What Rayleigh-Bénard, Taylor-Couette and Pipe Flows have in Common

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The very close correspondence between the three types of thermally or shear driven fluid flows is elucidated. Expressions for the relevant currents of temperature, angular momentum, and axial velocity by transverse convective flow in the profile direction are derived from the Navier-Stokes equations. Also the dissipation rates of the advective flows are calculated. Exact relations between the respective currents, the dissipation rates and the external control parameters are presented.

1 Introduction

Although Rayleigh-Bénard, Taylor-Couette, and pipe flow have rather different driving mechanisms – externally controlled temperature differences, concentric cylinders rotating at different speeds, or an externally controlled pressure drop or mean flow – the also different physical quantities of interest – the heat flow, the torque, or the skin friction – have a remarkable feature in common. Instead of the expected scaling behavior of these quantities of interest in terms of the driving forces, the corresponding scaling exponents turn out to be valid only locally. In fact they depend on the driving force and change for increasing forcing. This striking correspondence prompts the idea, that the corresponding mechanism might be very similar. In this paper we report on this. Section 2 deals with the main ideas to calculate the Nusselt number in Rayleigh-Bénard flow. In the next Sect. 3 we show that the corresponding ideas can be developed also for flow in the gap between independently rotating cylinders. In the last step, Sect. 4, we derive the very corresponding relations

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also for pipe flow. We close (Sect. 5) by indicating the answer to the question of a forcing dependent scaling: It is the varying weight of the boundary layer relative to the bulk contributions that changes the relevant scaling exponent with increasing external forcing!

2 Rayleigh-Bénard Convetion

Thermal heat convection in fluid layers heated from below, known as Rayleigh-Bénard (RB) flow, has been intensively studied in the recent years, see [1, 2, 3] and others. Considerable, impressive experimental progress has been obtained. The standards of precision today are very high, with a scatter $\leq 0.1\%$ and a temperature stabilization of $\approx 0.001^o$ C or better.

The quantities of main interest are the heat flux Q and the amplitude U of the convection field. The corresponding dimensionless quantities are the Nusselt number $Nu = Q/\Lambda\Delta L^{-1} = J^\theta/\kappa\Delta L^{-1}$ and the Reynolds number $Re = U/\nu L^{-1}$. Here Λ and κ are the heat and thermal conductivity, Δ is the temperature surplus of the hotter bottom plate relative to the colder top plate of the container, L is the height of the fluid layer (or container) and ν the fluid's kinematic viscosity. The flow is assumed to be incompressible, the density of the liquid ρ_{fluid} is constant, $\kappa = \Lambda/c_p\rho_{fluid}$, with c_p the isobaric heat capacity per mass.

The external flow parameters are the Rayleigh number $Ra = g\alpha_p\Delta L^3/(\nu\kappa)$ and the material property $Pr = \nu/\kappa$ is the Prandtl number. Ra measures the buoyancy due to the gravitational acceleration g, with α_p the isobaric thermal expansion coefficient. The geometry of the container is parametrized by the aspect ratio $\Gamma = D/L$, lateral extension D in multiples of the height L. In experiment Γ is in the range of about $1/2, \ldots, 20$, in theory we consider the ideal limit $\Gamma \to \infty$.

The physical understanding of Nu, Re as functions of the varying control parameters Ra and Pr has been developed in a series of papers [4, 5, 6, 7]. It basically consists of two arguments. First, there are close and exact relations between the thermal current J^θ and the dissipation rates ε_u and ε_θ of the velocity and the temperature fields $\mathbf{u}$ and θ. Second, the dissipation rates can be modelled in terms of their dimensional amplitudes.

The exact relations, which can be derived from the Oberbeck-Boussinesq equations of motion, are

$$\varepsilon_u/(\nu^3 L^{-4}) \equiv \tilde{\varepsilon}_u = Pr^{-2} Ra(Nu - 1) \tag{1}$$

$$\text{and} \qquad \varepsilon_\theta/(\kappa\Delta^2 L^{-2}) \equiv \tilde{\varepsilon}_\theta = Nu \ . \tag{2}$$

They are valid in the case $\Gamma \to \infty$ or with laterally periodic boundary conditions.

In order to derive them, a well defined expression for the thermal current J^θ is needed,

$$J^\theta = -\kappa \partial_z \langle \theta \rangle_{A,t} + \langle u_z \theta \rangle_{A,t} = Nu \cdot \kappa \Delta L^{-1} \ . \tag{3}$$

This definition is a unique consequence of the θ-equation, if averaged over planes A parallel to the bottom and top plates at any height z, with $0 < z < L$ from bottom to top. Also the time average is taken (or stationarity assumed). The defining property to identify J^θ is that it is **independent** of z. This expresses the conservation of heat, the current must be the same at each height z.

While J^θ consists of area averages, the dissipation rates are volume averages, $\varepsilon_u = \nu \langle u_{i|j}^2 \rangle_{V,t}$ and $\varepsilon_\theta = \kappa \langle \mathrm{grad}^2 \theta \rangle_{V,t}$. The average convective dissipation rate $\tilde{\varepsilon}_u$ is decomposed now into its isotropic bulk contribution and the anisotropic boundary layer part. The thermal dissipation, being equal to the **area** averaged current, is decomposed according to the dimensional amplitudes of the two J^θ-terms.

$$\tilde{\varepsilon}_u = \ \sim U^2/(LU^{-1}) \ + \ \sim \nu (U^2/\lambda_u^2) \cdot (\lambda_u/L) \ , \tag{4}$$

$$Nu = \ \sim L/\lambda_\theta \ + \ \sim U\Delta \ . \tag{5}$$

The Ansatz for the width λ_u of the kinematic boundary layer is the Prandtl scaling

$$\lambda_u / L = a/\sqrt{Re} \ , \tag{6}$$

the one for the thermal boundary layer width λ_θ is either obtained from (3) taken at the plates or from the Pohlhausen theory

$$\lambda_\theta / L = 1/(2Nu) \approx \ \sim \sqrt{Re \ Pr} \ . \tag{7}$$

Depending on the size of Pr, the velocity U which is relevant in the expression for Nu either is the full amplitude U itself, namely if $\lambda_u \ll \lambda_\theta$, or it is only the fraction $U\lambda_\theta/\lambda_u$ at the edge of the thermal boundary layer, if $\lambda_u \gg \lambda_\theta$. These two limiting cases, known as "lower" and "upper" ranges, are connected by a switch function $f(\lambda_\theta/\lambda_u) = \left(\frac{(\lambda_\theta/\lambda_u)^n}{1+(\lambda_\theta/\lambda_u)^n} \right)^{1/n}$; that $n = 4$ is chosen, is not important.

These ideas lead to the following set of two equations,

$$Pr^{-2} Ra (Nu - 1) = c_1 Re^{5/2} + c_2 Re^3 \ , \tag{8}$$

$$Nu = c_3 \sqrt{Re Pr f\left(\frac{\sqrt{Re}}{Nu} \right)} + c_4 Re Pr f \left(\frac{\sqrt{Re}}{Nu} \right) \ . \tag{9}$$

Its solution leads to the quantities of interest, $Nu(Ra, Pr)$ and $Re(Ra, Pr)$ (after fitting the constants c_i to one available data set), and allows to calculate the heat current as well as the convection amplitude for any value of the external control parameters Ra and Pr. The agreement with an increasing amount of data (experimental as well as numerical) is up to now very pleasing; details are given in [4, 5, 6, 7].

The most remarkable observation is that there are **no global** scaling exponents

$$Nu \sim Ra^{\beta} Pr^{\beta'} , \tag{10}$$

but that these β's are local quantities only, changing as functions of Ra and Pr in a ball park (for β) between about $\frac{1}{5}$, $\frac{1}{4}$ and $\frac{1}{3}$ or even $\frac{1}{2}$. More precisely, Nu and Re are varying superpositions of boundary layer and bulk contributions to the dissipation rate and the current.

3 Taylor-Couette Flow

We now show that there is a very close correspondence between thermal RB flow and Taylor-Couette (TC) flow in the gap between concentrically rotating cylinders and (in the next section) also with pipe flow. The details will be published elsewhere, cf. [8, 9, 10]. Here only the main ideas of this interesting correspondence are discussed. We even restrict ourselves to the exact relations; the modelling can be (and has been) also worked out, of course, see the detailed publications.

In TC flow two cylinders with radii r_1, r_2 rotate independently with angular velocities ω_1, ω_2. The gap width between the cylinders is $d = r_2 - r_1$, the aspect ratio Γ is ℓ_{cyl}/d and in practice of the order of 10. The nondimensional control parameters are $Re_1 = r_1\omega_1 d/\nu$ and Re_2 correspondingly. Another important geometrical parameter is the radius ratio $\eta = r_1/r_2$, varying between 0 (large gap) and 1 (small gap). With decreasing η the effects of curvature become more relevant, while for $\eta \to 1$ the TC system approaches the flat narrow channel. Another interesting degree of freedom is counter rotation, $\omega_2 < 0$ and $\omega_1 > 0$. Then a neutral surface develops at some intermediate radius r_n, defined by $u_\varphi(r_n) = 0$. The results presented in the following hold in all these various cases. They are exact consequences of the Navier-Stokes equations for incompressible fluid flow.

$$\partial_t \mathbf{u} = -(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla} p + \nu \Delta \mathbf{u} \ , \qquad \mathrm{div}\mathbf{u} = 0 \ . \tag{11}$$

These have to be used, of course, in cylindrical coordinates, as found e.g. in [11].

Consider at first laminar TC flow, which apparently corresponds to the RB system at rest. As there is a profile in the RB case, namely the temperature profile, also in laminar TC flow one has a profile, this time of angular velocity, $\omega_1 < \omega(r) < \omega_2$. It is $\omega(r) = A + B/r^2$. The corresponding velocity is purely azimuthal, $u_\varphi = r\omega(r)$ and the angular momentum field is $L = ru_\varphi = r^2\omega(r)$. Due to the r-gradient of the profile there is a transverse molecular ω-current $\propto -\nu\partial_r\omega$, similar to $\propto -\kappa\partial_z\theta$ in RB. Note that it is not the state of resting cylinder which corresponds to the RB state with $\mathbf{u} = 0$; this latter still has a θ-profile. The resting TC system has no profile anymore.

If the rotation rates are sufficiently large, convection transverse to the cylinder surfaces develops, $\mathbf{u}_\perp$, having velocity components u_r and u_z. These of course couple to u_φ via (11). This transverse convection enhances the radial ω-transport.

The ω-transport leads to forces $F_{1,2}$ and thus to torques $T_1 \equiv T$ and $T_2 = -T$ on the cylinders, which can be measured [12, 13, 14, 15, 16] and more. One expects $T \propto Re_1^\alpha$, similar as in RB we wrote $J^\theta \propto Ra^\beta$. It again turns out that also this exponent α depends on Re_1, i.e. is only local but not universal.

How can one define the ω-current? In perfect analogy to the argument in RB convection we perform area and time averages the relevant equation of motion for TC flow. In RB this is the θ-equation because the driving mechanism is the θ-profile. Since in TC the driving profile is in ω or $u_\varphi = r\omega$, we start from the u_φ-equation, i.e. the φ-component of the Navier-Stokes (11). The relevant area here are cylinders $A(r) = 2\pi r \cdot \ell_{cyl}$ concentric to the TC cylinders, for any $r_1 \leq r \leq r_2$. After some algebra one finds that the following quantity has to be the same for all r,

$$r^3 \left(-\nu \partial_r \langle \omega \rangle_{A(r),t} + \langle u_r \omega \rangle_{A(r),t} \right) \equiv J^\omega \ . \tag{12}$$

Because of its r-independence we consider this J^ω as the angular momentum current and as the very analog of J^θ. Of course, an proportionality factor is still free. To prove the physical meaning of just this expression J^ω, we consider its relation to the torque. It is $T = r_1 F_1 = r_1 \cdot \sigma_{r\varphi}(r_1) \cdot 2\pi r_1 \ell_{cyl}$, with the stress tensor in cylindrical coordinates $\sigma_{r\varphi}(r_1) = -\rho_{fluid} \nu (\partial_r u_\varphi - \frac{u_\varphi}{r})_{r_1}$. Thus $T = 2\pi \ell_{cyl} \rho_{fluid} J^\omega$.

J^ω, as J^θ in RB, consists of a molecular contribution to the transport, $\propto -\nu \partial_r \langle \omega \rangle_{A(r),t}$ and a convective part $\propto \langle u_r \omega \rangle_{A(r),t}$, proportional to the transverse amplitude u_r. The latter term is missing in the laminar case. Thus the convective enhancement of the angular momentum transport is

$$N^\omega = \frac{J^\omega}{J^\omega_{lam}} = \frac{J^\omega}{2\nu B} \ , \tag{13}$$

which we call the ω-Nusselt number. This leads to

$$T \propto J^\omega_{lam} \cdot N^\omega \propto Re_1 \cdot N^\omega \ . \tag{14}$$

There is a significant influence of the curvature on J^ω, namely the explicit factor of r^3 in (12). It is only *with* this r^3 that J^ω is r-constant. It means that the molecular gradient contribution as well as the convective correlation function are much weaker near the outer than near the inner cylinder. In particular, the differences between the ω-slopes near the inner and outer cylinder increase dramatically with η,

$$\partial_{r_2} \langle \omega \rangle_2 = \eta^3 \partial_{r_1} \langle \omega \rangle_1 \ . \tag{15}$$

For decreasing radius ratio η or increasing gap width $1-\eta$ the boundary layer at the inner cylinder is (much) thinner than at the outer one.

Having identified the corrrespondence of the relevant currents, we now consider the energy dissipation rate in TC flow. It reads $\varepsilon = \frac{\nu}{2}\langle(\partial_i u_j + \partial_j u_i)^2\rangle_V$ and is calculated from the Navier-Stokes (11) by multiplying with $\mathbf{u}$ and volume averaging. After some algebra (see [9]) one finds

$$\varepsilon = \frac{2}{r_2^2 - r_1^2}(\omega_1 - \omega_2) J^\omega \ . \tag{16}$$

This total dissipation rate clearly does **not** correspond to ε_u in RB flow, cf. (1); namely, we have $\varepsilon_u = 0$ in the case of no convective transport, but ε in (16), instead, reduces to $\varepsilon_{lam} \propto J^\omega_{lam} \neq 0$ in case of missing convective transport. Therefore, the convective dissipation rate in TC flow corresponding to ε_u in RB is

$$\tilde{\varepsilon}_w = \frac{\varepsilon - \varepsilon_{lam}}{\nu^3 d^{-4}} = \frac{\varepsilon_{lam}}{\nu^3 d^{-4}} \left(N^\omega - 1\right) \ . \tag{17}$$

One can introduce a purely geometrical analog of the material property $Pr = \nu/\kappa$ by defining a quasi-Prandtl number

$$\sigma(\eta) = \left(\frac{r_a}{r_g}\right)^4 = \left(\frac{\frac{1}{2}(1+\eta)}{\sqrt{\eta}}\right)^4 \ , \tag{18}$$

$r_{a,g}$ denoting the arithmetic and geometric mean radii of $r_{1,2}$. A quasi-thermal conductivity then is $\kappa = \nu/\sigma$. One can also introduce a Taylor number

$$Ta = \sigma \frac{d^2 r_a^2 (\omega_1 - \omega_2)^2}{\nu^2} = \frac{d^2 r_a^2 (\omega_1 - \omega_2)^2}{\nu\kappa} \ . \tag{19}$$

In terms of these control parameters the convective dissipation rate (17) becomes

$$\tilde{\varepsilon}_w = \sigma^{-2} Ta (N^\omega - 1) \ , \tag{20}$$

the perfect analogon of $\tilde{\varepsilon}_u$ from (1). These two exact relations, (20) for the convective dissipation rate, and (12),(13) for the ω-current between the cylinders, establish the complete correspondence between RB and TC flows. It is, incidentally, precisely with the prefactors included in the definition (19) of the Taylor number, that this analogy is perfect. This suggests which of the various different, though dimensionally all correct Taylor numbers should be preferred, namely the definition (19). The current or ω-Nusselt number (13) can be written as

$$N^\omega = \left(\frac{r}{r_a}\right)^3 \frac{\langle u_r \omega\rangle_{A(r),t} - \nu \partial_r \langle\omega\rangle_{A(r),t}}{\kappa(\omega_1 - \omega_2)/d} \ . \tag{21}$$

The correspondence to (3) is striking. In a simplified version we had shown in [17] that the torque data can be rather well described along this RB-TC-correspondence. For more details in the present frame of complete correspondence, see [9].

4 Pipe Flow

We finally analyse the correspondence not only of RB with TC but now also with pipe flow, being rather brief only and giving reference to [8, 10] for more details.

The counter part to the RB state of rest but with a temperature θ profile, corresponding to laminar TC flow with an ω proffile, in the case of pipe flow is the laminar Hagen-Poiseuille flow. It also has a profile, namely of the axial component $u_z(r)$, leading to a molecular current $\propto -\nu\partial_r u_z(r)$, which in turn gives rise to a skin friction at the pipe's wall that has to be overcome by the applied external pressure drop $\Delta p/\ell_{pipe}$. To define the proper current in pipe flow we perform area and time averages of the u_z-component of the Navier-Stokes (11). Here the appropriate surfaces are virtual circular surfaces concentric to the pipe, of any radius $r \leq a$, the pipe's radius: $A^{(=)} = 2\pi r \cdot \ell_{pipe} = A(r)^{(=)}$. After some arguments one arrives at the following definition for the u_z-current, which is independent of r, constant for all radii smaller than a,

$$J^{u_z} = \frac{2}{r}\left(\langle u_r u_z\rangle_{A^{(=)},t} - \nu\partial_r\langle u_z\rangle_{A^{(=)},t}\right) \ . \tag{22}$$

It is only including the factor r^{-1} that this expression is constant. The u_z-Nusselt number is $N^u = J^{u_z}/J^{u_z}_{lam}$, where $J^{u_z}_{lam} = 8\nu a^{-2}U$. Here U is the cross-sectional mean flow velocity and $a = d/2$ the radius and diameter of the pipe.

Convective enhancement of the u_z-current, which flows from the interior to the pipe wall is provided by the radial component u_r (and, of course, u_φ), which develop in the turbulent state beyond laminarity. This gives rise to a convective dissipation rate. The full dissipation rate in the pipe is $\varepsilon = UJ^{u_z}$. The contribution due to the transverse u_r-convection is obtained by subtracting the laminar contribution. This leads to the following definition of the convective dissipation rate

$$\tilde{\varepsilon}_w = \frac{\varepsilon - \varepsilon_{lam}}{\nu^3 d^{-4}} \propto Re^2(N^u - 1) \ . \tag{23}$$

This corresponds perfectly to (1) in RB and (17),(20) in TC. We would like to draw the readers' attention to the fact, that there is one possibly severe difference between pipe and the other two flows. While in RB and TC the driving profiles are externally controlled as the bottom and top plates temperatures or as the cylinders rotation frequencies, in pipe flow the profile underlies dynamical fluctuations. The center velocity which determines the u_z-profile is a **dynamical** response of the flow already, either to the externally applied pressure drop $\Delta p/\ell_{pipe}$ or to the controlled *mean* flow velocity U.

5 Summarizing Conclusions

Having calculated Nu, N^{ω}, or N^{u}, all along the same lines, one finds the local exponents of the quantities of physical interest. In RB convection this is the Nusselt number exponent β itself. In TC flow between independently rotating cylinders it is the torque exponent α from $T \propto Re_1^{\alpha}$; one then has with (14) $T \propto Re_1 N^{\omega} \propto Re_1^{1+\beta}$, i.e. $\alpha(Re_1) = 1 + \beta(Re_1)$. In pipe flow the skin friction coefficient is of interest, $c_f = (16/Re)N^u \propto Re^{-1+\beta}$, so that the relation $\alpha(Re) = -1 + \beta(Re)$ holds. This establishes the close correspondence between RB, TC, and pipe flow. In all three cases exact relations connecting the currents and the convective dissipation rates hold.They are consequences of the Navier-Stokes equations and are valid in any situation. The further steps, namely the modelling of both the currents N and the dissipation rates $\tilde{\varepsilon}_w$ in terms of the dimensional amplitudes, need more details about the respective flows and will be discussed in [8, 9, 10].

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