d}x_6 - 2mx_1x_6\mathrm{d}x_2 - 2mx_1x_2\mathrm{d}x_6\right] \\ &= * + \alpha_1 \mathrm{d}x_2 + \alpha_2 \mathrm{d}x_4 + \alpha_3 \mathrm{d}x_6 - 2mx_1x_6\alpha_4 \mathrm{d}x_2 \\ &= * + (\alpha_1 - 2mx_1x_6\alpha_4)\mathrm{d}x_2 + \alpha_2 \mathrm{d}x_4 + \alpha_3 \mathrm{d}x_6\end{aligned}$$

The constraint $\dot{\omega} \in \mathcal{H}_2$ has to independent solutions:

- $\alpha_1 = 0$, $\alpha_4 = 0$, $\alpha_2 = J$, and $\alpha_3 = mx_1^2$
- $\alpha_1 = 2mx_1x_6$, $\alpha_4 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 0$

and thus,

$$\begin{aligned}\mathcal{H}_3 &= \mathrm{span}\{J\mathrm{d}x_3 + mx_1^2\mathrm{d}x_5, 2mx_1x_6\mathrm{d}x_1 + J\mathrm{d}x_4 + mx_1^2\mathrm{d}x_6\} \\ &= \mathrm{span}\{J\mathrm{d}x_3 + mx_1^2\mathrm{d}x_5, \mathrm{d}(Jx_4 + mx_1^2x_6)\}\end{aligned}$$

As it has been noted in Section 3.8, $\mathrm{d}(Jx_4 + mx_1^2x_6) \in \mathcal{H}_\infty$; thus, either $\mathcal{H}_4 = \mathcal{H}_3$ or $\mathcal{H}_4 = \mathcal{H}_\infty$. The time derivative of $J\mathrm{d}x_3 + mx_1^2\mathrm{d}x_5$ equals $J\mathrm{d}x_4 + 2mx_1x_2\mathrm{d}x_5 + mx_1^2\mathrm{d}x_6$ which does not belong to $\mathcal{H}_3$, and consequently $\mathcal{H}_4 = \mathcal{H}_\infty$.
Finally, the two controllability indices are $\{2, 3\}$.

3.3 Linear systems
Consider the linear system $\dot{x} = Ax + Bu$:

$$\mathcal{H}_2 = B^\perp$$

where $B^\perp$ denotes the left kernel of matrix B. More generally,

$$\mathcal{H}_k = [B, AB, ..., A^{k-2}B]^\perp$$

Due to Caley-Hamilton Theorem, $\mathcal{H}_\infty = [B, AB, ..., A^{n-1}B]^\perp$. By duality,

$$\mathcal{H}_\infty = 0 \Leftrightarrow \mathrm{rank}[B, AB, ..., A^{n-1}B] = n$$

Problems of Chapter 4

4.1 Given a system Σ of the form

$$\Sigma = \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) \end{cases}$$

$\mathcal{O}_\infty$ is the limit of the observability filtration, $0 \subset \mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}_2 \subset \ldots \subset \mathcal{O}_k \subset \ldots$, where $\mathcal{O}_k := \mathcal{X} \cap (\mathcal{Y}^k + \mathcal{U})$ and $\mathcal{Y}^{(k} = span\{\mathrm{d}y^{(k)},\ k \geq 0\}$.

4.2 As seen above, $\mathcal{O}_\infty$ is spanned by the differential of $\mathrm{d}y^{(k)}$, $k \geq 0$, when the contribution of $\mathrm{d}u^{(j)}$, $j \geq 0$, has been discarded, namely by $\dfrac{\partial y^{(k)}}{\partial x}$. See, for instance the following example. Consider the system

$$\begin{cases} \dot{x}_1 = x_2 + x_3 u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \ \ y = x_1 \end{cases}$$

for which

$$\begin{aligned} &\dot{y} = x_2 + x_3 u \\ &y^{(2)} = x_1 + x_2 u + x_3 \dot{u} \\ &y^{(3)} = x_2 + (x_1 + x_3)u + 2x_2\dot{u} + x_3\ddot{u},\ \ldots \end{aligned}$$

and

$$\begin{aligned} &dy = dx_1 \\ &d\dot{y} = dx_2 + udx_3 + x_3 du \\ &d\ddot{y} = dx_1 + udx_2 + \dot{u}dx_3 + x_1 du + x_3 d\dot{u} \\ &dy^{(3)} = udx_1 + (1 + 2\dot{u})dx_2 + (u + \ddot{u})dx_3 + (x_1 + x_3)du + \\ &\quad +(2x_2 d\dot{u} + x_3 d\ddot{u},\ \ldots \end{aligned}$$

Therefore we have

$$\begin{aligned} &\mathcal{O}_0 = \{dx_1\} \\ &\mathcal{O}_1 = span\{dx_1, dx_2 + udx_3\} \\ &\mathcal{O}_2 = span\{dx_1, dx_2 + udx_3, udx_2 + \dot{u}dx_3\} \end{aligned}$$

and $\mathcal{O}_\infty$ is spanned by the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u \\ 1 & u & \dot{u} \end{bmatrix}$$

and its dimension coincides with

$$\mathrm{rank}_{\mathcal{K}} \left[\frac{\partial(y, \dot{y}, \ldots, y^{(n-1)})}{\partial x} \right].$$

4.3 Given the linear system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

we have

$$\dot{y} = CAx + CBu$$
$$\ddot{y} = CA^2x + CABu + CB\dot{u}, \ \ldots$$

and

$$d\dot{y} = CAdx + CBdu$$
$$d\ddot{y} = CA^2dx + CABdu + CBd\dot{u}, \ \ldots$$

Then,

$$\mathcal{Y}^0 = span\{Cdx\}$$
$$\mathcal{Y}^1 = span\{Cdx, CAdx, CBdu\}$$
$$\mathcal{Y}^2 = span\{Cdx, CAdx, CA^2dx, CABdu, CBd\dot{u}\}, \ \ldots$$

and

$$\mathcal{O}_0 = span\{dx\} \bigcap span\{Cdx\} = span\{Cdx\}$$
$$\mathcal{O}_1 = span\{dx\} \bigcap span\{Cdx, CAdx, CBdu\} = span\{Cdx, CAdx\}$$
$$\mathcal{O}_2 = span\{dx\} \bigcap span\{Cdx, CAdx, CA^2dx, CABdu, CBd\dot{u}\} =$$
$$= span\{Cdx, CAdx, CA^2dx\}, \ \ldots$$

and

$$\mathcal{O}_\infty = span\{Cdx, CAdx, CA^2dx, ..., CA^{n-1}dx\}$$

Problems of Chapter 5

5.1
The observability assumption does not play any role here. By contradiction, assume that the relative degree of the output is infinite, then $\mathrm{d}h \in \mathcal{H}_\infty$, thus $\mathrm{d}h = 0$, which stands in contradiction with the assumption that $h(x)$ is non-constant. **5.2**

1. Compute $\dot{y}_1 = u_1$; thus, the order n_1 of the zero at infinity of output y_1 is $n_1 = 1$. In a similar vein, $\dot{y}_2 = x_3u_1$ and $n_2 = 1$.
2. The list of orders of the zeros at infinity may be obtained applying the inversion, or structure, algorithm:
   Step 1:
   $$\dot{y}_1 = u_1$$
   $$\dot{y}_2 = x_3\dot{y}_1$$
   The system thus has one zero at infinity of order 1.
   Step 2:
   $$\ddot{y}_2 = x_3\ddot{y}_1 + x_4\dot{y}_1^2$$
   Step k, for $3 \leq k \leq n-2$:
   $$y_2^{(k)} = x_3y_1^{(k)} + ... + x_{k+2}\dot{y}_1^k$$
   Step n-1:

$$y_2^{(n-1)} = x_3 y_1^{(n-1)} + ... + \dot{y}_1^{n-1} u_2$$

The system thus has one zero at infinity of order $n-1$. The list of orders of the zeros at infinity is $\{1, n-1\}$.

3. The structure at infinity of the system is defined by one zero at infinity of order 1 and one zero at infinity of order $n-1$.
4. The rank of the systems equals 2, the total number of zeros at infinity.

5.3

Since the system has 2 inputs and 1 output, its rank is at most 1 and the system can not be left-invertible. By inversion, the first time derivative of y depends on the input and the system is thus right-invertible:

$$\dot{y} = x_2 u_1 + u_2$$

A right-inverse is obtained as

$$\begin{cases} \dot{\eta}_1 = \dot{y} \\ \dot{\eta}_2 = u_1 \\ u_2 = \dot{y} - \eta_2 u_1 \end{cases}$$

5.4

Since the system has 1 input and 2 outputs, its rank is at most 1 and the system can not be right-invertible. The first time-derivative of y_1 depends on u, the system is thus left-invertible:

$$\dot{y}_1 = x_2 u$$

A (dynamics free) left-inverse system as

$$u = \dot{y}_1 / y_2$$

An alternative solution is provided by the computation of the time-derivative of y_2 as

$$\ddot{y}_2 = u$$

which represents obviously a dynamics free left-inverse.

5.5

Compute

$$\dot{y} = x_2 + u$$

The reduced order inverse system is obtained as

$$\begin{aligned} \dot{\eta} &= -\eta + \dot{y} \\ u &= -\eta + \dot{y} \end{aligned}$$

Constraining $y \equiv 0$ in the above dynamic yields the zero dynamic

$$\dot{\eta} = -\eta$$

Note that the given system is linear, its transfer function is $(s+1)/s^2$, and its transmission zero equals -1 which is consistent with the results obtained from the state equations.

5.6
Compute

$$\dot{y} = x_2 - u$$

The reduced order inverse system is obtained as

$$\begin{aligned} \dot{\eta} &= \eta - \dot{y} \\ u &= \eta - \dot{y} \end{aligned}$$

Constraining $y \equiv 0$ in the above dynamic yields the zero dynamic

$$\dot{\eta} = \eta$$

Note that the given system is linear, its transfer function is $-(s-1)/s^2$, and its transmission zero equals $+1$ which is consistent with the results obtained from the state equations.

5.7
Compute

$$\dot{y} = x_2^2 + x_2 u$$

The reduced order inverse system is obtained as

$$\begin{aligned} \dot{\eta} &= -\eta^3 + \eta\dot{y} \\ u &= -\eta + \dot{y}/\eta \end{aligned}$$

Constraining $y \equiv 0$ in the above dynamic yields the zero dynamic

$$\dot{\eta} = -\eta^3$$

Problem of Chapter 6

6.1
The generalized transformations are obtained from the inversion algorithm:

$$\begin{aligned} y_1 &= x_1 \\ &=: \zeta_1 \\ y_2 &= x_2 \\ &=: \zeta_2 \\ \dot{y}_1 &= \cos x_3 u_1 \\ &=: v_1 \\ \dot{y}_2 &= \sin x_3 u_1 \\ &=: \zeta_3 \\ \ddot{y}_2 &= \sin x_3 \dot{u}_1 + \cos x_3 u_1 u_2 \\ &=: v_2 \end{aligned}$$

They yield the canonical form

$$\begin{cases} \dot{\zeta}_1 = v_1 \\ \dot{\zeta}_2 = \zeta_3 \\ \dot{\zeta}_3 = v_2 \\ y_1 = \zeta_1 \\ y_2 = \zeta_2 \end{cases}$$

The generalized transformation has the following inverse transformation

$$\begin{aligned} x_1 &= \zeta_1 \\ x_2 &= \zeta_2 \\ x_3 &= \arctan(\zeta_3/v_1) \\ u_1 &= v_1/\cos[\arctan(\zeta_3/v_1)] \\ u_2 &= \tfrac{v_2}{v_1} + \tfrac{\zeta_3}{v_1^2}(\zeta_3 - \dot{v}_1) \end{aligned}$$

Problems of Chapter 7

7.1 The unicycle
Both relative degrees are 1 since

$$\begin{aligned} \dot{y}_1 &= u_1 \cos x_3 \\ \dot{y}_2 &= u_1 \sin x_3 \end{aligned}$$

The decoupling matrix is then obtained as

$$\begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \end{bmatrix}$$

and its rank equals 1. The condition (7.3) of Theorem 7.3 is not fulfilled.

7.2 The unicycle continued
Both relative degrees are 2 since

$$\begin{aligned} \ddot{y}_1 &= v_1 \cos x_3 - v_2 x_4 \sin x_3 \\ \ddot{y}_2 &= v_1 \sin x_3 + v_2 x_4 \cos x_3 \end{aligned}$$

The decoupling matrix is then obtained as

$$\begin{bmatrix} \cos x_3 & -x_4 \sin x_3 \\ \sin x_3 & x_4 \cos x_3 \end{bmatrix}$$

and its rank equals 2. The condition (7.3) of Theorem 7.3 is fulfilled.

Problem of Chapter 8

8.1 The unicycle
A dynamic state feedback is derived from the inversion equations.

$$\begin{aligned}\dot y_1 &= u_1 \cos x_3 \\ \ddot y_2 &= \ddot y_1 \tan x_3 + \dot y_1 (1 + \tan^2 x_3) u_2\end{aligned}$$

Set $\eta = u_1 \cos x_3$, $\ddot y_2 = v_2$, and $\ddot y_1 = v_1$ which yields the definition of the following dynamic state feedback

$$\begin{cases} \dot\eta = v_1 \\ u_1 = \eta / \cos x_3 \\ u_2 = \frac{v_2 - v_1 \tan x_3}{\eta(1+\tan^2 x_3)} \end{cases}$$

A quasi-static state feedback that solves the noninteracting control problem is obtained by setting $w_1 = \dot y_1$, and $w_2 = \ddot y_2$:

$$\begin{cases} u_1 = w_1 / \cos x_3 \\ u_2 = \frac{w_2 - \dot w_1 \tan x_3}{w_1(1+\tan^2 x_3)} \end{cases}$$

Problem of Chapter 9

9.1 The ball and beam
Recall the considered state equations

$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = \left(\dfrac{J}{R^2} + m\right)^{-1} \left[m x_1 x_4^2 - mg \sin x_3\right] \\ \dot x_3 = x_4 \\ \dot x_4 = u \end{cases}$$

1. The $\mathcal{H}_k$ spaces have been computed in Problem 3.1. Obviously, $\mathcal{H}_3$ is not fully integrable; thus, the dynamics are not fully linearizable.
2. $\mathcal{H}_4$ is not fully integrable either, so, there is no function of the state which has relative degree 4. The exact one-form $\mathrm{d}x_1$ can be picked in $\mathcal{H}_3$ that has relative degree 3. Set

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= \left(\frac{J}{mR^2} + 1\right)^{-1} \left[x_1 x_4^2 - g \sin x_3\right] \\ z_4 &= x_4 \\ v &= \left(\frac{J}{mR^2} + 1\right)^{-1} \left[x_2 x_4^2 - g x_4 \cos x_3 + 2 x_1 x_4 u\right] \end{aligned}$$

Solving the last equation in u, yields the state feedback

$$u = \left[g x_4 \cos x_3 - x_2 x_4^2 + \left(\frac{J}{mR^2} + 1 \right) v \right] / 2 x_1 x_4$$

The closed-loop systems displays the subsystem of order 3:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = v \end{cases}$$

Problem of Chapter 12

9.1 Virus dynamics
Recall the virus dynamics model from Section 2.10.2

$$\begin{aligned} \dot{T} &= s - \delta T - \beta T v \\ \dot{T}^* &= \beta T v - \mu T^* \\ \dot{v} &= k T^* - c v \\ y_1 &= T \\ y_2 &= v \end{aligned}$$

This model is linear up to the single input-output injection $\beta T v$. Considering the input β, on defines the static output feedback

$$\beta = w / y_1 y_2$$

and the closed loop reads

$$\begin{aligned} \dot{T} &= s - \delta T - w \\ \dot{T}^* &= w - \mu T^* \\ \dot{v} &= k T^* - c v \\ y_1 &= T \\ y_2 &= v \end{aligned}$$

that is fully linear.

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