4

Theory of the Static Behavior of Piezoelectric Beam Bending Actuators

In the previous chapter, the energy density of the elastic deformation and the electrical energy density of a solid body have been described and derived. They provide a basis for the description of the deformation and stress state of piezoelectric materials within the scope of thermodynamics. This approach directly results in the piezoelectric constitutive equations being essential for the description of the static behavior of piezoelectric multilayer beam bending actuators. With respect to the constitutive equations, the consideration of the crystal symmetry of PZT provides a basis for the description of the static behavior of n-layered beam benders. The extensive state variables $(\mathbf{T}, \mathbf{E})$ are the starting point for the static behavior modeling. In combination with the linear piezoelectric constitutive equations, the total stored energy of the bending actuator can be formulated.
The *theorem of minimum total potential energy* provides the combination of the extensive parameters such as mechanical moment M, force F, pressure load p and driving voltage U with the intensive parameters angular deflection α, deflection ξ, volume displacement V and the electric charge Q as functions of any point x over the entire length of the bending actuator.

4.1 Sectional Quantities of a Bending Beam

A beam can be described by its main inertia axis (also called neutral axis or beam axis, respectively) and the cross sections assigned to each point of the main inertia axis [92]. Thereby, the beam axis connects the centroids of the beam's cross sections aligned perpendicularly to the beam axis. Furthermore, a flexural resistant beam is assumed, i.e. it opposes a resistance concerning a bending movement.
If external loads affect a beam, internal mechanical loads will arise. In order to determine the arising internal loads, a beam is intersected in two seperated bodies just at this position, internal forces and moments are of interest. Each separated body must be in equilibrium concerning its remaining outside loads

and affecting sectional loads, thus the equilibrium of the total system is assured. In order to define the *positive* sectional loads a right-handed Cartesian coordinate system is carried along the beam's cross-sectional area (see Fig. 4.1).

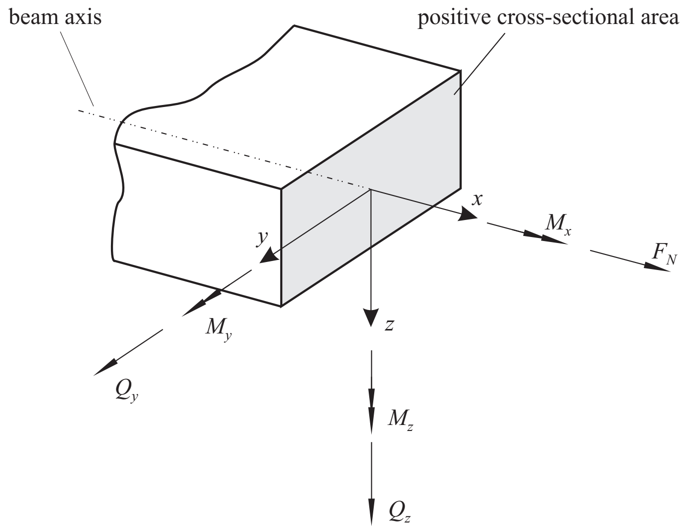

Fig. 4.1. Sectional loads within a beam

At the intersection position, the *positive* and *negative* cross-sectional areas are distinguished. A cross-sectional area is called positive, if the x-axis points out of the sectional plane. If the x-axis points into the sectional plane, a cross-sectional area is called negative. For the further considerations, planar loads of planar beam cantilevers are assumed, i.e. the principal axis of the planar loads is located in the x-z plane. Thus, in figure 4.1 only the sectional quantities like axial force F_N in x-direction, transverse force F_Q in z-direction and bending moment M_b around the y-axis remain (see Fig. 4.2).

Sectional loads resulting from distributed forces in the cross section (mechanical stresses) are referred to the centroid, thus they affect the beam axis.

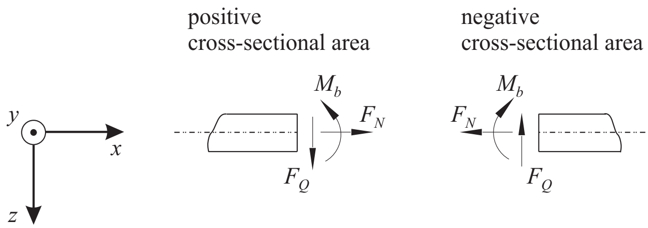

Fig. 4.2. Sectional loads and associated cross-sectional areas

4.2 Bernoulli Hypothesis of Beam Bending Theory

In the following, the kinematics of the deformation of the planar beam bender is analyzed in more detail. The *Bernoulli hypothesis of beam bending theory* provides a basis for the kinematics of deformation [93]:

All points of a planar plane perpendicular to the main inertia axis are still located in a planar plane after a pure bending deformation. Even so, the planar plane is aligned perpendicularly to the deformed main inertia axis.

In order to describe the kinematics of the planar beam, figure 4.3 is considered:

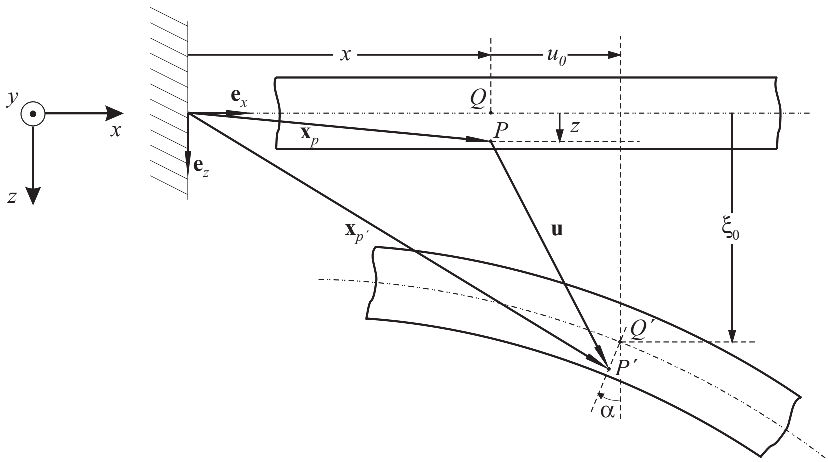

Fig. 4.3. Kinematics concerning the deformation of a beam bender

Figure 4.3 shows the location of the point P in the non-deformed state and the location of the same point P' in the deformed state. The coordinate system is chosen in such a way, that the unit vector $\mathbf{e}_x$ coincides with the main inertia axis. The Cartesian coordinate system is defined in accordance with the definition of the sectional quantities affecting the positive cross-sectional area of a beam bender. u_0 and ξ are the displacements in x- and z-direction, respectively. $\mathbf{x}_P$ and $\mathbf{x}_{P'}$ are the position vectors of the points P and P'.

The total displacement $\mathbf{u}$ from P to P' results in

$$\mathbf{u} = \mathbf{x}_{P'} - \mathbf{x}_P, \tag{4.1}$$

with

$$\mathbf{x}_P = x\mathbf{e}_x + z\mathbf{e}_z \tag{4.2}$$

$$\mathbf{x}_{P'} = (x + u_0 - z\sin\alpha)\,\mathbf{e}_x + (\xi_0 + z\cos\alpha)\,\mathbf{e}_z \tag{4.3}$$

For small angles α, following simplifications can be made applying Taylor's theorem:

$$\sin\alpha \approx \alpha \qquad \text{und} \qquad \cos\alpha \approx 1 \tag{4.4}$$

Thus, in combination with (4.2) - (4.4), the displacement $\mathbf{u}$ in equation (4.1) yields:

$$\mathbf{u} = \underbrace{(u_0 - z\alpha)}_{u}\mathbf{e}_x + \underbrace{\ \xi\ }_{w}\mathbf{e}_z \tag{4.5}$$

The components of the displacement vector in the x-z plane (see Fig. 4.3) and the displacement w in z-direction result directly from (4.5):

$$u = u_0 - z\alpha \tag{4.6}$$

$$v = 0 \tag{4.7}$$

$$w = \xi \tag{4.8}$$

Considering the definitions for the mechanical strains (3.4) and (3.7), the individual strain components in the x-z plane can be calculated according to the equations (4.6) - (4.8):

$$S_{xx} = \frac{\partial u_0}{\partial x} - z\frac{\partial \alpha}{\partial x} \tag{4.9}$$

$$S_{yy} = \frac{\partial v}{\partial z} = 0 \tag{4.10}$$

$$2S_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial \xi}{\partial x} - \alpha \tag{4.11}$$

However, within the scope of Bernoulli's hypothesis of beam bending theory no transverse strains can develop. Thus, it can be written:

$$S_{xz} = 0$$

Finally, from equation (4.11) it follows:

$$\alpha = \frac{\partial \xi}{\partial x} \tag{4.12}$$

The strain S_{xx} of a planar beam results from insertion of equation (4.12) in (4.9) [94]:

$$S_{xx}(x, z) = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 \xi}{\partial x^2} \tag{4.13}$$

For further considerations, the following quantities are defined:

$$\varepsilon^0 = \tfrac{\partial u_0}{\partial x} \quad \text{(strain of the neutral axis)}$$

$$\kappa^0 = \tfrac{\partial^2 \xi}{\partial x^2} \quad \text{(bend of the neutral axis)}$$

Finally, the strain S_{xx} of a planar beam can be formulated in compressed tensor notation (see Table 3.3):

$$S_1(x, z) = \varepsilon^0 - z\kappa^0 \tag{4.14}$$

It should be noted, that a strain of the neutral axis only develops, if an axial force F_N affects the positive cross-sectional area. In further considerations, no external affecting axial forces are assumed, thus the neutral axis is not expanded and it can generally be written $\varepsilon^0 = 0$. Equation (4.14) denotes a linear behavior of the strain S_1 over the entire cross section of the beam bender, whereas z defines the distance from the neutral axis [94].
The strain can only be determined if the neutral axis position is known. For a *homogeneous* and *planar* beam, this position directly results from its symmetry characteristics. By contrast, the neutral axis position of a multilayered structure has to be calculated explicitly.

4.3 Neutral Axis Position of a Multilayered Beam Bender

A piezoelectric bending actuator consisting of n layers can also be described as a beam. However, the difference consists in the different geometrical dimensions of the individual layers and their material properties generally resulting in a change of the neutral axis position $\overline{z}$. Moreover, it is suitable to define

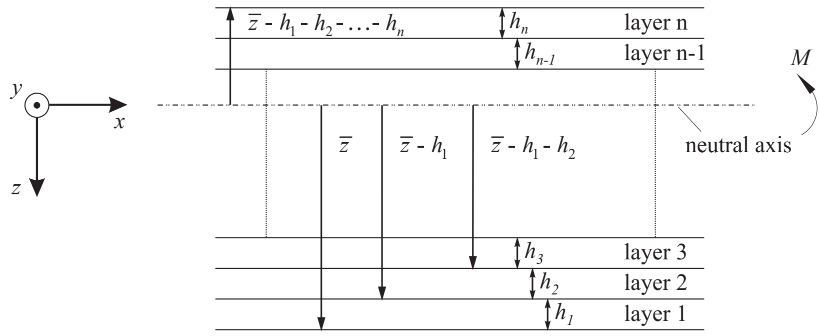

Fig. 4.4. Derivation of the neutral axis position - schematics of a multilayered beam segment

the neutral axis position as the distance to the lower edge of the multilayered beam bender (see Fig. 4.4) [95].

Following three conditions are necessary for the derivation of the neutral axis position:

1. In the case of an affecting external bending moment M, the sum of all forces in x-direction is zero [93].

$$\sum_{i=1}^{n} F_{1,i} = 0 \tag{4.15}$$

2. Assuming equation (4.14) and taking the neutral axis position into account, the strain results in:

$$S_1\left(x,z\right) = -z\kappa^0 \tag{4.16}$$

3. For elastic materials, Hooke's law is valid, thus the mechanical stresses $T_{1,i}$ arising in x-direction can be calculated according to equation (4.16):

$$T_{1,i}\left(x,z\right) = -\frac{z\kappa^0}{s_{11,i}} \tag{4.17}$$

   Compared to the mechanical strain, equation (4.17) indicates a linear behavior of the mechanical stress only in sections.

The mechanical stress of the ith layer results in:

$$T_{1,i} = \frac{dF_{1,i}}{dA_i} \tag{4.18}$$

In equation (4.18), dA_i denotes a surface segment of the cross-sectional area of the ith layer in the y-z plane.

Rearranging (4.18) with respect to $dF_{1,i}$ and following integration with respect to the width w_i and the layer thickness of the ith layer yields in combination with (4.17) the axial force in the ith layer:

$$F_{1,i} = -\int\limits_{h_{i,u}}^{h_{i,o}} \frac{w_i \kappa^0}{s_{11,i}} z dz \tag{4.19}$$

Considering equation (4.15) yields in combination with (4.19):

$$\sum_{i=1}^{n} \frac{w_i}{s_{11,i}} \int\limits_{h_{i,u}}^{h_{i,o}} z dz = 0 \tag{4.20}$$

The variables $h_{i,u}$ and $h_{i,o}$ represent the lower and upper integration limits of the ith layer. They can be defined according to figure 4.4.

1. Lower integration limit $h_{i,u}$:

$$h_{i,u} = \overline{z} - \sum_{j=1}^{i} h_j \tag{4.21}$$

2. Upper integration limit $h_{i,o}$:

$$h_{i,o} = \overline{z} - \sum_{j=1}^{i-1} h_j \tag{4.22}$$

The evaluation of the integral sum (4.20) results with the definitions of the integration limits (4.21) and (4.22) in

$$\sum_{i=1}^{n} \frac{w_i}{s_{11,i}} \left[h_{i,o}^2 - h_{i,u}^2\right] = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \frac{w_i}{s_{11,i}} \left[\left(\overline{z} - \sum_{j=1}^{i-1} h_j\right)^2 - \left(\overline{z} - \sum_{j=1}^{i} h_j\right)^2\right] = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \frac{w_i}{s_{11,i}} \left[2\overline{z}\left(\sum_{j=1}^{i} h_j - \sum_{j=1}^{i-1} h_j\right) + \left(\sum_{j=1}^{i} h_j - h_i\right)^2 - \left(\sum_{j=1}^{i} h_j\right)^2\right] = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \frac{w_i}{s_{11,i}} \left[2\overline{z} h_i - 2h_i \sum_{j=1}^{i} h_j + h_i^2\right] = 0,$$

and the neutral axis position can be calculated according to:

$$\overline{z} = -\frac{\sum_{i=1}^{n} \frac{w_i}{s_{11,i}} h_i^2 - 2\sum_{i=1}^{n} \frac{w_i}{s_{11,i}} h_i \sum_{j=1}^{i} h_j}{2\sum_{i=1}^{n} \frac{w_i}{s_{11,i}} h_i} \tag{4.23}$$

In the further considerations, equation (4.23) will play an important role. In order to ensure the transition to an energetic consideration of piezoelectric multilayer beam bending actuators, it necessitates the representation of the neutral axis bend κ^0 dependent on external and internal moments. Thus, it is possible to define the mechanical stresses in each individual layer by externally and internally affecting moments and to turn to the energetic description of a piezoelectric multilayered system.

4.4 Forces and Moments within a Multilayer System

In the following considerations, each individual layer of the bending actuator is assumed to consist either of a flexible or a piezoelectric material. Thus, in the further formalism it is necessary to consider the electromechanical interconnections by means of the piezoelectric constitutive equations.

At first, the following conditions are defined:

1. The vector of the electric field $\mathbf{E}$ develops only in z-direction.

$$E_1 = E_2 = 0 \tag{4.24}$$

2. Only one mechanical stress component develops along the x-direction (transverse piezoelectric effect; only the bending along the neutral axis is of interest).

$$T_2 = \ldots = T_6 = 0 \tag{4.25}$$

3. According to the pair of variates $(\mathbf{T}, \mathbf{E})$, the constitutive equations (3.57) and (3.58) are used.

$$D_3 = \varepsilon_{33}^T E_3 + d_{31} T_1 \tag{4.26}$$

$$S_1 = d_{31} E_3 + s_{11}^E T_1 \tag{4.27}$$

The mechanical stress within the ith layer can be calculated with respect to (4.14) and (4.27). Taking $\varepsilon^0 = 0$ into account, it can be written:

$$T_{1,i} = \frac{1}{s_{11,i}^E} \left[-z\kappa^0 - d_{31,i} E_{3,i} \right] \tag{4.28}$$

The resulting bending moment M is calculated according to [93]:

$$M = \sum_{i=1}^{n} w_i \int_{h_{i,u}}^{h_{i,o}} T_{1,i} z dz \tag{4.29}$$

Insertion of (4.28) into (4.29) yields:

$$M = -\kappa^0 \underbrace{\frac{1}{3}\sum_{i=1}^{n} \frac{w_i}{s_{11,i}^E}\left[h_{i,o}^3 - h_{i,u}^3\right]}_{\equiv C} - \underbrace{\frac{1}{2}\sum_{i=1}^{n} \frac{w_i}{s_{11,i}^E} d_{31,i} E_{3,i}\left[h_{i,o}^2 - h_{i,u}^2\right]}_{\equiv M_{Piezo}} \tag{4.30}$$

With the definition of the kinematic quantities C and M_{Piezo}, equation (4.30) results in:

$$M = -C\kappa^0 - M_{Piezo} \tag{4.31}$$

After some algebraic calculations and taking the definition of the integration limits (4.21) and (4.22) into account, the kinematic quantities can be formulated as follows :

$$C = \frac{1}{3}\sum_{i=1}^{n} \frac{w_i}{s_{11,i}^E}\left[3h_i\left(\overline{z} - \sum_{j=1}^{i} h_j\right)\left(\overline{z} - \sum_{j=1}^{i-1} h_j\right) + h_i^3\right] \tag{4.32}$$

$$M_{Piezo} = \frac{1}{2}\sum_{i=1}^{n} \frac{w_i}{s_{11,i}^E} d_{31,i} E_{3,i}\left[2\overline{z}h_i - 2h_i\sum_{j=1}^{i} h_j + h_i^2\right] \tag{4.33}$$

The quantities C and M_{Piezo} represent the *total flexural rigidity* and the *piezoelectric bending moment* of the multilayered system, respectively. The bend κ^0 results with respect to equation (4.31) in:

$$\kappa^0 = -\frac{M + M_{Piezo}}{C} \tag{4.34}$$

Equation (4.34) shows, that the kinematics of the beam bender is described by the external moment M, the piezoelectric bending moment M_{Piezo} and the total flexural rigidity C (4.32).

4.5 Total Stored Energy within a Multilayer System

Since the mechanical stress of each individual layer can now be described with respect to externally and internally affecting moments, it is possible to turn to the energetic consideration of a multilayered system.

The total energy density w_{tot} can be determined by means of the sum of the energy density of the elastic deformation w_m (3.11) and the energy density of the electrostatic field w_e (3.22):

$$w_{tot} = \frac{1}{2}E_i D_i + \frac{1}{2}T_{ij}S_{ij} \tag{4.35}$$

Considering the conditions (4.24) and (4.25), equation (4.35) results in

$$w_{tot} = \frac{1}{2}E_3 D_3 + \frac{1}{2}T_1 S_1. \tag{4.36}$$

Insertion of the piezoelectric constitutive equations (4.26) and (4.27) into (4.36) yields

$$w_{tot} = \frac{1}{2}\varepsilon_{33}^T E_3^2 + d_{31}E_3 T_1 + \frac{1}{2}s_{11}^E T_1^2. \tag{4.37}$$

4.5.1 Total Energy in a Single Layer

The energy density of the ith layer of the multilayer beam bending actuator results with respect to equation (4.37) in:

$$w_{tot,i} = \frac{1}{2}\varepsilon_{33,i}^T E_{3,i}^2 + d_{31,i}E_{3,i}T_{1,i} + \frac{1}{2}s_{11,i}^E T_{1,i}^2 \tag{4.38}$$

The volume integration of a single layer yields the total stored energy $W_{tot,i}$:

$$W_{tot,i} = \int_{h_{i,u}}^{h_{i,o}} \int_0^{w_i} \int_0^l w_{tot,i}\,dxdydz \tag{4.39}$$

Applying equation (4.38) provides $W_{tot,i}$:

$$W_{tot,i} = \int_{h_{i,u}}^{h_{i,o}} \int_0^{w_i} \int_0^l \left[\frac{1}{2}\varepsilon_{33,i}^T E_{3,i}^2 + d_{31,i}E_{3,i}T_{1,i} + \frac{1}{2}s_{11,i}^E T_{1,i}^2\right] dxdydz \tag{4.40}$$

After some algebraic transformations of equation (4.40), it can be written in combination with (4.28):

$$\begin{aligned} W_{tot,i} = {} & \frac{1}{2}\int_0^l \left[\varepsilon_{33,i}^T E_{3,i}^2 w_i \left(h_{i,o} - h_{i,u}\right)\right] dx \\ & + \frac{1}{6}\int_0^l \left[\frac{w_i}{s_{11,i}^E}\left(h_{i,o}^3 - h_{i,u}^3\right)\left(\kappa^0\right)^2\right] dx \\ & - \frac{1}{2}\int_0^l \left[\frac{w_i}{s_{11,i}^E} d_{31,i}^2 E_{3,i}^2 \left(h_{i,o} - h_{i,u}\right)\right] dx \end{aligned} \tag{4.41}$$

Integration with respect to the beam bender's length l is not effected yet. Thus, the *theorem of minimum total potential energy* will be applied subsequently. In the following, the total stored energy of the multilayered system can be determined with respect to equation (4.41).

4.5.2 Energy in an n-layered System

The energy within the multilayered system results from summation of the individual energy amounts (4.41) of each individual layer.

$$W_{tot} = \sum_{i=1}^{n} W_{tot,i} \tag{4.42}$$

Insertion of (4.41) into (4.42) yields:

$$\begin{aligned} W_{tot} = &\frac{1}{2}\sum_{i=1}^{n}\int_0^l \left[\varepsilon_{33,i}^T E_{3,i}^2 w_i \left(h_{i,o} - h_{i,u}\right)\right] dx \\ &+\frac{1}{6}\sum_{i=1}^{n}\int_0^l \left[\frac{w_i}{s_{11,i}^E}\left(h_{i,o}^3 - h_{i,u}^3\right)\left(\kappa^0\right)^2\right] dx \\ &-\frac{1}{2}\sum_{i=1}^{n}\int_0^l \left[\frac{w_i}{s_{11,i}^E} d_{31,i}^2 E_{3,i}^2 \left(h_{i,o} - h_{i,u}\right)\right] dx \end{aligned} \tag{4.43}$$

Using the definition for the total flexural rigidity C in (4.30), (4.43) can be formulated as follows:

$$\begin{aligned} W_{tot} = &\frac{1}{2}\sum_{i=1}^{n}\int_0^l \left[\varepsilon_{33,i}^T E_{3,i}^2 w_i h_i\right] dx - \frac{1}{2}\sum_{i=1}^{n}\int_0^l \left[\frac{w_i}{s_{11,i}^E} d_{31,i}^2 E_{3,i}^2 \left(h_{i,o} - h_{i,u}\right)\right] dx \\ &+\frac{1}{2}\int_0^l \underbrace{\left[C\left(\kappa^0\right)^2\right]}_{(*)} dx \end{aligned} \tag{4.44}$$

The term $(*)$ in the last integral of equation (4.44) is determined by applying the definition (4.34) for the bend of the neutral axis. After some algebraic calculations, the total energy W_{tot} can be formulated in its final form:

$$\begin{aligned} W_{tot} = &\frac{1}{2}\sum_{i=1}^{n}\int_0^l \left[\varepsilon_{33,i}^T E_{3,i}^2 w_i h_i\right] dx - \frac{1}{2}\sum_{i=1}^{n}\int_0^l \left[\frac{w_i h_i}{s_{11,i}^E} d_{31,i}^2 E_{3,i}^2\right] dx \\ &+\int_0^l \left[\frac{M^2}{2C} + \frac{M M_{Piezo}}{C} + \frac{M_{Piezo}^2}{2C}\right]^2 dx \end{aligned} \tag{4.45}$$

Equation (4.45) provides a basis for the calculation of the coupling matrix connecting the intensive and extensive quantities. In the next section, these quantities are considered in more detail.

4.6 Canonical Conjugates and Coupling Matrix

The term of the *canonical conjugates* is originally used within the context of Hamilton's mechanics [96, 97]. Concerning the description of the static behavior of a piezoelectric bending actuator based on energetic considerations, canonical conjugates are directly connected with the concept of the total stored deformation energy W_{tot} [15].
In the further considerations the extensive quantities like bending moment M, force F, pressure load p and driving voltage U are assumed to affect a clamped-free piezoelectric beam bending actuator (see Fig. 4.5) with respect to the following boundary conditions:

1. The multilayer beam bender is subjected to an external static moment M at the tip.
2. The beam bending actuator is subjected to an external static force F perpendicularly to the tip.
3. The beam bender is subjected to a uniform pressure load p applied over the entire length l and width w of the bender.
4. The active piezoelectric layers are subjected to the same driving voltage U over their entire length l and width w (electrical parallel connection).

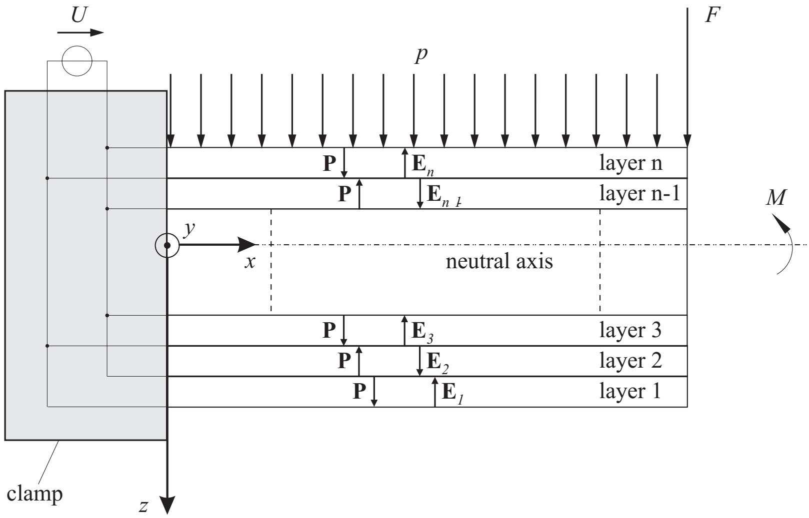

Fig. 4.5. Extensive parameters M, F, p and U

Subjecting the beam bending actuator to the external loads mentioned above results in a change of its total stored deformation energy W_{tot}. The following partial derivation of W_{tot} with respect to the individual extensive quantitiy yields the appropriate canonical conjugate. In Table 4.1, the extensive parameters and their corresponding intensive parameters (canonical conjugates) are represented.

Table 4.1. Extensive parameters and their corresponding intensive parameters

extensive parameters		intensive parameters
bending moment	$M \longleftrightarrow \alpha$	bending angle
force	$F \longleftrightarrow \xi$	deflection
pressure load	$p \longleftrightarrow V$	volume displacement
electrical voltage	$U \longleftrightarrow Q$	charge

In each case, the multiplication of two corresponding pairs of variates results in a physical quantity with the dimensions of an energy. In further considerations, the canonical conjugates designate the *intensive quantities* [15]. In figure 4.6 the corresponding intensive quantities like bending angle α, deflection ξ, volume displacement V and generated charge Q are illustrated.

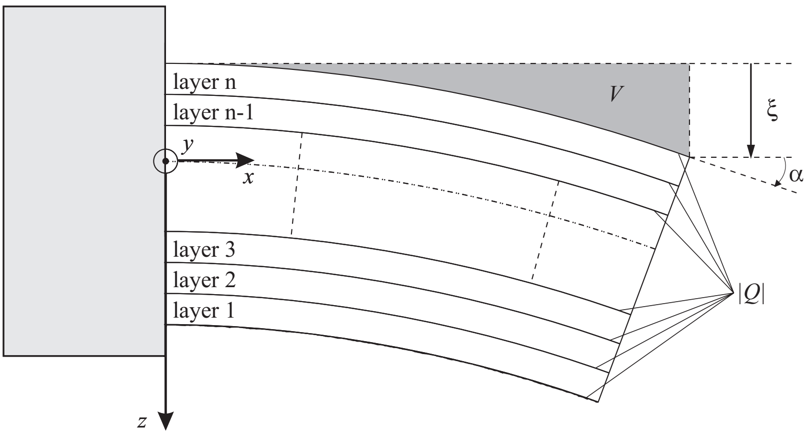

Fig. 4.6. Resulting intensive quantities α, ξ, V und Q

By means of partial derivation of the total stored deformation energy W_{tot} with respect to an extensive size, the appropriate intensive size can only be determined at the *affecting point* x_0. These facts are also summarized in CASTIGLIANO's theorem, which is only suitable for the determination of deformations of an elastic system with respect to especially chosen points [18, 98]. Thus, no information can be obtained concerning the behavior of the bending angle α, the deflection ξ, the volume displacement V and the generated charge Q at arbitrary points $x \neq x_0$.

By means of the *theorem of minimum total potential energy* in combination with the *Ritz method*, the behavior of the intensive quantities can be described for arbitrary points x over the entire length of the beam bender. Furthermore, the dependence of the individual intensive sizes on the non-corresponding extensive sizes is of interest. In order to allow for a general consideration, this dependence can be represented by means of a *coupling matrix* $\mathbf{M}$ connecting the intensive and extensive quantities:

$$\underbrace{\begin{pmatrix} \alpha(x) \\ \xi(x) \\ V(x) \\ Q(x) \end{pmatrix}}_{\text{intensive sizes}} = \underbrace{\begin{pmatrix} m_{11}(x) & m_{12}(x) & m_{13}(x) & m_{14}(x) \\ m_{21}(x) & m_{22}(x) & m_{23}(x) & m_{24}(x) \\ m_{31}(x) & m_{32}(x) & m_{33}(x) & m_{34}(x) \\ m_{41}(x) & m_{42}(x) & m_{43}(x) & m_{44}(x) \end{pmatrix}}_{\text{coupling matrix } \mathbf{M}} \underbrace{\begin{pmatrix} M \\ F \\ p \\ U \end{pmatrix}}_{\text{extensive sizes}} \tag{4.46}$$

In the two following sections, the relevant principles of mechanics are discussed in more detail, which allow for the determination of the matrix elements m_{ij} of the coupling matrix $\mathbf{M}$ formulated in equation (4.46).

4.7 Principle of Virtual Work

In order to achieve a generalized representation between the canonically conjugated pair of variates M and α, F and ξ, p and V as well as U and Q, some preliminary considerations have to be made.

Generally, the physical work W is defined as scalar product of a displacement $d\mathbf{x}$ and the corresponding force $\mathbf{F}$. In order to apply the term of physical work also to static considerations, where no displacements develop, virtual displacements $\delta\mathbf{x}$ are defined with following characteristics [99]:

1. Virtual displacements (conceivable ones, not necessarily actually developing ones) and virtual twists, respectively are infinitesimally small and can be considered like differentials!
2. Virtual displacements must be compatible with the geometrical system boundaries!

$$\Longrightarrow \text{virtual work done by a force:} \quad \delta W = \mathbf{F} \cdot \delta\mathbf{x}$$

$$\Longrightarrow \text{virtual work done by a moment:} \quad \delta W = \mathbf{M} \cdot \delta\boldsymbol{\varphi}$$

The principle of virtual work means:

A mechanical system will be in equilibrium, if the virtual work done by external forces and moments corresponding with virtual displacements and twists regarding equilibrium position disappears:

$$\delta W = \sum_j \mathbf{F}_{j,ex} \cdot \delta \mathbf{x}_j + \sum_j \mathbf{M}_{j,ex} \cdot \delta \boldsymbol{\varphi}_j = 0 \tag{4.47}$$

In the following, the principle of the virtual work will be used for the formulation of the theorem of total potential energy.

4.8 Theorem of Minimum Total Potential Energy

Conservative forces (e.g. weight, spring force) distinguish themselves from the path independence of the performed work concerning the movement from a point 1 to a point 2. The performed work only depends on the position of the points. For example, if a particle mass m performs a virtual displacement δz from point 1 to point 2 in the gravitational field (representing a conservative field), thus the performed work of the weight results in $\delta W = -mg\mathbf{e}_z \cdot \delta z \mathbf{e}_z = -mg\delta z$ (see Fig. 4.7).

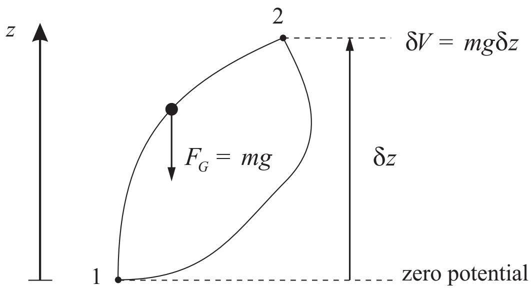

Fig. 4.7. Virtual work in a conservative field

However, the work performed by the weight is associated to an increase of the potential energy according to $\delta V = mg\delta z$. Thus, it can be written:

$$\delta W = -\delta V \tag{4.48}$$

If a system is solely loaded by potential forces, then the principle of virtual work (4.47) can also be formulated by means of the potential energy according to equation (4.48):

$$\delta V = 0 \tag{4.49}$$

In case of a clamped-free beam bending actuator, it can not be assumed any longer, that externally affecting forces are exclusively potential forces. Nevertheless, in order to apply the principle of virtual work, in the following the stored deformation energy W_{tot} (4.45) within the multilayered system is to be expressed by the potential energy, whereas the work performed by external forces is taken into consideration separately using the quantity W_a.

If only a part of the load can be described by the potential energy, thus equation (4.49) can be modified in the following way:

$$\delta \left(V - W \right) = 0 \tag{4.50}$$

With the separate representation of W_{tot} and W_a, it has to be written:

$$\delta \left(W_{tot} - W_a \right) = 0 \tag{4.51}$$

The difference $W_{tot} - W_a$ is called *total potential energy Π*. With the knowledge, that the potential energy possesses a relative minimum in a stable steady state, following theorem can be formulated [100]:

Theorem of total potential energy
A system is in a steady state, if the variation of the total potential energy vanishes.

$$\delta \left(W_{tot} - W_a \right) = \delta \Pi = 0 \tag{4.52}$$

or

$$\Pi = W_{tot} - W_a \Longrightarrow \text{minimum} \tag{4.53}$$

It should be noted, that W_a is to be formulated in such a way, as if the force has obtained its maximum value along the entire deformation path. In this case, the quantity W_a is also called *final value work*. In the following, the matrix elements m_{ij} of the coupling matrix $\mathbf{M}$ (4.46) are derived by means of the theorem of minimum total potential energy.

4.9 Derivation of the Coupling Matrix

Using equation (4.53) in combination with the total stored deformation energy W_{tot} (4.45) within the piezoelectric beam bender, the total potential energy Π results in

$$\begin{aligned} \Pi = & \frac{1}{2} \sum_{i=1}^{n} \int_0^l \left[\varepsilon_{33,i}^T E_{3,i}^2 w_i h_i \right] dx - \frac{1}{2} \sum_{i=1}^{n} \int_0^l \left[\frac{w_i h_i}{s_{11,i}^E} d_{31,i}^2 E_{3,i}^2 \right] dx \\ & + \int_0^l \left[\frac{M^2}{2C} + \frac{M M_{Piezo}}{C} + \frac{M_{Piezo}^2}{2C} \right] dx - W_a . \end{aligned} \tag{4.54}$$

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