CHAPTER 1

Constructions and extensions of measures

> I compiled these lectures not assuming from the reader any knowledge other than is found in the under-graduate programme of all departments; I can even say that not assuming anything except for acquaintance with the definition and the most elementary properties of integrals of continuous functions. But even if there is no necessity to know much before reading these lectures, it is yet necessary to have some practice of thinking in such matters.
>
> *H. Lebesgue.* Intégration et la récherche des fonctions primitives.

1.1. Measurement of length: introductory remarks

Many problems discussed in this book grew from the following question: which sets have length? This question clear at the first glance leads to two other questions: what is a "set" and what is a "number" (since one speaks of a qualitative measure of length)? We suppose throughout that some answers to these questions have been given and do not raise them further, although even the first constructions of measure theory lead to situations requiring greater certainty. We assume that the reader is familiar with the standard facts about real numbers, which are given in textbooks of calculus, and for "set theory" we take the basic assumptions of the "naive set theory" also presented in textbooks of calculus; sometimes the axiom of choice is employed. In the last section the reader will find a brief discussion of major set-theoretic problems related to measure theory. We use throughout the following set-theoretic relations and operations (in their usual sense): $A \subset B$ (the inclusion of a set A to a set B), $a \in A$ (the inclusion of an element a in a set A), $A \cup B$ (the union of sets A and B), $A \cap B$ (the intersection of sets A and B), $A \backslash B$ (the complement of B in A, i.e., the set of all points from A not belonging to B). Finally, let $A \triangle B$ denote the symmetric difference of two sets A and B, i.e., $A \triangle B = (A \cup B) \backslash (A \cap B)$. We write $A_n \uparrow A$ if $A_n \subset A_{n+1}$ and $A = \bigcup_{n=1}^{\infty} A_n$; we write $A_n \downarrow A$ if $A_{n+1} \subset A_n$ and $A = \bigcap_{n=1}^{\infty} A_n$.

The restriction of a function f to a set A is denoted by $f|_A$.

The standard symbols $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}^n$ denote, respectively, the sets of all natural, integer, rational numbers, and the n-dimensional Euclidean space. The term "positive" means "strictly positive" with the exception of some special situations with the established terminology (e.g., the positive part of a function may be zero); similarly with "negative".

The following facts about the set $\mathbb{R}^1$ of real numbers are assumed to be known.

1) The sets $U \subset \mathbb{R}^1$ such that every point x from U belongs to U with some interval of the form $(x-\varepsilon, x+\varepsilon)$, where $\varepsilon > 0$, are called open; every open set is the union of a finite or countable collection of pairwise disjoint intervals or rays. The empty set is open by definition.

2) The closed sets are the complements to open sets; a set A is closed precisely when it contains all its limit points. We recall that a is called a limit point for A if every interval centered at a contains a point $b \neq a$ from A. It is clear that any unions and finite intersections of open sets are open. Thus, the real line is a topological space (more detailed information about topological spaces is given in Chapter 6).

It is clear that any intersections and finite unions of closed sets are closed. An important property of $\mathbb{R}^1$ is that the intersection of any decreasing sequence of nonempty bounded closed sets is nonempty. Depending on the way in which the real numbers have been introduced, this claim is either an axiom or is derived from other axioms. The principal concepts related to convergence of sequences and series are assumed to be known.

Let us now consider the problem of measurement of length. Let us aim at defining the length λ of subsets of the interval $I = [0,1]$. For an interval J of the form (a,b), $[a,b)$, $[a,b]$ or $(a,b]$, we set $\lambda(J) = |b-a|$. For a finite union of disjoint intervals $J_1,\ldots,J_n$, we set $\lambda\bigl(\bigcup_{i=1}^n J_i\bigr) = \sum_{i=1}^n \lambda(J_i)$. The sets of the indicated form are called *elementary*. We now have to make a non-trivial step and extend measure to non-elementary sets. A natural way of doing this, which goes back to antiquity, consists of approximating non-elementary sets by elementary ones. But how to approximate? The construction that leads to the so-called *Jordan measure* (which should be more precisely called the *Peano–Jordan measure* following the works Peano [**741**], Jordan [**472**]), is this: a set $A \subset I$ is Jordan measurable if for any $\varepsilon > 0$, there exist elementary sets A_ε and B_ε such that $A_\varepsilon \subset A \subset B_\varepsilon$ and $\lambda(B_\varepsilon \backslash A_\varepsilon) < \varepsilon$. It is clear that when $\varepsilon \to 0$, the lengths of A_ε and B_ε have a common limit, which one takes for $\lambda(A)$. Are all the sets assigned lengths after this procedure? No, not at all. For example, the set $\mathbb{Q} \cap I$ of rational numbers in the interval is not Jordan measurable. Indeed, it contains no elementary set of positive measure. On the other hand, any elementary set containing $\mathbb{Q} \cap I$ has measure 1. The question arises naturally about extensions of λ to larger domains. It is desirable to preserve the nice properties of length, which it possesses on the class of Jordan measurable sets. The most important of these properties are the additivity (i.e., $\lambda(A \cup B) = \lambda(A) + \lambda(B)$ for any disjoint sets A and B in the domain) and the invariance with respect to translations. The first property is even fulfilled in the following stronger form of countable additivity: if disjoint sets A_n together with their union $A = \bigcup_{n=1}^\infty A_n$ are Jordan measurable, then $\lambda(A) = \sum_{n=1}^\infty \lambda(A_n)$. As we shall see later, this problem admits solutions. The most important of them suggested by Lebesgue a century ago and leading to Lebesgue measurability consists of changing the way of approximating by elementary sets. Namely, by analogy with the ancient construction one introduces the outer measure λ^* for *every* set $A \subset I$ as the infimum of sums of measures of elementary sets forming countable covers of A. Then a set A is called Lebesgue measurable if the equality $\lambda^*(A) + \lambda^*(I\backslash A) = \lambda(I)$ holds, which can also be expressed in the form of the equality $\lambda^*(A) = \lambda_*(A)$, where the inner measure λ_* is defined *not* by means of inscribed sets as in the case of the Jordan measure, but by the equality $\lambda_*(A) = \lambda(I) - \lambda^*(I\backslash A)$. An equivalent description of the Lebesgue measurability in terms of approximations by elementary sets is this: for any $\varepsilon > 0$ there exists an elementary set A_ε such that $\lambda^*(A \triangle A_\varepsilon) < \varepsilon$. Now, unlike the Jordan measure, no inclusion of sets is required, i.e., "skew approximations" are admissible. This minor nuance leads to a substantial enlargement of the class of measurable sets. The enlargement is so great that the question of the existence of sets to which no measure is assigned becomes dependent on accepting or not accepting certain special set-theoretic axioms. We shall soon verify that the collection of Lebesgue measurable sets is closed with respect to countable unions, countable intersections, and complements. In addition, if we define the measure of a set A as the limit of measures of elementary sets approximating it in the above sense, then the extended measure turns out to be countably additive. All these claims will be derived from more general results. The role of the countable additivity is obvious from the very beginning: if one approximates a disc by unions of rectangles or triangles, then countable unions arise with necessity.

It follows from what has been said above that in the discussion of measures the key role is played by issues related to domains of definition and extensions. So the next section is devoted to principal classes of sets connected with domains of measures. It turns out in this discussion that the specifics of length on subsets of the real line play no role and it is reasonable from the very beginning to speak of measures of an arbitrary nature. Moreover, this point of view becomes necessary for considering measures on general spaces, e.g., manifolds or functional spaces, which is very important for many branches of mathematics and theoretical physics.

1.2. Algebras and σ-algebras

One of the principal concepts of measure theory is an algebra of sets.

1.2.1. Definition. *An algebra of sets $\mathcal{A}$ is a class of subsets of some fixed set X (called the space) such that*

(i) *X and the empty set belong to $\mathcal{A}$;*

(ii) *if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$, $A\backslash B \in \mathcal{A}$.*

In place of the condition $A\backslash B \in \mathcal{A}$ one could only require that $X\backslash B \in \mathcal{A}$ whenever $B \in \mathcal{A}$, since $A\backslash B = A \cap (X\backslash B)$ and $A \cup B = X\backslash\bigl((X\backslash A) \cap (X\backslash B)\bigr)$. It is sufficient as well to require in (ii) only that $A\backslash B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$, since $A \cap B = A\backslash(A\backslash B)$.

Sometimes in the definition of an algebra the inclusion $X \in \mathcal{A}$ is replaced by the following wider assumption: there exists a set $E \in \mathcal{A}$ called the unit of the algebra such that $A \cap E = A$ for all $A \in \mathcal{A}$. It is clear that replacing X by E we arrive at our definition on a smaller space. It should be noted that not all of the results below extend to this wider concept.

1.2.2. Definition. *An algebra of sets $\mathcal{A}$ is called a σ-algebra if for any sequence of sets A_n in $\mathcal{A}$ one has $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.*

1.2.3. Definition. *A pair $(X, \mathcal{A})$ consisting of a set X and a σ-algebra $\mathcal{A}$ of its subsets is called a measurable space.*

The basic set (space) on which a σ-algebra or measure are given is most often denoted in this book by X; other frequent symbols are E, M, S (from "ensemble", "Menge", "set"), and Ω, a generally accepted symbol in probability theory. For denoting a σ-algebra it is traditional to use script Latin capitals (e.g., $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{S}$), Gothic capitals $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{F}$, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{S}$ (i.e., A, B, F, L, M and S) and Greek letters (e.g., Σ, Λ, Γ, Ξ), although when necessary other symbols are used as well.

In the subsequent remarks and exercises some other classes of sets are mentioned such as semialgebras, rings, semirings, σ-rings, etc. These classes slightly differ in the operations they admit. It is clear that in the definition of a σ-algebra in place of stability with respect to countable unions one could require stability with respect to countable intersections. Indeed, by the formula $\bigcup_{n=1}^{\infty} A_n = X \backslash \bigcap_{n=1}^{\infty} (X \backslash A_n)$ and the stability of any algebra with respect to complementation it is seen that both properties are equivalent.

1.2.4. Example. The collection of finite unions of all intervals of the form $[a, b]$, $[a, b)$, $(a, b]$, (a, b) in the interval $[0, 1]$ is an algebra, but not a σ-algebra.

Clearly, the collection 2^X of all subsets of a fixed set X is a σ-algebra. The smallest σ-algebra is $(X, \varnothing)$. Any other σ-algebra of subsets of X is contained between these two trivial examples.

1.2.5. Definition. *Let $\mathcal{F}$ be a family of subsets of a space X. The smallest σ-algebra of subsets of X containing $\mathcal{F}$ is called the σ-algebra generated by $\mathcal{F}$ and is denoted by the symbol $\sigma(\mathcal{F})$. The algebra generated by $\mathcal{F}$ is defined as the smallest algebra containing $\mathcal{F}$.*

The smallest σ-algebra and algebra mentioned in the definition exist indeed.

1.2.6. Proposition. *Let X be a set. For any family $\mathcal{F}$ of subsets of X there exists a unique σ-algebra generated by $\mathcal{F}$. In addition, there exists a unique algebra generated by $\mathcal{F}$.*

Proof. Set $\sigma(\mathcal{F}) = \bigcap_{\mathcal{F} \subset \mathcal{A}} \mathcal{A}$, where the intersection is taken over all σ-algebras of subsets of the space X containing all sets from $\mathcal{F}$. Such σ-algebras exist: for example, 2^X; their intersection by definition is the collection of all sets that belong to each of such σ-algebras. By construction, $\mathcal{F} \subset \sigma(\mathcal{F})$. If we are given a sequence of sets $A_n \in \sigma(\mathcal{F})$, then their intersection, union and complements belong to any σ-algebra $\mathcal{A}$ containing $\mathcal{F}$, hence belong to $\sigma(\mathcal{F})$, i.e., $\sigma(\mathcal{F})$ is a σ-algebra. The uniqueness is obvious from the fact that the existence of a σ-algebra $\mathcal{B}$ containing $\mathcal{F}$ but not containing $\sigma(\mathcal{F})$ contradicts the definition of $\sigma(\mathcal{F})$, since $\mathcal{B}\cap\sigma(\mathcal{F})$ contains $\mathcal{F}$ and is a σ-algebra. The case of an algebra is similar. $\square$

Note that it follows from the definition that the class of sets formed by the complements of sets in $\mathcal{F}$ generates the same σ-algebra as $\mathcal{F}$. It is also clear that a countable class may generate an uncountable σ-algebra. For example, the intervals with rational endpoints generate the σ-algebra containing all single-point sets.

The algebra generated by a family of sets $\mathcal{F}$ can be easily described explicitly. To this end, let us add to $\mathcal{F}$ the empty set and denote by $\mathcal{F}_1$ the collection of all sets of this enlarged collection together with their complements. Then we denote by $\mathcal{F}_2$ the class of all finite intersections of sets in $\mathcal{F}_1$. The class $\mathcal{F}_3$ of all finite unions of sets in $\mathcal{F}_2$ is the algebra generated by $\mathcal{F}$. Indeed, it is clear that $\mathcal{F}\subset\mathcal{F}_1\subset\mathcal{F}_2\subset\mathcal{F}_3$ and that $\varnothing\in\mathcal{F}_3$. The class $\mathcal{F}_3$ admits any finite intersections, since if $A=\bigcup_{i=1}^n A_i$, $B=\bigcup_{j=1}^k B_j$, where $A_i,B_j\in\mathcal{F}_2$, then we have $A\cap B=\bigcup_{i\le n,j\le k} A_i\cap B_j$ and $A_i\cap B_j\in\mathcal{F}_2$. In addition, $\mathcal{F}_3$ is stable under complements. Indeed, if $E=E_1\cup\cdots\cup E_n$, where $E_i\in\mathcal{F}_2$, then $X\backslash E=\bigcap_{i=1}^n(X\backslash E_i)$. Since $E_i=E_{i,1}\cap\cdots\cap E_{i,k_i}$, where $E_{i,j}\in\mathcal{F}_1$, one has $X\backslash E_i=\bigcup_{j=1}^{k_i}(X\backslash E_{i,j})$, where $D_{i,j}:=X\backslash E_{i,j}\in\mathcal{F}_1$. Hence $X\backslash E=\bigcap_{i=1}^n\bigcup_{j=1}^{k_i} D_{i,j}$, which belongs to $\mathcal{F}_3$ by the stability of $\mathcal{F}_3$ with respect to finite unions and intersections. On the other hand, it is clear that $\mathcal{F}_3$ belongs to the algebra generated by $\mathcal{F}$.

One should not attempt to imagine the elements of the σ-algebra generated by the class $\mathcal{F}$ in a constructive form by means of countable unions, intersections or complements of the elements in $\mathcal{F}$. The point is that the above-mentioned operations can be repeated in an unlimited number of steps in any order. For example, one can form the class $\mathcal{F}_\sigma$ of countable unions of closed sets in the interval, then the class $\mathcal{F}_{\sigma\delta}$ of countable intersections of sets in $\mathcal{F}_\sigma$, and continue this process inductively. One will be obtaining new classes all the time, but even their union does not exhaust the σ-algebra generated by the closed sets (the proof of this fact is not trivial; see Exercises 6.10.30, 6.10.31, 6.10.32 in Chapter 6). In §1.10 we study the so-called A-operation, which gives all sets in the σ-algebra generated by intervals, but produces also other sets. Let us give an example where one can explicitly describe the σ-algebra generated by a class of sets.

1.2.7. Example. Let $\mathcal{A}_0$ be a σ-algebra of subsets in a space X. Suppose that a set $S\subset X$ does not belong to $\mathcal{A}_0$. Then the σ-algebra $\sigma\bigl(\mathcal{A}_0\cup\{S\}\bigr)$, generated by $\mathcal{A}_0$ and the set S coincides with the collection of all sets of the form

$$E=(A\cap S)\cup\bigl(B\cap(X\backslash S)\bigr),\quad\text{where } A,B\in\mathcal{A}_0. \tag{1.2.1}$$

PROOF. All sets of the form (1.2.1) belong to the σ-algebra $\sigma\bigl(\mathcal{A}_0\cup\{S\}\bigr)$. On the other hand, the sets of the indicated type form a σ-algebra. Indeed,

$$X\backslash E = \bigl((X\backslash A)\cap S\bigr)\cup\bigl((X\backslash B)\cap(X\backslash S)\bigr),$$

since x does not belong to E precisely when either x belongs to S but not to A, or x belongs neither to S, nor to B. In addition, if the sets E_n are represented in the form (1.2.1) with some $A_n, B_n \in \mathcal{A}_0$, then $\bigcap_{n=1}^\infty E_n$ and $\bigcup_{n=1}^\infty E_n$ also have the form (1.2.1). For example, $\bigcap_{n=1}^\infty E_n$ has the form (1.2.1) with $A = \bigcap_{n=1}^\infty A_n$ and $B = \bigcap_{n=1}^\infty B_n$. Finally, all sets in $\mathcal{A}_0$ are obtained in the form (1.2.1) with $A = B$, and for obtaining S we take $A = X$ and $B = \varnothing$. □

In considerations involving σ-algebras the following simple properties of the set-theoretic operations are often useful.

1.2.8. Lemma. *Let $(A_\alpha)_{\alpha\in\Lambda}$ be a family of subsets of a set X and let $f\colon E\to X$ be an arbitrary mapping of a set E to X. Then*

$$X\backslash\bigcup_{\alpha\in\Lambda}A_\alpha = \bigcap_{\alpha\in\Lambda}(X\backslash A_\alpha),\quad X\backslash\bigcap_{\alpha\in\Lambda}A_\alpha = \bigcup_{\alpha\in\Lambda}(X\backslash A_\alpha), \tag{1.2.2}$$

$$f^{-1}\Bigl(\bigcup_{\alpha\in\Lambda}A_\alpha\Bigr) = \bigcup_{\alpha\in\Lambda}f^{-1}(A_\alpha),\quad f^{-1}\Bigl(\bigcap_{\alpha\in\Lambda}A_\alpha\Bigr) = \bigcap_{\alpha\in\Lambda}f^{-1}(A_\alpha). \tag{1.2.3}$$

PROOF. Let $x\in X\backslash\bigcup_{\alpha\in\Lambda}A_\alpha$, i.e., $x\notin A_\alpha$ for all $\alpha\in\Lambda$. The latter is equivalent to the inclusion $x\in\bigcap_{\alpha\in\Lambda}(X\backslash A_\alpha)$. Other relationships are proved in a similar manner. □

1.2.9. Corollary. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X and f an arbitrary mapping from a set E to X. Then the class $f^{-1}(\mathcal{A})$ of all sets of the form $f^{-1}(A)$, where $A\in\mathcal{A}$, is a σ-algebra in E.*

In addition, for an arbitrary σ-algebra $\mathcal{B}$ of subsets of E, the class of sets $\{A\subset X\colon f^{-1}(A)\in\mathcal{B}\}$ is a σ-algebra. Furthermore, for any class of sets $\mathcal{F}$ in X, one has $\sigma\bigl(f^{-1}(\mathcal{F})\bigr) = f^{-1}\bigl(\sigma(\mathcal{F})\bigr)$.

PROOF. The first two assertions are clear from the lemma. Since the class $f^{-1}\bigl(\sigma(\mathcal{F})\bigr)$ is a σ-algebra by the first assertion, we obtain the inclusion $\sigma\bigl(f^{-1}(\mathcal{F})\bigr)\subset f^{-1}\bigl(\sigma(\mathcal{F})\bigr)$. Finally, by the second assertion, we have $f^{-1}\bigl(\sigma(\mathcal{F})\bigr)\subset\sigma\bigl(f^{-1}(\mathcal{F})\bigr)$ because $f^{-1}(\mathcal{F})\subset\sigma\bigl(f^{-1}(\mathcal{F})\bigr)$. □

Simple examples show that the class $f(\mathcal{B})$ of all sets of the form $f(B)$, where $B\in\mathcal{B}$, is not always an algebra.

1.2.10. Definition. *The Borel σ-algebra of $\mathbb{R}^n$ is the σ-algebra $\mathcal{B}(\mathbb{R}^n)$ generated by all open sets. The sets in $\mathcal{B}(\mathbb{R}^n)$ are called Borel sets. For any set $E\subset\mathbb{R}^n$, let $\mathcal{B}(E)$ denote the class of all sets of the form $E\cap B$, where $B\in\mathcal{B}(\mathbb{R}^n)$.*

The class $\mathcal{B}(E)$ can also be defined as the σ-algebra generated by the intersections of E with open sets in $\mathbb{R}^n$. This is clear from the following: if the latter σ-algebra is denoted by $\mathcal{E}$, then the family of all sets $B \in \mathcal{B}(\mathbb{R}^n)$ such that $B \cap E \in \mathcal{E}$ is a σ-algebra containing all open sets, i.e., it coincides with $\mathcal{B}(\mathbb{R}^n)$. The sets in $\mathcal{B}(E)$ are called Borel sets of the space E and $\mathcal{B}(E)$ is called the Borel σ-algebra of the space E. One should keep in mind that such sets may not be Borel in $\mathbb{R}^n$ unless, of course, E itself is Borel in $\mathbb{R}^n$. For example, one always has $E \in \mathcal{B}(E)$, since $E \cap \mathbb{R}^n = E$.

It is clear that $\mathcal{B}(\mathbb{R}^n)$ is also generated by the class of all closed sets.

1.2.11. Lemma. *The Borel σ-algebra of the real line is generated by any of the following classes of sets*:

(i) *the collection of all intervals*;

(ii) *the collection of all intervals with rational endpoints*;

(iii) *the collection of all rays of the form $(-\infty, c)$, where c is rational*;

(iv) *the collection of all rays of the form $(-\infty, c]$, where c is rational*;

(v) *the collection of rays of the form $(c, +\infty)$, where c rational*;

(vi) *the collection of all rays of the form $[c, +\infty)$, where c is rational. Finally, the same is true if in place of rational numbers one takes points of any everywhere dense set.*

Proof. It is clear that all the sets indicated above are Borel, since they are either open or closed. Therefore, the σ-algebras generated by the corresponding families are contained in $\mathcal{B}(\mathbb{R}^1)$. Since every open set on the real line is the union of an at most countable collection of intervals, it suffices to show that any interval (a, b) is contained in the σ-algebras corresponding to the classes (i)–(vi). This follows from the fact that (a, b) is the union of intervals of the form (a_n, b_n), where a_n and b_n are rational, and also is the union of intervals of the form $[a_n, b_n)$ with rational endpoints, whereas such intervals belong to the σ-algebra generated by the rays $(-\infty, c)$, since they can be written as differences of rays. In a similar manner, the differences of the rays of the form (c, ∞) give the intervals $(a_n, b_n]$, from which by means of unions one constructs the intervals (a, b). $\square$

It is clear from the proof that the Borel σ-algebra is generated by the closed intervals with rational endpoints. It is seen from this, by the way, that disjoint classes of sets may generate one and the same σ-algebra.

1.2.12. Example. The collection of all single-point sets in a space X generates the σ-algebra consisting of all sets that are either at most countable or have at most countable complements. In addition, this σ-algebra is strictly smaller than the Borel one if $X = \mathbb{R}^1$.

Proof. Denote by $\mathcal{A}$ the family of all sets $A \subset X$ such that either A is at most countable or $X \backslash A$ is at most countable. Let us verify that $\mathcal{A}$ is a σ-algebra. Since X is contained in $\mathcal{A}$ and $\mathcal{A}$ is closed under complementation, it suffices to show that $A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ whenever $A_n \in \mathcal{A}$. If all A_n are at most countable, this is obvious. Suppose that among the sets A_n there is at least one set A_{n_1} whose complement is at most countable. The complement of A is contained in the complement of A_{n_1}, hence is at most countable as well, i.e., $A \in \mathcal{A}$. All one-point sets belong to $\mathcal{A}$, hence the σ-algebra $\mathcal{A}_0$ generated by them is contained in $\mathcal{A}$. On the other hand, it is clear that any set in $\mathcal{A}$ is an element of $\mathcal{A}_0$, whence it follows that $\mathcal{A}_0 = \mathcal{A}$. □

Let us give definitions of several other classes of sets employed in measure theory.

1.2.13. Definition. (i) *A family $\mathcal{R}$ of subsets of a set X is called a ring if it contains the empty set and the sets $A \cap B$, $A \cup B$ and $A \backslash B$ belong to $\mathcal{R}$ for all $A, B \in \mathcal{R}$;*

(ii) *A family $\mathcal{S}$ of subsets of a set X is called a semiring if it contains the empty set, $A \cap B \in \mathcal{S}$ for all $A, B \in \mathcal{S}$ and, for every pair of sets $A, B \in \mathcal{S}$ with $A \subset B$, the set $B \backslash A$ is the union of finitely many disjoint sets in $\mathcal{S}$. If $X \in \mathcal{S}$, then $\mathcal{S}$ is called a semialgebra*;

(iii) *A ring is called a σ-ring if it is closed with respect to countable unions. A ring is called a δ-ring if it is closed with respect to countable intersections.*

As an example of a ring that is not an algebra, let us mention the collection of all bounded sets on the real line. The family of all intervals in the interval $[a, b]$ gives an example of a semiring that is not a ring. According to the following lemma, the collection of all finite unions of elements of a semiring is a ring (called the ring generated by the given semiring). It is clear that this is the minimal ring containing the given semiring.

1.2.14. Lemma. *For any semiring $\mathcal{S}$, the collection of all finite unions of sets in $\mathcal{S}$ forms a ring $\mathcal{R}$. Every set in $\mathcal{R}$ is a finite union of pairwise disjoint sets in $\mathcal{S}$. If $\mathcal{S}$ is a semialgebra, then $\mathcal{R}$ is an algebra.*

PROOF. It is clear that the class $\mathcal{R}$ admits finite unions. Suppose that $A = A_1 \cup \cdots \cup A_n$, $B = B_1 \cup \cdots \cup B_k$, where $A_i, B_j \in \mathcal{S}$. Then we have $A \cap B = \bigcup_{i \le n, j \le k} A_i \cap B_j \in \mathcal{R}$. Hence $\mathcal{R}$ admits finite intersections. In addition,

$$A \backslash B = \bigcup_{i=1}^{n} \Bigl(A_i \backslash \bigcup_{j=1}^{k} B_j \Bigr) = \bigcup_{i=1}^{n} \bigcap_{j=1}^{k} (A_i \backslash B_j).$$

Since the set $A_i \backslash B_j = A_i \backslash (A_i \cap B_j)$ is a finite union of sets in $\mathcal{S}$, one has $A \backslash B \in \mathcal{R}$. Clearly, A can be written as a union of a finitely many disjoint sets in $\mathcal{S}$ because $\mathcal{S}$ is closed with respect to intersections. The last claim of the lemma is obvious. □

Note that for any σ-algebra $\mathcal{B}$ in a space X and any set $A \subset X$, the class $\mathcal{B}_A := \{B \cap A \colon B \in \mathcal{B}\}$ is a σ-algebra in the space A. This σ-algebra is called the trace σ-algebra.

1.3. Additivity and countable additivity of measures

Functions with values in $(-\infty,+\infty)$ will be called real or real-valued. In the cases where we discuss functions with values in the extended real line $[-\infty,+\infty]$, this will always be specified.

1.3.1. Definition. *A real-valued set function μ defined on a class of sets $\mathcal{A}$ is called additive* (*or finitely additive*) *if*

$$\mu\Bigl(\bigcup_{i=1}^n A_i\Bigr)=\sum_{i=1}^n \mu(A_i) \tag{1.3.1}$$

for all n and all disjoint sets $A_1,\ldots,A_n\in\mathcal{A}$ such that $\bigcup_{i=1}^n A_i\in\mathcal{A}$.

In the case where $\mathcal{A}$ is closed with respect to finite unions, the finite additivity is equivalent to the equality

$$\mu(A\cup B)=\mu(A)+\mu(B) \tag{1.3.2}$$

for all disjoint sets $A,B\in\mathcal{A}$.

If the domain of definition of an additive real-valued set function μ contains the empty set $\varnothing$, then $\mu(\varnothing)=0$. In particular, this is true for any additive set function on a ring or an algebra.

It is also useful to consider the property of *subadditivity* (also called the *semiadditivity*):

$$\mu\Bigl(\bigcup_{i=1}^n A_i\Bigr)\le\sum_{i=1}^n \mu(A_i) \tag{1.3.3}$$

for all $A_i\in\mathcal{A}$ with $\bigcup_{i=1}^n A_i\in\mathcal{A}$. Any additive nonnegative set function on an algebra is subadditive (see below).

1.3.2. Definition. *A real-valued set function μ on a class of sets $\mathcal{A}$ is called countably additive if*

$$\mu\Bigl(\bigcup_{n=1}^\infty A_n\Bigr)=\sum_{n=1}^\infty \mu(A_n) \tag{1.3.4}$$

for all pairwise disjoint sets A_n in $\mathcal{A}$ such that $\bigcup_{n=1}^\infty A_n\in\mathcal{A}$. A countably additive set function defined on an algebra is called a measure.

It is readily seen from the definition that the series in (1.3.4) converges absolutely because its sum is independent of rearrangements of its terms.

1.3.3. Proposition. *Let μ be an additive real set function on an algebra* (*or a ring*) *of sets $\mathcal{A}$. Then the following conditions are equivalent*:

(i) *the function μ is countably additive,*

(ii) *the function μ is continuous at zero in the following sense: if $A_n\in\mathcal{A}$, $A_{n+1}\subset A_n$ for all $n\in\mathbb{N}$ and $\bigcap_{n=1}^\infty A_n=\varnothing$, then*

$$\lim_{n\to\infty}\mu(A_n)=0, \tag{1.3.5}$$

(iii) *the function μ is continuous from below, i.e., if $A_n \in \mathcal{A}$ are such that $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then*

$$\mu\Bigl(\bigcup_{n=1}^{\infty} A_n\Bigr) = \lim_{n\to\infty} \mu(A_n). \tag{1.3.6}$$

PROOF. (i) Let μ be countably additive and let the sets $A_n \in \mathcal{A}$ decrease monotonically to the empty set. Set $B_n = A_n \backslash A_{n+1}$. The sets B_n belong to $\mathcal{A}$ and are disjoint and their union is A_1. Hence the series $\sum_{n=1}^{\infty} \mu(B_n)$ converges. Then $\sum_{n=N}^{\infty} \mu(B_n)$ tends to zero as $N \to \infty$, but the sum of this series is $\mu(A_N)$, since $\bigcup_{n=N}^{\infty} B_n = A_N$. Hence we arrive at condition (ii).

Suppose now that condition (ii) is fulfilled. Let $\{B_n\}$ be a sequence of pairwise disjoint sets in $\mathcal{A}$ whose union B is an element of $\mathcal{A}$ as well. Set $A_n = B \backslash \bigcup_{k=1}^{n} B_k$. It is clear that $\{A_n\}$ is a sequence of monotonically decreasing sets in $\mathcal{A}$ with the empty intersection. By hypothesis, $\mu(A_n) \to 0$. By the finite additivity this means that $\sum_{k=1}^{n} \mu(B_k) \to \mu(B)$ as $n \to \infty$. Hence μ is countably additive. Clearly, (iii) follows from (ii), for if the sets $A_n \in \mathcal{A}$ increase monotonically and their union is the set $A \in \mathcal{A}$, then the sets $A \backslash A_n \in \mathcal{A}$ decrease monotonically to the empty set. Finally, by the finite additivity (iii) yields the countable additivity of μ. $\square$

The reader is warned that there is no such equivalence for semialgebras (see Exercise 1.12.75).

1.3.4. Definition. *A countably additive measure μ on a σ-algebra of subsets of a space X is called a probability measure if $\mu \geq 0$ and $\mu(X) = 1$.*

1.3.5. Definition. *A triple $(X, \mathcal{A}, \mu)$ is called a measure space if μ is a nonnegative measure on a σ-algebra $\mathcal{A}$ of subset of a set X. If μ is a probability measure, then $(X, \mathcal{A}, \mu)$ is called a probability space.*

Nonnegative not identically zero measures are called *positive measures*.

Additive set functions are also called additive measures, but to simplify the terminology we use the term measure only for *countably additive measures on algebras or rings*. Countably additive measures are also called σ-additive measures.

1.3.6. Definition. *A measure defined on the Borel σ-algebra of the whole space $\mathbb{R}^n$ or its subset is called a Borel measure.*

It is clear that if $\mathcal{A}$ is an algebra, then the additivity is just equality (1.3.2) for arbitrary disjoint sets in $\mathcal{A}$. Similarly, if $\mathcal{A}$ is a σ-algebra, then the countable additivity is equality (1.3.4) for arbitrary sequences of disjoint sets in $\mathcal{A}$. The above given formulations are convenient for two reasons. First, the validity of the corresponding equalities is required only for those collections of sets for which both parts make sense. Second, as we shall see later, under natural hypotheses, additive (or countably additive) set functions admit additive (respectively, countably additive) extensions to larger classes of sets that admit unions of the corresponding type.

1.3.7. Example. Let $\mathcal{A}$ be the algebra of sets $A \subset \mathbb{N}$ such that either A or $\mathbb{N}\backslash A$ is finite. For finite A, let $\mu(A) = 0$, and for A with a finite complement let $\mu(A) = 1$. Then μ is an additive, but not countably additive set function.

PROOF. It is clear that $\mathcal{A}$ is indeed an algebra. Relation (1.3.2) is obvious for disjoint sets A and B if A is finite. Finally, A and B in $\mathcal{A}$ cannot be infinite simultaneously being disjoint. If μ were countably additive, we would have had $\mu(\mathbb{N}) = \sum_{n=1}^{\infty} \mu(\{n\}) = 0$. □

There exist additive, but not countably additive set functions on σ-algebras (see Example 1.12.28). The simplest countably additive set function is identically zero. Another example: let X be a nonempty set and let $a \in X$; Dirac's measure δ_a at the point a is defined as follows: for every $A \subset X$, $\delta_a(A) = 1$ if $a \in A$ and $\delta_a(A) = 0$ otherwise. Let us give a slightly less trivial example.

1.3.8. Example. Let $\mathcal{A}$ be the σ-algebra of all subsets of $\mathbb{N}$. For every set $A = \{n_k\}$, let $\mu(A) = \sum_k 2^{-n_k}$. Then μ is a measure on $\mathcal{A}$.

In order to construct less trivial examples (say, Lebesgue measure), we need auxiliary technical tools discussed in the next section.

Note several simple properties of additive and countably additive set functions.

1.3.9. Proposition. *Let μ be a nonnegative additive set function on an algebra or a ring $\mathcal{A}$.*

(i) *If $A, B \in \mathcal{A}$ and $A \subset B$, then $\mu(A) \le \mu(B)$.*

(ii) *For any collection $A_1, \ldots, A_n \in \mathcal{A}$ one has*

$$\mu\Bigl(\bigcup_{i=1}^{n} A_i\Bigr) \le \sum_{i=1}^{n} \mu(A_i).$$

(iii) *The function μ is countably additive precisely when in addition to the additivity it is countably subadditive in the following sense: for any sequence $\{A_n\} \subset \mathcal{A}$ with $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ one has*

$$\mu\Bigl(\bigcup_{n=1}^{\infty} A_n\Bigr) \le \sum_{n=1}^{\infty} \mu(A_n).$$

PROOF. Assertion (i) follows, since $\mu(B\backslash A) \ge 0$. Assertion (ii) is easily verified by induction taking into account the nonnegativity of μ and the relation $\mu(A \cup B) = \mu(A\backslash B) + \mu(B\backslash A) + \mu(A \cap B)$.

If μ is countably additive and the union of sets $A_n \in \mathcal{A}$ belongs to $\mathcal{A}$, then according to Proposition 1.3.3 one has

$$\mu\Bigl(\bigcup_{i=1}^{n} A_i\Bigr) \to \mu\Bigl(\bigcup_{i=1}^{\infty} A_i\Bigr),$$

which by (ii) gives the estimate indicated in (iii). Finally, such an estimate combined with the additivity yields the countable additivity. Indeed, let B_n be pairwise disjoint sets in $\mathcal{A}$ whose union B belongs to $\mathcal{A}$ as well. Then for any $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \mu(B_k) = \mu\Bigl(\bigcup_{k=1}^{n} B_k\Bigr) \leq \mu(B) \leq \sum_{k=1}^{\infty} \mu(B_k),$$

whence it follows that $\sum_{k=1}^{\infty} \mu(B_k) = \mu(B)$. □

1.3.10. Proposition. *Let $\mathcal{A}_0$ be a semialgebra (see Definition 1.2.13). Then every additive set function μ on $\mathcal{A}_0$ uniquely extends to an additive set function on the algebra $\mathcal{A}$ generated by $\mathcal{A}_0$ (i.e., the family of all finite unions of sets in $\mathcal{A}_0$). This extension is countably additive provided that μ is countably additive on $\mathcal{A}_0$. The same is true in the case of a semiring $\mathcal{A}$ and the ring generated by it.*

PROOF. By Lemma 1.2.14 the collection of all finite unions of elements of $\mathcal{A}_0$ is an algebra (or a ring when $\mathcal{A}_0$ is a semiring). It is clear that any set in $\mathcal{A}$ can be represented as a union of disjoint elements of $\mathcal{A}_0$. Set

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i)$$

if $A_i \in \mathcal{A}_0$ are pairwise disjoint and their union is A. The indicated extension is obviously additive, but we have to verify that it is well-defined, i.e., is independent of partitioning A into parts in $\mathcal{A}_0$. Indeed, if $B_1, \ldots, B_m$ are pairwise disjoint sets in $\mathcal{A}_0$ whose union is A, then by the additivity of μ on the algebra $\mathcal{A}_0$ one has the equality $\mu(A_i) = \sum_{j=1}^{m} \mu(A_i \cap B_j)$, $\mu(B_j) = \sum_{i=1}^{n} \mu(A_i \cap B_j)$, whence the desired conclusion follows. Let us verify the countable additivity of the indicated extension in the case of the countable additivity on $\mathcal{A}_0$. Let $A, A_n \in \mathcal{A}$, $A = \bigcup_{n=1}^{\infty} A_n$ be such that $A_n \cap A_k = \varnothing$ if $n \neq k$. Then

$$A = \bigcup_{j=1}^{N} B_j, \quad A_n = \bigcup_{i=1}^{N_n} B_{n,i},$$

where $B_j, B_{n,i} \in \mathcal{A}_0$. Set $C_{n,i,j} := B_{n,i} \cap B_j$. The sets $C_{n,i,j}$ are pairwise disjoint and

$$B_j = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{N_n} C_{n,i,j}, \quad B_{n,i} = \bigcup_{j=1}^{N} C_{n,i,j}.$$

By the countable additivity of μ on $\mathcal{A}_0$ we have

$$\mu(B_j) = \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu(C_{n,i,j}), \quad \mu(B_{n,i}) = \sum_{j=1}^{N} \mu(C_{n,i,j}),$$

and by the definition of μ on $\mathcal{A}$ one has the following equality:

$$\mu(A) = \sum_{j=1}^{N} \mu(B_j), \quad \mu(A_n) = \sum_{i=1}^{N_n} \mu(B_{n,i}).$$

We obtain from these equalities that $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$, since both quantities equal the sum of all $\mu(C_{n,i,j})$. That it is possible to interchange the summations in n and j is obvious from the fact that the series in n converge and the sums in j and i are finite. □

1.4. Compact classes and countable additivity

In this section, we give a sufficient condition for the countable additivity, which is satisfied for most of the measures encountered in real applications.

1.4.1. Definition. *A family $\mathcal{K}$ of subsets of a set X is called a compact class if, for any sequence K_n of its elements with $\bigcap_{n=1}^{\infty} K_n = \varnothing$, there exists N such that $\bigcap_{n=1}^{N} K_n = \varnothing$.*

The terminology is explained by the following basic example.

1.4.2. Example. An arbitrary family of compact sets in $\mathbb{R}^n$ (more generally, in a topological space) is a compact class.

PROOF. Indeed, let K_n be compact sets whose intersection is empty. Suppose that for every n the set $E_n = \bigcap_{i=1}^{n} K_i$ contains some element x_n. We may assume that no element of the sequence $\{x_n\}$ is repeated infinitely often, since otherwise it is a common element of all E_n. By the compactness of K_1 there exists a point x each neighborhood of which contains infinitely many elements of the sequence $\{x_n\}$. All sets E_n are compact and $x_i \in E_n$ whenever $i \geq n$, hence the point x belongs to all E_n, which is a contradiction. □

Note that some authors call the above-defined compact classes countably compact or semicompact and in the definition of compact classes require the following stronger property: if the intersection of a (possibly uncountable) collection of sets in $\mathcal{K}$ is empty, then the intersection of some its finite subcollection is empty as well. See Exercise 1.12.105 for an example distinguishing the two properties. Although such a terminology is more consistent from the point of view of topology (see Exercise 6.10.66 in Chapter 6), we shall not follow it.

1.4.3. Theorem. *Let μ be a nonnegative additive set function on an algebra $\mathcal{A}$. Suppose that there exists a compact class $\mathcal{K}$ approximating μ in the following sense: for every $A \in \mathcal{A}$ and every $\varepsilon > 0$, there exist $K_\varepsilon \in \mathcal{K}$ and $A_\varepsilon \in \mathcal{A}$ such that $A_\varepsilon \subset K_\varepsilon \subset A$ and $\mu(A \backslash A_\varepsilon) < \varepsilon$. Then μ is countably additive. In particular, this is true if the compact class $\mathcal{K}$ is contained in $\mathcal{A}$ and for any $A \in \mathcal{A}$ one has the equality*

$$\mu(A) = \sup_{K \subset A,\ K \in \mathcal{K}} \mu(K).$$

Proof. Suppose that the sets $A_n \in \mathcal{A}$ are decreasing and their intersection is empty. Let us show that $\mu(A_n) \to 0$. Let us fix $\varepsilon > 0$. By hypothesis, there exist $K_n \in \mathcal{K}$ and $B_n \in \mathcal{A}$ such that $B_n \subset K_n \subset A_n$ and $\mu(A_n \backslash B_n) < \varepsilon 2^{-n}$. It is clear that $\bigcap_{n=1}^{\infty} K_n \subset \bigcap_{n=1}^{\infty} A_n = \varnothing$. By the definition of a compact class, there exists N such that $\bigcap_{n=1}^{N} K_n = \varnothing$. Then $\bigcap_{n=1}^{N} B_n = \varnothing$. Note that one has

$$A_N = \bigcap_{n=1}^{N} A_n \subset \bigcup_{n=1}^{N} (A_n \backslash B_n).$$

Indeed, let $x \in A_N$, i.e., $x \in A_n$ for all $n \leq N$. If x does not belong to $\bigcup_{n=1}^{N} (A_n \backslash B_n)$, then $x \notin A_n \backslash B_n$ for all $n \leq N$. Then $x \in B_n$ for every $n \leq N$, whence we obtain $x \in \bigcap_{n=1}^{N} B_n$, which is a contradiction. The above proved equality yields the estimate

$$\mu(A_N) \leq \sum_{n=1}^{N} \mu(A_n \backslash B_n) \leq \sum_{n=1}^{N} \varepsilon 2^{-n} \leq \varepsilon.$$

Hence $\mu(A_n) \to 0$, which implies the countable additivity of μ. $\square$

1.4.4. Example. Let I be an interval in $\mathbb{R}^1$, $\mathcal{A}$ the algebra of finite unions of intervals in I (closed, open and half-open). Then the usual length λ_1, which assigns the value $b - a$ to the interval with the endpoints a and b and extends by additivity to their finite disjoint unions, is countably additive on the algebra $\mathcal{A}$.

Proof. Finite unions of closed intervals form a compact class and approximate from within finite unions of arbitrary intervals. $\square$

1.4.5. Example. Let I be a cube in $\mathbb{R}^n$ of the form $[a, b]^n$ and let $\mathcal{A}$ be the algebra of finite unions of the parallelepipeds in I that are products of intervals in $[a, b]$. Then the usual volume λ_n is countably additive on $\mathcal{A}$. We call λ_n *Lebesgue measure.*

Proof. As in the previous example, finite unions of closed parallelepipeds form a compact approximating class. $\square$

It is shown in Theorem 1.12.5 below that the compactness property can be slightly relaxed.

The previous results justify the introduction of the following concept.

1.4.6. Definition. *Let m be a nonnegative function on a class $\mathcal{E}$ of subsets of a set X and let $\mathcal{P}$ be a class of subsets of X, too. We say that $\mathcal{P}$ is an approximating class for m if, for every $E \in \mathcal{E}$ and every $\varepsilon > 0$, there exist $P_\varepsilon \in \mathcal{P}$ and $E_\varepsilon \in \mathcal{E}$ such that $E_\varepsilon \subset P_\varepsilon \subset E$ and $|m(E) - m(E_\varepsilon)| < \varepsilon$.*

1.4.7. Remark. (i) The reasoning in Theorem 1.4.3 actually proves the following assertion. Let μ be a nonnegative additive set function on an algebra $\mathcal{A}$ and let $\mathcal{A}_0$ be a subalgebra in $\mathcal{A}$. Suppose that there exists a compact class $\mathcal{K}$ approximating μ on $\mathcal{A}_0$ with respect to $\mathcal{A}$ in the following sense: for any $A \in \mathcal{A}_0$ and any $\varepsilon > 0$, there exist $K_\varepsilon \in \mathcal{K}$ and $A_\varepsilon \in \mathcal{A}$ such that $A_\varepsilon \subset K_\varepsilon \subset A$ and $\mu(A\backslash A_\varepsilon) < \varepsilon$. Then μ is countably additive on $\mathcal{A}_0$.

(ii) The compact class $\mathcal{K}$ in Theorem 1.4.3 need not be contained in $\mathcal{A}$. For example, if $\mathcal{A}$ is the algebra generated by all intervals in $[0, 1]$ with rational endpoints and μ is Lebesgue measure, then the class $\mathcal{K}$ of all finite unions of closed intervals with irrational endpoints is approximating for μ and has no intersection with $\mathcal{A}$. However, it will be shown in §1.12(ii) that one can always replace $\mathcal{K}$ by a compact class $\mathcal{K}'$ that is contained in $\sigma(\mathcal{A})$ and approximates the countably additive extension of μ on $\sigma(\mathcal{A})$. It is worth noting that there exists a countably additive extension of μ to the σ-algebra generated by $\mathcal{A}_0$ and $\mathcal{K}$ (see Theorem 1.12.34).

Note that so far in the considered examples we have been concerned with the countable additivity on algebras. However, as we shall see below, any countably additive measure on an algebra automatically extends (in a unique way) to a countably additive measure on the σ-algebra generated by this algebra.

We shall see in Chapter 7 that the class of measures possessing a compact approximating class is very large (so that it is not easy even to construct an example of a countably additive measure without compact approximating classes). Thus, the described sufficient condition of countable additivity has a very universal character. Here we only give the following result.

1.4.8. Theorem. *Let μ be a nonnegative countably additive measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ in the space $\mathbb{R}^n$. Then, for any Borel set $B \subset \mathbb{R}^n$ and any $\varepsilon > 0$, there exist an open set U_ε and a compact set K_ε such that $K_\varepsilon \subset B \subset U_\varepsilon$ and $\mu(U_\varepsilon\backslash K_\varepsilon) < \varepsilon$.*

PROOF. Let us show that for any $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subset B$ such that

$$\mu(B\backslash F_\varepsilon) < \varepsilon/2.$$

Then, by the countable additivity of μ, the set F_ε itself can be approximated from within up to $\varepsilon/2$ by $F_\varepsilon \cap U$, where U is a closed ball of a sufficiently large radius. Denote by $\mathcal{A}$ the class of all sets $A \in \mathcal{B}(\mathbb{R}^n)$ such that, for any $\varepsilon > 0$, there exist a closed set F_ε and an open set U_ε with $F_\varepsilon \subset A \subset U_\varepsilon$ and $\mu(U_\varepsilon\backslash F_\varepsilon) < \varepsilon$. Every closed set A belongs to $\mathcal{A}$, since one can take for F_ε the set A itself, and for U_ε one can take some open δ-neighborhood A^δ of the set A, i.e., the union of all open balls of radius δ with centers at the points in A. When δ is decreasing to zero, the open sets A^δ are decreasing to A, hence their measures approach the measure of A. Let us show that $\mathcal{A}$ is a σ-algebra. If this is done, then the theorem is proven, for the closed sets generate the Borel σ-algebra. By construction, the class $\mathcal{A}$ is closed with respect to the operation of complementation. Hence it remains to verify the stability of $\mathcal{A}$ with respect to countable unions. Let $A_j \in \mathcal{A}$ and let $\varepsilon > 0$. Then there exist a closed set F_j and an open set U_j such that $F_j \subset A_j \subset U_j$ and $\mu(U_j\backslash F_j) < \varepsilon 2^{-j}$, $j \in \mathbb{N}$.

The set $U = \bigcup_{j=1}^{\infty} U_j$ is open and the set $Z_k = \bigcup_{j=1}^{k} F_j$ is closed for any $k \in \mathbb{N}$. It remains to observe that $Z_k \subset \bigcup_{j=1}^{\infty} A_j \subset U$ and for k large enough one has the estimate $\mu(U \backslash Z_k) < \varepsilon$. Indeed, $\mu\bigl(\bigcup_{j=1}^{\infty}(U_j \backslash F_j)\bigr) < \sum_{j=1}^{\infty} \varepsilon 2^{-j} = \varepsilon$ and by the countable additivity $\mu(Z_k) \to \mu\bigl(\bigcup_{j=1}^{\infty} F_j\bigr)$ as $k \to \infty$. □

This result shows that the measurability can be defined (as it is actually done in some textbooks) in the spirit of the Jordan–Peano construction via inner approximations by compact sets and outer approximations by open sets. Certainly, it is necessary for this to define first the measure of open sets, which determines the measures of compacts. In the case of an interval this creates no problem, since open sets are built from disjoint intervals, which by virtue of the countable additivity uniquely determines its measure from the measures of intervals. However, already in the case of a square there is no such disjoint representation of open sets, and the aforementioned construction is not as effective here.

Finally, it is worth mentioning that Lebesgue measure considered above on the algebra generated by cubes could be defined at once on the Borel σ-algebra by the equality $\lambda_n(B) := \inf \sum_{j=1}^{\infty} \lambda_n(I_j)$, where inf is taken over all at most countable covers of B by cubes I_j. In fact, exactly this will be done below, however, a justification of the fact that the indicated equality gives a countably additive measure is not trivial and will be given by some detour, where the principal role will be played by the idea of compact approximations and the construction of outer measure, with which the next section is concerned.

1.5. Outer measure and the Lebesgue extension of measures

It is shown in this section how to extend countably additive measures from algebras to σ-algebras. Extensions from rings are considered in §1.11.

For any nonnegative set function μ that is defined on a certain class $\mathcal{A}$ of subsets in a space X and contains X itself, the formula

$$\mu^*(A) = \inf\biggl\{\sum_{n=1}^{\infty} \mu(A_n)\Big| A_n \in \mathcal{A},\ A \subset \bigcup_{n=1}^{\infty} A_n\biggr\}$$

defines a new set function defined already for every $A \subset X$. The same construction is applicable to set functions with values in $[0, +\infty]$. If X does not belong to $\mathcal{A}$, then μ^* is defined by the above formula on all sets A that can be covered by a countable sequence of elements of $\mathcal{A}$, and all other sets are assigned the infinite value. An alternative definition of μ^* on a set A that cannot be covered by a sequence from $\mathcal{A}$ is to take the supremum of the values of μ^* on the sets contained in A and covered by sequences from $\mathcal{A}$ (see Example 1.12.130). The function μ^* is called the outer measure, although it need not be additive. In Section 1.11 below we discuss in more detail Carathéodory outer measures, not necessarily originated from additive set functions.

1.5.1. Definition. *Suppose that μ is a nonnegative set function on domain $\mathcal{A} \subset 2^X$. A set A is called μ-measurable (or Lebesgue measurable with respect to μ) if, for any $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ such that*

$$\mu^*(A \triangle A_\varepsilon) < \varepsilon.$$

The class of all μ-measurable sets is denoted by $\mathcal{A}_\mu$.

We shall be interested in the case where μ is a countably additive measure on an algebra $\mathcal{A}$.

Note that the definition of measurability given by Lebesgue (for an interval X) was the equality $\mu^*(A) + \mu^*(X\backslash A) = \mu(X)$. It is shown below that for additive functions on algebras this definition (possibly not so intuitively transparent) is equivalent to the one given above (see Theorem 1.11.8 and also Proposition 1.5.11 for countably additive measures). In addition, we discuss below the definition of the Carathéodory measurability, which is also equivalent to the above definition in the case of nonnegative additive set functions on algebras, but is much more fruitful in the general case.

1.5.2. Example. (i) Let $\varnothing \in \mathcal{A}$ and $\mu(\varnothing) = 0$. Then $\mathcal{A} \subset \mathcal{A}_\mu$ (if $A \in \mathcal{A}$, one can take $A_\varepsilon = A$). In addition, any set A with $\mu^*(A) = 0$ is μ-measurable, for one can take $A_\varepsilon = \varnothing$.

(ii) Let $\mathcal{A}$ be the algebra of finite unions of intervals from Example 1.4.4 with the usual length λ. Then, the λ-measurability of A is equivalent to the following: for each $\varepsilon > 0$, one can find a set E that is a finite union of intervals and two sets A'_ε and A''_ε with

$$A = (E \cup A'_\varepsilon)\backslash A''_\varepsilon, \ \lambda^*(A'_\varepsilon) \le \varepsilon, \ \lambda^*(A''_\varepsilon) \le \varepsilon.$$

(iii) Let $X = [0, 1]$, $\mathcal{A} = \{\varnothing, X\}$, $\mu(X) = 1$, $\mu(\varnothing) = 0$. Then μ is a countably additive measure on $\mathcal{A}$ and $\mathcal{A}_\mu = \mathcal{A}$. Indeed, $\mu^*(E) = 1$ for any $E \neq \varnothing$. Hence the whole interval is the only nonempty set that can be approximated up to $\varepsilon < 1$ by a set from $\mathcal{A}$.

Note that μ^* is *monotone*, i.e., $\mu^*(A) \le \mu^*(B)$ if $A \subset B$. However, even if μ is a countably additive measure on a σ-algebra $\mathcal{A}$, the corresponding outer measure μ^* may not be countably additive on the class of all sets.

1.5.3. Example. Let X be a two-point set $\{0, 1\}$ and let $\mathcal{A} = \{\varnothing, X\}$. Set $\mu(\varnothing) = 0$, $\mu(X) = 1$. Then $\mathcal{A}$ is a σ-algebra and μ is countably additive on $\mathcal{A}$, but μ^* is not additive on the σ-algebra of all sets, since $\mu^*(\{0\}) = 1$, $\mu^*(\{1\}) = 1$, and $\mu^*(\{0\} \cup \{1\}) = 1$.

1.5.4. Lemma. *Let μ be a nonnegative set function on a class $\mathcal{A}$. Then the function μ^* is countably subadditive, i.e.,*

$$\mu^*\Bigl(\bigcup_{n=1}^\infty A_n\Bigr) \le \sum_{n=1}^\infty \mu^*(A_n) \tag{1.5.1}$$

for any sets A_n.

PROOF. Let $\varepsilon > 0$ and $\mu^*(A_n) < \infty$. For any n, there exists a collection $\{B_{n,k}\}_{k=1}^\infty \subset \mathcal{A}$ such that $A_n \subset \bigcup_{k=1}^\infty B_{n,k}$ and

$$\sum_{k=1}^\infty \mu(B_{n,k}) \le \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

Then $\bigcup_{n=1}^\infty A_n \subset \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty B_{n,k}$ and hence

$$\mu^*\Bigl(\bigcup_{n=1}^\infty A_n\Bigr) \le \sum_{n=1}^\infty \sum_{k=1}^\infty \mu(B_{n,k}) \le \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

Since ε is arbitrary, we arrive at (1.5.1). □

1.5.5. Lemma. *In the situation of the previous lemma, for any sets A and B such that $\mu^*(B) < \infty$ one has the inequality*

$$|\mu^*(A) - \mu^*(B)| \le \mu^*(A \triangle B). \tag{1.5.2}$$

PROOF. We observe that $A \subset B \cup (A \triangle B)$, whence by the subadditivity of μ^* we obtain the estimate

$$\mu^*(A) \le \mu^*(B) + \mu^*(A \triangle B),$$

i.e., $\mu^*(A) - \mu^*(B) \le \mu^*(A \triangle B)$. The estimate $\mu^*(B) - \mu^*(A) \le \mu^*(A \triangle B)$ is obtained in a similar manner. □

1.5.6. Theorem. *Let μ be a nonnegative countably additive set function on an algebra $\mathcal{A}$. Then:*

(i) *one has $\mathcal{A} \subset \mathcal{A}_\mu$, and the outer measure μ^* coincides with μ on $\mathcal{A}$;*

(ii) *the collection $\mathcal{A}_\mu$ of all μ-measurable sets is a σ-algebra and the restriction of μ^* to $\mathcal{A}_\mu$ is countably additive;*

(iii) *the function μ^* is a unique nonnegative countably additive extension of μ to the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$ and a unique nonnegative countably additive extension of μ to $\mathcal{A}_\mu$.*

PROOF. (i) It has already been noted that $\mathcal{A} \subset \mathcal{A}_\mu$. Let $A \in \mathcal{A}$ and $A \subset \bigcup_{n=1}^\infty A_n$, where $A_n \in \mathcal{A}$. Then $A = \bigcup_{n=1}^\infty (A \cap A_n)$. Hence by Proposition 1.3.9(iii) we have

$$\mu(A) \le \sum_{n=1}^\infty \mu(A \cap A_n) \le \sum_{n=1}^\infty \mu(A_n),$$

whence we obtain $\mu(A) \le \mu^*(A)$. By definition, $\mu^*(A) \le \mu(A)$. Therefore, $\mu(A) = \mu^*(A)$.

(ii) First we observe that the complement of a measurable set A is measurable. This is seen from the formula $(X\backslash A) \triangle (X\backslash A_\varepsilon) = A \triangle A_\varepsilon$. Next, the union of two measurable sets A and B is measurable. Indeed, let $\varepsilon > 0$ and let A_ε, $B_\varepsilon \in \mathcal{A}$ be such that $\mu^*(A \triangle A_\varepsilon) < \varepsilon/2$ and $\mu^*(B \triangle B_\varepsilon) < \varepsilon/2$. Since

$$(A \cup B) \triangle (A_\varepsilon \cup B_\varepsilon) \subset (A \triangle A_\varepsilon) \cup (B \triangle B_\varepsilon),$$

one has

$$\mu^*\Big((A \cup B) \triangle (A_\varepsilon \cup B_\varepsilon)\Big) \le \mu^*\Big((A \triangle A_\varepsilon) \cup (B \triangle B_\varepsilon)\Big) < \varepsilon.$$

Therefore, $A \cup B \in \mathcal{A}_\mu$. In addition, by what has already been proven, we have $A \cap B = X\backslash\big((X\backslash A) \cup (X\backslash B)\big) \in \mathcal{A}_\mu$. Hence $\mathcal{A}_\mu$ is an algebra.

Let us now establish two less obvious properties of the outer measure. First we verify its additivity on $\mathcal{A}_\mu$. Let $A, B \in \mathcal{A}_\mu$, where $A \cap B = \varnothing$. Let us fix $\varepsilon > 0$ and find $A_\varepsilon, B_\varepsilon \in \mathcal{A}$ such that

$$\mu^*(A \triangle A_\varepsilon) < \varepsilon/2 \quad \text{and} \quad \mu^*(B \triangle B_\varepsilon) < \varepsilon/2.$$

By Lemma 1.5.5, taking into account that μ^* and μ coincide on $\mathcal{A}$, we obtain

$$\mu^*(A \cup B) \ge \mu(A_\varepsilon \cup B_\varepsilon) - \mu^*\Big((A \cup B) \triangle (A_\varepsilon \cup B_\varepsilon)\Big). \tag{1.5.3}$$

By the inclusion $(A \cup B) \triangle (A_\varepsilon \cup B_\varepsilon) \subset (A \triangle A_\varepsilon) \cup (B \triangle B_\varepsilon)$ and the subadditivity of μ^* one has the inequality

$$\mu^*\Big((A \cup B) \triangle (A_\varepsilon \cup B_\varepsilon)\Big) \le \mu^*(A \triangle A_\varepsilon) + \mu^*(B \triangle B_\varepsilon) \le \varepsilon. \tag{1.5.4}$$

By the inclusion $A_\varepsilon \cap B_\varepsilon \subset (A \triangle A_\varepsilon) \cup (B \triangle B_\varepsilon)$ we have

$$\mu(A_\varepsilon \cap B_\varepsilon) = \mu^*(A_\varepsilon \cap B_\varepsilon) \le \mu^*(A \triangle A_\varepsilon) + \mu^*(B \triangle B_\varepsilon) \le \varepsilon.$$

Hence the estimates $\mu(A_\varepsilon) \ge \mu^*(A) - \varepsilon/2$ and $\mu(B_\varepsilon) \ge \mu^*(B) - \varepsilon/2$ yield

$$\mu(A_\varepsilon \cup B_\varepsilon) = \mu(A_\varepsilon) + \mu(B_\varepsilon) - \mu(A_\varepsilon \cap B_\varepsilon) \ge \mu^*(A) + \mu^*(B) - 2\varepsilon.$$

Taking into account relationships (1.5.3) and (1.5.4) we obtain

$$\mu^*(A \cup B) \ge \mu^*(A) + \mu^*(B) - 3\varepsilon.$$

Since ε is arbitrary, one has $\mu^*(A \cup B) \ge \mu^*(A) + \mu^*(B)$. By the reverse inequality $\mu^*(A \cup B) \le \mu^*(A) + \mu^*(B)$, we conclude that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

The next important step is a verification of the fact that countable unions of measurable sets are measurable. It suffices to prove this for disjoint sets $A_n \in \mathcal{A}_\mu$. Indeed, in the general case one can write $B_n = A_n \backslash \bigcup_{k=1}^{n-1} A_k$. Then the sets B_n are pairwise disjoint and measurable according to what we have already proved; they have the same union as the sets A_n. Dealing now with disjoint sets, we observe that by the finite additivity of μ^* on $\mathcal{A}_\mu$ the following relations are valid:

$$\sum_{k=1}^{n} \mu^*(A_k) = \mu^*\Big(\bigcup_{k=1}^{n} A_k\Big) \le \mu^*\Big(\bigcup_{k=1}^{\infty} A_k\Big) \le \mu(X).$$

Hence $\sum\limits_{k=1}^{\infty} \mu^*(A_k) < \infty$. Let $\varepsilon > 0$. We can find n such that

$$\sum_{k=n+1}^{\infty} \mu^*(A_k) < \frac{\varepsilon}{2}.$$

By using the measurability of finite unions one can find a set $B \in \mathcal{A}$ such that $\mu^*\Bigl(\bigl(\bigcup_{k=1}^n A_k\bigr) \triangle B\Bigr) < \varepsilon/2$. Since

$$\Bigl(\bigcup_{k=1}^\infty A_k\Bigr) \triangle B \subset \Bigl(\Bigl(\bigcup_{k=1}^n A_k\Bigr) \triangle B\Bigr) \cup \Bigl(\bigcup_{k=n+1}^\infty A_k\Bigr),$$

we obtain

$$\begin{aligned}\mu^*\Bigl(\Bigl(\bigcup_{k=1}^\infty A_k\Bigr) \triangle B\Bigr) &\le \mu^*\Bigl(\Bigl(\bigcup_{k=1}^n A_k\Bigr) \triangle B\Bigr) + \mu^*\Bigl(\bigcup_{k=n+1}^\infty A_k\Bigr)\\ &\le \frac{\varepsilon}{2} + \sum_{k=n+1}^\infty \mu^*(A_k) < \varepsilon.\end{aligned}$$

Thus, $\bigcup_{k=1}^\infty A_k$ is measurable. Therefore, $\mathcal{A}_\mu$ is a σ-algebra. It remains to note that the additivity and countable subadditivity of μ^* on $\mathcal{A}_\mu$ yield the countable additivity (see Proposition 1.3.9).

(iii) We observe that $\sigma(\mathcal{A}) \subset \mathcal{A}_\mu$, since $\mathcal{A}_\mu$ is a σ-algebra containing $\mathcal{A}$. Let ν be some nonnegative countably additive extension of μ to $\sigma(\mathcal{A})$. Let $A \in \sigma(\mathcal{A})$ and $\varepsilon > 0$. It has been proven that $A \in \mathcal{A}_\mu$, hence there exists $B \in \mathcal{A}$ with $\mu^*(A \triangle B) < \varepsilon$. Therefore, there exist sets $C_n \in \mathcal{A}$ such that $A \triangle B \subset \bigcup_{n=1}^\infty C_n$ and $\sum_{n=1}^\infty \mu(C_n) < \varepsilon$. Then we obtain

$$|\nu(A) - \nu(B)| \le \nu(A \triangle B) \le \sum_{n=1}^\infty \nu(C_n) = \sum_{n=1}^\infty \mu(C_n) < \varepsilon.$$

Since $\nu(B) = \mu(B) = \mu^*(B)$, we finally obtain

$$\begin{aligned}|\nu(A) - \mu^*(A)| &= |\nu(A) - \nu(B) + \mu^*(B) - \mu^*(A)|\\ &\le |\nu(A) - \nu(B)| + |\mu^*(B) - \mu^*(A)| \le 2\varepsilon.\end{aligned}$$

We arrive at the equality $\nu(A) = \mu^*(A)$ because ε is arbitrary. This reasoning also shows the uniqueness of a nonnegative countably additive extension of μ to $\mathcal{A}_\mu$, since we have only used that $A \in \mathcal{A}_\mu$ (however, as noted below, it is important that we deal with nonnegative extensions). $\square$

A control question: where does the above proof employ the countable additivity of μ?

1.5.7. Example. Let $\mathcal{A}$ be the algebra of all finite subsets of $\mathbb{N}$ and their complements and let μ equal 0 on finite sets and 1 on their complements. Then μ is additive and the single-point sets $\{n\}$ cover $\mathbb{N}$, hence $\mu^*(\mathbb{N}) = 0 < \mu(\mathbb{N})$.

It is worth noting that in the above theorem μ has no signed countably additive extensions from $\mathcal{A}$ to $\sigma(\mathcal{A})$, which follows by (iii) and the Jordan decomposition constructed in Chapter 3 (see §3.1), but it may have signed extensions to $\mathcal{A}_\mu$. For example, this happens if we take $X = \{0, 1\}$ and let $\mathcal{A} = \sigma(\mathcal{A}) = \{\varnothing, X\}$, $\mu \equiv 0$, $\nu(\{0\}) = 1$, $\nu(\{1\}) = -1$, $\nu(X) = 0$.

An important special case, to which the extension theorem applies, is the situation of Example 1.4.5. Since the σ-algebra generated by the cubes with edges parallel to the coordinate axes is the Borel σ-algebra, we obtain a countably additive Lebesgue measure λ_n on the Borel σ-algebra of the cube (and even on a larger σ-algebra), which extends the elementary volume. This measure is considered in greater detail in §1.7. By Theorem 1.5.6, the Lebesgue measure of any Borel (as well as any measurable) set B in the cube is $\lambda_n^*(B)$. Now the question arises why we do not define at once the measure on the Borel σ-algebra of the cube by this formula. The point is that there is a difficulty in the verification of the additivity of the obtained set function. This difficulty is circumvented by considering the algebra generated by the parallelepipeds, where the additivity is obvious.

With the aid of the proven theorem one can give a new description of measurable sets.

1.5.8. Corollary. *Let μ be a nonnegative countably additive set function on an algebra $\mathcal{A}$. A set A is μ-measurable precisely when there exist two sets $A', A'' \in \sigma(\mathcal{A})$ such that*

$$A' \subset A \subset A'' \quad \textit{and} \quad \mu^*(A''\backslash A') = 0.$$

Moreover, one can take for A' a set of the form $\bigcup_{n=1}^\infty \bigcap_{k=1}^\infty A_{n,k}$, $A_{n,k} \in \mathcal{A}$, and for A'' a set of the form $\bigcap_{n=1}^\infty \bigcup_{k=1}^\infty B_{n,k}$, $B_{n,k} \in \mathcal{A}$.

Proof. Let $A \in \mathcal{A}_\mu$. Then, for any $\varepsilon > 0$, there exists a set $A_\varepsilon \in \sigma(\mathcal{A})$ such that $A \subset A_\varepsilon$ and $\mu^*(A) \geq \mu^*(A_\varepsilon) - \varepsilon$. Indeed, by definition there exist sets $A_n \in \mathcal{A}$ with $A \subset \bigcup_{n=1}^\infty A_n$ and $\mu^*(A) \geq \sum_{n=1}^\infty \mu(A_n) - \varepsilon$. Let $A_\varepsilon = \bigcup_{n=1}^\infty A_n$. It is clear that $A \subset A_\varepsilon$, $A_\varepsilon \in \sigma(\mathcal{A}) \subset \mathcal{A}_\mu$ and by the countable additivity of μ^* on $\mathcal{A}_\mu$ we have $\mu^*(A_\varepsilon) \leq \sum_{n=1}^\infty \mu(A_n)$. Set

$$A'' = \bigcap_{n=1}^\infty A_{1/n}.$$

Then $A \subset A'' \in \sigma(\mathcal{A}) \subset \mathcal{A}_\mu$ and $\mu^*(A) = \mu^*(A'')$, since

$$\mu^*(A) \geq \mu^*(A_{1/n}) - 1/n \geq \mu^*(A'') - 1/n$$

for all n. Note that for constructing A'' the measurability of A is not needed. Let us apply this to the complement of A and find a set $B \in \sigma(\mathcal{A}) \subset \mathcal{A}_\mu$ such that $X\backslash A \subset B$ and $\mu(B) = \mu^*(X\backslash A)$. Set $A' = X\backslash B$. Then we obtain $A' \subset A$, and by the additivity of μ^* on the σ-algebra $\mathcal{A}_\mu$ and the inclusion $A, B \in \mathcal{A}_\mu$ we have

$$\mu^*(A') = \mu(X) - \mu^*(B) = \mu(X) - \mu^*(X\backslash A) = \mu^*(A),$$

which is the required relation. Conversely, suppose that such sets A' and A'' exist. Since A is the union of A' and a subset of $A''\backslash A'$, it suffices to verify that every subset C in $A''\backslash A'$ belongs to $\mathcal{A}_\mu$. This is indeed true because $\mu^*(C) \leq \mu^*(A''\backslash A') = \mu^*(A'') - \mu^*(A') = 0$ by the additivity of μ^* on $\mathcal{A}_\mu$ and the inclusion $A'', A' \in \sigma(\mathcal{A}) \subset \mathcal{A}_\mu$. $\square$

The uniqueness of extension yields the following useful result.

1.5.9. Corollary. *For the equality of two nonnegative Borel measures μ and ν on the real line it is necessary and sufficient that they coincide on all open intervals* (*or all closed intervals*).

Proof. Any closed interval is the intersection of a decreasing sequence of open intervals and any open interval is the union of an increasing sequence of closed intervals. By the countable additivity the equality of μ and ν on open intervals is equivalent to their equality on closed intervals and implies the equality of both measures on the algebra generated by intervals in $\mathbb{R}^1$. Since this algebra generates $\mathcal{B}(\mathbb{R}^1)$, our assertion follows by the uniqueness of a countably additive extension from an algebra to the generated σ-algebra. $\square$

The countably additive extension described in Theorem 1.5.6 is called the *Lebesgue extension* or the *Lebesgue completion* of the measure μ, and the measure space $(X, \mathcal{A}_\mu, \mu)$ is called the Lebesgue completion of $(X, \mathcal{A}, \mu)$. In addition, $\mathcal{A}_\mu$ is called the Lebesgue completion of the σ-algebra $\mathcal{A}$ with respect to μ. This terminology is related to the fact that the measure μ on $\mathcal{A}_\mu$ is complete in the sense of the following definition.

1.5.10. Definition. *A nonnegative countably additive measure μ on a σ-algebra $\mathcal{A}$ is called complete if $\mathcal{A}$ contains all subsets of every set in $\mathcal{A}$ with μ-measure zero. In this case we say that the σ-algebra $\mathcal{A}$ is complete with respect to the measure μ.*

It is clear from the definition of outer measure that if $A \subset B \in \mathcal{A}_\mu$ and $\mu(B) = 0$, then $A \in \mathcal{A}_\mu$ and $\mu(A) = 0$. It is easy to construct an example of a countably additive measure on a σ-algebra that is not complete: it suffices to take the identically zero measure on the σ-algebra consisting of the empty set and the interval $[0, 1]$. As a less trivial example let us mention Lebesgue measure on the σ-algebra of all Borel subsets of the interval constructed according to Example 1.4.4. This measure is considered below in greater detail; we shall see that there exist compact sets of zero Lebesgue measure containing non-Borel subsets.

Let us note the following simple but useful criterion of measurability of a set in terms of outer measure (which is, as already remarked, the original Lebesgue definition).

1.5.11. Proposition. *Let μ be a nonnegative countably additive measure on an algebra $\mathcal{A}$. Then, a set A belongs to $\mathcal{A}_\mu$ if and only if one has*

$$\mu^*(A) + \mu^*(X \backslash A) = \mu(X).$$

This is also equivalent to the equality $\mu^*(E \cap A) + \mu^*(E \backslash A) = \mu^*(E)$ for all sets $E \subset X$.

Proof. Let us verify the sufficiency of the first condition (then the stronger second one is sufficient too). Let us find μ-measurable sets B and C such that $A \subset B$, $X \backslash A \subset C$, $\mu(B) = \mu^*(A)$, $\mu(C) = \mu^*(X \backslash A)$. The existence of such sets has been established in the proof of Corollary 1.5.8. Clearly, $D = X\backslash C \subset A$ and

$$\mu(B) - \mu(D) = \mu(B) + \mu(C) - \mu(X) = 0.$$

Hence $\mu^*(A \bigtriangleup B) = 0$, whence the measurability of A follows.

Let us now prove that the second condition above is necessary. By the subadditivity of the outer measure it suffices to verify that $\mu^*(E\cap A)+\mu^*(E\backslash A) \leq \mu^*(E)$ for any $E \subset X$ and any measurable A. It follows from (1.5.2) that it suffices to establish this inequality for all $A \in \mathcal{A}$. Let $\varepsilon > 0$ and let sets $A_n \in \mathcal{A}$ be such that $E \subset \bigcup_{n=1}^\infty A_n$ and $\mu^*(E) \geq \sum_{n=1}^\infty \mu(A_n) - \varepsilon$. Then $E \cap A \subset \bigcup_{n=1}^\infty (A_n \cap A)$ and $E\backslash A \subset \bigcup_{n=1}^\infty (A_n\backslash A)$, whence we obtain

$$\begin{aligned}\mu^*(E \cap A) + \mu^*(E\backslash A) &\leq \sum_{n=1}^\infty \mu(A_n \cap A) + \sum_{n=1}^\infty \mu(A_n\backslash A)\\ &= \sum_{n=1}^\infty \mu(A_n) \leq \mu^*(E) + \varepsilon.\end{aligned}$$

Since ε is arbitrary, our claim is proven. □

Note that this criterion of measurability can be formulated as the equality $\mu^*(A) = \mu_*(A)$ if we define the inner measure by the equality

$$\mu_*(A) := \mu(X) - \mu^*(X\backslash A),$$

as Lebesgue actually did. It is important that in this case one must not use the definition of inner measure in the spirit of the Jordan measure as the supremum of measures of the sets from $\mathcal{A}$ inscribed in A. Below we shall return to the discussion of outer measures and see that the last property in Proposition 1.5.11 can be taken for a definition of measurability, which leads to very interesting results. In turn, this proposition will be extended to finitely additive set functions.

Let us observe that any set $A \in \mathcal{A}_\mu$ can be made a measure space by restricting μ to the class of μ-measurable subsets of A, which is a σ-algebra in A. The obtained measure μ_A (or $\mu|_A$) is called the restriction of μ to A. Restrictions to arbitrary sets are considered in §1.12(iv).

We close this section by proving the following property of continuity from below for outer measure.

1.5.12. Proposition. *Let μ be a nonnegative measure on a σ-algebra $\mathcal{A}$. Suppose that sets A_n are such that $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$. Then, one has*

$$\mu^*\Bigl(\bigcup_{n=1}^\infty A_n\Bigr) = \lim_{n\to\infty} \mu^*(A_n). \tag{1.5.5}$$

PROOF. According to Corollary 1.5.8, there exist μ-measurable sets B_n such that $A_n \subset B_n$ and $\mu(B_n) = \mu^*(A_n)$. Set

$$B = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty B_k.$$

One has $A_n \subset B_k$ if $k \geq n$, hence $A_n \subset B$ and $\bigcup_{n=1}^{\infty} A_n \subset B$. Therefore,

$$\mu^*\Bigl(\bigcup_{n=1}^{\infty} A_n\Bigr) \leq \mu(B) = \lim_{n\to\infty} \mu\Bigl(\bigcap_{k=n}^{\infty} B_k\Bigr) \leq \limsup_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu^*(A_n).$$

Since the reverse inequality is also true, the claim is proven. □

1.6. Infinite and σ-finite measures

We have so far been discussing finite measures, but one has to deal with infinite measures as well. The simplest (and most important) example is Lebesgue measure on $\mathbb{R}^n$. There are several ways of introducing set functions with infinite values. The first one is to admit set functions with values in the extended real line. For simplicity let us confine ourselves to nonnegative set functions. Let $c + \infty = \infty$ for any $c \in [0, +\infty]$. Now we can define the finite or countable additivity of set functions on algebras and σ-algebras (or rings, semirings, semialgebras) in the same way as above. In particular, we keep the definitions of outer measure and measurability. In this situation we use the term "a countably additive measure with values in $[0, +\infty]$". Similarly, one can consider measures with values in $(-\infty, +\infty]$ or $[-\infty, +\infty)$. A certain drawback of this approach is that rather pathological measures arise such as the countably additive measure that assigns $+\infty$ to all nonempty sets.

1.6.1. Definition. *Let $\mathcal{A}$ be a σ-algebra in a space X and let μ be a set function on $\mathcal{A}$ with values in $[0, +\infty]$ that satisfies the condition $\mu(\varnothing) = 0$ and is countably additive in the sense that $\mu\Bigl(\bigcup_{j=1}^{\infty} A_j\Bigr) = \sum_{j=1}^{\infty} \mu(A_j)$ for all pairwise disjoint sets $A_j \in \mathcal{A}$, where infinite values are admissible as well. Then μ is called a measure with values in $[0, +\infty]$. We call μ a σ-finite measure if $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \in \mathcal{A}$, $\mu(X_n) < \infty$.*

A desire to consider only measures with real but possibly unbounded values leads to modification of requirements on domains of definitions of measures; this is the second option. Here the concepts of a ring and δ-ring of sets introduced in Definition 1.2.13 become useful. For example, a natural domain of definition of Lebesgue measure on $\mathbb{R}^n$ could be the collection $\mathcal{L}_n^0$ of all sets of finite Lebesgue measure, i.e., all sets $E \subset \mathbb{R}^n$ such that measures of the sets $E_k := E \cap \{x\colon |x_i| \leq k,\ i = 1, \ldots, n\}$ in cubes (where we have already defined Lebesgue measure) are uniformly bounded in k. Lebesgue measure on $\mathcal{L}_n^0$ is given by the formula $\lambda_n(E) = \lim_{k\to\infty} \lambda_n(E_k)$. It is clear that the class $\mathcal{L}_n^0$ is a δ-ring. Lebesgue measure is countably additive on $\mathcal{L}_n^0$ (see below). In the next section we discuss the properties of Lebesgue measure on $\mathbb{R}^n$ in greater detail.

In what follows when considering infinite measures we always specify which definition we have in mind. Some additional information about measures with values in the extended real line (including their extensions and measurability with respect to such measures) is given in the final section and exercises.

1.6.2. Lemma. *Let $\mathcal{R}$ be a ring of subsets of a space X (i.e., $\mathcal{R}$ is closed with respect to finite intersections and unions, $\varnothing \in \mathcal{R}$ and $A \backslash B \in \mathcal{R}$ for all $A, B \in \mathcal{R}$). Let μ be a countably additive set function on $\mathcal{R}$ with values in $[0, +\infty]$ such that there exist sets $X_n \in \mathcal{R}$ with $X = \bigcup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$. Denote by μ_n the Lebesgue extension of the measure μ regarded on the set $S_n := \bigcup_{j=1}^{n} X_j$ equipped with the algebra of sets consisting of the intersections of elements in $\mathcal{R}$ with S_n. Let $\mathcal{L}_{\mu_n}$ denote the class of all μ_n-measurable sets. Let*

$$\mathcal{A} = \bigl\{A \subset X\colon\ A \cap S_n \in \mathcal{L}_{\mu_n}\ \forall\, n \in \mathbb{N},\ \overline{\mu}(A) := \lim_{n\to\infty} \mu_n(A \cap S_n) < \infty\bigr\}.$$

Then $\mathcal{A}$ is a ring closed with respect to countable intersections (i.e., a δ-ring) and $\overline{\mu}$ is a σ-additive measure whose restriction to every set S_n coincides with μ.

Proof. Let $A_i \in \mathcal{A}$ be pairwise disjoint sets with union in $\mathcal{A}$. We denote this union by A. For every n, the sets $A_i \cap S_n$ are disjoint too, hence

$$\mu_n(A \cap S_n) = \sum_{i=1}^{\infty} \mu_n(A_i \cap S_n).$$

Since $A \in \mathcal{A}$, the left-hand side of this equality is increasing to $\overline{\mu}(A)$. Therefore, $\sum_{i=1}^{\infty} \mu_n(A_i \cap S_n) \le \overline{\mu}(A)$ for all n, whence it follows by the equality $\lim\limits_{n\to\infty} \mu_n(A_i \cap S_n) = \overline{\mu}(A_i)$ for every i that $\sum_{i=1}^{\infty} \overline{\mu}(A_i) \le \overline{\mu}(A)$. This yields that $\overline{\mu}$ is a countably additive measure. Let $E \in \mathcal{R}$. Then the sets $E \cap \bigcup_{i=1}^{n} X_i$ belong to $\mathcal{R}$ and increase to E, which gives $\mu(E) = \overline{\mu}(E)$. Other claims are obvious. □

1.6.3. Remark. Suppose that in the situation of Lemma 1.6.2 the space X is represented as the union of another sequence of sets X_n' in $\mathcal{R}$ with finite measures. Then, as is clear from the lemma, this sequence yields the same extension of μ and the same class $\mathcal{A}$.

1.6.4. Example. Let $\mathcal{L}_n$ be the class of all sets $E \subset \mathbb{R}^n$ such that all the sets $E_k := E \cap \{x\colon |x_i| \le k,\ i = 1, \dots, n\}$ are Lebesgue measurable. Then $\mathcal{L}_n$ is a σ-algebra, on which the function $\lambda_n(E) = \lim\limits_{k\to\infty} \lambda_n(E_k)$ is a σ-finite measure (called Lebesgue measure on $\mathbb{R}^n$). The σ-algebra $\mathcal{L}_n$ contains the above-considered δ-ring $\mathcal{L}_n^0$. If we apply the previous lemma to the ring of all bounded Lebesgue measurable sets, then we arrive at the δ-ring $\mathcal{L}_n^0$.

In addition to Lebesgue measure, σ-finite measures arise as Haar measures on locally compact groups and Riemannian volumes on manifolds. Sometimes in diverse problems of analysis, algebra, geometry and probability theory one has to deal with products of finite and σ-finite measures. Although the list of infinite measures encountered in real problems is not very large, it is useful to have a terminology which enables one to treat various concrete examples in a unified way. Many of our earlier-obtained assertions remain valid for infinite measures. We only give the following result extending Theorem 1.5.6, which is directly seen from the reasoning there (the details of proof are left as Exercise 1.12.78); this result also follows from Theorem 1.11.8 below.

1.6.5. Proposition. *Let μ be a countably additive measure on an algebra $\mathcal{A}$ with values in $[0, +\infty]$. Then $\mathcal{A}_\mu$ is a σ-algebra, $\mathcal{A} \subset \mathcal{A}_\mu$, and the function μ^* is a countably additive measure on $\mathcal{A}_\mu$ with values in $[0, +\infty]$ and coincides with μ on $\mathcal{A}$.*

However, there are exceptions. For example, for infinite measures, the countable additivity does not imply that the measures of sets A_n monotonically decreasing to the empty set approach zero. The point is that all the sets A_n may have infinite measures. In many books measures are defined from the very beginning as functions with values in $[0, +\infty]$. Then, in theorems, one has often to impose various additional conditions (moreover, different in different theorems; the reader will find a lot of examples in the exercises on infinite measures in Chapters 1–4). It appears that at least in a graduate course it is better to first establish all theorems for bounded measures, then observe that most of them remain valid for σ-finite measures, and finally point out that further generalizations are possible, but they require additional hypotheses. Our exposition will be developed according to this principle.

1.7. Lebesgue measure

Let us return to the situation considered in Example 1.4.5 and briefly discussed after Theorem 1.5.6. Let I be a cube in $\mathbb{R}^n$ of the form $[a, b]^n$, $\mathcal{A}_0$ the algebra of finite unions of parallelepipeds in I with edges parallel to the coordinate axes. As we know, the usual volume λ_n is countably additive on $\mathcal{A}_0$. Therefore, one can extend λ_n to a countably additive measure, also denoted by λ_n, on the σ-algebra $\mathcal{L}_n(I)$ of all λ_n-measurable sets in I, which contains the Borel σ-algebra. We write $\mathbb{R}^n$ as the union of the increasing sequence of cubes $I_k = \{|x_i| \le k, i = 1, \ldots, n\}$ and denote by λ_n the σ-finite measure generated by Lebesgue measures on the cubes I_k according to the construction of the previous section (see Example 1.6.4). Let

$$\mathcal{L}_n = \{E \subset \mathbb{R}^n \colon\ E \cap I_k \in \mathcal{L}_n(I_k),\ \forall\, k \in \mathbb{N}\}.$$

1.7.1. Definition. *The above-defined measure λ_n on $\mathcal{L}_n$ is called Lebesgue measure on $\mathbb{R}^n$. The sets in $\mathcal{L}_n$ are called Lebesgue measurable.*

In the case where a subset of $\mathbb{R}^n$ is regarded with Lebesgue measure, it is customary to use the terms "measure zero set", "measurable set" etc. without explicitly mentioning Lebesgue measure. We also follow this tradition.

For defining Lebesgue measure of a set $E \in \mathcal{L}_n$ one can use the formula

$$\lambda_n(E) = \lim_{k\to\infty} \lambda_n(E \cap I_k)$$

as well as the formula

$$\lambda_n(E) = \sum_{j=1}^{\infty} \lambda_n(E \cap Q_j),$$

where Q_j are pairwise disjoint cubes that are translations of $[-1,1)^n$ and whose union is all of $\mathbb{R}^n$. Since the σ-algebra generated by the parallelepipeds of the above-mentioned form is the Borel σ-algebra $\mathcal{B}(I)$ of the cube I, we see that all Borel sets in the cube I, hence in $\mathbb{R}^n$ as well, are Lebesgue measurable.

Lebesgue measure can also be regarded on the δ-ring $\mathcal{L}_n^0$ of all sets of finite Lebesgue measure.

In the case of $\mathbb{R}^1$ Lebesgue measure of a set E is the sum of the series of $\lambda_1\bigl(E\cap(n,n+1]\bigr)$ over all integer numbers n.

The translation of a set A by a vector h, i.e., the set of all points of the form $a+h$, where $a\in A$, is denoted by $A+h$.

1.7.2. Lemma. *Let W be an open set in the cube $I=(-1,1)^n$. Then there exists an at most countable family of open pairwise disjoint cubes Q_j in W of the form $Q_j=c_jI+h_j$, $c_j>0$, $h_j\in W$, such that the set $W\backslash\bigcup_{j=1}^\infty Q_j$ has Lebesgue measure zero.*

PROOF. Let us employ Exercise 1.12.48 and write W as $W=\bigcup_{j=1}^\infty W_j$, where W_j are open cubes whose edges are parallel to the coordinate axes and have lengths $q2^{-p}$ with positive integer p,q, and whose centers have the coordinates of the form $l2^{-m}$ with integer l and positive integer m. Next we restructure the cubes W_j as follows. We delete all cubes W_j that are contained in W_1 and set $Q_1=W_1$. Let us take the first cube W_{n_2} in the remaining sequence and represent the interior of the body $W_{n_2}\backslash Q_1$ as the finite union of open pairwise disjoint cubes $Q_2,\ldots,Q_{m_2}$ of the same type as the cubes W_j and some pieces of the boundaries of these new cubes. This is possible by our choice of the initial cubes. Next we delete all the cubes W_j that are contained in $\bigcup_{i=1}^{m_2} Q_i$, take the first cube in the remaining sequence and construct a partition of its part that is not contained in the previously constructed cubes in the same way as explained above. Continuing the described process, we obtain pairwise disjoint cubes that cover W up to a measure zero set, namely, up to a countable union of boundaries of these cubes. $\square$

In Exercise 1.12.72, it is suggested that the reader modify this reasoning to make it work for any Borel measure. We have only used above that the boundaries of our cubes have measure zero. Note that the lengths of the edges of the constructed cubes are rational.

1.7.3. Theorem. *Let A be a Lebesgue measurable set of finite measure. Then*:

(i) $\lambda_n(A+h)=\lambda_n(A)$ *for any vector* $h\in\mathbb{R}^n$;

(ii) $\lambda_n\bigl(U(A)\bigr)=\lambda_n(A)$ *for any orthogonal linear operator* U *on* $\mathbb{R}^n$;

(iii) $\lambda_n(\alpha A)=|\alpha|^n\lambda_n(A)$ *for any real number* α.

PROOF. It follows from the definition of Lebesgue measure that it suffices to prove the listed properties for bounded measurable sets.

(i) Let us take a cube I centered at the origin such that the sets A and $A+h$ are contained in some cube inside I. Let $\mathcal{A}_0$ be the algebra generated by all cubes in I with edges parallel to the coordinate axes. When evaluating the outer measure of A it suffices to consider only sets $B \in \mathcal{A}_0$ with $B+h \subset I$. Since the volumes of sets in $\mathcal{A}_0$ are invariant under translations, the sets $A+h$ and A have equal outer measures. For every $\varepsilon > 0$, there exists a set $A_\varepsilon \in \mathcal{A}_0$ with $\lambda_n^*(A \bigtriangleup A_\varepsilon) < \varepsilon$. Then

$$\lambda_n^*\big((A+h) \bigtriangleup (A_\varepsilon + h)\big) = \lambda_n^*\big((A \bigtriangleup A_\varepsilon) + h\big) = \lambda_n^*(A \bigtriangleup A_\varepsilon) < \varepsilon,$$

whence we obtain the measurability of $A+h$ and the desired equality.

(ii) As in (i), it suffices to prove our claim for sets in $\mathcal{A}_0$. Hence it remains to show that, for any closed cube K with edges parallel to the coordinate axes, one has the equality

$$\lambda_n\big(U(K)\big) = \lambda_n(K). \tag{1.7.1}$$

Suppose that this is not true for some cube K, i.e.,

$$\lambda_n\big(U(K)\big) = r\lambda_n(K),$$

where $r \neq 1$. Let us show that for every ball $Q \subset I$ centered at the origin one has

$$\lambda_n\big(U(Q)\big) = r\lambda_n(Q) \quad \text{if} \quad U(Q) \subset I. \tag{1.7.2}$$

Let d be the length of the edge of K. Let us take an arbitrary natural number p and partition the cube K into p^n equal smaller closed cubes K_j that have equal edges of length d/p and disjoint interiors (i.e., may have in common only parts of faces). The cubes $U(K_j)$ are translations of each other and have equal measures as proved above. It is readily seen that faces of any cube have measure zero. Hence $\lambda_n\big(U(K)\big) = p^n\lambda_n\big(U(K_1)\big)$. Therefore, $\lambda_n\big(U(K_1)\big) = r\lambda_n(K_1)$. Then (1.7.2) is true for any cube of the form $qK+h$, where q is a rational number. This yields equality (1.7.2) for the ball Q. Indeed, by additivity this equality extends to finite unions of cubes with edges parallel to the coordinate axes. Next, for any $\varepsilon > 0$, one can find two such unions E_1 and E_2 with $E_1 \subset Q \subset E_2$ and $\lambda_n(E_2 \backslash E_1) < \varepsilon$. To this end, it suffices to take balls Q' and Q'' centered at the origin such that $Q' \subset Q \subset Q''$ with strict inclusions and a small measure of $Q'' \backslash Q'$. Then one can find a finite union E_1 of cubes of the indicated form with $Q' \subset E_1 \subset Q$ and an analogous union E_2 with $Q \subset E_2 \subset Q''$. It remains to observe that $U(Q) = Q$, and (1.7.2) leads to contradiction.

(iii) The last claim is obvious for sets in $\mathcal{A}_0$, hence as claims (i) and (ii), it extends to arbitrary measurable sets. □

It is worth noting that property (iii) of Lebesgue measure is a corollary of property (i), since by (i) it is valid for all cubes and $\alpha = 1/m$, where m is any natural number, then it extends to all rational α, which yields the general case by continuity. It is seen from the proof that property (ii) also follows from property (i). Property (i) characterizes Lebesgue measure up to a constant factor (see Exercise 1.12.74). There is an alternative derivation of property (ii) from properties (i) and (iii), employing the invariance of the ball with respect to rotations and the following theorem, which is very interesting in its own right.

1.7.4. Theorem. *Let W be a nonempty open set in $\mathbb{R}^n$. Then, there exists a countable collection of pairwise disjoint open balls $U_j \subset W$ such that the set $W \backslash \bigcup_{j=1}^{\infty} U_j$ has measure zero.*

PROOF. It suffices to prove the theorem in the case where $\lambda_n(W) < \infty$ (we may even assume that W is contained in a cube). Let $K = (-1,1)^n$ and let V be the open ball inscribed in K. It is clear that $\lambda_n(V) = \alpha\lambda_n(K)$, where $0 < \alpha < 1$. Set $q = 1 - \alpha$. Let us take a number $\beta > 1$ such that $q\beta < 1$. By Lemma 1.7.2, the set W can be written as the union of a measure zero set and a sequence of open pairwise disjoint cubes K_j of the form $K_j = c_j K + h_j$, where $c_j > 0$ and $h_j \in \mathbb{R}^n$. In every cube K_j we inscribe the open ball $V_j = c_j V + h_j$. Since $\lambda_n(V_j)/\lambda_n(K_j) = \alpha$, we obtain

$$\lambda_n(K_j \backslash V_j) = \lambda_n(K_j) - \lambda_n(V_j) = q\lambda_n(K_j).$$

Hence

$$\lambda_n\Bigl(W \backslash \bigcup_{j=1}^{\infty} V_j\Bigr) = \sum_{j=1}^{\infty} \lambda_n(K_j \backslash V_j) = q\sum_{j=1}^{\infty} \lambda_n(K_j) = q\lambda_n(W).$$

Let us take a finite number of these cubes such that

$$\lambda_n\Bigl(W \backslash \bigcup_{j=1}^{N_1} V_j\Bigr) \le \beta q \lambda_n(W).$$

Set $V_j^{(1)} = V_j$, $j \le N_1$. Let us repeat the described construction for the open set W_1 obtained from W by deleting the closures of the balls $V_1,\ldots,V_{N_1}$ (we observe that a finite union of closed sets is closed). We obtain pairwise disjoint open balls $V_j^{(2)} \subset W_1$, $j \le N_2$, such that

$$\lambda_n\Bigl(W_1 \backslash \bigcup_{j=1}^{N_2} V_j^{(2)}\Bigr) \le \beta q \lambda_n(W_1) \le (\beta q)^2 \lambda_n(W).$$

By induction, we obtain a countable family of pairwise disjoint open balls $V_j^{(k)}$, $j \le N_k$, with the following property: if Z_k is the union of the closures of the balls $V_1^{(k)},\ldots,V_{N_k}^{(k)}$ and $W_k = W_{k-1} \backslash Z_k$, where $W_0 = W$, then

$$\lambda_n\Bigl(W_k \backslash \bigcup_{j=1}^{N_{k+1}} V_j^{(k+1)}\Bigr) \le (\beta q)^{k+1} \lambda_n(W).$$

Since $(\beta q)^k \to 0$, the set $W \backslash \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{N_k} V_j^{(k)}$ has measure zero. □

It is clear that in the formulation of this theorem the balls U_j can be replaced by any sets of the form $c_j S + h_j$, where S is a fixed bounded set of positive measure. Indeed, the proof only employed the translation invariance of Lebesgue measure and the relation $\lambda_n(rA) = r^n \lambda_n(A)$ for $r > 0$. In

Chapter 5 (Corollary 5.8.3) this theorem will be extended to arbitrary Borel measures.

Note that it follows by Theorem 1.7.3 that Lebesgue measure of any rectangular parallelepiped $P \subset I$ (not necessarily with edges parallel to the coordinate axes) equals the product of lengths of its edges. Clearly, any countable set has Lebesgue measure zero. As the following example of the Cantor set (named after the outstanding German mathematician Georg Cantor) shows, there exist uncountable sets of Lebesgue measure zero as well.

1.7.5. Example. Let $I = [0, 1]$. Denote by $J_{1,1}$ the interval $(1/3, 2/3)$. Let $J_{2,1}$ and $J_{2,2}$ denote the intervals $(1/9, 2/9)$ and $(7/9, 8/9)$, which are the middle thirds of the intervals obtained after deleting $J_{1,1}$. Continue this process inductively by deleting the open middle intervals. After the nth step we obtain 2^n closed intervals; at the next step we delete their open middle thirds $J_{n+1,1}, \ldots, J_{n+1,2^n}$, after which there remains 2^{n+1} closed intervals, and the process continues. The set $C = I \backslash \bigcup_{n,j} J_{n,j}$ is called the Cantor set. It is compact, has cardinality of the continuum, but its Lebesgue measure is zero.

PROOF. The set C is compact, since its complement is open. In order to see that C has cardinality of the continuum, we write the points in $[0, 1]$ in the ternary expansion, i.e., $x = \sum_{j=1}^{\infty} x_j 3^{-j}$, where x_j takes values $0, 1, 2$. As in the decimal expansion, this representation is not unique, since, for example, the sequence $(1, 1, 2, 2, \ldots)$ corresponds to the same number as the sequence $(1, 2, 0, 0, \ldots)$. However, this non-uniqueness is only possible for points of some countable set, which we denote by M. It is verified by induction that after the nth step of deleting there remain the points x such that $x_j = 0$ or $x_j = 2$ if $j \leq n$. Thus, $C \backslash M$ consists of all points whose ternary expansion involves only 0 and 2, whence it follows that C has cardinality of the set of all reals. Finally, in order to show that C has zero measure, it remains to verify that the complement of C in $[0, 1]$ has measure 1. By induction one verifies that the measure of the set $J_{n,1} \cup \cdots \cup J_{n,2^{n-1}}$ equals $2^{n-1}3^{-n}$. Since $\sum_{n=1}^{\infty} 2^{n-1}3^{-n} = 1$, our claim is proven. □

1.7.6. Example. Let $\varepsilon > 0$ and let $\{r_n\}$ be the set of all rational numbers in $[0, 1]$. Set $K = [0, 1] \backslash \bigcup_{n=1}^{\infty} (r_n - \varepsilon 4^{-n}, r_n + \varepsilon 4^{-n})$. Then K is a compact set without inner points and its Lebesgue measure is not less than $1 - \varepsilon$ because the measure of the complement does not exceed $2\varepsilon \sum_{n=1}^{\infty} 4^{-n}$.

Thus, a compact set of positive measure may have the empty interior. A similar example (but with some additional interesting properties) can be constructed by a modification of the construction of the Cantor set. Namely, at every step one deletes a bit less than the middle third so that the sum of the deleted intervals becomes $1 - \varepsilon$.

Note that any subset of the Cantor set has measure zero, too. Therefore, the family of all measurable sets has cardinality equal to that of the class of all subsets of the real line. As we shall see below, the Borel σ-algebra has cardinality of the continuum. Hence among subsets of the Cantor set there are non-Borel Lebesgue measurable sets. The existence of non-Borel Lebesgue measurable sets will be established below in a more constructive way by means of the Souslin operation.

Now the question naturally arises how large the class of all Lebesgue measurable sets is and whether it includes all the sets. It turns out that an answer to this question depends on additional set-theoretic axioms and cannot be given in the framework of the "naive set theory" without the axiom of choice. In any case, as the following example due to Vitali shows, by means of the axiom of choice it is easy to find an example of a nonmeasurable (in the Lebesgue sense) set.

1.7.7. Example. Let us declare two points x and y in $[0,1]$ equivalent if the number $x-y$ is rational. It is clear that the obtained relation is indeed an equivalence relation, i.e., 1) $x \sim x$, 2) $y \sim x$ if $x \sim y$, 3) $x \sim z$ if $x \sim y$ and $y \sim z$. Hence we obtain the equivalence classes each of which contains points with rational mutual differences, and the differences between any representatives of different classes are irrational. Let us now choose in every class exactly one representative and denote the constructed set by E. It is the axiom of choice that enables one to construct such a set. The set E cannot be Lebesgue measurable. Indeed, if its measure equals zero, then the measure of $[0,1]$ equals zero as well, since $[0,1]$ is covered by countably many translations of E by rational numbers. The measure of E cannot be positive, since for different rational p and q, the sets $E+p$ and $E+q$ are disjoint and have equal positive measures. One has $E+p \subset [0,2]$ if $p \in [0,1]$, hence the interval $[0,2]$ would have infinite measure.

However, one should have in mind that the axiom of choice may be replaced by a proposition (added to the standard set-theoretic axioms) that makes all subsets of the real line measurable. Some remarks about this are made in §1.12(x).

Note also that even if we use the axiom of choice, there still remains the question: does there exist *some* extension of Lebesgue measure to a countably additive measure on the class of all subsets of the interval? The above example only says that such an extension cannot be obtained by means of the Lebesgue completion. An answer to this question also depends on additional set-theoretic axioms (see §1.12(x)). In any case, the Lebesgue extension is not maximal: by Theorem 1.12.14, for every set $E \subset [0,1]$ that is not Lebesgue measurable, one can extend Lebesgue measure to a countably additive measure on the σ-algebra generated by all Lebesgue measurable sets in $[0,1]$ and the set E.

Closing our discussion of the properties of Lebesgue measure let us mention the Jordan (Peano–Jordan) measure.

1.7.8. Definition. *A bounded set E in $\mathbb{R}^n$ is called Jordan measurable if, for each $\varepsilon > 0$, there exist sets U_ε and V_ε that are finite unions of cubes such that $U_\varepsilon \subset E \subset V_\varepsilon$ and $\lambda_n(V_\varepsilon \backslash U_\varepsilon) < \varepsilon$.*

It is clear that when $\varepsilon \to 0$, there exists a common limit of the measures of U_ε and V_ε, called the Jordan measure of the set E. It is seen from the definition that every Jordan measurable set E is Lebesgue measurable and its Lebesgue measure coincides with its Jordan measure. However, the converse is false: for example, the set of rational numbers in the interval is not Jordan measurable. The collection of all Jordan measurable sets is a ring (see Exercise 1.12.77), on which the Jordan measure coincides with Lebesgue measure. Certainly, the Jordan measure is countably additive on its domain and its Lebesgue extension is Lebesgue measure. In Exercise 3.10.75 one can find a useful sufficient condition of the Jordan measurability.

1.8. Lebesgue–Stieltjes measures

Let μ be a nonnegative Borel measure on $\mathbb{R}^1$. Then the function

$$t \mapsto F(t) = \mu\bigl((-\infty, t)\bigr)$$

is bounded, nondecreasing (i.e., $F(t) \le F(s)$ whenever $t \le s$; such functions are also called increasing), left continuous, i.e., $F(t_n) \to F(t)$ as $t_n \uparrow t$, which follows by the countable additivity μ, and one has $\lim\limits_{t\to-\infty} F(t) = 0$. These conditions turn out also to be sufficient in order that the function F be generated by some measure according to the above formula. The function F is called the *distribution function* of the measure μ. Note that the distribution function is often defined by the formula $F(t) = \mu\bigl((-\infty, t]\bigr)$, which leads to different values at the points of positive μ-measure (the jumps of the function F are exactly the points of positive μ-measure).

1.8.1. Theorem. *Let F be a bounded, nondecreasing, left continuous function with $\lim\limits_{t\to-\infty} F(t) = 0$. Then, there exists a unique nonnegative Borel measure on $\mathbb{R}^1$ such that*

$$F(t) = \mu\bigl((-\infty, t)\bigr) \quad \textit{for all } t \in \mathbb{R}^1.$$

PROOF. It is known from the elementary calculus that the function F has an at most countable set D of points of discontinuity. Clearly, there is a countable set S in $\mathbb{R}^1\backslash D$ that is everywhere dense in $\mathbb{R}^1$. Let us consider the class $\mathcal{A}$ of all sets of the form $A = \bigcup_{i=1}^n J_i$, where J_i is an interval of one of the following four types: (a, b), $[a, b]$, $(a, b]$ or $[a, b)$, where a and b either belong to S or coincide with $-\infty$ or $+\infty$. It is readily seen that $\mathcal{A}$ is an algebra. Let us define the set function μ on $\mathcal{A}$ as follows: if A is an interval with endpoints a and b, where $a \le b$, then $\mu(A) = F(b) - F(a)$, and if A is a finite union of disjoint intervals J_i, then $\mu(A) = \sum_i \mu(J_i)$. It is clear that the function μ is well-defined and additive. For the proof of countable additivity μ on $\mathcal{A}$, it suffices to observe that the class of finite unions of compact intervals is compact and is approximating. Indeed, if J is an open or semiopen interval, e.g., $J = (a, b)$, where a and b belong to S (or coincide with the points $+\infty$, $-\infty$), then, by the continuity of F at the points of S, we have

$F(b)-F(a) = \lim_{i\to\infty}[F(b_i)-F(a_i)]$, where $a_i \downarrow a$, $b_i \uparrow b$, $a_i,\ b_i \in S$. If $a=-\infty$, then the same follows by the condition $\lim_{t\to-\infty} F(t) = 0$. Let us extend μ to a countably additive measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^1)$ (note that $\mathcal{B}(\mathbb{R}^1)$ is generated by the algebra $\mathcal{A}$, since S is dense). We have $F(t) = \mu\big((-\infty,t)\big)$ for all t (and not only for $t\in S$). This follows by the left continuity of both functions and their coincidence on a countable everywhere dense set. The uniqueness of μ is clear from the fact that the function F uniquely determines the values of μ on intervals.

We observe that due to Proposition 1.3.10, we could also use the semialgebra of semiclosed intervals of the form $(-\infty,b)$, $[a,b)$, $[a,+\infty)$, where $a,b\in S$. □

The measure μ constructed from the function F as described above is called the Lebesgue–Stieltjes measure with distribution function F. Similarly, by means of the distribution functions of n variables (representing measures of sets $(-\infty,x_1)\times\cdots\times(-\infty,x_n)$) one defines Lebesgue–Stieltjes measures on $\mathbb{R}^n$ (see Exercise 1.12.156).

1.9. Monotone and σ-additive classes of sets

In this section, we consider two more classes of sets that are frequently used in measure theory.

1.9.1. Definition. *A family $\mathcal{E}$ of subsets of a set X is called a monotone class if $\bigcup_{n=1}^\infty E_n \in \mathcal{E}$ for every increasing sequence of sets $E_n\in\mathcal{E}$ and $\bigcap_{n=1}^\infty E_n \in \mathcal{E}$ for every decreasing sequence of sets $E_n \in \mathcal{E}$.*

1.9.2. Definition. *A family $\mathcal{E}$ of subsets of a set X is called a σ-additive class if the following conditions are fulfilled:*

(i) *$X\in\mathcal{E}$,*

(ii) *$E_2\backslash E_1 \in \mathcal{E}$ provided that $E_1,\ E_2\in\mathcal{E}$ and $E_1\subset E_2$,*

(iii) *$\bigcup_{n=1}^\infty E_n\in\mathcal{E}$ provided that $E_n\in\mathcal{E}$ are pairwise disjoint.*

Note that in the presence of conditions (i) and (ii), condition (iii) can be restated as follows: $E_1\cup E_2\in\mathcal{E}$ for every disjoint pair $E_1,\ E_2\in\mathcal{E}$ and $\bigcup_{n=1}^\infty E_n\in\mathcal{E}$ whenever $E_n\in\mathcal{E}$ and $E_n\subset E_{n+1}$ for all $n\in\mathbb{N}$.

Given a class $\mathcal{E}$ of subsets of X, we have the smallest monotone class containing $\mathcal{E}$ (called the monotone class generated by $\mathcal{E}$), and the smallest σ-additive class containing $\mathcal{E}$ (called the σ-additive class generated by $\mathcal{E}$). These minimal classes are, respectively, the intersections of all monotone and all σ-additive classes containing $\mathcal{E}$.

The next result called the monotone class theorem is frequently used in measure theory.

1.9.3. Theorem. (i) *Let $\mathcal{A}$ be an algebra of sets. Then the σ-algebra generated by $\mathcal{A}$ coincides with the monotone class generated by $\mathcal{A}$.*

(ii) *If the class $\mathcal{E}$ is closed under finite intersections, then the σ-additive class generated by $\mathcal{E}$ coincides with the σ-algebra generated by $\mathcal{E}$.*

PROOF. (i) Denote by $\mathcal{M}(\mathcal{A})$ the monotone class generated by $\mathcal{A}$. Since $\sigma(\mathcal{A})$ is a monotone class, one has $\mathcal{M}(\mathcal{A}) \subset \sigma(\mathcal{A})$. Let us prove the inverse inclusion. To this end, let us show that $\mathcal{M}(\mathcal{A})$ is a σ-algebra. It suffices to prove that $\mathcal{M}(\mathcal{A})$ is an algebra. We show first that the class $\mathcal{M}(\mathcal{A})$ is closed with respect to complementation. Let

$$\mathcal{M}_0 = \{B\colon\ B, X\backslash B \in \mathcal{M}(\mathcal{A})\}.$$

The class $\mathcal{M}_0$ is monotone, which is obvious, since $\mathcal{M}(\mathcal{A})$ is a monotone class and one has the equalities

$$X\backslash \bigcap_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} (X\backslash B_n), \quad X\backslash \bigcup_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} (X\backslash B_n).$$

Since $\mathcal{A} \subset \mathcal{M}_0 \subset \mathcal{M}(\mathcal{A})$, one has $\mathcal{M}_0 = \mathcal{M}(\mathcal{A})$.

Let us verify that $\mathcal{M}(\mathcal{A})$ is closed with respect to finite intersections. Let $A \in \mathcal{M}(\mathcal{A})$. Set

$$\mathcal{M}_A = \bigl\{B \in \mathcal{M}(\mathcal{A})\colon\ A \cap B \in \mathcal{M}(\mathcal{A})\bigr\}.$$

If $B_n \in \mathcal{M}_A$ are monotonically increasing sets, then

$$A \cap \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} (A \cap B_n) \in \mathcal{M}(\mathcal{A}).$$

The case where the sets B_n are decreasing is similar. Hence $\mathcal{M}_A$ is a monotone class. If $A \in \mathcal{A}$, then we have $\mathcal{A} \subset \mathcal{M}_A \subset \mathcal{M}(\mathcal{A})$, whence we obtain that $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$. Now let $A \in \mathcal{A}$ and $B \in \mathcal{M}(\mathcal{A})$. Then, according to the equality $\mathcal{M}(\mathcal{A}) = \mathcal{M}_A$, we have $A \cap B \in \mathcal{M}(\mathcal{A})$, which gives $A \in \mathcal{M}_B$. Thus, $\mathcal{A} \subset \mathcal{M}_B \subset \mathcal{M}(\mathcal{A})$. Therefore, $\mathcal{M}_B = \mathcal{M}(\mathcal{A})$ for all $B \in \mathcal{M}(\mathcal{A})$, which means that $\mathcal{M}(\mathcal{A})$ is closed with respect to finite intersections. It follows that $\mathcal{M}(\mathcal{A})$ is an algebra as required.

(ii) Denote by $\mathcal{S}$ the σ-additive class generated by $\mathcal{E}$. It is clear that $\mathcal{S} \subset \sigma(\mathcal{E})$, since $\sigma(\mathcal{E})$ is a σ-additive class. Let us show the inverse inclusion. To this end, we show that $\mathcal{S}$ is a σ-algebra. It suffices to verify that the class $\mathcal{S}$ is closed with respect to finite intersections. Set

$$\mathcal{S}_0 = \{A \in \mathcal{S}\colon\ A \cap E \in \mathcal{S} \ \text{ for all } E \in \mathcal{E}\}.$$

Note that $\mathcal{S}_0$ is a σ-additive class. Indeed, $X \in \mathcal{S}_0$. Let $A, B \in \mathcal{S}_0$ and $A \subset B$. Then, for any $E \in \mathcal{E}$, we have $(B\backslash A) \cap E = (B \cap E)\backslash(A \cap E) \in \mathcal{S}$, since the intersections $A \cap E, B \cap E$ belong to $\mathcal{S}$ and $\mathcal{S}$ is a σ-additive class. Similarly, it is verified that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}_0$ for any pairwise disjoint sets $A_n \in \mathcal{S}_0$. Since $\mathcal{E} \subset \mathcal{S}_0$, one has $\mathcal{S}_0 = \mathcal{S}$. Thus, $A \cap E \in \mathcal{S}$ for all $A \in \mathcal{S}$ and $E \in \mathcal{E}$. Now set

$$\mathcal{S}_1 = \{A \in \mathcal{S}\colon\ A \cap B \in \mathcal{S} \ \text{ for all } B \in \mathcal{S}\}.$$

Let us show that $\mathcal{S}_1$ is a σ-additive class. Indeed, $X \in \mathcal{S}_1$. If $A_1, A_2 \in \mathcal{S}_1$, $A_1 \subset A_2$, then $A_2\backslash A_1 \in \mathcal{S}_1$, since for all $B \in \mathcal{S}$, by the definition of $\mathcal{S}_1$, we obtain $(A_2\backslash A_1) \cap B = (A_2 \cap B)\backslash(A_1 \cap B) \in \mathcal{S}$. Similarly, it is verified that $\bigcup_{n=1}^{\infty} B_n \in \mathcal{S}_1$ for any sequence of disjoint sets in $\mathcal{S}_1$. Since $\mathcal{E} \subset \mathcal{S}_1$ as proved above, one has $\mathcal{S}_1 = \mathcal{S}$. Therefore, $A \cap B \in \mathcal{S}$ for all $A, B \in \mathcal{S}$. Thus, $\mathcal{S}$ is a σ-algebra. □

As an application of Theorem 1.9.3 we prove the following useful result.

1.9.4. Lemma. *If two probability measures μ and ν on a measurable space $(X, \mathcal{A})$ coincide on some class of sets $\mathcal{E} \subset \mathcal{A}$ that is closed with respect to finite intersections, then they coincide on the σ-algebra generated by $\mathcal{E}$.*

Proof. Let $\mathcal{B} = \bigl\{A \in \mathcal{A}\colon\ \mu(A) = \nu(A)\bigr\}$. By hypothesis, $X \in \mathcal{B}$. If $A, B \in \mathcal{B}$ and $A \subset B$, then $B\backslash A \in \mathcal{B}$. In addition, if sets A_i in $\mathcal{B}$ are pairwise disjoint, then their union also belongs to $\mathcal{B}$. Hence $\mathcal{B}$ is a σ-additive class. Therefore, the σ-additive class $\mathcal{S}$ generated by $\mathcal{E}$ is contained in $\mathcal{B}$. By Theorem 1.9.3(ii) one has $\mathcal{S} = \sigma(\mathcal{E})$. Therefore, $\sigma(\mathcal{E}) \subset \mathcal{B}$. □

1.10. Souslin sets and the A-operation

Let B be a Borel set in the plane and let A be its projection to one of the axes. Is A a Borel set? One can hardly imagine that the correct answer to this question is negative. This answer was found due to efforts of several eminent mathematicians investigating the structure of Borel sets. A result of those investigations was the creation of descriptive set theory, in particular, the invention of the A-operation. It was discovered that the continuous images of the Borel sets coincide with the result of application of the A-operation to the closed sets. This section is an introduction to the theory of Souslin sets discussed in greater detail in Chapter 6. In spite of an introductory and relatively elementary character of this section, it contains complete proofs of two deep facts of measure theory: the measurability of Souslin sets and, as a consequence, the measurability of sets that are images of Borel sets under continuous mappings.

Denote by $\mathbb{N}^\infty$ the set of all infinite sequences (n_i) with natural components.

1.10.1. Definition. *Let X be a nonempty set and let $\mathcal{E}$ be some class of its subsets. We say that we are given a Souslin scheme (or a table of sets) $\{A_{n_1,\dots,n_k}\}$ with values in $\mathcal{E}$ if, to every finite sequence $(n_1, \dots, n_k)$ of natural numbers, there corresponds a set $A_{n_1,\dots,n_k} \in \mathcal{E}$. The A-operation (or the Souslin operation) over the class $\mathcal{E}$ is a mapping that to every Souslin scheme $\{A_{n_1,\dots,n_k}\}$ with values in $\mathcal{E}$ associates the set*

$$A = \bigcup_{(n_i)\in\mathbb{N}^\infty} \bigcap_{k=1}^{\infty} A_{n_1,\dots,n_k}. \tag{1.10.1}$$

The sets of this form are called $\mathcal{E}$-Souslin or $\mathcal{E}$-analytic. The collection of all such sets along with the empty set is denoted by $S(\mathcal{E})$.

Certainly, if $\varnothing \in \mathcal{E}$ (or if $\mathcal{E}$ contains disjoint sets), then $\varnothing \in S(\mathcal{E})$ automatically.

1.10.2. Example. By means of the A-operation one can obtain any countable unions and countable intersections of elements in the class $\mathcal{E}$.

Proof. In the first case, it suffices to take $A_{n_1,\ldots,n_k} = A_{n_1}$, and in the second, $A_{n_1,\ldots,n_k} = A_k$. □

A Souslin scheme is called *monotone* (or regular) if

$$A_{n_1,\ldots,n_k,n_{k+1}} \subset A_{n_1,\ldots,n_k}.$$

If the class $\mathcal{E}$ is closed under finite intersections, then any Souslin scheme with values in $\mathcal{E}$ can be replaced by a monotone one giving the same result of the A-operation. Indeed, set

$$A^*_{n_1,\ldots,n_k} = A_{n_1} \cap A_{n_1,n_2} \cap \cdots \cap A_{n_1,\ldots,n_k}.$$

We need the following technical assertion. Let $\bigl(\mathbb{N}^\infty\bigr)^\infty$ denote the space of all sequences $\eta = (\eta^1, \eta^2, \ldots)$ with $\eta^i \in \mathbb{N}^\infty$.

1.10.3. Lemma. *There exist bijections*

$$\beta\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N} \quad \textit{and} \quad \Psi\colon \mathbb{N}^\infty \times \bigl(\mathbb{N}^\infty\bigr)^\infty \to \mathbb{N}^\infty$$

with the property: for all $m, n \in \mathbb{N}$, $\sigma = (\sigma_i) \in \mathbb{N}^\infty$ and $(\tau^i) \in \bigl(\mathbb{N}^\infty\bigr)^\infty$, where $\tau^i = (\tau^i_j) \in \mathbb{N}^\infty$, the collections $\sigma_1, \ldots, \sigma_m$ and $\tau^m_1, \ldots, \tau^m_n$ are uniquely determined by the first $\beta(m,n)$ components of the element $\Psi\bigl(\sigma, (\tau^i)\bigr)$.

Proof. Set $\beta(m,n) = 2^{m-1}(2n-1)$. It is clear that β is a bijection of $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$, since, for any $l \in \mathbb{N}$, there exists a unique pair of natural numbers (m,n) with $l = 2^{m-1}(2n-1)$. Set also $\varphi(l) := m$, $\psi(l) := n$, where $\beta(m,n) = l$. Let $\sigma = (\sigma_i) \in \mathbb{N}^\infty$ and $(\tau^i) \in \bigl(\mathbb{N}^\infty\bigr)^\infty$, where $\tau^i = (\tau^i_j) \in \mathbb{N}^\infty$. Finally, set

$$\Psi\bigl(\sigma, (\tau^i)\bigr) = \Bigl(\beta\bigl(\sigma_1, \tau^{\varphi(1)}_{\psi(1)}\bigr), \ldots, \beta\bigl(\sigma_l, \tau^{\varphi(l)}_{\psi(l)}\bigr), \ldots\Bigr).$$

For every $\eta = (\eta_i) \in \mathbb{N}^\infty$, the equation $\Psi\bigl(\sigma, (\tau^i)\bigr) = \eta$ has a unique solution $\sigma_i = \varphi(\eta_i)$, $\tau^i_j = \psi(\eta_{\beta(i,j)})$. Hence Ψ is bijective. Since $m \le \beta(m,n)$ and $\beta(m,k) \le \beta(m,n)$ whenever $k \le n$, it follows from the form of the solution that the first $\beta(m,n)$ components of $\Psi\bigl(\sigma, (\tau^i)\bigr)$ uniquely determine the first m components of σ and the first n components of τ^m. □

The next theorem describes a number of important properties of Souslin sets.

1.10.4. Theorem. (i) *One has $S\bigl(S(\mathcal{E})\bigr) = S(\mathcal{E})$. In particular, the class $S(\mathcal{E})$ is closed under countable unions and countable intersections.*

(ii) *If the complement of every set in $\mathcal{E}$ belongs to $S(\mathcal{E})$ (for example, is an at most countable union of elements of $\mathcal{E}$) and $\varnothing \in \mathcal{E}$, then the σ-algebra $\sigma(\mathcal{E})$ generated by $\mathcal{E}$ is contained in the class $S(\mathcal{E})$.*

PROOF. (i) Let $A^{\nu_1,\ldots,\nu_m}_{n_1,\ldots,n_k} \in \mathcal{E}$ and let

$$A = \bigcup_{(n_i)\in\mathbb{N}^\infty} \bigcap_{k=1}^\infty A_{n_1,\ldots,n_k}, \quad A_{n_1,\ldots,n_k} = \bigcup_{\nu\in\mathbb{N}^\infty} \bigcap_{m=1}^\infty A^{\nu_1,\ldots,\nu_m}_{n_1,\ldots,n_k}.$$

Keeping the notation of the above lemma, for any natural numbers $\eta_1, \ldots, \eta_l$ we find $\sigma \in \mathbb{N}^\infty$ and $\tau = (\tau^m) \in \bigl(\mathbb{N}^\infty\bigr)^\infty$ such that $\eta_1 = \Psi(\sigma,\tau)_1, \ldots, \eta_l = \Psi(\sigma,\tau)_l$. Certainly, σ and τ are not uniquely determined, but according to the lemma, the collections $\sigma_1,\ldots,\sigma_{\varphi(l)}$ and $\tau_1^{\varphi(l)},\ldots,\tau_{\psi(l)}^{\varphi(l)}$ are uniquely determined by the numbers $\eta_1, \ldots, \eta_l$. Hence we may set

$$B(\eta_1, \ldots, \eta_l) = A^{\tau_1^{\varphi(l)},\ldots,\tau_{\psi(l)}^{\varphi(l)}}_{\sigma_1,\ldots,\sigma_{\varphi(l)}} \in \mathcal{E}.$$

Then, denoting by $\eta = (\eta_l)$ and $\sigma = (\sigma_m)$ elements of $\mathbb{N}^\infty$ and by (τ^m) with $\tau^m = (\tau^m_n)$ elements of $\bigl(\mathbb{N}^\infty\bigr)^\infty$, we have

$$\begin{aligned}
\bigcup_\eta \bigcap_{l=1}^\infty B(\eta_1, \ldots, \eta_l) &= \bigcup_{\sigma,(\tau^m)} \bigcap_{l=1}^\infty B\Bigl(\Psi\bigl(\sigma,(\tau^m)\bigr)_1, \ldots, \Psi\bigl(\sigma,(\tau^m)\bigr)_l\Bigr) \\
&= \bigcup_{\sigma,(\tau^m)} \bigcap_{l=1}^\infty A^{\tau_1^{\varphi(l)},\ldots,\tau_{\psi(l)}^{\varphi(l)}}_{\sigma_1,\ldots,\sigma_{\varphi(l)}} = \bigcup_{\sigma,(\tau^m)} \bigcap_{m,n=1}^\infty A^{\tau_1^m,\ldots,\tau_n^m}_{\sigma_1,\ldots,\sigma_m} \\
&= \bigcup_\sigma \bigcup_{(\tau^m)} \bigcap_{m=1}^\infty \bigcap_{n=1}^\infty A^{\tau_1^m,\ldots,\tau_n^m}_{\sigma_1,\ldots,\sigma_m} = \bigcup_\sigma \bigcap_{m=1}^\infty \bigcup_{\tau^m} \bigcap_{n=1}^\infty A^{\tau_1^m,\ldots,\tau_n^m}_{\sigma_1,\ldots,\sigma_m} \\
&= \bigcup_\sigma \bigcap_{m=1}^\infty A_{\sigma_1,\ldots,\sigma_m} = A.
\end{aligned}$$

Thus, $S\bigl(S(\mathcal{E})\bigr) \subset S(\mathcal{E})$. The inverse inclusion is obvious.

(ii) Set

$$\mathcal{F} = \bigl\{B \in S(\mathcal{E})\colon\ X\backslash B \in S(\mathcal{E})\bigr\}.$$

Let us show that $\mathcal{F}$ is a σ-algebra. By construction, $\mathcal{F}$ is closed under complementation. Let $B_n \in \mathcal{F}$. Then $\bigcap_{n=1}^\infty B_n \in S(\mathcal{E})$ according to assertion (i). Similarly, $X\backslash\bigcap_{n=1}^\infty B_n = \bigcup_{n=1}^\infty (X\backslash B_n) \in S(\mathcal{E})$. By hypothesis, $\varnothing \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a σ-algebra. Since by hypothesis $\mathcal{E} \subset \mathcal{F}$, we obtain $\sigma(\mathcal{E}) \subset \mathcal{F} \subset S(\mathcal{E})$. □

It is clear that the condition $X\backslash E \in S(\mathcal{E})$ for $E \in \mathcal{E}$ is also necessary in order that $\sigma(\mathcal{E}) \subset S(\mathcal{E})$. The class $S(\mathcal{E})$ may not be closed with respect to complementation even in the case where $\mathcal{E}$ is a σ-algebra. As we shall see later, this happens, for example, with $\mathcal{E} = \mathcal{B}(\mathbb{R}^1)$. If we apply the A-operation to the class of all compact (or closed) sets in $\mathbb{R}^n$, then the hypothesis in assertion (ii) of the above theorem is satisfied, since every nonempty open set in $\mathbb{R}^n$ is a countable union of closed cubes. Below we consider this example more carefully.

The next fundamental result shows that the A-operation preserves measurability. This assertion is not at all obvious and, moreover, it is very surprising, since the A-operation involves uncountable unions.

1.10.5. Theorem. *Suppose that μ is a finite nonnegative measure on a σ-algebra $\mathcal{M}$. Then, the class $\mathcal{M}_\mu$ of all μ-measurable sets is closed with respect to the A-operation. Moreover, given a family of sets $\mathcal{E} \subset \mathcal{M}$ that is closed with respect to finite unions and countable intersections, one has*

$$\mu^*(A) = \sup\bigl\{\mu(E)\colon\ E \subset A,\ E \in \mathcal{E}\bigr\}$$

for every $\mathcal{E}$-Souslin set A. In particular, every $\mathcal{E}$-Souslin set is μ-measurable.

Proof. The first claim is a simple corollary of the second one applied to the family $\mathcal{E} = \mathcal{M}_\mu$. So we prove the second claim. Let a set A be constructed by means of a monotone table of sets $E_{n_1,\ldots,n_k} \in \mathcal{E}$. Let $\varepsilon > 0$. For every collection $m_1, \ldots, m_k$ of natural numbers, denote by $D_{m_1,\ldots,m_k}$ the union of the sets $E_{n_1,\ldots,n_k}$ over all $n_1 \le m_1, \ldots, n_k \le m_k$. Let

$$M_{m_1,\ldots,m_k} := \bigcup_{(n_i)\in\mathbb{N}^\infty,\ n_1\le m_1,\ldots,n_k\le m_k} \bigcap_{j=1}^{\infty} E_{n_1,\ldots,n_j}.$$

It is clear that as $m \to \infty$, the sets M_m monotonically increase to A, and the sets $M_{m_1,\ldots,m_k,m}$ with fixed $m_1, \ldots, m_k$ monotonically increase to $M_{m_1,\ldots,m_k}$. By Proposition 1.5.12, there is a number m_1 with $\mu^*(M_{m_1}) > \mu^*(A) - \varepsilon 2^{-1}$. Then we can find a number m_2 with $\mu^*(M_{m_1,m_2}) > \mu^*(M_{m_1}) - \varepsilon 2^{-2}$. Continuing this construction by induction, we obtain a sequence of natural numbers m_k such that

$$\mu^*(M_{m_1,m_2,\ldots,m_k}) > \mu^*(M_{m_1,m_2,\ldots,m_{k-1}}) - \varepsilon 2^{-k}.$$

Therefore, for all k one has

$$\mu^*(M_{m_1,m_2,\ldots,m_k}) > \mu^*(A) - \varepsilon.$$

By the stability of $\mathcal{E}$ with respect to finite unions we have $D_{m_1,\ldots,m_k} \in \mathcal{E}$, and the stability of $\mathcal{E}$ with respect to countable intersections yields the inclusion $E := \bigcap_{k=1}^\infty D_{m_1,\ldots,m_k} \in \mathcal{E}$. Since $M_{m_1,\ldots,m_k} \subset D_{m_1,\ldots,m_k}$, we obtain by the previous estimate $\mu^*(D_{m_1,m_2,\ldots,m_k}) > \mu^*(A) - \varepsilon$, whence it follows that $\mu(E) \ge \mu^*(A) - \varepsilon$, since the sets $D_{m_1,m_2,\ldots,m_k}$ decrease to E.

It remains to prove that $E \subset A$. Let $x \in E$. Then, for all k we have $x \in D_{m_1,\ldots,m_k}$. Hence $x \in E_{n_1,\ldots,n_k}$ for some collection $n_1, \ldots, n_k$ such that $n_1 \le m_1, \ldots, n_k \le m_k$. Such collections will be called admissible. Our task is to construct an infinite sequence $n_1, n_2, \ldots$ such that all its initial intervals $n_1, \ldots, n_k$ are admissible. In this case $x \in \bigcap_{k=1}^\infty E_{n_1,\ldots,n_k} \subset A$. In order to construct such a sequence let us observe that, for any $k > 1$, we have admissible collections of k numbers. An admissible collection $n_1, \ldots, n_k$ is called extendible if, for every $l \ge k$, there exists an admissible collection $p_1, \ldots, p_l$ with $p_1 = n_1, \ldots, p_k = n_k$. Let us now observe that there exists at least one extendible collection n_1 of length 1. Indeed, suppose the contrary. Since every initial interval $n_1, \ldots, n_k$ in any admissible collection $n_1, \ldots, n_k, \ldots, n_l$ is admissible by the inclusion $E_{n_1,\ldots,n_l} \subset E_{n_1,\ldots,n_k}$, we obtain that for every $n \leq m_1$ there exists the maximal length $l(n)$ of admissible collections with the number n at the first position. Therefore, the lengths of all admissible collections are uniformly bounded and we arrive at a contradiction. Similarly, the extendible collection n_1 is contained in some extendible collection n_1, n_2 and so on. The obtained sequence possesses the desired property. □

1.10.6. Corollary. *If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces and a mapping $f\colon X \to Y$ be such that $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$, then for every set $E \in S(\mathcal{B})$, the set $f^{-1}(E)$ belongs to $S(\mathcal{A})$ and hence is measurable with respect to every measure on $\mathcal{A}$.*

PROOF. It follows from (1.10.1) that $f^{-1}(E) \in S(\mathcal{A})$. □

Another method of proof of Theorem 1.10.5 is described in Exercise 6.10.60 in Chapter 6. A thorough study of Souslin sets and related problems in measure theory is accomplished in Chapters 6 and 7. However, even now we are able to derive from Theorem 1.10.5 very useful corollaries.

1.10.7. Definition. *The sets obtained by application of the A-operation to the class of closed sets in $\mathbb{R}^n$ are called the Souslin sets in the space $\mathbb{R}^n$.*

It is clear that the same result is obtained by applying the A-operation to the class of all compact sets in $\mathbb{R}^n$. Indeed, if A is contained in a cube K, then closed sets $A_{\nu_1,\ldots,\nu_k}$ that generate A can be replaced by the compacts $A_{\nu_1,\ldots,\nu_k} \cap K$. Any unbounded Souslin set A can be written as the union of its intersections $A \cap K_j$ with increasing cubes K_j. It remains to use that the class of sets constructed by the A-operation from compact sets admits countable unions.

As was mentioned above, it follows by Theorem 1.10.4 that Borel sets in $\mathbb{R}^n$ are Souslin. Note also that if L is a linear subspace in $\mathbb{R}^n$ of dimension $k < n$, then the intersection of L with any Souslin set A in $\mathbb{R}^n$ is Souslin in the space L. This follows by the fact that the intersection of any closed set with L is closed in L. Conversely, any Souslin set in L is Souslin in $\mathbb{R}^n$ as well.

1.10.8. Proposition. *The image of any Souslin set under a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^d$ is Souslin.*

PROOF. Let a set A have the form (1.10.1), where the sets $A_{n_1,\ldots,n_k}$ are compact (as we know, such a representation is possible for every Souslin set). As noted above, we may assume that $A_{n_1,\ldots,n_k,n_{k+1}} \subset A_{n_1,\ldots,n_k}$ for all k. Let $f\colon \mathbb{R}^n \to \mathbb{R}^d$ be a continuous mapping. It is clear that

$$f(A) = \bigcup_{(n_i)\in\mathbb{N}^\infty} f\Bigl(\bigcap_{k=1}^{\infty} A_{n_1,\ldots,n_k}\Bigr).$$

It remains to observe that the sets $B_{n_1,\ldots,n_k} = f(A_{n_1,\ldots,n_k})$ are compact by the continuity of f and that

$$f\Bigl(\bigcap_{k=1}^{\infty} A_{n_1,\ldots,n_k}\Bigr) = \bigcap_{k=1}^{\infty} f(A_{n_1,\ldots,n_k}).$$

Indeed, the left-hand side of this equality is contained in the right-hand side for any sets and mappings. Let $y \in \bigcap_{k=1}^{\infty} f(A_{n_1,\ldots,n_k})$. Then, for every k, there exists $x_k \in A_{n_1,\ldots,n_k}$ with $f(x_k) = y$. If for infinitely many indices k the points x_k coincide with one and the same point x, then $x \in \bigcap_{k=1}^{\infty} A_{n_1,\ldots,n_k}$ by the monotonicity of $A_{n_1,\ldots,n_k}$. Clearly, $f(x) = y$. Hence it remains to consider the case where the sequence $\{x_k\}$ contains infinitely many distinct points. Since this sequence is contained in the compact set A_{n_1}, there exists a limit point x of $\{x_k\}$. Then $x \in A_{n_1,\ldots,n_k}$ for all k, since $x_j \in A_{n_1,\ldots,n_k}$ for all $j \geq k$ and $A_{n_1,\ldots,n_k}$ is a closed set. Thus, $x \in \bigcap_{k=1}^{\infty} A_{n_1,\ldots,n_k}$. By the continuity of f we obtain $f(x) = y$. □

1.10.9. Corollary. *The image of any Borel set $B \subset \mathbb{R}^n$ under a continuous mapping $f\colon \mathbb{R}^n \to \mathbb{R}^d$ is a Souslin set. In particular, the set $f(B)$ is Lebesgue measurable.*

In particular, the orthogonal projection of a Borel set is Souslin, hence measurable. We shall see in Chapter 6 that Souslin sets in $\mathbb{R}^n$ coincide with the orthogonal projections of Borel sets in $\mathbb{R}^{n+1}$ (thus, Souslin sets can be defined without the A-operation) and that there exist non-Borel Souslin sets. It is easily verified that the product of two Borel sets in $\mathbb{R}^n$ is Borel in $\mathbb{R}^{2n}$. Indeed, it suffices to check that $A\times\mathbb{R}^n \in \mathcal{B}(\mathbb{R}^{2n})$ if $A \in \mathcal{B}(\mathbb{R}^n)$. This is true for any open set A, hence for any Borel set A, since the class of all Borel sets A with such a property is obviously a σ-algebra.

1.10.10. Example. Let A and B be nonempty Borel sets in $\mathbb{R}^n$. Then the vector sum of the sets A and B defined by the equality

$$A + B := \{a + b\colon\ a \in A, b \in B\}$$

is a Souslin set. In addition, the convex hull $\operatorname{conv} A$ of the set A, i.e., the smallest convex set containing A, is Souslin as well. Indeed, $A+B$ is the image of the Borel set $A\times B$ in $\mathbb{R}^{2n}$ under the continuous mapping $(x, y) \mapsto x + y$. The convex hull of A consists of all sums of the form

$$\sum_{i=1}^{k} t_i a_i,\ \text{where } t_i \geq 0,\ \textstyle\sum_{i=1}^{k} t_i = 1,\ a_i \in A,\ k \in \mathbb{N}.$$

For every fixed k, the set S of all points $(t_1, \ldots, t_k) \in \mathbb{R}^k$ such that $\sum_{i=1}^{k} t_i = 1$ and $t_i \geq 0$ is Borel. Hence the set $A^k\times S$ in $(\mathbb{R}^n)^k\times\mathbb{R}^k$ is Borel as well and its image under the mapping $(a_1, \ldots, a_k, t_1, \ldots, t_k) \mapsto \sum_{i=1}^{k} t_i a_i$ is Souslin.

1.11. Carathéodory outer measures

In this section, we discuss in greater detail constructions of measures by means of the so-called Carathéodory outer measures. We have already encountered the principal idea in the consideration of extensions of countably additive measures from an algebra to a σ-algebra, but now we do not assume that an "outer measure" is generated by an additive measure.

1.11.1. Definition. *A set function* $\mathfrak{m}$ *defined on the class of all subsets of a set* X *and taking values in* $[0, +\infty]$ *is called an outer measure on* X (*or a Carathéodory outer measure*) *if*:
(i) $\mathfrak{m}(\varnothing) = 0$;
(ii) $\mathfrak{m}(A) \le \mathfrak{m}(B)$ *whenever* $A \subset B$, *i.e.,* $\mathfrak{m}$ *is monotone*;
(iii) $\mathfrak{m}\Bigl(\bigcup_{n=1}^{\infty} A_n\Bigr) \le \sum_{n=1}^{\infty} \mathfrak{m}(A_n)$ *for all* $A_n \subset X$.

An important example of a Carathéodory outer measure is the function μ^* discussed in §1.5.

1.11.2. Definition. *Let* $\mathfrak{m}$ *be a set function with values in* $[0, +\infty]$ *defined on the class of all subsets of a space* X *such that* $\mathfrak{m}(\varnothing) = 0$. *A set* $A \subset X$ *is called Carathéodory measurable with respect to* $\mathfrak{m}$ (*or Carathéodory* $\mathfrak{m}$*-measurable*) *if, for every set* $E \subset X$, *one has the equality*

$$\mathfrak{m}(E \cap A) + \mathfrak{m}(E \backslash A) = \mathfrak{m}(E). \tag{1.11.1}$$

The class of all Carathéodory $\mathfrak{m}$*-measurable sets is denoted by* $\mathfrak{M}_{\mathfrak{m}}$.

Thus, a measurable set splits every set according to the requirement of additivity of $\mathfrak{m}$ (see also Exercise 1.12.150 in this relation).

Let us note at once that in general the measurability does not follow from the equality

$$\mathfrak{m}(A) + \mathfrak{m}(X \backslash A) = \mathfrak{m}(X) \tag{1.11.2}$$

even in the case of an outer measure with $\mathfrak{m}(X) < \infty$. Let us consider the following example.

1.11.3. Example. Let $X = \{1, 2, 3\}$, $\mathfrak{m}(\varnothing) = 0$, $\mathfrak{m}(X) = 2$, and let $\mathfrak{m}(A) = 1$ for all other sets A. It is readily verified that $\mathfrak{m}$ is an outer measure. Here every subset $A \subset X$ satisfies (1.11.2), but for $A = \{1\}$ and $E = \{1, 2\}$ equality (1.11.1) does not hold (its left-hand side equals 2 and the right-hand side equals 1). It is easy to see that only two sets $\varnothing$ and X are $\mathfrak{m}$-measurable.

In this example the class $\mathfrak{M}_{\mathfrak{m}}$ of all Carathéodory $\mathfrak{m}$-measurable sets is smaller than the class $\mathcal{A}_{\mathfrak{m}}$ from Definition 1.5.1, since for the outer measure $\mathfrak{m}$ on the class of all sets the family $\mathcal{A}_{\mathfrak{m}}$ is the class of all sets. However, we shall see later that in the case where $\mathfrak{m} = \mu^*$ is the outer measure generated by a countably additive measure μ with values in $[0, +\infty]$ defined on a σ-algebra, the class $\mathfrak{M}_{\mathfrak{m}}$ may be larger than $\mathcal{A}_{\mu}$ (Exercise 1.12.129). On the other hand, under reasonable assumptions, the classes $\mathfrak{M}_{\mu^*}$ and $\mathcal{A}_{\mu}$ coincide.

Below a class of outer measures is singled out such that the corresponding measurability is equivalent to (1.11.2). This class embraces all outer measures generated by countably additive measures on algebras (see Proposition 1.11.7 and Theorem 1.11.8).

1.11.4. Theorem. *Let $\mathfrak{m}$ be a set function with values in $[0,+\infty]$ on the class of all sets in a space X such that $\mathfrak{m}(\varnothing)=0$. Then:*

(i) *$\mathfrak{M}_{\mathfrak{m}}$ is an algebra and the function $\mathfrak{m}$ is additive on $\mathfrak{M}_{\mathfrak{m}}$.*

(ii) *For every sequence of pairwise disjoint sets $A_i \in \mathfrak{M}_{\mathfrak{m}}$ one has*

$$\mathfrak{m}\Bigl(E \cap \bigcup_{i=1}^{n} A_i\Bigr) = \sum_{i=1}^{n} \mathfrak{m}(E \cap A_i), \quad \forall\, E \subset X,$$

$$\mathfrak{m}\Bigl(E \cap \bigcup_{i=1}^{\infty} A_i\Bigr) = \sum_{i=1}^{\infty} \mathfrak{m}(E \cap A_i) + \lim_{n\to\infty} \mathfrak{m}\Bigl(E \cap \bigcup_{i=n}^{\infty} A_i\Bigr), \quad \forall\, E \subset X.$$

(iii) *If the function $\mathfrak{m}$ is an outer measure on the set X, then the class $\mathfrak{M}_{\mathfrak{m}}$ is a σ-algebra and the function $\mathfrak{m}$ with values in $[0,+\infty]$ is countably additive on $\mathfrak{M}_{\mathfrak{m}}$. In addition, the measure $\mathfrak{m}$ is complete on $\mathfrak{M}_{\mathfrak{m}}$.*

Proof. (i) It is obvious from (1.11.1) that $\varnothing \in \mathfrak{M}_{\mathfrak{m}}$ and that the class $\mathfrak{M}_{\mathfrak{m}}$ is closed with respect to complementation. Suppose that sets A_1, A_2 belong to $\mathfrak{M}_{\mathfrak{m}}$ and let $E \subset X$. By the measurability of A_1 and A_2 we have

$$\begin{aligned}\mathfrak{m}(E) &= \mathfrak{m}(E \cap A_1) + \mathfrak{m}(E\backslash A_1)\\ &= \mathfrak{m}(E \cap A_1) + \mathfrak{m}\bigl((E\backslash A_1) \cap A_2\bigr) + \mathfrak{m}\bigl((E\backslash A_1)\backslash A_2\bigr)\\ &= \mathfrak{m}(E \cap A_1) + \mathfrak{m}\bigl((E\backslash A_1) \cap A_2\bigr) + \mathfrak{m}\bigl(E\backslash(A_1 \cup A_2)\bigr).\end{aligned}$$

According to the equality $E\cap A_1 = E \cap (A_1\cup A_2)\cap A_1$ and the measurability of A_1 one has

$$\mathfrak{m}\bigl(E \cap (A_1 \cup A_2)\bigr) = \mathfrak{m}(E \cap A_1) + \mathfrak{m}\bigl((E\backslash A_1) \cap A_2\bigr). \tag{1.11.3}$$

Hence we obtain

$$\mathfrak{m}(E) = \mathfrak{m}\bigl(E \cap (A_1 \cup A_2)\bigr) + \mathfrak{m}\bigl(E\backslash(A_1 \cup A_2)\bigr).$$

Thus, $A_1 \cup A_2 \in \mathfrak{M}_{\mathfrak{m}}$, i.e., $\mathfrak{M}_{\mathfrak{m}}$ is an algebra. For disjoint sets A_1 and A_2 by taking $E = X$ in (1.11.3) we obtain the equality $\mathfrak{m}(A_1\cup A_2) = \mathfrak{m}(A_1)+\mathfrak{m}(A_2)$.

(ii) Let $A_i \in \mathfrak{M}_{\mathfrak{m}}$ be disjoint. Set

$$S_n = \bigcup_{i=1}^{n} A_i, \quad R_n = \bigcup_{i=n}^{\infty} A_i.$$

Then by equality (1.11.3) we have

$$\mathfrak{m}(E \cap S_n) = \mathfrak{m}(E \cap A_n) + \mathfrak{m}(E \cap S_{n-1}).$$

By induction this yields the first equality in assertion (ii). Next, by the equalities $R_1 \cap S_{n-1} = S_{n-1}$ and $R_1\backslash S_{n-1} = R_n$ one has

$$\mathfrak{m}(E \cap R_1) = \mathfrak{m}(E \cap S_{n-1}) + \mathfrak{m}(E \cap R_n) = \sum_{i=1}^{n-1} \mathfrak{m}(E \cap A_i) + \mathfrak{m}(E \cap R_n).$$

This gives the second equality in assertion (ii), since the sequence $\mathfrak{m}(E \cap R_n)$ is decreasing by the equality

$$\mathfrak{m}(E \cap R_n) = \mathfrak{m}(E \cap R_{n+1}) + \mathfrak{m}(E \cap A_n),$$

which follows from the measurability of A_n and the relations $R_n \backslash A_n = R_{n+1}$ and $R_n \cap A_n = A_n$.

(iii) Suppose now that $\mathfrak{m}$ is countably subadditive and that sets $A_i \in \mathfrak{M}_\mathfrak{m}$ are disjoint. Let $A = \bigcup_{i=1}^\infty A_i$. The second equality in (ii) yields that for any $E \subset X$ one has $\mathfrak{m}(E \cap A) \geq \sum_{i=1}^\infty \mathfrak{m}(E \cap A_i)$, which by the countable subadditivity gives

$$\mathfrak{m}(E \cap A) = \sum_{i=1}^\infty \mathfrak{m}(E \cap A_i). \tag{1.11.4}$$

We already know that $S_n = A_1 \cup \cdots \cup A_n \in \mathfrak{M}_\mathfrak{m}$. It follows by the first equality in assertion (ii) that

$$\mathfrak{m}(E) = \mathfrak{m}(E \cap S_n) + \mathfrak{m}(E \backslash S_n) \geq \sum_{i=1}^n \mathfrak{m}(E \cap A_i) + \mathfrak{m}(E \backslash A).$$

By (1.11.4) we obtain $\mathfrak{m}(E) \geq \mathfrak{m}(E \cap A) + \mathfrak{m}(E \backslash A)$. By subadditivity the reverse inequality is true as well, i.e., $A \in \mathfrak{M}_\mathfrak{m}$. Hence $\mathfrak{M}_\mathfrak{m}$ is an algebra closed with respect to countable unions of disjoint sets. This means that $\mathfrak{M}_\mathfrak{m}$ is a σ-algebra. By taking $E = X$ in (1.11.4) we obtain the countable additivity of $\mathfrak{m}$ on $\mathfrak{M}_\mathfrak{m}$. We verify that $\mathfrak{m}$ is complete on $\mathfrak{M}_\mathfrak{m}$. Let $\mathfrak{m}(A) = 0$. Then, for any set E, we have $\mathfrak{m}(E \cap A) + \mathfrak{m}(E \backslash A) = \mathfrak{m}(E)$, as $0 \leq \mathfrak{m}(E \cap A) \leq \mathfrak{m}(A) = 0$, and $\mathfrak{m}(E \backslash A) = \mathfrak{m}(E)$, as $\mathfrak{m}(E \backslash A) \leq \mathfrak{m}(E) \leq \mathfrak{m}(E \backslash A) + \mathfrak{m}(A) = \mathfrak{m}(E \backslash A)$. □

Note that the countably additive measure $\mu := \mathfrak{m}|_{\mathfrak{M}_\mathfrak{m}}$ on $\mathfrak{M}_\mathfrak{m}$, where $\mathfrak{m}$ is an outer measure, gives rise to a usual outer measure μ^* as we did before. However, this outer measure may differ from the original function $\mathfrak{m}$ (certainly, on the sets in $\mathfrak{M}_\mathfrak{m}$ both outer measures coincide). Say, in Example 1.11.3 we obtain $\mu^*(A) = 2$ for any nonempty set A different from X. Some additional information is given in Exercises 1.12.125 and 1.12.126.

In applications, outer measures are often constructed by the so-called Method I described in the following example and already employed in §1.5, where in Lemma 1.5.4 the countable subadditivity has been established.

1.11.5. Example. Let $\mathfrak{X}$ be a family of subsets of a X such that $\varnothing \in \mathfrak{X}$. Suppose that we are given a function $\tau\colon\ \mathfrak{X} \to [0, +\infty]$ with $\tau(\varnothing) = 0$. Set

$$\mathfrak{m}(A) = \inf\Bigl\{\sum_{n=1}^\infty \tau(X_n)\colon\ X_n \in \mathfrak{X}, A \subset \bigcup_{n=1}^\infty X_n\Bigr\}, \tag{1.11.5}$$

where in the case of absence of such sets X_n we set $\mathfrak{m}(A) := \infty$. Then $\mathfrak{m}$ is an outer measure. It is denoted by τ^*.

This construction will be used in §3.10(iii) for defining the so-called Hausdorff measures. Exercise 1.12.130 describes a modification of the construction of $\mathfrak{m}$ that differs as follows: if there are no sequences of sets in $\mathfrak{X}$ covering A, then the value $\mathfrak{m}(A)$ is defined as $\sup \mathfrak{m}(A')$ over those $A' \subset A$ for which such sequences exist.

It should be emphasized that it is not claimed in the above example that the constructed outer measure extends τ. In general, this may be false. In addition, sets in the original family $\mathfrak{X}$ may be nonmeasurable with respect to $\mathfrak{m}$. Let us consider the corresponding counter-examples. Let us take for X the set $\mathbb{N}$ and for $\mathfrak{X}$ the family of all singletons and the whole set X. Let $\tau(n) = 2^{-n}$, $\tau(X) = 2$. Then $\mathfrak{m}(X) = 1$ and X is measurable with respect to $\mathfrak{m}$. If we take for X the interval $[0,1]$ and for τ the outer Lebesgue measure defined on the class $\mathfrak{X}$ of all sets, then the obtained function $\mathfrak{m}$ coincides with the initial function τ and the collection of $\mathfrak{m}$-measurable sets coincides with the class of the usual Lebesgue measurable sets, which is smaller than $\mathfrak{X}$. In Exercise 1.12.121 it is suggested to construct a similar example with an additive function τ on a σ-algebra of all sets in the interval.

Let us now specify one important class of outer measures.

1.11.6. Definition. *An outer measure $\mathfrak{m}$ on X is called regular if, for every set $A \subset X$, there exists an $\mathfrak{m}$-measurable set B such that $A \subset B$ and $\mathfrak{m}(A) = \mathfrak{m}(B)$.*

For example, the outer measure λ^* constructed from Lebesgue measure on the interval is regular, since one can take for B the set $\bigcap_{n=1}^{\infty} A_n$, where the sets A_n are measurable, $A \subset A_n$ and $\lambda(A_n) < \lambda^*(A) + 1/n$ (such a set is called a measurable envelope of A, see §1.12(iv)). More general examples are given below.

1.11.7. Proposition. *Let $\mathfrak{m}$ be a regular outer measure on X with $\mathfrak{m}(X) < \infty$. Then, the $\mathfrak{m}$-measurability of a set A is equivalent to the equality*

$$\mathfrak{m}(A) + \mathfrak{m}(X \backslash A) = \mathfrak{m}(X). \tag{1.11.6}$$

PROOF. The necessity of (1.11.6) is obvious. Let us verify its sufficiency. Let E be an arbitrary set in X, $C \in \mathfrak{M}_{\mathfrak{m}}$, $E \subset C$, $\mathfrak{m}(C) = \mathfrak{m}(E)$. It suffices to show that

$$\mathfrak{m}(E) \geq \mathfrak{m}(E \cap A) + \mathfrak{m}(E \backslash A), \tag{1.11.7}$$

since the reverse inequality follows by the subadditivity. Note that

$$\mathfrak{m}(A \backslash C) + \mathfrak{m}\big((X \backslash A) \backslash C\big) \geq \mathfrak{m}(X \backslash C). \tag{1.11.8}$$

By the measurability of C one has

$$\mathfrak{m}(A) = \mathfrak{m}(A \cap C) + \mathfrak{m}(A \backslash C), \tag{1.11.9}$$

$$\mathfrak{m}(X \backslash A) = \mathfrak{m}\big(C \cap (X \backslash A)\big) + \mathfrak{m}\big((X \backslash A) \backslash C\big). \tag{1.11.10}$$

It follows by (1.11.6), (1.11.9) and (1.11.10) combined with the subadditivity of $\mathfrak{m}$ that

$$\begin{aligned}\mathfrak{m}(X) &= \mathfrak{m}(A \cap C) + \mathfrak{m}(A \backslash C) + \mathfrak{m}\big(C \cap (X \backslash A)\big) + \mathfrak{m}\big((X \backslash A) \backslash C\big) \\ &\geq \mathfrak{m}(C) + \mathfrak{m}(X \backslash C) = \mathfrak{m}(X).\end{aligned}$$

Therefore, the inequality in the last chain is in fact an equality. Subtracting from it (1.11.8), which is possible, since $\mathfrak{m}$ is finite, we arrive at the estimate

$$\mathfrak{m}(C \cap A) + \mathfrak{m}(C \backslash A) \leq \mathfrak{m}(C).$$

Finally, the last estimate along with the inclusion $E \subset C$ and monotonicity of $\mathfrak{m}$ yields

$$\mathfrak{m}(E \cap A) + \mathfrak{m}(E \backslash A) \leq \mathfrak{m}(C) = \mathfrak{m}(E).$$

Hence we have proved (1.11.7). □

Example 1.11.3 shows that Method I from Example 1.11.5 does not always yield regular outer measures. According to Exercise 1.12.122, if $\mathfrak{X} \subset \mathfrak{M}_{\mathfrak{m}}$, then Method I gives a regular outer measure. Yet another useful result in this direction is contained in the following theorem.

1.11.8. Theorem. *Let X, $\mathfrak{X}$, τ, and $\mathfrak{m}$ be the same as in Example* 1.11.5*. Suppose, in addition, that $\mathfrak{X}$ is an algebra* (*or a ring*) *and the function τ is additive. Then, the outer measure $\mathfrak{m}$ is regular and all sets in the class $\mathfrak{X}$ are measurable with respect to $\mathfrak{m}$. If τ is countably additive, then $\mathfrak{m}$ coincides with τ on $\mathfrak{X}$.*

Finally, if $\tau(X) < \infty$, then $\mathfrak{M}_{\mathfrak{m}} = \mathfrak{X}_\tau$, i.e., in this case the definition of the Carathéodory measurability is equivalent to Definition 1.5.1*.*

PROOF. It suffices to verify that all sets in $\mathfrak{X}$ are measurable with respect to $\mathfrak{m}$; then the regularity will follow by Exercise 1.12.122. Let $A \in \mathfrak{X}$. In order to prove the inclusion $A \in \mathfrak{M}_{\mathfrak{m}}$, it suffices to show that, for every set E with $\mathfrak{m}(E) < \infty$, one has the estimate

$$\mathfrak{m}(E) \geq \mathfrak{m}(E \cap A) + \mathfrak{m}\big(E \cap (X \backslash A)\big).$$

Let $\varepsilon > 0$. There exist sets $X_n \in \mathfrak{X}$ with $E \subset \bigcup_{n=1}^{\infty} X_n$ and

$$\sum_{n=1}^{\infty} \tau(X_n) < \mathfrak{m}(E) + \varepsilon.$$

The condition that $\mathfrak{X}$ is a ring yields $X_n \cap A \in \mathfrak{X}$ and $X_n \cap (X \backslash A) = X_n \backslash A \in \mathfrak{X}$. Hence by the additivity of τ on $\mathfrak{X}$ we have for all n

$$\tau(X_n) = \tau(X_n \cap A) + \tau\big(X_n \cap (X \backslash A)\big).$$

Since

$$E \cap A \subset \bigcup_{n=1}^{\infty} (X_n \cap A), \quad E \cap (X \backslash A) \subset \bigcup_{n=1}^{\infty} \big(X_n \cap (X \backslash A)\big),$$

we obtain

$$\begin{aligned}\mathfrak{m}(E)+\varepsilon &> \sum_{n=1}^\infty \tau(X_n) = \sum_{n=1}^\infty \tau(X_n\cap A) + \sum_{n=1}^\infty \tau\bigl(X_n\cap (X\backslash A)\bigr)\\ &\ge \sum_{n=1}^\infty \mathfrak{m}(X_n\cap A) + \sum_{n=1}^\infty \mathfrak{m}\bigl(X_n\cap (X\backslash A)\bigr)\\ &\ge \mathfrak{m}(E\cap A)+\mathfrak{m}\bigl(E\cap (X\backslash A)\bigr).\end{aligned}$$

The required inequality is established, since ε is arbitrary. In the general case, one has $\mathfrak{m}\le\tau$ on $\mathfrak{X}$, but for a countably additive function τ it is easy to obtain the reverse inequality.

Let us now verify that in the case $\tau(X)<\infty$, Definition 1.5.1 gives the same class of τ-measurable sets as Definition 1.11.2 applied to the outer measure $\mathfrak{m}=\tau^*$. Let $A\in\mathfrak{M}_\mathfrak{m}$ and $\varepsilon>0$. There exist sets $A_n\in\mathfrak{X}$ with $A\subset\bigcup_{n=1}^\infty A_n$ and $\mathfrak{m}(A)\ge\sum_{n=1}^\infty \tau(A_n)-\varepsilon$. Since $\mathfrak{m}(A_n)\le\tau(A_n)$, taking into account the countable additivity of $\mathfrak{m}$ on the σ-algebra $\mathfrak{M}_\mathfrak{m}$, which contains $\mathfrak{X}$, we obtain

$$\mathfrak{m}(A)\ge\sum_{n=1}^\infty \mathfrak{m}(A_n)-\varepsilon\ge\mathfrak{m}\Bigl(\bigcup_{n=1}^\infty A_n\Bigr)-\varepsilon.$$

Therefore, $\mathfrak{m}\bigl(\bigcup_{n=1}^\infty A_n\backslash A\bigr)\le\varepsilon$. By using the countable additivity of $\mathfrak{m}$ once again, we obtain $\mathfrak{m}\bigl(A\bigtriangleup\bigcup_{n=1}^k A_n\bigr)\le 2\varepsilon$ for k sufficiently large. Since ε is arbitrary it follows that $A\in\mathfrak{X}_\tau$. Conversely, if $A\in\mathfrak{X}_\tau$, then, for every $\varepsilon>0$, there exists a set $A_\varepsilon\in\mathfrak{X}$ with $\mathfrak{m}(A\bigtriangleup A_\varepsilon)\le\varepsilon$. One has $\mathfrak{X}\subset\mathfrak{M}_\mathfrak{m}$. By the countable additivity of $\mathfrak{m}$ on $\mathfrak{M}_\mathfrak{m}$, we obtain that A belongs to the Lebesgue completion of $\mathfrak{M}_\mathfrak{m}$. The completeness of $\mathfrak{M}_\mathfrak{m}$ yields the inclusion $A\in\mathfrak{M}_\mathfrak{m}$. □

1.11.9. Corollary. *If a countably additive set function with values in* $[0,+\infty]$ *is defined on a ring, then it has a countably additive extension to the* σ*-algebra generated by the given ring.*

Unlike the case of an algebra, the aforementioned extension is not always unique (as an example, consider the space $X=\{0\}$ with the zero measure on the ring $\mathfrak{X}=\{\varnothing\}$). It is easy to prove the uniqueness of a countably additive extension of a σ-finite measure τ from a ring $\mathfrak{X}$ to the generated σ-ring (see Exercise 1.12.159); if a measure τ on a ring $\mathfrak{X}$ is such that the corresponding outer measure $\mathfrak{m}$ on $\mathfrak{M}_\mathfrak{m}$ is σ-finite, then $\mathfrak{m}$ is a unique countably extension of τ also to $\sigma(\mathfrak{X})$ (see Exercise 1.12.159). In the above example the measure $\mathfrak{m}$ is not σ-finite because $\mathfrak{m}(\{0\})=\infty$.

Let us stress again that in general the outer measure $\mathfrak{m}$ may differ from τ on $\mathfrak{X}$ (see Exercise 1.12.121). Finally, we recall that if a function τ on an algebra $\mathfrak{X}$ is countably additive, then the associated outer measure $\mathfrak{m}$ coincides with τ on $\mathfrak{X}$. For infinite measures, it may happen that the class $\mathfrak{X}_\tau$ is strictly contained in $\mathfrak{M}_{\tau^*}$ (see Exercise 1.12.129).

Closing our discussion of Carathéodory outer measures let us prove a criterion of $\mathfrak{m}$-measurability of all Borel sets for an outer measure on $\mathbb{R}^n$. We recall that the distance from a point a to a set B is the number

$$\operatorname{dist}(a,B) := \inf_{b\in B} |b-a|.$$

1.11.10. Theorem. *Let $\mathfrak{m}$ be a Carathéodory outer measure on $\mathbb{R}^n$. In order that all Borel sets be $\mathfrak{m}$-measurable, it is necessary and sufficient that the following condition be fulfilled:*

$$\mathfrak{m}(A\cup B) = \mathfrak{m}(A) + \mathfrak{m}(B) \quad \textit{whenever} \ \ d(A,B) > 0, \tag{1.11.11}$$

where $d(A,B) := \inf_{a\in A, b\in B} |a-b|$, and $d(A,\varnothing) := +\infty$.

PROOF. Let $\mathfrak{M}_{\mathfrak{m}}$ contain all closed sets and $d(A,B) = d > 0$. We take disjoint closed sets

$$C_1 = \{x\colon\ \operatorname{dist}(x,A) \le d/4\} \supset A \quad \text{and} \quad C_2 = \{x\colon\ \operatorname{dist}(x,B) \le d/4\} \supset B$$

and observe that by Theorem 1.11.4(ii) one has

$$\mathfrak{m}\bigl((A\cup B)\cap(C_1\cup C_2)\bigr) = \mathfrak{m}\bigl((A\cup B)\cap C_1\bigr) + \mathfrak{m}\bigl((A\cup B)\cap C_2\bigr),$$

which yields (1.11.11), since

$$(A\cup B)\cap C_1 = A,\ (A\cup B)\cap C_2 = B,\ (A\cup B)\cap(C_1\cup C_2) = A\cup B.$$

Let (1.11.11) be fulfilled. It suffices to verify that every closed set C is $\mathfrak{m}$-measurable. Due to the subadditivity of $\mathfrak{m}$, the verification reduces to proving the estimate

$$\mathfrak{m}(A) \ge \mathfrak{m}(A\cap C) + \mathfrak{m}(A\backslash C), \quad \forall\, A\subset \mathbb{R}^n. \tag{1.11.12}$$

If $\mathfrak{m}(A) = \infty$, then (1.11.12) is true. So we assume that $\mathfrak{m}(A) < \infty$. The sets $C_n := \{x\colon\ \operatorname{dist}(x,C) \le n^{-1}\}$ monotonically decrease to C. Obviously, one has $d(A\backslash C_n, A\cap C) \ge n^{-1}$. Therefore,

$$\mathfrak{m}(A\backslash C_n) + \mathfrak{m}(A\cap C) = \mathfrak{m}\bigl((A\backslash C_n)\cup(A\cap C)\bigr) \le \mathfrak{m}(A). \tag{1.11.13}$$

Let us show that

$$\lim_{n\to\infty} \mathfrak{m}(A\backslash C_n) = \mathfrak{m}(A\backslash C). \tag{1.11.14}$$

Let us consider the sets $D_k := \bigl\{x\in A\colon\ (k+1)^{-1} < \operatorname{dist}(x,C) \le k^{-1}\bigr\}$. Then $A\backslash C = \bigcup_{k=n}^{\infty} D_k \bigcup (A\backslash C_n)$. Hence

$$\mathfrak{m}(A\backslash C_n) \le \mathfrak{m}(A\backslash C) \le \mathfrak{m}(A\backslash C_n) + \sum_{k=n}^{\infty} \mathfrak{m}(D_k).$$

Now, for proving (1.11.14), it suffices to observe that the series of $\mathfrak{m}(D_k)$ converges. Indeed, one has $d(D_k, D_j) > 0$ if $j \ge k+2$. By (1.11.11) and induction this gives the relation $\sum_{k=1}^{N} \mathfrak{m}(D_{2k}) = \mathfrak{m}\Bigl(\bigcup_{k=1}^{N} D_{2k}\Bigr) \le \mathfrak{m}(A)$ and a similar relation for odd numbers. According to (1.11.13) and (1.11.14) we obtain

$$\mathfrak{m}(A\backslash C) + \mathfrak{m}(A\cap C) = \lim_{n\to\infty} \mathfrak{m}(A\backslash C_n) + \mathfrak{m}(A\cap C) \le \mathfrak{m}(A).$$

The proof of (1.11.12) is complete. So the theorem is proven. $\square$

It is seen from our reasoning that it applies to any metric space in place of $\mathbb{R}^n$. We shall return to this subject in §7.14(x).

1.12. Supplements and exercises

1.12(i). Set operations

The following result of Sierpiński contains several useful modifications of Theorem 1.9.3 on monotone classes.

Let us consider the following list of operations on sets in a given set X and indicate the corresponding notation:

a finite union $\cup f$, a countable union $\cup c$, the union of an increasing sequence of sets $\lim\uparrow$, a disjoint union $\sqcup f$, a countable disjoint union $\sqcup c$, a finite intersection $\cap f$, a countable intersection $\cap c$, the intersection of a decreasing sequence of sets $\lim\downarrow$, the difference of sets $\backslash$, the difference of a set and its subset $-$.

Note that the symbols f and c indicate the finite and countable character of the corresponding operations and that in the operation $A\backslash B$ the set B may not belong to A, unlike the operation $-$. Every operation O in this list has the dual operation denoted by the symbol O^d and defined as follows:

$$(\cup f)^d := \cap f,\ (\cup c)^d := \cap c,\ (\lim\uparrow)^d := \lim\downarrow,\ (\sqcup f)^d := -,\ (\sqcup c)^d := -, \tag{1.12.1}$$
$$(\cap f)^d := \cup f,\ (\cap c)^d := \cup c, (\lim\downarrow)^d := \lim\uparrow,\ (\backslash)^d := \cup f,\ (-)^d := \sqcup f.$$

The property of a family $\mathcal{F}$ of subsets of X to be closed with respect to some of the above operations is understood in the natural way; for example, "$\mathcal{F}$ is closed with respect to $\lim\uparrow$" means that if sets $F_n \in \mathcal{F}$ increase, then their union belongs to $\mathcal{F}$ as well. It is readily verified that if we are given a class $\mathcal{F}$ of subsets of X and a collection of operations from the above list, then there is the smallest class of sets that contains $\mathcal{F}$ and is closed with respect to the given operations.

1.12.1. Theorem. *Let $\mathcal{F}$ and $\mathcal{G}$ be two classes of subsets of X such that $\mathcal{G} \subset \mathcal{F}$ and the class $\mathcal{F}$ is closed with respect to some collection of operations $\mathcal{O} = (O_1, O_2, \ldots)$ from* (1.12.1). *Denote by $\mathcal{F}_0$ the smallest class of sets that contains $\mathcal{G}$ and is closed with respect to the operations from the same collection $\mathcal{O}$. Then the following assertions are true*:

(i) *if $G\cap G' \in \mathcal{F}_0$ for all $G, G' \in \mathcal{G}$, then the class $\mathcal{F}_0$ is closed with respect to finite intersections*;

(ii) *if* $O^d \in \mathcal{O}$ *for every operation* $O \in \mathcal{O}$ *and* $X\backslash G \in \mathcal{F}_0$ *for all* $G \in \mathcal{G}$, *then the class* $\mathcal{F}_0$ *is closed with respect to complementation*; *in particular, if* $\mathcal{O} = (\cup c, \cap c)$, *then* $\mathcal{F}_0 = \sigma(\mathcal{G})$;

(iii) *if all the conditions in* (i) *and* (ii) *are fulfilled, then the algebra generated by* $\mathcal{G}$ *is contained in* $\mathcal{F}$, *and if* $\mathcal{O} = (\lim\uparrow, \lim\downarrow)$, *then* $\mathcal{F}_0 = \sigma(\mathcal{G})$.

A proof analogous to that of the monotone class theorem is left as Exercise 1.12.100. Another result due to Sierpiński gives a modification of the theorem on σ-additive classes.

1.12.2. Theorem. *Let* $\mathcal{E}$ *be a class of subsets in a space* X *containing the empty set. Denote by* $\mathcal{E}_{\sqcup,\delta}$ *the smallest class of sets in* X *that contains* $\mathcal{E}$ *and is closed with respect to countable unions of pairwise disjoint sets and any countable intersections. If* $X\backslash E \in \mathcal{E}_{\sqcup,\delta}$ *for all* $E \in \mathcal{E}$, *then* $\mathcal{E}_{\sqcup,\delta} = \sigma(\mathcal{E})$.

Proof. Let $\mathcal{A} := \{A \in \mathcal{E}_{\sqcup,\delta}\colon\ X\backslash A \in \mathcal{E}_{\sqcup,\delta}\}$. It suffices to show that the class $\mathcal{A}$ is closed with respect to countable unions of pairwise disjoint sets and any countable intersections, since it will coincide then with the class $\mathcal{E}_{\sqcup,\delta}$, hence the latter will be closed under complementation, i.e., will be a σ-algebra. If sets $A_n \in \mathcal{A}$ are disjoint, then their union belongs to $\mathcal{E}_{\sqcup,\delta}$ by the definition of $\mathcal{E}_{\sqcup,\delta}$, and the complement of their union is $\bigcap_{n=1}^\infty (X\backslash A_n)$, which also belongs to $\mathcal{E}_{\sqcup,\delta}$, since $X\backslash A_n \in \mathcal{E}_{\sqcup,\delta}$. Hence $\mathcal{A}$ admits countable unions of disjoint sets. If $B_n \in \mathcal{A}$, then $\bigcap_{n=1}^\infty B_n \in \mathcal{E}_{\sqcup,\delta}$. Finally, observe that $X\backslash\bigcap_{n=1}^\infty B_n$ can be written in the form

$$\bigcup_{n=1}^\infty (X\backslash B_n) = \bigcup_{n=1}^\infty \Bigl[(X\backslash B_n) \cap \Bigl(\bigcap_{k=1}^{n-1} B_k\Bigr)\Bigr]. \tag{1.12.2}$$

Indeed, the right-hand side obviously belongs to the left one. If x belongs to the left-hand side, then, for some n, we have $x \notin B_n$. If x does not belong to the right-hand side, then $x \notin \bigcap_{k=1}^{n-1} B_k$ and $x \in B_1$. Hence there exists a number m between 1 and $n-2$ such that $x \in \bigcap_{k=1}^m B_k$ and $x \notin \bigcap_{k=1}^{m+1} B_k$. Then $x \in (X\backslash B_{m+1}) \cap \bigl(\bigcap_{k=1}^m B_k\bigr)$, which belongs to the right-hand side of (1.12.2), contrary to our assumption. It is clear that the sets whose union is taken in the right-hand side of (1.12.2) are pairwise disjoint and belong to $\mathcal{E}_{\sqcup,\delta}$ because we have $X\backslash B_n, B_k \in \mathcal{E}_{\sqcup,\delta}$. Thus, $\mathcal{E}_{\sqcup,\delta}$ admits countable intersections. □

1.12.3. Example. The smallest class of subsets of the real line that contains all open sets and is closed under countable unions of pairwise disjoint sets and any countable intersections is the Borel σ-algebra. The same is true if in place of all open sets one takes all closed sets.

Proof. If $\mathcal{E}$ is the class of all open sets, then the theorem applies directly, since the complement of any open set is closed and hence is the countable intersection of a sequence of open sets.

Now let $\mathcal{E}$ be the class of all closed sets. Let us verify that the complements of sets in $\mathcal{E}$ belong to the class $\mathcal{E}_{\sqcup,\delta}$. These complements are open, hence are disjoint unions of intervals or rays. Hence it remains to show that every open interval (a,b) belongs to $\mathcal{E}_{\sqcup,\delta}$. This is not completely obvious, since the open interval cannot be represented in the form of a disjoint union of a sequence of closed intervals. However, one can find a sequence of pairwise disjoint nondegenerate closed intervals $I_n \subset (a,b)$ such that their union S is everywhere dense in (a,b). Let us now verify that $B := (a,b)\backslash S \in \mathcal{E}_{\sqcup,\delta}$. We observe that the closure $\overline{B}$ of the set B consists of B and the countable set $M = \{x_k\}$ formed by the points a and b and the endpoints of the intervals I_n. Hence $B = \bigcap_{m=1}^{\infty} \overline{B}\backslash\{x_1,\ldots,x_m\}$. The set $\overline{B}$ is nowhere dense compact. This enables us to represent each of the sets $\overline{B}\backslash\{x_1,\ldots,x_m\}$ in the form of the union of disjoint compact sets. Let us do this for $\overline{B}\backslash\{x_1\}$, the reasoning for other sets is similar. Since $\overline{B}$ has no interior, the open complement of $\overline{B}$ contains a sequence of points l_j increasing to x_1 and a sequence of points r_j decreasing to x_1. We may assume that $l_1 < a$, $r_1 > b$. The sets $(l_j, l_{j+1}) \cap \overline{B}$ and $(r_{j+1}, r_j) \cap \overline{B}$ are compact, since the points $l_j, l_{j+1}, r_{j+1}, r_j$ belong to the complement of $\overline{B}$ with some neighborhoods. These sets give the desired decomposition of $\overline{B}\backslash\{x_1\}$. □

In Chapter 6 one can find some additional information related to the results in this subsection.

1.12(ii). Compact classes

A compact class approximating a measure may not consist of measurable sets. For example, if $\mathcal{A}$ is the σ-algebra on $[0,1]^2$ consisting of the sets $B\times[0,1]$, where $B \in \mathcal{B}([0,1])$, μ is the restriction of Lebesgue measure to $\mathcal{A}$, and $\mathcal{K}$ is the class of all compact sets in $[0,1]^2$, then $\mathcal{K}$ is approximating for μ, but the interval $I := [0,1]\times\{0\}$ does not belong to $\mathcal{A}_\mu$, since $\mu^*(I) = 1$ and I does not contain nonempty sets from $\mathcal{A}$. In addition, a compact approximating class may not be closed with respect to unions and intersections. The next result shows that one can always "improve" the original approximating compact class by replacing it with a compact class that consists of measurable sets, approximates the measure, and is stable under finite unions and countable intersections.

1.12.4. Proposition. (i) *Let $\mathcal{K}$ be a compact class of subsets of a set X. Then, the minimal class $\mathcal{K}_{s\delta}$ which contains $\mathcal{K}$ and is closed with respect to finite unions and countable intersections, is compact as well* (*more precisely, $\mathcal{K}_{s\delta}$ coincides with the class of at most countable intersections of finite unions of elements of $\mathcal{K}$*).

(ii) *In addition, if $\mathcal{E}$ is a compact class of subsets of a set Y, then the class of products $K\times E$, $K \in \mathcal{K}$, $E \in \mathcal{E}$, is compact as well.*

(iii) *If a nonnegative measure μ on an algebra* (*or semialgebra*) *$\mathcal{A}_0$ has an approximating compact class $\mathcal{K}$, then there exists a compact class $\mathcal{K}'$ that is contained in $\sigma(\mathcal{A}_0)$, approximates μ on $\sigma(\mathcal{A}_0)$, and is stable under finite unions and countable intersections.*

PROOF. (i) We show first that the class $\mathcal{K}_s$ of finite unions of sets in $\mathcal{K}$ is compact. Let $A_i = \bigcup_{n=1}^{m_i} K_i^n$, where $K_i^n \in \mathcal{K}$, be such that $\bigcap_{i=1}^k A_i \neq \varnothing$ for all $k \in \mathbb{N}$. Denote by M the set of all sequences $\nu = (\nu_i)$ such that $\nu_i \leq m_i$ for all $i \geq 1$. Let M_k be the collection of all sequences ν in M such that $\bigcap_{i=1}^k K_i^{\nu_i} \neq \varnothing$. Note that the sets M_k are nonempty for all k. This follows from the relation

$$\bigcup_{\nu \in M} \bigcap_{i=1}^{k} K_i^{\nu_i} = \bigcap_{i=1}^{k} A_i \neq \varnothing,$$

which is easily seen from the fact that $x \in \bigcap_{i=1}^k A_i$ precisely when there exist $\nu_i \leq m_i$, $i = 1, \ldots, k$, with $x \in K_i^{\nu_i}$. In addition, the sets M_k are decreasing. We prove that there is a sequence ν in their intersection. This means that the intersection $\bigcap_{n=1}^\infty A_n$ is nonempty, since it contains the set $\bigcap_{n=1}^\infty K_n^{\nu_n}$, which is nonempty by the compactness of the class $\mathcal{K}$ and the fact that the sets $\bigcap_{n=1}^k K_n^{\nu_n}$ are nonempty.

In order to prove the relation $\bigcap_{k=1}^\infty M_k \neq \varnothing$ let us choose an element $\nu^{(k)} = (\nu_n^{(k)})_{n=1}^\infty$ in every set M_k. Since $\nu_n^{(k)} \leq m_n$ for all n and k, there exist infinitely many indices k such that the numbers $\nu_1^{(k)}$ coincide with one and the same number ν_1. By induction, we construct a sequence of natural numbers $\nu = (\nu_i)$ such that, for every n, there exist infinitely many indices k with the property that $\nu_i^{(k)} = \nu_i$ for all $i = 1, \ldots, n$. This means that $\nu \in M_n$, since the membership in M_n is determined by the first n coordinates of a sequence, and for all $k > n$ we have $\nu^{(k)} \in M_n$ by the inclusion $\nu^{(k)} \in M_k \subset M_n$. Thus, ν belongs to all M_n.

The compactness of the class $\mathcal{K}_s$ obviously yields the compactness of the class $\mathcal{K}_{s\delta}$ of all at most countable intersections of sets in $\mathcal{K}_s$. It is clear that this is the smallest class that contains $\mathcal{K}$ and is closed with respect to finite unions and at most countable intersections (observe that a finite union of several countable intersections of finite unions of sets in $\mathcal{K}$ can be written as a countable intersection of finite unions).

(ii) If the intersections $\bigcap_{n=1}^N (K_n \times E_n)$, where $K_n \in \mathcal{K}$, $E_n \in \mathcal{E}$, are nonempty, then $\bigcap_{n=1}^N K_n$ and $\bigcap_{n=1}^N E_n$ are nonempty as well, which by the compactness of $\mathcal{K}$ and $\mathcal{E}$ gives points $x \in \bigcap_{n=1}^\infty K_n$ and $y \in \bigcap_{n=1}^\infty E_n$. Then $(x, y) \in \bigcap_{n=1}^\infty (K_n \times E_n)$.

(iii) According to (i) we can assume that $\mathcal{K}$ is stable under finite unions and countable intersections. Let $\mathcal{K}' = \mathcal{K} \cap \sigma(\mathcal{A}_0)$. Clearly, $\mathcal{K}'$ is a compact class. Let us show that $\mathcal{K}'$ approximates μ on $\mathcal{A}_0$. Given $A \in \mathcal{A}_0$ and $\varepsilon > 0$, we can construct inductively sets $A_n \in \mathcal{A}_0$ and $K_n \in \mathcal{K}$ such that

$$A \supset K_1 \supset A_1 \supset K_2 \supset A_2 \supset \cdots \quad \text{and} \quad \mu(A_n \backslash A_{n+1}) < \varepsilon 2^{-n-1}, \ A_0 := A.$$

We observe that $\bigcap_{n=1}^\infty A_n = \bigcap_{n=1}^\infty K_n$. Denoting this set by K we have $K \in \mathcal{K}'$, since $\sigma(\mathcal{A}_0)$ and $\mathcal{K}$ admit countable intersections. In addition, $K \subset A$ and $\mu(A \backslash K) < \varepsilon$. Finally, $\mathcal{K}'$ approximates μ on $\sigma(\mathcal{A}_0)$. Indeed, for every $A \in \sigma(\mathcal{A}_0)$ and every $\varepsilon > 0$, one can find sets $A_n \in \mathcal{A}$ such that $A_0 :=$

$\bigcap_{n=1}^\infty A_n \subset A$ and $\mu(A\backslash A_0) < \varepsilon$. To this end, it suffices to find sets $B_n \in \mathcal{A}$ covering $X\backslash A$ such that the measure of their union is less than $\mu(X\backslash A) + \varepsilon$ and take $A_n = X\backslash B_n$. There exist sets $K_n \in \mathcal{K}'$ such that $K_n \subset A_n$ and $\mu(A_n\backslash K_n) < \varepsilon 2^{-n}$. Let $K := \bigcap_{n=1}^\infty K_n$. Then $K \subset A_0$, $\mu(A_0\backslash K) < \mu(K) + \varepsilon$ and $K \in \mathcal{K}'$ because $\mathcal{K}'$ is stable under countable intersections. □

Assertion (ii) will be reinforced in Lemma 3.5.3. The class of sets of the form $K{\times}E$, where $K \in \mathcal{K}$, $E \in \mathcal{E}$, is denoted by $\mathcal{K}{\times}\mathcal{E}$ (the usual understanding of the product of sets $\mathcal{K}$ and $\mathcal{E}$ as the collection of pairs (K, E) does not lead to confusion here).

It is worth noting that if μ is a finite nonnegative measure on a σ-algebra $\mathcal{A}$, then, by assertion (iii) above, the existence of a compact approximating class for μ does not depend on whether we consider μ on $\mathcal{A}$ or on its completion $\mathcal{A}_\mu$. We know that an approximating compact class $\mathcal{K}$ need not be contained in $\mathcal{A}_\mu$. However, according to Theorem 1.12.34 stated below, there is a countably additive extension of μ to the σ-algebra generated by $\mathcal{A}$ and $\mathcal{K}$.

A property somewhat broader than compactness is monocompactness, considered in the following result of Mallory [**647**], which strengthens Theorem 1.4.3.

1.12.5. Theorem. *Let $\mathcal{R}$ be a semiring and let μ be an additive nonnegative function on $\mathcal{R}$ such that there exists a class of sets $\mathcal{M} \subset \mathcal{R}$ with the following property: if sets $M_n \in \mathcal{M}$ are nonempty and decreasing, then $\bigcap_{n=1}^\infty M_n$ is nonempty (such a class is called monocompact). Suppose that*

$$\mu(R) = \sup\{\mu(M)\colon\ M \in \mathcal{M}, M \subset R\} \quad \textit{for all } R \in \mathcal{R}.$$

Then μ is countably additive on $\mathcal{R}$.

Proof. Let $R = \bigcup_{n=1}^\infty R_n$, where $R_n \in \mathcal{R}$. It suffices to show that

$$\mu(R) \le \sum_{n=1}^\infty \mu(R_n).$$

Suppose the opposite. Then there exists a number c such that

$$\sum_{n=1}^\infty \mu(R_n) < c < \mu(R).$$

Let us take $M \in \mathcal{M}$ with $M \subset R$ and $\mu(M) > c$. We can write $M\backslash R_1$ as a disjoint union

$$M\backslash R_1 = \bigcup_{j=1}^{m_1} R^j, \quad R^j \in \mathcal{R}.$$

Let us find $M_1, \ldots, M_{m_1} \in \mathcal{M}$ with $M_j \subset R^j$ and $\sum_{j=1}^{m_1} \mu(M_j) + \mu(R_1) > c$. By induction, we construct sets $M_{j_1,\ldots,j_n} \in \mathcal{M}$ as follows. If $M_{j_1,\ldots,j_n}$ are already constructed, then we find finitely many disjoint sets $R^{j_1,\ldots,j_n,j} \in \mathcal{R}$

whose union is $M_{j_1,\ldots,j_n}\backslash R_{n+1}$, and also a set $M_{j_1,\ldots,j_n,j} \in \mathcal{M}$ such that one has $M_{j_1,\ldots,j_n,j} \subset R^{j_1,\ldots,j_n,j}$ and

$$\sum_{j_1,\ldots,j_n,j} \mu(M_{j_1,\ldots,j_n,j}) + \sum_{i=1}^{n} \mu(R_i) > c.$$

Note that $\sum\limits_{j_1,\ldots,j_n,j} \mu(M_{j_1,\ldots,j_n,j}) > 0$ due to our choice of c. Hence there exists a sequence of indices j_i such that $M_{j_1,\ldots,j_k} \neq \varnothing$ for all k (such a sequence is found by induction by choosing $j_1,\ldots,j_{k-1}$ with $\mu(M_{j_1,\ldots,j_{k-1}}) > 0$). Thus, $\bigcap_{k=1}^{\infty} M_{j_1,\ldots,j_k}$ is nonempty, whence it follows that $R \neq \bigcup_{n=1}^{\infty} R_n$, which is a contradiction. □

Fremlin [**326**] constructed an example that distinguishes compact and monocompact measures, i.e., there is a probability measure possessing a monocompact approximating class, but having no compact (countably compact by the terminology of the cited work) approximating classes.

1.12(iii). Metric Boolean algebra

Let $(X, \mathcal{A}, \mu)$ be a measure space with a finite nonnegative measure μ. In this subsection we discuss a natural metric structure on the set of all μ-measurable sets.

Suppose first that μ is a bounded nonnegative additive set function on an algebra $\mathcal{A}$. Set

$$d(A, B) = \mu(A \bigtriangleup B), \quad A, B \in \mathcal{A}.$$

The function d is called the Fréchet–Nikodym metric. Let us introduce the following relation on $\mathcal{A}$: $A \sim B$ if $d(A, B) = 0$. Clearly, $A \sim B$ if and only if A and B differ in a measure zero set. This is an equivalence relation:

1) $A \sim A$, 2) if $A \sim B$, then $B \sim A$, 3) if $A \sim B$ and $B \sim C$, then $A \sim C$. Denote by $\mathcal{A}/\mu$ the set of all equivalence classes for this relation. The function d has a natural extension to $\mathcal{A}/\mu \times \mathcal{A}/\mu$:

$$d(\widetilde{A}, \widetilde{B}) = d(A, B)$$

if A and B represent the classes $\widetilde{A}$ and $\widetilde{B}$, respectively. By the additivity of μ, this definition does not depend on our choice of representatives in the equivalence classes. The function d makes the set $\mathcal{A}/\mu$ a metric space. The triangle inequality follows, since for all $A, B, C \in \mathcal{A}$ one has the inclusion $A \triangle C \subset (A \triangle B) \cup (B \triangle C)$, whence we obtain $\mu(A \triangle C) \leq \mu(A \triangle B) + \mu(B \triangle C)$. By means of representatives of classes, one introduces the operations of intersection, union, and complementation on $\mathcal{A}/\mu$. The metric space $(\mathcal{A}/\mu, d)$ is called the metric Boolean algebra generated by $(\mathcal{A}, \mu)$. Note that the function μ is naturally defined on $\mathcal{A}/\mu$ and is Lipschitzian on $(\mathcal{A}/\mu, d)$. This follows by the inequality $|\mu(A) - \mu(B)| \leq \mu(A \bigtriangleup B) = d(A, B)$.

A measure μ is called separable if the metric space $(\mathcal{A}/\mu, d)$ is separable, i.e., contains a countable everywhere dense subset. The separability of μ is equivalent to the existence of an at most countable collection of sets $A_n \in \mathcal{A}$

such that, for every $A \in \mathcal{A}$ and $\varepsilon > 0$, there exists n with $\mu(A \triangle A_n) < \varepsilon$. The last property can be taken as a definition of separability for infinite measures. Lebesgue measure and many other measures encountered in applications are separable, but nonseparable measures exist as well. Concerning separable measures, see Exercises 1.12.102 and 4.7.63 and §7.14(iv).

1.12.6. Theorem. *Let μ be a bounded nonnegative additive set function on an algebra $\mathcal{A}$.*

(i) *The function μ is countably additive if and only if $d(A_n, \varnothing) \to 0$ as $A_n \downarrow \varnothing$.*

(ii) *If $\mathcal{A}$ is a σ-algebra and μ is countably additive, then the metric space $(\mathcal{A}/\mu, d)$ is complete.*

Proof. (i) It suffices to note that $A_n \triangle \varnothing = A_n$ and $d(A_n, \varnothing) = \mu(A_n)$. (ii) Let $\{\widetilde{A}_n\}$ be a Cauchy sequence in $(\mathcal{A}/\mu, d)$ and A_n a representative of the class $\widetilde{A}_n$. Let us show that there exists a set $A \in \mathcal{A}$ such that $d(A_n, A) \to 0$. It suffices to show that there is a convergent subsequence in $\{A_n\}$. Hence, passing to a subsequence, we may assume that $\mu(A_k \triangle A_n) < 2^{-n}$ for all n and $k \geq n$. Set

$$A = \limsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

We show that $d(A_n, A) \to 0$. Let $\varepsilon > 0$. The sets $\bigcap_{n=1}^{N} \bigcup_{k=n}^{\infty} A_k$ increase to A. By the countable additivity of μ there exists a number N such that

$$\mu\Bigl(\bigcup_{k=N}^{\infty} A_k \backslash A\Bigr) = \mu\Bigl(\bigcap_{n=1}^{N} \bigcup_{k=n}^{\infty} A_k \backslash A\Bigr) < \varepsilon.$$

Then, for all $m \geq N$, we have

$$\mu\Bigl(\bigcup_{k=m}^{\infty} A_k \backslash A\Bigr) < \varepsilon.$$

Since $\mu(A_m \triangle A_k) \geq \mu(A_k \backslash A_m)$, we obtain for all m sufficiently large that

$$\mu\Bigl(\bigcup_{k=m}^{\infty} A_k \backslash A_m\Bigr) \leq \sum_{k=m+1}^{\infty} \mu(A_k \backslash A_m) \leq \sum_{k=m+1}^{\infty} 2^{-k} < \varepsilon,$$

whence we have $\mu(A_m \triangle A) < 2\varepsilon$, since $A, A_m \subset \bigcup_{k=m}^{\infty} A_k$. □

We remark that in assertion (ii) the space $(\mathcal{A}/\mu, d)$ is complete even if $\mathcal{A}$ is not complete with respect to μ, which is natural, since every set in the completed σ-algebra $\mathcal{A}_\mu$ coincides up to a measure zero set with an element of $\mathcal{A}$, hence belongs to the same equivalence class. Note also that the consideration of $(\mathcal{A}/\mu, d)$ is simplified if we employ the concepts of the theory of integration developed in Chapters 2 and 4 and deal with the indicator functions of sets rather than with sets themselves.

Now let $\mathcal{A}$ be a σ-algebra and let μ be countably additive.

1.12.7. Definition. The set $A \in \mathcal{A}$ is called an atom of the measure μ if $\mu(A) > 0$ and every set $B \subset A$ from $\mathcal{A}$ has measure either 0 or $\mu(A)$.

If two atoms A_1 and A_2 are distinct in the sense that $d(A, B) > 0$ (i.e., A and B are not equivalent), then $\mu(A_1 \cap A_2) = 0$. Hence there exists an at most countable set $\{A_n\}$ of pairwise non-equivalent atoms. The measure μ is called purely atomic if $\mu\bigl(X \backslash \bigcup_{n=1}^{\infty} A_n\bigr) = 0$. If there are no atoms, then the measure μ is called atomless.

1.12.8. Example. Lebesgue measure λ is atomless on every measurable set A in $[a, b]$. Moreover, for any $\alpha \in [0, \lambda(A)]$, there exists a set $B \subset A$ such that $\lambda(B) = \alpha$.

PROOF. The function $F(x) = \lambda\bigl(A \cap [a, x)\bigr)$ is continuous on $[a, b]$ by the countable additivity of Lebesgue measure. It remains to apply the mean value theorem. $\square$

1.12.9. Theorem. *Let $(X, \mathcal{A}, \mu)$ be a measure space with a finite non-negative measure μ. Then, for every $\varepsilon > 0$, there exists a finite partition of X into pairwise disjoint sets $X_1, \dots, X_n \in \mathcal{A}$ with the following property: either $\mu(X_i) \le \varepsilon$, or X_i is an atom of measure greater than ε.*

PROOF. There exist only finitely many non-equivalent atoms $A_1, \dots, A_p$ of measure greater than ε. Then the space $Y = X \backslash \bigcup_{i=1}^{p} A_i$ has no atoms of measure greater than ε. Let us show that every set $B \in \mathcal{A}$, contained in Y and having positive measure, contains a set C such that $0 < \mu(C) \le \varepsilon$. Indeed, suppose that there exists a set B for which this is false. Then $\mu(B) > \varepsilon$ (otherwise we may take $C = B$) and hence B is not an atom. Therefore, there exists a set $B_1 \in \mathcal{A}$ with $\varepsilon < \mu(B_1) < \mu(B)$. Then $\mu(B \backslash B_1) > \varepsilon$ (otherwise we arrive at a contradiction with our choice of B) and for the same reason the set $C_1 = B \backslash B_1$ contains a subset $B_2 \in \mathcal{A}$ with $\varepsilon < \mu(B_2) < \mu(C_1)$. Note that $\mu(C_1 \backslash B_2) > \varepsilon$. Let $C_2 = C_1 \backslash B_2$ and in C_2 we find a set $B_3 \in \mathcal{A}$ with $\varepsilon < \mu(B_3) < \mu(C_2)$. Continuing by induction, we obtain an infinite sequence of pairwise disjoint sets B_n of measure greater than ε, which is impossible, since $\mu(Y) < \infty$.

Now for every $A \in \mathcal{A}$ we set

$$\eta(A) = \sup\bigl\{\mu(B)\colon\ B \subset A, B \in \mathcal{A}, \mu(B) \le \varepsilon\bigr\}.$$

According to what has been proven above, one has that $0 < \eta(A) \le \varepsilon$ if $A \subset Y$ and $\mu(A) > 0$. We may find a set $B_1 \in \mathcal{A}$ in Y such that $0 < \mu(B_1) \le \eta(Y)$, provided that $\mu(Y) > \varepsilon$; if $\mu(Y) \le \varepsilon$, then the proof is complete. By using the above established property of subsets of Y, we construct by induction a sequence of pairwise disjoint sets $B_n \in \mathcal{A}$ such that $B_n \subset Y$ and

$$\frac{1}{2}\eta\Bigl(Y \backslash \bigcup_{i=1}^{n} B_i\Bigr) \le \mu(B_{n+1}) \le \varepsilon.$$

If at some step it is impossible to continue this construction, then this completes the proof. Let $B_0 = Y\backslash\bigcup_{i=1}^\infty B_i$. Then

$$\eta(B_0) \le \eta\Bigl(Y\backslash\bigcup_{i=1}^n B_i\Bigr) \le 2\mu(B_{n+1})$$

for all n. The series of measures of B_n converges, hence $\mu(B_n) \to 0$, whence we have $\eta(B_0) = 0$. Therefore, $\mu(B_0) = 0$. It remains to take a number k such that $\sum_{i=k}^\infty \mu(B_i) < \varepsilon$. The sets $A_1, \dots, A_p$, $B_1, \dots, B_k$, $\bigcup_{i=k+1}^\infty B_i \bigcup B_0$ form a desired partition. $\square$

1.12.10. Corollary. *Let μ be an atomless measure. Then, for every $\alpha \in [0, \mu(X)]$, there exists a set $A \in \mathcal{A}$ such that $\mu(A) = \alpha$.*

Proof. By using the previous theorem one can construct an increasing sequence of sets $A_n \in \mathcal{A}$ such that $\mu(A_n) \to \alpha$. Indeed, let $\alpha > 0$. We can partition X into finitely many parts X_j with $\mu(X_j) < 1/2$. Let us take the biggest number m with $\mu\bigl(\bigcup_{j=1}^m X_j\bigr) \le \alpha$. Letting $A_1 := \bigcup_{j=1}^m X_j$ we have $\mu(A_1) \ge \alpha - 1/2$. In the same manner we find a set $B_1 \subset X\backslash A_1$ with $\mu(B_1) \ge \alpha - \mu(A_1) - 1/3$ and take $A_2 := A_1 \cup B_1$. We proceed by induction and obtain sets A_{n+1} of the form $A_n \cup B_n$, where $B_n \subset X\backslash A_n$ and $\mu(B_n) \ge \alpha - \mu(A_n) - (n+1)^{-1}$. Now we can take $A = \bigcup_{n=1}^\infty A_n$. $\square$

We remark that in the case of infinite measures the Fréchet–Nikodym metric can be considered on the class of sets of finite measure. Another related metric is considered in Exercise 1.12.152.

1.12(iv). Measurable envelope, measurable kernel and inner measure

Let $(X, \mathcal{B}, \mu)$ be a measure space with a finite nonnegative measure μ. We observe that the restriction of μ to a measurable subset A is again a measure defined on the trace σ-algebra $\mathcal{B}_A$ of the space A that consists of the sets $A \cap B$, where $B \in \mathcal{B}$. The following construction enables one to restrict μ to arbitrary sets A, possibly nonmeasurable, if we define $\mathcal{B}_A$ as above. The trace σ-algebra $\mathcal{B}_A$ is also called the restriction of the σ-algebra $\mathcal{B}$ to A and denoted by the symbol $\mathcal{B} \cap A$.

For any set $A \subset X$, there exists a set $\widetilde{A} \in \mathcal{B}$ (called a *measurable envelope* of A) with

$$A \subset \widetilde{A} \text{ and } \mu(\widetilde{A}) = \mu^*(A). \tag{1.12.3}$$

For such a set (which is not unique) we can take

$$\widetilde{A} = \bigcap_{n=1}^\infty A_n, \text{ where } A_n \in \mathcal{B},\ A_n \supset A \text{ and } \mu(A_n) \le \mu^*(A) + 1/n. \tag{1.12.4}$$

Informally speaking, $\widetilde{A}$ is a minimal measurable set containing A.

By (1.12.3) and the definition of outer measure it follows that if we have $A \subset B \subset \widetilde{A}$ and $B \in \mathcal{B}$, then $\mu(\widetilde{A} \triangle B) = 0$.

1.12.11. Definition. *The restriction* μ_A *(denoted also by* $\mu|_A$*) of the measure* μ *to* $\mathcal{B}_A$ *is defined by the formula*

$$\mu_A(B \cap A) := \mu|_A(B \cap A) := \mu(B \cap \widetilde{A}), \quad B \in \mathcal{B},$$

where $\widetilde{A}$ *is an arbitrary measurable envelope of* A.

It is easily seen that this definition does not depend on our choice of $\widetilde{A}$ and that the function μ_A is countably additive. If $A \in \mathcal{B}$, then we obtain the usual restriction.

1.12.12. Proposition. *The measure* μ_A *coincides with the restriction of the outer measure* μ^* *to* $\mathcal{B}_A$.

PROOF. Let $B \in \mathcal{B}$. Then

$$\mu^*(B \cap A) \leq \mu^*(B \cap \widetilde{A}) = \mu(B \cap \widetilde{A}) = \mu_A(B \cap A).$$

On the other hand, if $B \cap A \subset C$, where $C \in \mathcal{B}$, then

$$A \subset \widetilde{A}\backslash\bigl(B \cap (\widetilde{A}\backslash C)\bigr).$$

By the definition of a measurable envelope we obtain $\mu\bigl(B \cap (\widetilde{A}\backslash C)\bigr) = 0$. Hence

$$\mu(B \cap \widetilde{A}) \leq \mu(B \cap C) + \mu\bigl(B \cap (\widetilde{A}\backslash C)\bigr) = \mu(B \cap C) \leq \mu(C),$$

which yields by taking inf over C that $\mu(B \cap \widetilde{A}) \leq \mu^*(B \cap A)$. □

By analogy with a measurable envelope one can define a measurable kernel $\underline{A}$ of an arbitrary set A. Namely, let us first define the *inner measure* of a set A by the formula

$$\mu_*(A) = \sup\bigl\{\mu(B)\colon\ B \subset A, B \in \mathcal{B}\bigr\}.$$

A measurable kernel of a set A is a set $\underline{A} \in \mathcal{B}$ such that

$$\underline{A} \subset A \quad \text{and} \quad \mu(\underline{A}) = \mu_*(A).$$

For $\underline{A}$ one can take the union of a sequence of sets $B_n \in \mathcal{B}$ such that $B_n \subset A$ and $\mu(B_n) \geq \mu_*(A) - 1/n$. Obviously, a measurable kernel is not unique, but if a set C from $\mathcal{B}$ is contained in A, then $\mu(C\backslash\underline{A}) = 0$. Informally speaking, $\underline{A}$ is a maximal measurable subset of A.

Outer and inner measures are also denoted by the symbols μ_e and μ_i, respectively (from "mesure extérieure" and "mesure intérieure").

Note that the nonmeasurable set in Example 1.7.7 has inner measure 0 (otherwise E would contain a measurable set E_0 of positive measure, which gives disjoint sets $E_0 + r_n$ with equal positive measures). The following modification of this example produces an even more exotic set.

1.12.13. Example. The real line with Lebesgue measure λ contains a set E such that

$$\lambda_*(E) = 0 \quad \text{and} \quad \lambda^*(E \cap A) = \lambda(A) = \lambda^*(A\backslash E)$$

for any Lebesgue measurable set A. The same is true for the interval $[0,1]$.

PROOF. Similarly to Example 1.7.7, we find a set E_0 containing exactly one representative from every equivalence class for the following equivalence relation: $x \sim y$ if $x - y = n + m\sqrt{2}$, where $m, n \in \mathbb{Z}$. Set

$$E = \Bigl\{ e + 2n + m\sqrt{2} \colon \ e \in E_0, m, n \in \mathbb{Z} \Bigr\}.$$

In the case of the interval we consider the intersection of E with $[0,1]$. Let $A \subset E$ be a measurable set. Note that the set $A - A = \bigl\{a_1 - a_2 \colon \ a_1, a_2 \in A\bigr\}$ contains no points of the form $2n+1+m\sqrt{2}$ with integer n and m. Therefore, $A - A$ contains no intervals, hence $\lambda(A) = 0$ (see Exercise 1.12.62). Thus, $\lambda_*(E) = 0$. We observe that the complement of E coincides with $E+1$ (in the case of $[0,1]$ one has $[0,1]\backslash E \subset (E+1) \cup (E-1)$). Indeed, the difference between any point x and its representative in E_0 is a number of the form $n+m\sqrt{2}$. Hence $x = e+n+m\sqrt{2}$ is either in E (if n is even) or in $E+1$. On the other hand, $E \cap (E+1) = \varnothing$, since E_0 contains only one representative from every class. Therefore, the complement of E has inner measure 0. This means that $\lambda^*(A \cap E) = \lambda(A)$ for any Lebesgue measurable set A, since

$$\lambda^*(A \cap E) = \lambda(A) - \lambda_*\bigl(A \backslash (A \cap E)\bigr) = \lambda(A) - \lambda_*(A \backslash E),$$

where the number $\lambda_*(A\backslash E)$ does not exceed the inner measure of the complement of E, i.e., equals zero. Similarly, $\lambda^*(A\backslash E) = \lambda(A)$. □

1.12(v). Extensions of measures

The next result shows that one can always extend a measure whose domain does not coincide with the class of all subsets of the given space. It follows that a measure has no maximal countably additive extension unless it can be extended to all subsets.

1.12.14. Theorem. *Let μ be a finite nonnegative measure on a σ-algebra $\mathcal{B}$ in a space X and let S be a set such that $\mu_*(S) = \alpha < \mu^*(S) = \beta$, where $\mu_*(S) = \sup\bigl\{\mu(B) \colon \ B \subset S, B \in \mathcal{B}\bigr\}$. Then, for any $\gamma \in [\alpha, \beta]$, there exists a countably additive measure ν on the σ-algebra $\sigma(\mathcal{B} \cup S)$ generated by $\mathcal{B}$ and S such that $\nu = \mu$ on $\mathcal{B}$ and $\nu(S) = \gamma$.*

PROOF. Suppose first that $\mu_*(S) = 0$ and $\mu^*(S) = \mu(X)$. We may assume that $\mu(X) = 1$. Set

$$\mathcal{E}_S = \Bigl\{ E = (S \cap A) \cup \bigl((X\backslash S) \cap B\bigr) \colon \ A, B \in \mathcal{B} \Bigr\}. \tag{1.12.5}$$

As we have seen in Example 1.2.7, $\mathcal{E}_S$ is the σ-algebra generated by S and $\mathcal{B}$. Now we set

$$\nu\Bigl((S \cap A) \cup \bigl((X\backslash S) \cap B\bigr)\Bigr) = \gamma\mu(A) + (1-\gamma)\mu(B).$$

Let us show that the set function ν is well-defined, i.e., if

$$E = (S \cap A) \cup \bigl((X\backslash S) \cap B\bigr) = (S \cap A_0) \cup \bigl((X\backslash S) \cap B_0\bigr),$$

where $A_0, B_0 \in \mathcal{B}$, then $\nu(E)$ does not depend on which of the two representations of E we use. To this end, it suffices to note that $\mu(A_0) = \mu(A)$ and $\mu(B_0) = \mu(B)$. Indeed, $A \cap S = A_0 \cap S$. Then the measurable sets $A\backslash A_0$ and $A_0\backslash A$ are contained in $X\backslash S$ and have measure zero, since $\mu^*(S) = \mu(X)$. Therefore, one has $\mu(A \bigtriangleup A_0) = 0$. Similarly we obtain $\mu(B \bigtriangleup B_0) = 0$, since $\mu^*(X\backslash S) = \mu(X)$ by the equality $\mu_*(S) = 0$. By construction we have $\nu(S) = \gamma\mu(X) = \gamma$. If $A = B \in \mathcal{B}$, then $\nu(B) = \gamma\mu(B) + (1-\gamma)\mu(B) = \mu(B)$.

Let us show that ν is a countably additive measure. Let E_n be pairwise disjoint sets in $\mathcal{E}_S$, generated by pairs of sets $(A_n, B_n) \in \mathcal{B}$ according to (1.12.5). Then the sets $A_n \cap S$ are pairwise disjoint. Therefore, if $n \neq k$, the measurable sets $A_n \cap A_k$ are contained in $X\backslash S$ and hence have measure zero. Therefore, $\mu\bigl(\bigcup_{n=1}^\infty A_n\bigr) = \sum_{n=1}^\infty \mu(A_n)$. Similarly, $\mu\bigl(\bigcup_{n=1}^\infty B_n\bigr) = \sum_{n=1}^\infty \mu(B_n)$. This shows that $\nu\bigl(\bigcup_{n=1}^\infty E_n\bigr) = \sum_{n=1}^\infty \nu(E_n)$. Thus, in the considered case the theorem is proven.

In the general case, let us take a measurable envelope $\widetilde{S}$ of the set S (see (1.12.4). Let $\underline{S}$ be a measurable kernel of S. Then $\mu(\underline{S}) = \mu_*(S) = \alpha$. Set

$$X_0 = \widetilde{S}\backslash\underline{S}, \quad S_0 = S\backslash\underline{S}.$$

The restriction of the measure μ to X_0 is denoted by μ_0. Note that we have $\mu_0^*(S_0) = \mu_0(X_0) = \beta - \alpha$ and $(\mu_0)_*(S_0) = 0$. According to the previous step, there exists a measure ν_0 on the space X_0 with the σ-algebra $\mathcal{E}_{S_0}$ generated by S_0 and all sets $B \in \mathcal{B}$ with $B \subset X_0$ such that $\nu_0(S_0) = \gamma - \alpha$ and ν_0 coincides with μ_0 on all sets $B \subset X_0$ in $\mathcal{B}$. The collection of all sets of the form

$$E = A \cup E_0 \cup B, \quad \text{where } A, B \in \mathcal{B}, A \subset X\backslash\widetilde{S}, B \subset \underline{S}, E_0 \in \mathcal{E}_{S_0},$$

is the σ-algebra $\mathcal{E}$ generated by S and $\mathcal{B}$. Let us consider the measure

$$\nu(E) = \mu(A) + \nu_0(E_0) + \mu(B).$$

It is readily seen that ν is a countably additive measure on $\mathcal{E}$ equal to μ on $\mathcal{B}$, and that $\nu(S) = \mu(\varnothing) + \nu_0(S_0) + \mu(\underline{S}) = \gamma - \alpha + \alpha = \gamma$.

It is easily verified that the formula

$$\nu(E) := \mu^*(E \cap S) + \mu_*\bigl(E \cap (X\backslash S)\bigr), \quad E \in \mathcal{E}_S,$$

gives an extension of the measure μ with $\nu(S) = \mu^*(S)$. The closely related Nikodym's approach is described in Exercise 3.10.37. □

The assertion on existence of extensions can be generalized to arbitrary families of pairwise disjoint sets. For countable families of additional sets this is due to Bierlein [**89**]; the general case was considered in Ascherl, Lehn [**40**].

1.12.15. Theorem. *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\{Z_\alpha\}$ be a family of pairwise disjoint subsets in X. Then, there exists a probability measure ν that extends μ to the σ-algebra generated by $\mathcal{B}$ and $\{Z_\alpha\}$.*

PROOF. First we consider a countable family of pairwise disjoint sets Z_n. Let us choose measurable envelopes $\widetilde{Z}_n$ of the sets Z_n. Let

$$B_1 = \widetilde{Z}_1, \quad B_n = \widetilde{Z}_n \backslash \bigcup_{i=1}^{n-1} \widetilde{Z}_i, \quad n > 1.$$

Then the sets B_n belong to $\mathcal{B}$ and are disjoint. We shall show that the set $S = \bigcup_{n=1}^{\infty}(B_n \backslash Z_n)$ has inner measure zero. Note first that

$$\mu_*(B_n \backslash Z_n) \leq \mu_*(\widetilde{Z}_n \backslash Z_n) = 0$$

for all $n \geq 1$, since $B_n \subset \widetilde{Z}_n$. Now let $C \in \mathcal{B}$, $C \subset \bigcup_{n=1}^{\infty}(B_n \backslash Z_n)$. Then $\mu(C) = \sum_{n=1}^{\infty} \mu(C \cap B_n) = 0$, since $C \cap B_n \subset B_n \backslash Z_n$. Thus, $\mu_*(S) = 0$. By Theorem 1.12.14, there exists an extension of the measure μ to a countably additive measure ν_0 on the σ-algebra $\mathcal{A}$ generated by $\mathcal{B}$ and S such that $\nu_0(S) = 0$. Denote by ν the Lebesgue completion of ν_0. All subsets of the set S belong to $\mathcal{A}_{\nu_0}$ and the measure ν vanishes on them. In particular, $\nu(B_n \backslash Z_n) = 0$. Note that

$$Z_n \backslash B_n \subset \bigcup_{i=1}^{n-1} (B_i \backslash Z_i). \tag{1.12.6}$$

Indeed, if $x \in Z_n \backslash B_n$, then $x \in Z_n \bigcap \bigcup_{i=1}^{n-1} \widetilde{Z}_i \subset \widetilde{Z}_n \bigcap \bigcup_{i=1}^{n-1} B_i$. Then $x \in B_i$ for some $i < n$. Clearly, $x \notin Z_i$, since $Z_i \cap Z_n = \varnothing$. Hence $x \in B_i \backslash Z_i$. By (1.12.6) we obtain $\nu(Z_n \backslash B_n) = 0$. Thus, we have $\nu(B_n \bigtriangleup Z_n) = 0$, which means the ν-measurability of all sets Z_n.

In the case of an uncountable family we set

$$c = \sup\Bigl\{\mu_*(S)\colon\ S = \bigcup_{n=1}^{\infty} Z_{\alpha_n}\Bigr\},$$

where sup is taken over all countable subfamilies $\{Z_{\alpha_n}\}$ of the initial family of sets. By using the countable additivity of μ, it is readily verified that there exists a countable family $N = \{\alpha_n\}$ such that $\mu_*(S) = c$, where $S = \bigcup_{n=1}^{\infty} Z_{\alpha_n}$. According to the previous step, the measure μ extends to a countably additive measure ν_0 on the σ-algebra $\mathcal{A}$ generated by $\mathcal{B}$ and the sets Z_{α_n}. Denote by $\mathcal{E}$ the class of all sets of the form

$$E = A \bigtriangleup C, \quad \text{where } A \in \mathcal{A},\ C \subset \bigcup_{j=1}^{\infty} Z_{\beta_j},\ \beta_j \notin N.$$

It is readily verified that $\mathcal{E}$ is a σ-algebra. It is clear that $\mathcal{A} \subset \mathcal{E}$ (since one can take $C = \varnothing$) and that $Z_\alpha \in \mathcal{E}$ for all α (since for $\alpha \notin N$ one can take $A = \varnothing$). Finally, let $\nu(A \bigtriangleup C) := \nu_0(A)$. This definition is non-ambiguous, which follows from the above-established non-ambiguity of Definition 1.12.11. To this end, however, it is necessary to verify that if $E = A_1 \bigtriangleup C_1$ is another representation of the above form, then the set $A \bigtriangleup A_1$ has ν_0-measure zero. Since this set is contained in a countable union of the sets Z_{β_j}, $\beta_j \notin N$, we have to show that the set $Z = \bigcup_{j=1}^{\infty} Z_{\beta_j}$ has inner measure zero with respect to ν_0. This is not completely obvious: although Z has zero inner measure with respect to μ, in the process of extending a measure the inner measure may increase. In our case, however, this does not happen. Indeed, suppose that Z contains a set E of positive ν_0-measure. By the construction of ν_0 (the Lebesgue completion of the extension explicitly described above) it follows that for E one can take a set of the form $E = (A_1 \cap S) \cup \bigl(A_2 \cap (X\backslash S)\bigr)$, where $A_1, A_2 \in \mathcal{B}$, $S = \bigcup_{n=1}^\infty (B_n \backslash Z_{\alpha_n})$ with some sets $B_n \in \mathcal{B}$ constructed at the first step of our proof. We have $\nu_0(E) = \mu(A_2)$. Then, the set E and its subset $E_0 = A_2 \cap (X\backslash S)$ have equal ν_0-measures. Since the sets B_n are pairwise disjoint, the set $X\backslash S$ is the union of the sets $\bigcup_{n=1}^\infty (B_n \cap Z_{\alpha_n})$ and $X\backslash \bigcup_{n=1}^\infty B_n$. But A_2 does not meet the sets Z_{α_n}, for it is contained in Z. Therefore, we obtain $E_0 = A_2 \cap \bigl(X\backslash \bigcup_{n=1}^\infty B_n\bigr) \in \mathcal{B}$ and hence $\mu(E_0) = \nu_0(E_0) > 0$. This contradicts the equality $\mu_*(Z) = 0$. By the above reasoning we also obtain that ν is a countably additive measure that extends the measure ν_0, hence extends the measure μ as well. □

The question arises whether the assumption that the additional sets in the above theorem are disjoint is essential. Under the continuum hypothesis, there exists a countable family of sets $E_j \subset [0,1]$ such that Lebesgue measure has no extensions to a countably additive measure on a σ-algebra containing all E_j. This assertion goes back to Banach and Kuratowski [**57**], and its proof is found in Corollary 3.10.3. The same is true under Martin's axiom defined below in §1.12(x); see a short reasoning in Mauldin [**659**]. On the other hand, it is proved in Carlson [**168**] that if the system of axioms ZFC (the Zermelo–Fraenkel system with the axiom of choice) is consistent, then it remains consistent with the statement that Lebesgue measure is extendible to any σ-algebra obtained by adding any countable sequence of sets. For yet another extension result, see Exercise 1.12.149.

Generalizations of Theorem 1.12.15 are obtained in Weber [**1007**] and Lipecki [**616**], where disjoint collections are replaced by well-ordered collections.

In Chapter 7 we discuss extensions to σ-algebras not necessarily obtained by adding disjoint families.

1.12(vi). Some interesting sets

In this subsection, we consider several interesting examples of measurable and nonmeasurable sets on the real line.

1.12.16. Example. There exists a Borel set B on the real line such that, for every nonempty interval J, the sets $B \cap J$ and $(\mathbb{R}^1 \backslash B) \cap J$ have positive measures.

Proof. Let $\{I_n\}$ be all nondegenerate intervals in $[0,1]$ with rational endpoints. Let us find a nowhere dense compact set $A_1 \subset I_1$ of positive measure. The set $I_1 \backslash A_1$ contains an interval, hence there is a nowhere dense compact set $B_1 \subset I_1 \backslash A_1$ of positive measure. Similarly, there exist nowhere dense compact sets $A_2 \subset I_2 \backslash (A_1 \cup B_1)$ and $B_2 \subset I_2 \backslash (A_1 \cup B_1 \cup A_2)$ with $\lambda(A_2) > 0$ and $\lambda(B_2) > 0$. By induction, we construct in $[0,1]$ a sequence of pairwise disjoint nowhere dense compact sets A_n and B_n of positive measure such that $B_n \subset I_n \backslash A_n$. If A_i and B_i are already constructed for $i \leq n$, the set $I_{n+1} \backslash \bigcup_{i=1}^{n} (A_i \cup B_i)$ contains some interval, since the union of finitely many nowhere dense compact sets is a nowhere dense compact set. In this interval one can find disjoint nowhere dense compact sets A_{n+1} and B_{n+1} of positive measure and continue our construction. Let $E = \bigcup_{n=1}^{\infty} B_n$. If we are given an interval in $[0,1]$, then it contains the interval I_m for some m. According to our construction, I_m contains sets A_{m+1} and B_{m+1}, i.e., the intersections of I_m with E and $[0,1] \backslash E$ have positive measures. Finally, let us set $B = \bigcup_{z=-\infty}^{+\infty} (E + z)$. □

Let us introduce several concepts and facts related to ordered sets and ordinal numbers. A detailed exposition of these issues (including the transfinite induction) is given in the following books: Dudley [**251**], Jech [**459**], Kolmogorov, Fomin [**536**], Natanson [**707**]. A set T is called partially ordered if it is equipped with a partial order, i.e., some pairs $(t, s) \in T \times T$ are linked by a relation $t \leq s$ satisfying the conditions: 1) $t \leq t$, 2) if $t \leq s$ and $s \leq u$, then $t \leq u$ for all $s, t, u \in T$. Sometimes such a relation is called a partial pre-order, and the definition of a partial order includes the requirement of antisymmetry: if $t \leq s$ and $s \leq t$, then $t = s$. But we do not require this. We write $t < s$ if $t \leq s$ and $t \neq s$. The set T is called linearly ordered if all its elements are pairwise comparable and, in addition, if $t \leq s$ and $s \leq t$, then $t = s$. An element m of a partially ordered set is called maximal if there is no element x with $x > m$. A minimal element is defined by analogy.

A set is called well-ordered if it is linearly ordered and every nonempty subset of it has a minimal element. For example, the sets $\mathbb{N}$ and $\mathbb{R}^1$ with their natural orderings are linearly ordered, $\mathbb{N}$ is well-ordered, but $\mathbb{R}^1$ is not.

The interval (α, β) in a well-ordered set M is defined as the set of all points x such that $\alpha < x < \beta$. A set of the form $\{x \in M\colon\ x < \alpha\}$ is called an initial interval in M (the point α is not included). The closed interval $[\alpha, \beta]$ is the interval (α, β) with the added endpoints. Two well-ordered sets are called order-isomorphic if there is a one-to-one order-preserving correspondence between them. A class of order-isomorphic well-ordered sets is called an ordinal number or an ordinal. Ordinal numbers corresponding to infinite sets are called transfinite numbers or transfinites. If we are given two well-ordered sets A and B that represent distinct ordinal numbers α and β, then either A is order-isomorphic to some initial interval in B, or B is order-isomorphic to some initial interval in A. In the first case, we write $\alpha < \beta$, and in the second $\beta < \alpha$. Thus, given any two distinct ordinals, one is less than the other. Any set consisting of ordinal numbers is also well-ordered (unlike subsets of $\mathbb{R}^1$ with their usual ordering). The set $W(\alpha)$ of all ordinal numbers less than α is a well-ordered set of the type α. If we are given a set X of cardinality κ, then by means of the axiom of choice it can be well-ordered (Zermelo's theorem), i.e., there exist ordinals corresponding to sets of cardinality κ. Therefore, among such ordinals there is the smallest one $\omega(\kappa)$. Similarly, one defines the smallest uncountable ordinal number ω_1 (the smallest ordinal number corresponding to an uncountable set), which is sometimes used in measure theory for constructing various exotic examples. The least uncountable cardinality is denoted by $\aleph_1$. The continuum hypothesis is the equality $\aleph_1 = \mathfrak{c}$. The first (i.e., the smallest) infinite ordinal is denoted by ω_0.

The next example is a typical application of well-ordered sets.

1.12.17. Example. There exists a set $B \subset \mathbb{R}$ (called the *Bernstein set*) such that this set and its complement have nonempty intersections with all uncountable closed subsets of the real line. The intersection of B with every set of positive Lebesgue measure is nonmeasurable.

PROOF. It is clear that there exist the continuum of closed sets on the real line (since the complement of any closed set is a countable union of intervals) and that the collection of all uncountable closed sets has cardinality of the continuum $\mathfrak{c}$. Let us employ the following fact: the set of all ordinal numbers smaller than $\omega(\mathfrak{c})$ (the first ordinal number corresponding to sets of cardinality of the continuum) has cardinality of the continuum $\mathfrak{c}$. Hence the set of all uncountable closed sets on the real line can be parameterized by infinite ordinal numbers less than $\omega(\mathfrak{c})$, and represented in the form $\{F_\alpha, \alpha < \omega(\mathfrak{c})\}$. By means of transfinite induction, in every F_α we can choose two points x_α and y_α such that all selected points are distinct. Indeed, the sets F_α can be well-ordered. By using that the set of indices α is well-ordered, we pick the first (in the sense of the established order) elements $x_1, y_1 \in F_1$ for the first element in the index set. If $1 < \alpha < \mathfrak{c}$ and pairwise distinct elements x_β, y_β are already found for all $\beta < \alpha$, we take for x_α, y_α the first elements in the set $F_\alpha \backslash \bigcup_{\beta<\alpha}\{x_\beta, y_\beta\}$, which is infinite, since F_α has cardinality of the continuum according to Exercise 1.12.111, and the cardinality of the set of indices not exceeding α has cardinality less than $\mathfrak{c}$. By the transfinite induction principle, elements x_α, y_α are defined for all $\alpha < \omega(\mathfrak{c})$. It remains to take $B = \{x_\alpha, \alpha < \omega(\mathfrak{c})\}$. It is clear that $y_\alpha \in \mathbb{R}\backslash B$ and $x_\alpha \in F_\alpha \cap B$, $y_\alpha \in F_\alpha \cap (\mathbb{R}\backslash B)$. The last claim is obvious from the fact that any set of positive measure contains a compact set of positive measure. $\square$

It will be shown in Chapter 6 (Corollary 6.7.13) that every uncountable Souslin set contains an uncountable compact subset. Hence the Bernstein set contains no uncountable Souslin subsets. This is employed in the following lemma.

1.12.18. Lemma. *Let T be a set of cardinality of the continuum and let $E \subset \mathbb{R}\times T$. Suppose that, for any $x \in \mathbb{R}$, the section $E_x = \{t\colon (x,t) \in E\}$ is finite and that, for any $T' \subset T$, the set $\{x\colon E_x \cap T' \neq \varnothing\}$ is Lebesgue measurable. Then, there exist a set Z of Lebesgue measure zero and an at most countable set $S \subset T$ such that $E_x \subset S$ for all $x \in \mathbb{R}\backslash Z$.*

Proof. Without loss of generality we may take for T a set of cardinality of the continuum such that it contains no uncountable Souslin subsets (for example, the Bernstein set). Note that there exists a Borel set N of measure zero such that the set $D := E \cap ((\mathbb{R}\backslash N)\times\mathbb{R})$ has the following property: for any open set U, the set $\{x\colon\ D_x \cap U \neq \varnothing\}$ is Borel. Indeed, let $\{U_n\}$ be the sequence of all intervals with rational endpoints. By hypothesis, we have $\{x\colon\ U_n \cap E_x \neq \varnothing\} = B_n \cup N_n$, where $B_n \in \mathcal{B}(\mathbb{R})$ and $\lambda(N_n) = 0$. We find measure zero Borel sets N_n' with $N_n \subset N_n'$ and put $N = \bigcup_{n=1}^\infty N_n'$. An arbitrary nonempty open set U is the union of finitely or countably many sets U_n. Hence in order to establish the indicated property of the set N, it suffices to verify that the sets $\{x\colon\ D_x \cap U_n \neq \varnothing\}$ are Borel. To this end, we observe that $\{x\colon\ D_x \cap U_n \neq \varnothing\} = B_n \cup N_n\backslash N = B_n\backslash N$. Let us now show that D is Borel. It follows from our assumption that the sets D_x are finite. Hence

$$D = \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty \Bigl\{(x,r)\colon\ |r-r_m| < 1/n,\ D_x \cap (r_m - 1/n, r_m + 1/n) \neq \varnothing\Bigr\},$$

where $\{r_m\}$ are all rational numbers. Indeed, the left-hand side of this relation always belongs to the right-hand side, and if (x,r) does not belong to D, then, for some n, we have $|r-t| > (2n)^{-1}$ for all t from the finite set D_x, hence (x,r) does not belong to the right-hand side of this relation. Thus, D is the countable intersection of countable unions of the sets

$$(r_m - 1/n, r_m + 1/n)\times\bigl\{x\colon\ D_x \cap (r_m - 1/n, r_m + 1/n) \neq \varnothing\bigr\},$$

which are Borel as shown above. Thus, D is a Borel set. Let S be the projection of D to the second factor. Then S is a Souslin set. According to our choice of T, the set S is at most countable. It is clear that N and S are as required. □

Now we can prove the following interesting result.

1.12.19. Theorem. *Let $\{A_t\}_{t\in T}$ be some family of measure zero sets covering the real line such that every point belongs only to finitely many of them. Then, there exists a subfamily $T' \subset T$ such that the set $\bigcup_{t\in T'} A_t$ is nonmeasurable.*

Proof. Let $E = \{(x,t)\colon\ t \in T, x \in A_t\}$. If, for each $T' \subset T$, the set $\bigcup_{t\in T'} A_t$ is measurable, then E satisfies the hypotheses of the above lemma. Hence there exist a measure zero set Z and an at most countable set $S \subset T$ such that $E_x \subset S$ for all $x \in \mathbb{R}^1\backslash Z$. Then $\mathbb{R}^1\backslash Z \subset \bigcup_{s\in S} A_s$, which is a contradiction. □

Let us recall that a Hamel basis (or an algebraic basis) in a linear space L is a collection of linearly independent vectors v_α such that every vector in L is a finite linear combination of v_α. If $\mathbb{R}$ is regarded as a linear space over the real field, then any nonzero vector serves as a basis. However, the situation changes if we regard $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers: now there is no finite basis. But it is known (see Kolmogorov, Fomin [**536**]) that in this case there exists a Hamel basis as well and any basis has cardinality of the continuum. It is interesting that the metric properties of Hamel bases of the space $\mathbb{R}$ over $\mathbb{Q}$ may be very different.

1.12.20. Lemma. *Each Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ has inner Lebesgue measure zero, and there exist Lebesgue measurable Hamel bases.*

PROOF. Let H be a Hamel basis and $h \in H$. In the case $\lambda_*(H) > 0$, where λ is Lebesgue measure, the set H contains a compact set of positive measure. According to Exercise 1.12.62, the set $\{h_1 - h_2, h_1, h_2 \in H\}$ contains a nonempty interval. Hence there exist $h_1, h_2 \in H$ and nonzero $q \in \mathbb{Q}$ such that $h_1 - h_2 = qh$, which contradicts the linear independence of vectors of our basis over $\mathbb{Q}$.

In order to construct a measurable Hamel basis, we apply Exercise 1.12.61 and take two sets A and B of measure zero such that $\{a+b, a \in A, b \in B\} = \mathbb{R}$. Let $M = A \cup B$. Then M has measure zero. It remains to observe that there exists a Hamel basis consisting of elements of M. As in the proof of the existence of a Hamel basis, it suffices to take a set $H \subset M$ that is a maximal (in the sense of inclusion) linearly independent set over $\mathbb{Q}$. Then H is a Hamel basis, since the linear span of H over $\mathbb{Q}$ contains M, hence it equals $\mathbb{R}$. $\square$

1.12.21. Example. There exists a Lebesgue nonmeasurable Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$.

PROOF. We give a proof under the assumption of the continuum hypothesis, although this hypothesis is not necessary (Exercise 1.12.66). Let us take any Hamel basis H. By using that it has cardinality of the continuum we can establish a one-to-one correspondence $\alpha \mapsto h_\alpha$ between ordinal numbers $\alpha < \mathfrak{c}$ and elements of H. For any $\alpha < \mathfrak{c}$ and any nonzero $q \in \mathbb{Q}$, we denote by $V_{\alpha,q}$ the collection of all numbers of the form $q_1 h_{\alpha_1} + \cdots + q_n h_{\alpha_n} + q h_\alpha$, where $q_i \in \mathbb{Q}$ and $\alpha_i < \alpha$. According to the continuum hypothesis, every set $V_{\alpha,q}$ is countable (since its cardinality is less than $\mathfrak{c}$), and their union gives $\mathbb{R}\backslash\{0\}$. Let us write $V_{\alpha,q}$ as a countable sequence $\{h^n_{\alpha,q}\}$ and, for every $k \in \mathbb{N}$, consider $M_{k,q} = \bigcup_{\alpha<\mathfrak{c}} h^k_{\alpha,q}$. If we prove that the sets $M_{k,q}$ are linearly independent, then they can be complemented to Hamel bases $H_{k,q}$. The union of the latter sets contains the union of the sets $M_{k,q}$ and hence equals $\mathbb{R}\backslash\{0\}$, whence it follows that a countable collection of bases $H_{k,q}$ contains nonmeasurable sets because they all have inner measure zero. For the proof of linear independence of $M_{k,q}$ we consider a collection of distinct elements $h^k_{\alpha_1,q}, \ldots, h^k_{\alpha_n,q} \in M_{k,q}$, where $\alpha_1 < \cdots < \alpha_n < \mathfrak{c}$. Let $q_1, \ldots, q_n \in \mathbb{Q}$ and let $j \geq 1$ be the maximum of the indices of nonzero q_i. The expansion of $q_j h^k_{\alpha_j,q}$ with respect to the basis H contains the element $q_j q h_{\alpha_j}$, whereas the expansions of all other $q_i h^k_{\alpha_i,q}$ do not involve h_{α_j}, whence it follows that $q_1 h^k_{\alpha_1,q} + \cdots + q_n h^k_{\alpha_n,q} \neq 0$. $\square$

The next example is a deep theorem due to Besicovitch; its compact proof can be found in Stein [**906**, Chapter X]. Let R be a rectangle in the plane with the longer side length 1. Denote by $\widetilde{R}$ its translation to 2 in the positive direction parallel to the longer side, i.e., if e is the unit vector in the right half-plane giving the direction of the longer side, then $\widetilde{R} = R+2e$. The known methods of constructing the Besicovitch set (see Stein [**906**]) are based on the following assertions.

1.12.22. Lemma. *For any $\varepsilon > 0$, there exist a number $N = N_\varepsilon \in \mathbb{N}$ and 2^N rectangles $R_1, \ldots, R_{2^N} \subset \mathbb{R}^2$ with the side lengths 1 and 2^{-N} such that $\lambda_2\bigl(\bigcup_{j=1}^{2^N} R_j\bigr) < \varepsilon$, and the above-defined rectangles $\widetilde{R}_j$ are pairwise disjoint, so that $\lambda_2\bigl(\bigcup_{j=1}^{2^N} \widetilde{R}_j\bigr) = 1$, where λ_2 is Lebesgue measure on $\mathbb{R}^2$.*

1.12.23. Lemma. *Let P be a parallelogram in the plane with two sides in the lines $y = 0$ and $y = 1$. Then, for any $\varepsilon > 0$, one can find a number $N = N_\varepsilon \in \mathbb{N}$ and N parallelograms $P_1, \ldots, P_N$ in P such that each of them has two sides in the lines $y = 0$ and $y = 1$, $\lambda_2\bigl(\bigcup_{i=1}^{N} P_i\bigr) < \varepsilon$, and every interval in P with the endpoints in the lines $y = 0$ and $y = 1$ can be parallely translated to one of P_i.*

1.12.24. Example. There exists a compact set $K \subset \mathbb{R}^2$ (the *Besicovitch set*) of measure zero such that, for any straight line l in $\mathbb{R}^2$, the set K contains a unit interval parallel to l.

PROOF. Consequently applying the previous lemma, we obtain a sequence of compact sets $K_1 \supset K_2 \supset \cdots \supset K_j \supset \cdots$, where K_1 is the square $0 \leq x, y \leq 1$, with the following properties: $\lambda_2(K_j) \leq 1/j$ and, for any closed interval I joining the horizontal sides of K_1, the set K_j contains a closed interval obtained by a parallel transport of I. The set $\bigcap_{j=1}^{\infty} K_j$ has measure zero and contains a parallel transport of every interval of length 1 whose angle with the axis of ordinates lies between $-\pi/4$ and $\pi/4$. The union of two sets of such a type is a desired compact set. $\square$

Sets of the indicated type give a solution to the so-called Kakeya problem: what is a minimal measure of a set that contains unit intervals in all directions? Concerning this problem, see Wolff [**1024**].

Kahane [**479**] considered the set F of all line segments joining the points of the compact set E in the interval $[0, 1]$ of the axis of abscissas described in Exercise 1.12.155 and the points of the form $(-2x, 1)$, $x \in E$. This set has zero measure, but contains translations of line segments of unit length whose angles with the axis of ordinates fill in some interval, so that a suitable union of finitely many sets of this type is a Besicovitch set. It is possible to prove the existence of a Besicovitch type set without any explicit construction. A class of random Besicovitch sets is described in Alexander [**11**]. Körner [**542**] considered the set $\mathcal{P}$ of all compact subsets $P \subset [-1, 1]\times[0, 1]$ with the following two properties: (i) P is a union of line segments joining points of the interval $[-1, 1]$ in the axis of abscissas and points of the interval $[0, 1]$ in the axis of ordinates, (ii) P contains a translation of each line segment of unit length. It is shown that $\mathcal{P}$ is closed in the space $\mathcal{K}$ of all compact sets in the plane equipped with the Hausdorff metric, and the collection of all compact sets in $\mathcal{P}$ of measure zero is a second category set in $\mathcal{P}$, hence is not empty.

Finally, let us mention the following surprising example due to Nikodym. Its construction is quite involved and may be read in the books by Guzmán [**386**] and Falconer [**277**].

1.12.25. Example. There exists a Borel set $A \subset [0,1]\times[0,1]$ (the *Nikodym set*) of Lebesgue measure 1 such that, for every point $x \in A$, there exists a straight line l_x whose intersection with A is exactly the point x.

The Nikodym set is especially surprising in connection with Fubini's theorem discussed in Chapter 3; see also Exercise 3.10.59, where the discussion concerns interesting Davies sets that are related to the Nikodym set.

1.12(vii). Additive, but not countably additive measures

In this subsection, it is explained how to construct additive measures on σ-algebras that are not countably additive. Unlike our constructive example on an algebra, here one has to employ non-constructive methods based on the axiom of choice. More precisely, we need the following Hahn–Banach theorem, which is proven in courses on functional analysis by means of the axiom of choice (see Kolmogorov, Fomin [**536**]).

1.12.26. Theorem. *Let L be a real linear space and let p be a real function with the following properties*:

(a) $p(\alpha x) = \alpha p(x)$ *for all* $\alpha \geq 0$ *and* $x \in L$;

(b) $p(x+y) \leq p(x) + p(y)$ *for all* $x, y \in L$.

Suppose that L_0 is a linear subspace in L and that l is a linear function on L_0 such that $l(x) \leq p(x)$ for all $x \in L_0$. Then l extends to a linear function $\widehat{l}$ on all of L such that $\widehat{l}(x) \leq p(x)$ for all $x \in L$.

Functions p with properties (a) and (b) are called sublinear. If, in addition, $p(-x) = p(x)$, then p is called a seminorm. For example, the norm of a normed space (see Chapter 4) is sublinear. Let us give less trivial examples that are employed for constructing some interesting linear functions.

1.12.27. Example. The following functions p are sublinear:

(i) let L be the space of all bounded real sequences $x = (x_n)$ with its natural linear structure (the operations are defined coordinate-wise) and let

$$p(x) = \inf S(x, a_1, \ldots, a_n), \quad S(x, a_1, \ldots, a_n) := \sup_{k\geq 1} \frac{1}{n}\sum_{i=1}^{n} x_{k+a_i},$$

where inf is taken over all natural n and all finite sequences $a_1, \ldots, a_n \in \mathbb{N}$;

(ii) let L be the space of all bounded real functions on the real line with its natural linear structure and let

$$p(f) = \inf S(f, a_1, \ldots, a_n), \quad S(f, a_1, \ldots, a_n) := \sup_{t\in\mathbb{R}} \frac{1}{n}\sum_{i=1}^{n} f(t + a_i),$$

where inf is taken over all natural n and all finite sequences $a_1, \ldots, a_n \in \mathbb{R}$;

(iii) let L be the space of all bounded real functions on the real line and let

$$p(f) = \inf \left\{ \limsup_{t\to+\infty} \frac{1}{n} \sum_{i=1}^{n} f(t+a_i) \right\},$$

where inf is taken over all natural n and all finite sequences $a_1, \ldots, a_n \in \mathbb{R}$;

(iv) let L be the space of all bounded real sequences $x = (x_n)$ and let

$$p(x) = \inf S(x, a_1, \ldots, a_n), \quad S(x, a_1, \ldots, a_n) := \limsup_{k\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_{k+a_i},$$

where inf is taken over all natural n and all finite sequences $a_1, \ldots, a_n \in \mathbb{N}$.

PROOF. Claim (i) follows from (ii). Let us show (ii). It is clear that $|p(f)| < \infty$ and $p(\alpha f) = \alpha p(f)$ if $\alpha \geq 0$. Let $f, g \in L$. Take $\varepsilon > 0$ and find $a_1, \ldots, a_n, b_1, \ldots, b_m$ such that

$$\sup_{t\in\mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} f(t+a_i) < p(f) + \varepsilon, \quad \sup_{t\in\mathbb{R}} \frac{1}{m} \sum_{i=1}^{m} g(t+b_i) < p(g) + \varepsilon.$$

We observe that

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} (f+g)(t+a_i+b_j) \\ &\leq \sup_{t\in\mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m} \sum_{j=1}^{m} f(t+a_i+b_j) + \sup_{t\in\mathbb{R}} \frac{1}{m} \sum_{j=1}^{m} \frac{1}{n} \sum_{i=1}^{n} g(t+a_i+b_j). \end{aligned}$$

For fixed t and b_j we have $n^{-1} \sum\limits_{i=1}^{n} f(t+a_i+b_j) \leq S(f, a_1, \ldots, a_n)$, whence it follows that

$$\sup_{t\in\mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m} \sum_{j=1}^{m} f(t+a_i+b_j) \leq S(f, a_1, \ldots, a_n).$$

A similar estimate for g yields

$$p(f+g) \leq S(f, a_1, \ldots, a_n) + S(g, b_1, \ldots, b_m) < p(f) + p(g) + 2\varepsilon,$$

which shows that $p(f+g) \leq p(f) + p(g)$, since ε is arbitrary. The proof of (iii) is similar, and (iv) follows from (iii). □

Let us now consider applications to constructing some interesting set functions.

1.12.28. Example. On the σ-algebra of all subsets in $\mathbb{N}$, there exists a nonnegative additive function ν that vanishes on all finite sets and equals 1 on $\mathbb{N}$; in particular, ν is not countably additive.

PROOF. Let us consider the space L of all bounded sequences with the function p from assertion (iv) in the previous example and take the subspace L_0 of all convergent sequences. Set $l(x) = \lim\limits_{n\to\infty} x_n$ if $x \in L_0$. Note that $l(x) = p(x)$, since for fixed a_i and n we have $\limsup\limits_{k\to\infty} n^{-1} \sum\limits_{i=1}^{n} x_{k+a_i} = \lim\limits_{k\to\infty} x_k$. Let us extend l to a linear function $\widehat{l}$ on L with $\widehat{l} \le p$. If $x \in L$ and $x_n \le 0$ for all n, then $p(x) \le 0$ and hence $\widehat{l}(x) \le 0$. Therefore, $\widehat{l}(x) \ge 0$ if $x_n \ge 0$. If $x = (x_1, \ldots, x_n, 0, 0, \ldots)$, then $\widehat{l}(x) = l(x) = 0$. Finally, $\widehat{l}(1, 1, \ldots) = 1$. For every set $E \subset \mathbb{N}$, let $\nu(E) := \widehat{l}(I_E)$, where I_E is the indicator of the set E, i.e., the sequence having in the nth position either 1 or 0 depending on whether n is in E or not. Finite sets are associated with finite sequences, hence ν vanishes on them. The value of ν on $\mathbb{N}$ is 1, and the additivity of ν follows by the additivity of $\widehat{l}$ and the fact that $I_{E_1\cup E_2} = I_{E_1} + I_{E_2}$ for disjoint E_1 and E_2. It is obvious that ν is not countably additive. □

The following assertion is justified in a similar manner (its proof is delegated to Exercise 2.12.102 in the next chapter because it is naturally related to the concept of the integral, although can be given without it).

1.12.29. Example. On the σ-algebra of all subsets in $[0, 1)$, there exists a nonnegative additive set function ζ that coincides with Lebesgue measure on all Lebesgue measurable sets and $\zeta(E + h) = \zeta(E)$ for all $E \subset [0, 1)$ and $h \in [0, 1)$, where in the formation of $E + h$ the sum $e + h \ge 1$ is replaced by $e + h - 1$.

If we do not require that the additive function ζ should extend Lebesgue measure, then there is a simpler example.

1.12.30. Example. There exists an additive nonnegative set function ζ defined on all bounded sets on the real line and invariant with respect to translations such that $\zeta\bigl([0, 1)\bigr) = 1$.

PROOF. Let L be the space of bounded functions on the real line with the sublinear function p from Example 1.12.27(ii). By the Hahn–Banach theorem, there exists a linear function l on L with $l(f) \le p(f)$ for all $f \in L$. Indeed, on $L_0 = 0$ we set $l_0(0) = 0$. Note that $l(-f) = -l(f) \le p(-f)$, whence

$$-p(-f) \le l(f) \le p(f), \quad \forall\, f \in L.$$

If $f \ge 0$, then $p(-f) \le 0$ by the definition of p, hence $l(f) \ge 0$. Next, $p(1) = 1$, $p(-1) = -1$, which gives $l(1) = 1$. It is clear that $|l(f)| \le \sup\limits_t |f(t)|$, since $p(f) \le \sup\limits_t |f(t)|$. Finally, for all $h \in \mathbb{R}^1$ we have $l(f) = l(f(\cdot + h))$ for each $f \in L$. Indeed, let $g(t) = f(t + h) - f(t)$. We verify that $l(g) = 0$. Let $h_k = (k - 1)h$ if $k = 1, \ldots, n + 1$. Then

$$p(g) \le S(g, h_1, \ldots, h_{n+1}) = \sup_t \frac{1}{n+1}[f(t + (n + 1)h) - f(t)] \le \frac{2\sup\limits_s |f(s)|}{n+1},$$

which tends to zero as $n \to \infty$. Thus, $p(g) \le 0$. Similarly, we obtain the estimate $p(-g) \le 0$. Therefore, $l(g) = 0$. Now it remains to set $\zeta(A) = l(\overline{I}_A)$ for all $A \subset [0,1)$, where $\overline{I}_A$ is the 1-periodic extension of I_A to the real line. By the above-established properties of l we obtain a nonnegative additive set function on $[0,1)$ that is invariant with respect to translations within the set $[0,1)$. In addition, $\zeta\bigl([0,1)\bigr) = 1$, since $\overline{I}_{[0,1)} = 1$. For any bounded set A, we find n with $A \subset [-n,n)$ and set

$$\zeta(A) = \sum_{j=-n}^{n-1} \zeta\Bigl(\bigl(A \cap [j,j+1)\bigr) - j\Bigr).$$

It is readily verified that we obtain a desired function. $\square$

We observe that ζ coincides with Lebesgue measure on all intervals.

1.12(viii). Abstract inner measures

Having considered Carathéodory outer measures, it is natural to turn to superadditive functions. In this subsection, we present some results in this direction.

A set function η defined on the family of all subsets in a space X and taking values in $[0,+\infty]$ is called an abstract inner measure if $\eta(\varnothing) = 0$ and:

(a) $\eta(A \cup B) \ge \eta(A) + \eta(B)$ for all disjoint A and B,

(b) $\eta\bigl(\bigcap_{n=1}^\infty A_n\bigr) = \lim\limits_{n\to\infty} \eta(A_n)$ for every decreasing sequence of sets such that $\eta(A_1) < \infty$,

(c) if $\eta(A) = \infty$, then, for every number c, there exists $B \subset A$ such that $c \le \eta(B) < \infty$.

It follows from (a) that $\eta\bigl(\bigcup_{n=1}^\infty E_n\bigr) \ge \sum_{n=1}^\infty \eta(E_n)$ for all pairwise disjoint sets E_n. In addition, $\eta(B) \le \eta(A)$ whenever $B \subset A$ because we have $\eta(A\backslash B) \ge 0$, i.e., η is monotone.

If μ is a nonnegative countably additive measure on a σ-algebra $\mathcal{A}$, then the function μ_* has properties (a) and (b), which is readily verified (one can either directly verify property (b) by using measurable kernels of the sets E_n or refer to the properties of μ^* and the equality $\mu_*(A) = \mu(X) - \mu^*(X\backslash A)$ for finite measures). For finite (or semifinite) measures μ property (c) is fulfilled, too. In fact, this property will be fulfilled for any measure if we define μ_* by

$$\mu_*(A) := \sup\bigl\{\mu(B)\colon\ B \subset A,\, B \in \mathcal{A},\, \mu(B) < \infty\bigr\}. \tag{1.12.7}$$

Suppose that $\mathcal{F}$ is a family of subsets of a set X with $\varnothing \in \mathcal{F}$. Let $\tau\colon \mathcal{F} \to [0,+\infty]$ be a set function with $\tau(\varnothing) = 0$. We define the function τ_* on all sets $A \subset X$ by the formula

$$\tau_*(A) = \sup\biggl\{\sum_{j=1}^\infty \tau(F_j)\colon\ F_j \in \mathcal{F},\ F_j \subset A \text{ are disjoint}\biggr\}. \tag{1.12.8}$$

Note that τ_* can also be defined by the formula

$$\tau_*(A) = \sup\biggl\{\sum_{j=1}^{n}\tau(F_j)\colon\ n \in \mathbb{N},\ F_j \in \mathcal{F},\ F_j \subset A \text{ are disjoint}\biggr\}. \tag{1.12.9}$$

This follows by the equality $\tau(\varnothing) = 0$. Note the following obvious estimate:

$$\tau_*(F) \geq \tau(F), \quad \forall\, F \in \mathcal{F}.$$

It is seen from the definition that τ_* is superadditive. Certainly, this function (as any other one) generates the class $\mathfrak{M}_{\tau_*}$ (see Definition 1.11.2) that is an algebra, on which τ_* is additive by Theorem 1.11.4. The question arises of the countable additivity of the function τ_* on this algebra and its relation to τ. Obviously, if $\tau\colon 2^X \to [0, +\infty]$ with $\tau(\varnothing) = 0$ is superadditive on the family of all sets, then $\tau_* = \tau$ because $\sum_{j=1}^{\infty}\tau(F_j) \leq \tau\bigl(\bigcup_{j=1}^{\infty}F_j\bigr) \leq \tau(A)$ for all pairwise disjoint sets $F_j \subset A$.

1.12.31. Proposition. (i) *Let τ be an abstract inner measure on a space X. Then $\mathfrak{M}_\tau$ is a σ-algebra and τ is countably additive on $\mathfrak{M}_\tau$.*

(ii) *Suppose that on a σ-algebra $\mathcal{A}$ we are given a measure μ with values in $[0, +\infty]$. Then, the function $\tau = \mu_*$ defined by* (1.12.7) *is an abstract inner measure and if the measure μ is finite, then the measure τ on the domain $\mathfrak{M}_\tau$ extends μ.*

PROOF. (i) Under condition (b) the function τ is countably additive on the algebra $\mathfrak{M}_\tau$ by Theorem 1.11.4(ii) and this does not employ condition (a). Let us show that $\mathfrak{M}_\tau$ is a σ-algebra. For simplification of our reasoning we assume that τ has only finite values (the general case is similar and uses condition (c)). As noted above, condition (a) yields that $\tau(B) \leq \tau(A)$ if $B \subset A$, i.e., τ is monotone. Let $A_n \in \mathfrak{M}_\tau$ increase to A. For any $E \subset X$, by the monotonicity of τ and (b) we have

$$\tau(E \cap A) + \tau(E\backslash A) \geq \lim_{n\to\infty}\tau(E \cap A_n) + \lim_{n\to\infty}\tau(E\backslash A_n) = \tau(A).$$

Since (a) yields the converse, we obtain $A \in \mathfrak{M}_\tau$. Assertion (ii) has already been explained. Here one has $\mathcal{A} \subset \mathfrak{M}_{\mu_*}$ and if $\mu(X) < \infty$, then $\mu_*|_{\mathcal{A}} = \mu$. $\square$

It should be noted that for a measure μ on an algebra $\mathcal{A}$ that is not a σ-algebra, the function μ_* may fail to have property (b). For example, this is the case for the usual length on the algebra $\mathcal{A}$ generated by intervals in $[0, 1]$: the set $\mathcal{R}$ of irrational numbers has inner measure 0 (evaluated, of course, by means of $\mathcal{A}$!) and is the intersection of a sequence of decreasing sets with finite complements and inner measures 1. However, inner measures are a very efficient tool for constructing and extending measures. Here and in the next subsection, we consider rather abstract examples whose real content is seen when dealing with inner compact regular set functions on topological spaces (see Chapter 7).

1.12.32. Proposition. *Let $\mathcal{F}$ be a family of subsets of a space X and let $\mu\colon \mathcal{F} \to [0,+\infty]$ be such that $\varnothing \in \mathcal{F}$ and $\mu(\varnothing) = 0$. Suppose that we have the identity*

$$\mu(A) = \mu_*(A \cap B) + \mu_*(A\backslash B), \quad \forall\, A, B \in \mathcal{F},$$

and that there exists a compact class $\mathcal{K}$ such that

$$\mu(A) \le \sup\{\mu_*(K)\colon\ K \in \mathcal{K},\, K \subset A\}, \quad \forall\, A \in \mathcal{F}.$$

Then:

(i) *the class $\mathfrak{M}_{\mu_*}$ is an algebra, $\mathcal{F} \subset \mathfrak{M}_{\mu_*}$, the function μ_* is countably additive on $\mathfrak{M}_{\mu_*}$ and coincides with μ on $\mathcal{F}$*;

(ii) $\lim\limits_{n\to\infty} \mu_*(A_n) = 0$ *if $A_n \subset X$, $A_n \downarrow \varnothing$ and $\mu_*(A_1) < \infty$.*

Proof. (i) It is clear that μ_* extends μ, since we can take $A = B$ in the above equality. According to Exercise 1.12.127, we have $\mathcal{F} \subset \mathfrak{M}_{\mu_*}$. By Theorem 1.11.4, the class $\mathfrak{M}_{\mu_*}$ is an algebra and μ_* is additive on $\mathfrak{M}_{\mu_*}$. The countable additivity will be established below.

(ii) Let $A_n \downarrow \varnothing$, $\mu_*(A_1) < \infty$ and $\varepsilon > 0$. We may assume that the class $\mathcal{K}$ is closed with respect to finite unions and countable intersections, passing to the smallest compact class $\widetilde{\mathcal{K}} \supset \mathcal{K}$ with such a property. Let us find $C_n \in \mathcal{K}$ with

$$C_n \subset A_n, \quad \mu_*(A_n) \le \mu_*(C_n) + \varepsilon 2^{-n-1}.$$

For this purpose we take a number $c \in \bigl(\mu_*(A_n) - \varepsilon 2^{-n-1}, \mu_*(A_n)\bigr)$ and find disjoint sets $F_1, \ldots, F_m \in \mathcal{F}$ such that $F_1 \cup \cdots \cup F_m \subset A_n$ and $c < \mu(F_1) + \cdots + \mu(F_m)$. Then we find $K_j \subset F_j$ such that $c < \mu(K_1) + \cdots + \mu(K_m)$ and take $C_n = K_1 \cup \cdots \cup K_m$. Similarly one verifies that there exist sets $M_n \in \mathfrak{M}_{\mu_*}$ with

$$M_n \subset C_n \quad \text{and} \quad \mu_*(C_n) \le \mu_*(M_n) + \varepsilon 2^{-n-1}.$$

It is easy to see that $\mu_*(A_n\backslash M_n) \le \varepsilon 2^{-n}$. One has $\bigcap_{n=1}^\infty C_n = \varnothing$, as $C_n \subset A_n$. Hence $\bigcap_{n=1}^k C_n = \varnothing$ for some k. By using the additivity of μ_* and the relation $\bigcap_{n=1}^k M_n \subset \bigcap_{n=1}^k C_n = \varnothing$, we obtain

$$\begin{aligned}\mu_*(A_n) &\le \mu_*(C_n) + \varepsilon 2^{-n-1} \le \mu_*(M_n) + \varepsilon 2^{-n}\\ &= \mu_*\Bigl(M_n\backslash \bigcap_{i=1}^k M_i\Bigr) + \varepsilon 2^{-n} \le \sum_{i=1}^k \mu_*(M_n\backslash M_i) + \varepsilon 2^{-n}.\end{aligned}$$

For $n > k \ge i$ we have

$$\mu_*(M_n\backslash M_i) \le \mu_*(A_n\backslash M_i) \le \mu_*(A_i\backslash M_i) \le \varepsilon 2^{-i},$$

whence we obtain $\mu_*(A_n) \le \varepsilon$.

It remains to show the countable additivity of μ_* on $\mathfrak{M}_{\mu_*}$. To this end, it suffices to verify that if $M, M_n \in \mathfrak{M}_{\mu_*}$ and $M \subset \bigcup_{n=1}^\infty M_n$, then $\mu_*(M) \le \sum_{n=1}^\infty \mu_*(M_n)$. Let $B_1 = M_1$ and $B_n = M_n\backslash(M_1 \cup \cdots \cup M_{n-1})$, $n > 1$. Then the sets $B_n \in \mathfrak{M}_{\mu_*}$ are disjoint and $M \subset \bigcup_{n=1}^\infty B_n$. Let $R_n = \bigcup_{j=n}^\infty B_j$.

Suppose that the series of $\mu_*(M_n)$ converges to $c < \infty$. If $\mu_*(M) > c$, then, for any $C \subset M$ with $\mu_*(C) > c$, we have $\mu_*(C \cap R_n) = \infty$. This follows from what has already been proven, since by Theorem 1.11.4 we have

$$\mu_*(C) = \sum_{n=1}^{\infty} \mu_*(C \cap B_n) + \lim_{n\to\infty} \mu_*(C \cap R_n),$$

and $C \cap R_n \downarrow \varnothing$. As shown above, one can find $C_0 \in \mathcal{K}$ with $C_0 \subset M$ and $\mu_*(C_0) > c$. Then $\mu_*(C_0 \cap R_1) = \infty$. By induction we construct $C_n \in \mathcal{K}$ such that $C_{n+1} \subset C_n \cap R_{n+1}$ and $\mu_*(C_n) > c$. This leads to a contradiction, since $C_n \downarrow \varnothing$ and hence for some p we have $C_p = C_1 \cap \cdots \cap C_p = \varnothing$, whereas one has $\mu_*(\varnothing) = 0$. $\square$

1.12.33. Theorem. *Let $\mathcal{K}$ be a compact class of sets in X that contains the empty set and is closed with respect to formation of finite unions and countable intersections, and let $\mu\colon \mathcal{K} \to [0, +\infty)$ be a set function satisfying the condition*

$$\mu(A) = \mu_*(A \cap B) + \mu_*(A\backslash B), \quad \forall\, A, B \in \mathcal{K},$$

or, which is equivalent, the condition

$$\mu(A) = \mu(A \cap B) + \sup\{\mu(K)\colon\ K \in \mathcal{K},\, K \subset A\backslash B\}, \quad \forall\, A, B \in \mathcal{K}.$$

Then:

(i) $\mathfrak{M}_{\mu_*}$ *is a σ-algebra and μ_* is countably additive on $\mathfrak{M}_{\mu_*}$ as a function with values in* $[0, +\infty]$;

(ii) $\mathcal{K} \subset \mathfrak{M}_{\mu_*}$ *and μ_* extends μ*;

(iii) $\mu_*(A) = \sup\{\mu(K)\colon\ K \subset A,\, K \in \mathcal{K}\}$ *for all $A \subset X$*;

(iv) $M \in \mathfrak{M}_{\mu_*}$ *precisely when $M \cap K \in \mathfrak{M}_{\mu_*}$ for all $K \in \mathcal{K}$*;

(v) $\lim\limits_{n\to\infty} \mu_*(A_n) = \mu_*(A)$ *if $A_n \downarrow A$ and $\mu_*(A_1) < \infty$.*

Proof. Since $\mu(\varnothing) = 2\mu_*(\varnothing)$, one has $\mu(\varnothing) = \mu_*(\varnothing) = 0$. By the above proposition with $\mathcal{F} = \mathcal{K}$ we obtain that $\mathfrak{M}_{\mu_*}$ is an algebra, on which μ_* is countably additive and (ii) is true. In particular, μ is additive on $\mathcal{K}$, which gives (iii) (this also follows by Exercise 1.12.124). Let us verify (v). Let $\varepsilon > 0$. By (iii) we can find $K_1 \subset A_1$ with $K_1 \in \mathcal{K}$ and $\mu_*(A_1) \le \mu(K_1) + \varepsilon/2$. By induction we construct sets $K_n \in \mathcal{K}$ with

$$K_n \subset A_n \cap K_{n-1}, \quad \mu_*(A_n \cap K_{n-1}) \le \mu(K_n) + \varepsilon 2^{-n}.$$

By using the decrease of A_j and the inclusion $\mathcal{K} \subset \mathfrak{M}_{\mu_*}$, we obtain

$$\begin{aligned}\mu_*(A_{j+1}) + \mu(K_j) &\le \mu(K_{j+1}) + \mu_*(A_j\backslash K_j) + \mu(K_j) + \varepsilon 2^{-j-1}\\ &\le \mu(K_{j+1}) + \mu_*(A_j\backslash K_j) + \mu_*(A_j \cap K_j) + \varepsilon 2^{-j-1}\\ &\le \mu(K_{j+1}) + \mu_*(A_j) + \varepsilon 2^{-j-1}.\end{aligned}$$

Set $K = \bigcap_{n=1}^{\infty} K_n$. Then $K \subset A$ and $K \in \mathcal{K} \subset \mathfrak{M}_{\mu_*}$. Since $K_n \backslash K \downarrow \varnothing$, by the above proposition we have $\mu_*(K_n \backslash K) \to 0$. Therefore,

$$\begin{aligned}
\mu_*(A_n) &= \mu_*(A_1) + \sum_{j=1}^{n-1} [\mu_*(A_{j+1}) - \mu_*(A_j)] \\
&\le \mu(K_1) + \frac{\varepsilon}{2} + \sum_{j=1}^{n-1} [\mu_*(K_{j+1}) - \mu_*(K_j) + \varepsilon 2^{-j-1}] \\
&\le \mu(K_n) + \varepsilon \le \mu_*(A) + \mu_*(K_n \backslash K) + \varepsilon.
\end{aligned}$$

Hence $\mu_*(A) \le \lim\limits_{n\to\infty} \mu_*(A_n) \le \mu_*(A)$.

Let us verify that $\mathfrak{M}_{\mu_*}$ is a σ-algebra. It suffices to show that if $M_n \in \mathfrak{M}_{\mu_*}$ and $M_n \downarrow M$, then $M \in \mathfrak{M}_{\mu_*}$. Let $A \subset X$. If $K \in \mathcal{K}$ and $K \subset A$, then

$$\mu(K) = \mu_*(K \cap M_n) + \mu_*(K \backslash M_n) \le \mu_*(K \cap M_n) + \mu_*(A \backslash M).$$

By using (v) and taking into account that μ is finite on $\mathcal{K}$, we obtain passing to the limit as $n \to \infty$ that

$$\mu(K) \le \mu_*(K \cap M) + \mu_*(A \backslash M) \le \mu_*(A \cap M) + \mu_*(A \backslash M).$$

According to (iii) we have $\mu_*(A) \le \mu_*(A \cap M) + \mu_*(A \backslash M)$. Since the reverse inequality is true as well, one has $M \in \mathfrak{M}_{\mu_*}$. Thus, (i) is established.

It remains to show (iv). Clearly, if $M \in \mathfrak{M}_{\mu_*}$ and $K \in \mathcal{K}$, then we have $K \cap M \in \mathfrak{M}_{\mu_*}$, since $\mathcal{K}$ belongs to the algebra $\mathfrak{M}_{\mu_*}$. Conversely, let $K \cap M \in \mathfrak{M}_{\mu_*}$ for all $K \in \mathcal{K}$. For every $A \subset X$, we have whenever $K \subset A$ and $K \in \mathcal{K}$

$$\begin{aligned}
\mu(K) &= \mu_*\bigl(K \cap (M \cap K)\bigr) + \mu_*\bigl(K \backslash (M \cap K)\bigr) \\
&\le \mu_*(A \cap M) + \mu_*(A \backslash M) \le \mu_*(A).
\end{aligned}$$

Taking sup over K we obtain by (iii) that $M \in \mathfrak{M}_{\mu_*}$.

If we have the second condition of the theorem, then $\mu(\varnothing) = 0$, whence $\mu(A) = \sup\{\mu(K)\colon\ K \in \mathcal{K}, K \subset A\}$ if $A \in \mathcal{K}$. Hence $\mu(B \cup C) = \mu(B) + \mu(C)$ if $B, C \in \mathcal{K}$, $B \cap C = \varnothing$. Hence μ_* coincides with μ on $\mathcal{K}$. So we have (iii) and the first condition of the theorem. The converse is true as well. $\square$

The proof of the next theorem, which can be read in Fremlin [**327**, §413], combines the functions ν_* and ν^*.

1.12.34. Theorem. *Let $\mathcal{R}$ be a ring of subsets of a space X, let $\mathcal{K}$ be some class of subsets of X closed with respect to formation of finite intersections and finite disjoint unions, and let ν be a finite nonnegative additive function on $\mathcal{R}$ such that $\mathcal{K}$ is an approximating class for ν. Then the following assertions are true.*

(i) *If every element of $\mathcal{K}$ is contained in an element of $\mathcal{R}$, then ν extends to a finite nonnegative additive function $\widetilde{\nu}$ defined on a ring $\widetilde{\mathcal{R}}$ that contains $\mathcal{R}$ and $\mathcal{K}$, such that $\mathcal{K}$ is an approximating class for $\widetilde{\nu}$ and, for each $R \in \widetilde{\mathcal{R}}$ and $\varepsilon > 0$, there exists $R_\varepsilon \in \mathcal{R}$ with $\widetilde{\nu}(R \triangle R_\varepsilon) < \varepsilon$.*

(ii) *If $\mathcal{R}$ a σ-algebra, ν is countably additive, and $\mathcal{K}$ admits countable intersections, then ν extends to a measure $\widetilde{\nu}$ defined on a σ-algebra $\mathcal{A}$ containing $\mathcal{R}$ and $\mathcal{K}$, such that $\mathcal{K}$ remains an approximating class for $\widetilde{\nu}$ and, for each $R \in \mathcal{R}$, there exists $A \in \mathcal{A}$ with $\widetilde{\nu}(R \triangle A) = 0$.*

It is readily seen that unlike superadditive functions, a subadditive function $\mathfrak{m}$ may not be monotone, i.e., may not satisfy the condition $\mathfrak{m}(A) \le \mathfrak{m}(B)$ whenever $A \subset B$. A *submeasure* is a finite nonnegative monotone subadditive function $\mathfrak{m}$ on an algebra $\mathfrak{A}$ such that $\mathfrak{m}(\varnothing) = 0$. A submeasure $\mathfrak{m}$ is called exhaustive if, for each sequence of disjoint sets $A_n \in \mathfrak{A}$, one has the equality $\lim\limits_{n\to\infty} \mathfrak{m}(A_n) = 0$. A submeasure $\mathfrak{m}$ is called uniformly exhaustive if, for each $\varepsilon > 0$, there exists n such that, in every collection of disjoint sets $A_1, \ldots, A_n \in \mathfrak{A}$, there exists A_i with $\mathfrak{m}(A_i) < \varepsilon$. Clearly, a uniformly exhaustive submeasure is exhaustive. A submeasure $\mathfrak{m}$ is called Maharam if $\lim\limits_{n\to\infty} \mathfrak{m}(A_n) = 0$ as $A_n \downarrow \varnothing$, $A_n \in \mathfrak{A}$. Recently, Talagrand [**932**] has constructed a counter-example to a long-standing open problem (the so-called control measure problem) that asked whether for every Maharam submeasure $\mathfrak{m}$ on a σ-algebra $\mathfrak{A}$, there exists a finite nonnegative measure μ with the same class of zero sets as $\mathfrak{m}$. It is known that this problem is equivalent to the following one: is every exhaustive submeasure uniformly exhaustive? Thus, both questions are answered negatively.

1.12(ix). Measures on lattices of sets

In applications one often encounters set functions defined not on algebras or semirings, but on lattices of sets. The results in this subsection are employed in Chapter 10 in our study of disintegrations.

1.12.35. Definition. *A class $\mathfrak{R}$ of subsets in a space X is called a lattice of sets if it contains the empty set and is closed with respect to finite intersections and unions.*

Unlike an algebra, a lattice may not be closed under complementation. Typical examples are: (a) the collection of all compact sets in a topological space X, (b) the collection of all open sets in a given space X. Sometimes in the definition of a lattice it is required that $X \in \mathfrak{R}$. Certainly, this can be always achieved by simply adding X to $\mathfrak{R}$, which does not affect the stability with respect to formation of unions and intersections.

A finite nonnegative set function β on a lattice $\mathfrak{R}$ is called modular if one has $\beta(\varnothing) = 0$ and

$$\beta(R_1 \cup R_2) + \beta(R_1 \cap R_2) = \beta(R_1) + \beta(R_2), \quad \forall\, R_1,\, R_2 \in \mathfrak{R}. \qquad (1.12.10)$$

If in (1.12.10) we replace the equality sign by "$\le$", then we obtain the definition of a submodular function, and the change of "$=$" to "$\ge$" gives the definition of a supermodular function. If $\mathfrak{R}$ is an algebra, then the modular functions are precisely the additive ones. We recall that a set function β is called *monotone* if $\beta(R_1) \le \beta(R_2)$ whenever $R_1 \subset R_2$.

1.12.36. Proposition. *Let β be a monotone submodular function on a lattice $\mathfrak{R}$ and $X \in \mathfrak{R}$. Then, there exists a monotone modular function α on $\mathfrak{R}$ such that $\alpha \le \beta$ and $\alpha(X) = \beta(X)$.*

The proof is delegated to Exercise 1.12.148.

1.12.37. Corollary. *Suppose that β is a monotone supermodular function on a lattice $\mathfrak{R}$ and $X \in \mathfrak{R}$. Then, there exists a monotone modular function γ on $\mathfrak{R}$ such that $\gamma \ge \beta$ and $\gamma(X) = \beta(X)$.*

PROOF. Let us consider the set function

$$\beta_0(C) = \beta(X) - \beta(X\backslash C)$$

on the lattice $\mathfrak{R}_0 = \{C\colon\ X\backslash C \in \mathfrak{R}\}$. It is readily verified that β_0 is monotone and submodular. According to the above proposition, there exists a monotone modular function α_0 on $\mathfrak{R}_0$ with $\alpha_0 \le \beta_0$ and $\alpha_0(X) = \beta_0(X)$. Now set $\gamma(R) = \alpha_0(X) - \alpha_0(X\backslash R)$, $R \in \mathfrak{R}$. Then $\gamma(X) = \beta(X)$ and $\gamma(R) \ge \beta(R)$, since $\alpha_0(X\backslash R) \le \beta_0(X\backslash R)$. □

1.12.38. Lemma. *Let β be a monotone modular set function on a lattice $\mathfrak{R}$, $X \in \mathfrak{R}$, and $\beta(X) = 1$. Then, there exists a monotone modular set function ζ on $\mathfrak{R}$ such that $\beta \le \zeta$, $\zeta(X) = 1$, and*

$$\zeta(R) + \zeta_*(X\backslash R) = 1, \quad \forall\, R \in \mathfrak{R}. \tag{1.12.11}$$

PROOF. The set Ψ of all monotone modular set functions ψ on $\mathfrak{R}$ satisfying the conditions $\psi(X) = 1$ and $\psi \ge \beta$, is partially ordered by the relation $\le$. Each linearly ordered part of Ψ has an upper bound in Ψ given as the supremum of that part (this upper bound is modular, since the considered part is linearly ordered). By Zorn's lemma Ψ has a maximal element ζ. Corollary 1.12.37 yields (1.12.11), since otherwise the function ζ is not maximal. To see this, it suffices to show that for any fixed $R_0 \in \mathfrak{R}$, there is a function $\psi \in \Psi$ such that $\psi(R_0) + \psi_*(X\backslash R_0) = 1$. Let

$$\tau_1(R) := \sup\{\beta(R \cap S)\colon\ S \in \mathfrak{R},\, S \cap R_0 = \varnothing\}, \quad R \in \mathfrak{R}.$$

The function τ_1 is modular. Indeed, given $R_1, R_2 \in \mathfrak{R}$, for every $\varepsilon > 0$, one can find $S_i \in \mathfrak{R}$, $i = 1, \ldots, 4$, such that $S_i \cap R_0 = \varnothing$ and the sum of the quantities $\tau_1(R_1) - \beta(R_1 \cap S_1)$, $\tau_1(R_2) - \beta(R_2 \cap S_2)$, $\tau_1(R_1 \cap R_2) - \beta(R_1 \cap R_2 \cap S_3)$, $\tau_1(R_1 \cup R_2) - \beta\bigl((R_1 \cup R_2) \cap S_4\bigr)$ is less than ε. The same estimate holds if we replace all S_i by $S := S_1 \cup \cdots \cup S_4$. Then $\beta(R_1 \cap S) + \beta(R_2 \cap S)$ equals $\beta(R_1 \cap R_2 \cap S) + \beta\bigl((R_1 \cup R_2) \cap S\bigr)$, since β is modular and $(R_1 \cup R_2) \cap S = (R_1 \cap S) \cup (R_2 \cap S)$. The function $\beta - \tau_1$ is modular and monotone as well, which is seen from the fact that if $R_1 \subset R_2$, $R_i \in \mathfrak{R}$ and $S \in \mathfrak{R}$, then

$$\beta(R_1) + \beta(R_2 \cap S) = \beta(R_1 \cap S) + \beta\bigl(R_1 \cup (R_2 \cap S)\bigr) \le \beta(R_1 \cap S) + \beta(R_2).$$

Let

$$\tau_2(R) := \sup\{\beta(S) - \tau_1(S)\colon\ S \in \mathfrak{R},\, S \cap R_0 \subset R\}, \quad R \in \mathfrak{R}.$$

It is readily verified that the function τ_2 is monotone and supermodular. By the above corollary there exists a monotone modular function τ_3 on $\mathfrak{R}$ with

$\tau_3 \geq \tau_2$ and $\tau_3(X) = \tau_2(X) = 1 - \tau_1(X)$. Let $\psi = \tau_1 + \tau_3$. The function ψ is monotone and modular. For all $R \in \mathfrak{R}$, we have $\psi(R) \geq \tau_1(R)+\tau_2(R) \geq \beta(R)$, since $\tau_2(R) \geq \beta(R) - \tau_1(R)$. Finally, by the monotonicity of $\beta - \tau_1$ one has

$$\psi(R_0) \geq \tau_2(R_0) = \beta(X) - \tau_1(X) \geq 1 - \psi_*(X\backslash R_0).$$

Since $\psi(R_0) + \psi_*(X\backslash R_0) \leq 1$, we obtain the required equality. $\square$

1.12.39. Corollary. *Suppose that in the proven lemma $\mathfrak{R}$ is a compact class closed with respect to formation of countable intersections. Set*

$$\mathcal{E} = \{E \subset X:\ \zeta_*(E) + \zeta_*(X\backslash E) = 1\}.$$

Then $\mathcal{E}$ is a σ-algebra and the restriction of ζ_* to $\mathcal{E}$ is countably additive.

PROOF. Let us show that $\mathcal{E} = \mathfrak{M}_{\zeta_*}$. Let $E \in \mathcal{E}$ and $A \subset X$. Then $\zeta_*(A) \geq \zeta_*(A \cap E) + \zeta_*(A\backslash E)$. Let us verify the reverse inequality. Let $\varepsilon > 0$. We can find $R_1, R_2, R_3 \in \mathfrak{R}$ such that $R_1 \subset A$, $R_2 \subset E$, $R_3 \subset X\backslash E$ and $\zeta_*(A) \leq \zeta(R_1) + \varepsilon$, $\zeta_*(E) \leq \zeta(R_2) + \varepsilon$, $\zeta_*(X\backslash E) \leq \zeta(R_3) + \varepsilon$. Then $\zeta_*(A\cap E) \geq \zeta(R_1\cap R_2)$, $\zeta_*(A\backslash E) \geq \zeta(R_1\cap R_3)$. Since $\zeta(R_2)+\zeta(R_3) \geq 1-2\varepsilon$, by the modularity of ζ we obtain

$$\begin{aligned}\zeta_*(A \cap E) + \zeta_*(A\backslash E) &\geq \zeta(R_1 \cap R_2) + \zeta(R_1 \cap R_3) = \zeta\bigl(R_1 \cap (R_2 \cup R_3)\bigr)\\ &= \zeta(R_1) + \zeta(R_2 \cup R_3) - \zeta(R_1 \cup R_2 \cup R_3) \geq \zeta(R_1) - 2\varepsilon.\end{aligned}$$

Hence $E \in \mathfrak{M}_{\zeta_*}$. By Theorem 1.11.4 we obtain our assertion. $\square$

1.12(x). Set-theoretic problems in measure theory

We have already seen that constructions of nonmeasurable sets involve certain set-theoretic axioms such as the axiom of choice. The question arises whether this is indispensable and what the situation is in the framework of the naive set theory without the axiom of choice. In addition, one might also ask the following question: even if there exist sets that are nonmeasurable in the Lebesgue sense, is it possible to extend Lebesgue measure to a countably additive measure on all sets (i.e., not necessarily by means of the Lebesgue completion and not necessarily with the property of the translation invariance)? Here we present a number of results in this direction. First, by admitting the axiom of choice, we consider the problem of the existence of nontrivial measures defined on all subsets of a given set, and then several remarks are made on the role of the axiom of choice.

Let X be a set of cardinality $\aleph_1$, i.e., X is equipotent to the set of all ordinal numbers that are smaller than the first uncountable ordinal number. Note that X is uncountable and can be well-ordered in such a way that every element is preceded by an at most countable set of elements. The following theorem is due to Ulam [**967**].

1.12.40. Theorem. *If a finite countably additive measure μ is defined on all subsets of the set X of cardinality $\aleph_1$ and vanishes on all singletons, then it is identically zero.*

PROOF. It suffices to consider only nonnegative measures (see §3.1 in Chapter 3). By hypothesis, X can be well-ordered in such a way that, for every y, the set $\{x\colon\ x < y\}$ is at most countable. There is an injective mapping $x \mapsto f(x,y)$ of this set into $\mathbb{N}$. Thus, for every pair (x,y) with $x < y$ one has a natural number $f(x,y)$. For every $x \in X$ and every natural n, we have the set

$$A_x^n = \{y \in X\colon\ x < y, f(x,y) = n\}.$$

For fixed n, the sets A_x^n, $x \in X$, are pairwise disjoint. Indeed, let $y \in A_x^n \cap A_z^n$, where $x \neq z$. We may assume that $x < z$. This is, however, impossible, since $x < y$, $z < y$ and hence $f(x,y) \neq f(z,y)$ by the injectivity of the function $f(\,\cdot\,,y)$. Therefore, by the countable additivity of the measure, for every n, there can be an at most countable set of points x such that $\mu(A_x^n) > 0$. Since X is uncountable, there exists a point $x \in X$ such that $\mu(A_x^n) = 0$ for all n. Hence $A = \bigcup_{n=1}^\infty A_x^n$ has measure zero. It remains to observe that the set $X\backslash A$ is at most countable, since it is contained in the set $\{y\colon\ y \le x\}$, which is at most countable by hypothesis. Indeed, if $y > x$, then $y \in A_x^n$, where $n = f(x,y)$. Therefore, $\mu(X\backslash A) = 0$, which completes the proof. $\square$

Another proof will be given in Corollary 3.10.3 in Chapter 3.

We recall that one of the forms of the continuum hypothesis is the assertion that the cardinality of the continuum $\mathfrak{c}$ equals $\aleph_1$.

1.12.41. Corollary. *Assume the continuum hypothesis. Then, any finite countably additive measure that is defined on all subsets of a set of cardinality of the continuum and vanishes on all singletons is identically zero.*

One more set-theoretic axiom employed in this circle of problems is called Martin's axiom. A topological space X is said to satisfy the countable chain condition if every disjoint family of its open subsets is at most countable. Martin's axiom (MA) can be introduced as the assertion that, in every nonempty compact space satisfying the countable chain condition, the intersection of less than $\mathfrak{c}$ open dense sets is not empty. The continuum hypothesis (CH) is equivalent to the same assertion valid for all compacts (not necessarily satisfying the countable chain condition). Thus, CH implies MA. It is known that each of the axioms CH, MA and MA–CH (Martin's axiom with the negation of the continuum hypothesis) is consistent with the system of axioms ZFC (this is the notation for the Zermelo–Fraenkel system with the axiom of choice), i.e., if ZFC is consistent, then it remains consistent after adding any of these three axioms. In this book, none of these axioms is employed in main theorems, but sometimes they turn out to be useful for constructing certain exotic counter-examples or play some role in the situations where one is concerned with the validity of certain results in their maximal generality. Concerning the continuum hypothesis and Martin's axiom, see Jech [**458**], Kuratowski, Mostowski [**555**], Fremlin [**323**], Sierpiński [**879**].

Ulam's theorem leads to the notion of a measurable cardinal. For brevity, cardinal numbers are called cardinals. A cardinal κ is called *real measurable* if there exist a space of cardinality κ and a probability measure ν defined on the family of all its subsets and vanishing on all singletons. If ν assumes the values 0 and 1 only, then κ is called *two-valued measurable.* Real nonmeasurable cardinals (i.e., the ones that are not real measurable) are called Ulam numbers. The terminology here is opposite to the one related to the measurability of sets or functions: nonmeasurable cardinals are "nice". It is clear that the countable cardinality is nonmeasurable. Since every cardinal less than a nonmeasurable one is nonmeasurable as well, the nonmeasurable cardinals form some initial interval in the "collection of all cardinal numbers" (possibly embracing all cardinals as seen from what is said below). Anyway, this "interval" is very large, which is clear from the following Ulam–Tarski theorem (for a proof, see Federer [**282**, §2.1], Kharazishvili [**507**]).

1.12.42. Theorem. (i) *If a cardinal β is the immediate successor of a nonmeasurable cardinal α, then β is nonmeasurable.* (ii) *If the cardinality of a set M of nonmeasurable cardinals is nonmeasurable, then the supremum of M is nonmeasurable as well.*

A cardinal κ is called inaccessible if the class of all smaller cardinal numbers has no maximal element and there is no subset of cardinality less than κ whose supremum equals κ. The previous theorem means that if there exist measurable cardinals, then the smallest one is inaccessible. The cardinal $\aleph_1$ in Theorem 1.12.40 is the successor of the countable cardinal $\aleph_0$, which makes it nonmeasurable. The two-valued nonmeasurability of cardinality $\mathfrak{c}$ of the continuum is proved without use of the continuum hypothesis, which follows from Exercise 1.12.108 or from the following result (see Jech [**459**], Kuratowski, Mostowski [**555**, Ch. IX, §3], Kharazishvili [**507**]).

1.12.43. Proposition. *If a cardinal κ is two-valued nonmeasurable, then so is the cardinal 2^κ.*

This proposition yields that the cardinal $\mathfrak{c}$ is not two-valued measurable. Martin's axiom implies that the cardinal $\mathfrak{c}$ is not real measurable. If $\mathfrak{c}$ is not real measurable, then real measurable and two-valued measurable cardinals coincide. The following theorem (see Jech [**459**]) summarizes the basic facts related to measurable cardinals.

1.12.44. Theorem. *The supposition that measurable cardinals do not exist is consistent with the ZFC. In addition, if either of the following assertions is consistent with the ZFC, then so are all of them*:

(i) *two-valued measurable cardinals exist*;

(ii) *real measurable cardinals exist*;

(iii) *the cardinal $\mathfrak{c}$ is real measurable*;

(iv) *Lebesgue measure can be extended to a measure on the σ-algebra of all subsets in* $[0, 1]$.

Nonmeasurable cardinals will be encountered in Chapter 7 in our discussion of supports of measures in metric spaces. Some additional information about measurable and nonmeasurable cardinals can be found in Buldygin, Kharazishvili [**142**], Kharazishvili [**506**], [**507**], [**508**], [**511**], Fremlin [**323**], [**325**], Jech [**459**], Solovay [**898**].

We recall that the axiom of choice does not exclude countably additive extensions of Lebesgue measure to all sets, but only makes impossible the existence of such extensions with the property of translation invariance (in the next subsection there are remarks on invariant extensions), in particular, it does not enable one to exhaust all sets by means of the Lebesgue completion.

It is now natural to discuss what happens if we restrict the use of the axiom of choice. It is reasonable to admit the countable form of the axiom of choice, i.e., the possibility of choosing representatives from any countable collection of nonempty sets. At least, without it, there is no measure theory, nor even the theory of infinite series (see Kanovei [**490**]). It turns out that if we permit the use of the countable form of the axiom of choice, then, as shown by Solovay [**897**], there exists a model of set theory such that all sets on the real line are Lebesgue measurable (see also Jech [**458**, §20]). Certainly, the full axiom of choice is excluded here. Another interesting related result deals with the so-called axiom of determinacy. For the formulation, we have to define the following game G_A of two players I and II, associated with every set A consisting of infinite sequences $a = (a_0, a_1, , \ldots)$ of natural numbers a_n. The game proceeds as follows. Player I writes a number $b_0 \in \mathbb{N}$, then player II writes a number $b_1 \in \mathbb{N}$ and so on; the players know all the previous moves. If the obtained sequence $b = (b_0, b_1, \ldots)$ belongs to A, then I wins, otherwise II wins. The set A and game G_A are called determined if one of the players I or II has a winning strategy (i.e., a rule to make steps corresponding to the steps of the opposite side leading to victory). For example, if A consists of a single sequence $a = (a_i)$, then II has a winning strategy: it suffices to write $b_1 \neq a_1$ at the very first move. The axiom of determinacy (AD) is the statement that every set $A \subset \mathbb{N}^\infty$ is determined. In Kanovei [**490**] one can find interesting consequences of the axiom of determinacy, of which the most interesting for us are the measurability of all sets of reals (see also Martin [**657**]) and the real measurability of the cardinal $\aleph_1$. Thus, on the one hand, the axiom of determinacy excludes some paradoxical sets, but, on the other hand, it gives some objects impossible under the full axiom of choice.

1.12(xi). Invariant extensions of Lebesgue measure

We already know that Lebesgue measure can be extended to a countably additive measure on the σ-algebra obtained by adding a given nonmeasurable set to the class of Lebesgue measurable sets. However, such an extension may not be invariant with respect to translations. Szpilrajn-Marczewski [**928**] proved that there exists an extension of Lebesgue measure λ on the real line to a countably additive measure l that is defined on some σ-algebra $\mathfrak{L}$ strictly containing the σ-algebra of Lebesgue measurable sets, and is complete and invariant with respect to translations (i.e., if $A \in \mathfrak{L}$, then $A + t \in \mathfrak{L}$ and

$l(A+t) = l(A)$ for all t). It was proved in Kodaira, Kakutani [**525**] that there exists a countably additive extension of Lebesgue measure that is invariant with respect to translations and is nonseparable, i.e., there exists no countable collection of sets approximating all measurable sets in the sense of measure. It was shown in Kakutani, Oxtoby [**483**] that there also exist nonseparable extensions of Lebesgue measure that are invariant with respect to all isometries.

Besides countably additive, finitely additive extensions invariant with respect to translations or isometries have been considered, too. In this direction Banach [**49**] proved that on the class of all bounded sets in $\mathbb{R}^1$ and $\mathbb{R}^2$ there exist nontrivial additive set functions m invariant with respect to all isometries, i.e., translations and linear isometries (moreover, one can ensure the coincidence of m with Lebesgue measure on all measurable sets, but one can also obtain the equality $m(E) = 1$ for some set E of Lebesgue measure zero). There are no such functions on $\mathbb{R}^3$, which was first proved by F. Hausdorff. This negative result was investigated by Banach and Tarski [**60**], who proved the following theorem; a proof is found in Stromberg [**915**], Wise, Hall [**1022**, Example 6.1], and also in Wagon [**1001**].

1.12.45. Theorem. *Let A and B be bounded sets in $\mathbb{R}^3$ with nonempty interiors. Then, for some $n \in \mathbb{N}$, one can partition A into pieces $A_1, \ldots, A_n$ and B into pieces $B_1, \ldots, B_n$ such that, for every i, the set A_i is congruent to the set B_i.*

If A is a ball and B consists of two disjoint balls of the same radius, then $n = 5$ suffices in this theorem, but $n = 4$ is not enough.

Let $\mathcal{R}_n$ be the ring of bounded Lebesgue measurable sets in $\mathbb{R}^n$. Banach [**49**] investigated the following question (posed by Ruziewicz): is it true that every finitely additive measure on $\mathcal{R}_n$ that is invariant with respect to isometries is proportional to Lebesgue measure? Banach gave negative answers for $n = 1, 2$. G.A. Margulis [**655**] proved that for $n \geq 3$ the answer is positive. W. Sierpiński raised the following question (see Szpilrajn [**928**]): does there exist a maximal countably additive extension of Lebesgue measure on $\mathbb{R}^n$, invariant with respect to isometries? A negative answer to this question was given only half a century later in Ciesielski, Pelc [**182**] (see also Ciesielski [**180**]), where it was proved that, for any group G of isometries of the space $\mathbb{R}^n$ containing all parallel translations, one can write $\mathbb{R}^n$ as the union of a sequence of sets Z_n, each of which is absolutely G-null (earlier under the continuum hypothesis, a solution was given by S.S. Pkhakadze and A. Hulanicki, see references in [**182**]). Here an absolutely G-null set is a set Z such that, for each σ-finite G-invariant measure m, there exists a G-invariant extension defined on Z, and all such extensions vanish on Z (a countably additive σ-finite measure m is called G-invariant if it is defined on some σ-algebra $\mathcal{M}$ such that $g(A) \in \mathcal{M}$ and $m\bigl(g(A)\bigr) = m(A)$ for all $g \in G$, $A \in \mathcal{M}$). For the group of parallel translations, this result was obtained earlier by A.B. Kharazishvili, who proved under the continuum hypothesis a more general assertion (see [**507**]). On this subject and related problems, see Hadwiger [**392**], Kharazishvili [**507**], [**510**], [**512**], Lubotzky [**625**], von Neumann [**712**], Sierpiński [**880**], and Wagon [**1001**].

1.12(xii). Whitney's decomposition

In Lemma 1.7.2, we have represented any open set as a union of closed cubes with disjoint interiors. However, the behavior of diameters of such cubes could be quite irregular. It was observed by Whitney that one can achieve that these diameters be comparable with the distance to the boundary of the set. As above, for nonempty sets A and B we denote by $d(A,B)$ the infimum of the distances between the points in A and B.

1.12.46. Theorem. *Let Ω be an open set in $\mathbb{R}^n$ and let $Z := \mathbb{R}^n\backslash\Omega$ be nonempty. Then, there exists an at most countable family of closed cubes Q_k with edges parallel to the coordinate axes such that*:
(i) *the interiors of Q_k are disjoint and $\Omega = \bigcup_{k=1}^{\infty} Q_k$,*
(ii) $\operatorname{diam} Q_k \le d(Q_k, Z) \le 4\operatorname{diam} Q_k$.

PROOF. In the reasoning that follows we mean by cubes only closed cubes with edges parallel to the coordinate axes. Let $\mathcal{S}_k$ be a net of cubes obtained by translating the cube $[0,2^{-k}]^n$ by all vectors whose coordinates are multiples of 2^{-k}. The cubes in $\mathcal{S}_k$ have edges 2^{-k} and diameters $\sqrt{n}2^{-k}$. Set

$$\Omega_k := \Bigl\{x\in\Omega:\ 2\sqrt{n}2^{-k} < \operatorname{dist}(x,Z) \le 2\sqrt{n}2^{-k+1}\Bigr\},\quad k\in\mathbb{Z}.$$

It is clear that $\Omega = \bigcup_{k\in\mathbb{Z}} \Omega_k$. Now we can choose a preliminary collection $\mathcal{F}$ of cubes in the above nets. To this end, let us consider the cubes in $\mathcal{S}_k$. If a cube $Q\in\mathcal{S}_k$ meets Ω_k, then we include it in $\mathcal{F}$. Thus,

$$\mathcal{F} = \bigcup_{k=-\infty}^{\infty} \bigl\{Q\in\mathcal{S}_k:\ Q\cap\Omega_k\ne\varnothing\bigr\}.$$

It is clear that the union of all cubes in $\mathcal{F}$ covers Ω. Let us show that

$$\operatorname{diam} Q \le d(Q,Z) \le 4\operatorname{diam} Q,\quad \forall\, Q\in\mathcal{F}. \tag{1.12.12}$$

A cube Q from $\mathcal{F}$ belongs to $\mathcal{S}_k$ for some k. Hence it has the diameter $\sqrt{n}2^{-k}$ and there exists $x\in Q\cap\Omega_k$. Therefore,

$$d(Q,Z)\le \operatorname{dist}(x,Z)\le 2\sqrt{n}2^{-k+1}.$$

On the other hand,

$$d(Q,Z)\ge \operatorname{dist}(x,Z) - \operatorname{diam} Q > 2\sqrt{n}2^{-k} - \sqrt{n}2^{-k}.$$

It follows by (1.12.12) that all cubes Q are contained in Ω. However, cubes in $\mathcal{F}$ may not be disjoint. For this reason some further work on $\mathcal{F}$ is needed. Let us show that for every cube $Q\in\mathcal{F}$, there exists a unique cube from $\mathcal{F}$ that contains Q and is maximal in the sense that it is not contained in a larger cube from $\mathcal{F}$, and that such maximal cubes have disjoint interiors. Then the collection of such maximal cubes is a desired one: they have all the necessary properties, in particular, their union equals the union of cubes in $\mathcal{F}$, i.e., equals Ω. For the proof of the existence of maximal cubes, let us observe that two cubes $Q' \in \mathcal{S}_k$ and $Q'' \in \mathcal{S}_m$ may have common inner points only if one of them is entirely contained in the other (i.e., if there are common inner points and $k < m$, then we have $Q'' \subset Q'$). This is clear from the construction of $\mathcal{S}_k$. Now let $Q \in \mathcal{F}$. If $Q \subset Q' \in \mathcal{F}$, then we obtain by (1.12.12) that $\operatorname{diam} Q' \leq 4 \operatorname{diam} Q$. By the above observation we see that, for any two cubes $Q', Q'' \in \mathcal{F}$ containing Q, either $Q' \subset Q''$ or $Q'' \subset Q'$. Together with the previous estimate of diameter this proves the existence and uniqueness of a maximal cube $K(Q) \in \mathcal{F}$ containing Q. For the same reasons, maximal cubes $K(Q_1)$ and $K(Q_2)$, corresponding to distinct $Q_1, Q_2 \in \mathcal{F}$, either coincide or have disjoint interiors. Indeed, otherwise one of them would strictly belong to the other, say, $K(Q_1) \subset K(Q_2)$. Then $Q_1 \subset K(Q_2)$, contrary to the uniqueness of a maximal cube for Q_1. Deleting from the collection of cubes $K(Q)$ the repeating ones (if different Q' and Q'' give one and the same maximal cube), we obtain the required sequence. □

Exercises

1.12.47.° Suppose we are given a family of open sets in $\mathbb{R}^n$. Show that this family contains an at most countable subfamily with the same union.

Hint: consider a countable everywhere dense set of points x_k in the union W of the given sets W_α; for every point x_k, take all open balls $K(x_k, r_j)$ centered at x_k, having rational radii r_j and contained in at least one of the sets W_α; for every $U(x_k, r_j)$, pick a set $W_{\alpha_{k,j}} \supset U(x_k, r_j)$ and consider the obtained family.

1.12.48.° Let W be a nonempty open set in $\mathbb{R}^n$. Prove that W is the union of an at most countable collection of open cubes whose edges are parallel to the coordinate axes and have lengths of the form $p2^{-q}$, where $p, q \in \mathbb{N}$, and whose centers have coordinates of the form $m2^{-k}$, where $m \in \mathbb{Z}$, $k \in \mathbb{N}$.

Hint: observe that the union of all cubes in W of the indicated type is W.

1.12.49.° Let μ be a nonnegative measure on a ring $\mathcal{R}$. Prove that the class of all sets $Z \in \mathcal{R}$ of measure zero is a ring.

1.12.50.° Let μ be an arbitrary finite Borel measure on a closed interval I. Show that there exists a first category set E (i.e., a countable union of nowhere dense sets) such that $\mu(I \backslash E) = 0$.

Hint: it suffices to find, for each n, a compact set K_n without inner points such that $\mu(K_n) > \mu(I) - 2^{-n}$. By using that μ has an at most countable set of points a_j of nonzero measure, one can find a countable everywhere dense set of points s_j of μ-measure zero. Around every point s_j there is an interval $U_{n,j}$ with $\mu(U_{n,j}) < 2^{-j-n}$. Now we take the compact set $K_n = I \backslash \bigcup_{j=1}^{\infty} U_{n,j}$.

1.12.51.° Let $\mathcal{S}$ be some collection of subsets of a set X such that it is closed with respect to finite unions and finite intersections and contains the empty set (for example, the class of all closed sets or the class of all open sets in $[0,1]$). Show that the class of all sets of the form $A \backslash B$, $A, B \in \mathcal{S}$, $B \subset A$, is a semiring, and the class of all sets of the form $(A_1\backslash B_1)\cup\cdots\cup(A_n\backslash B_n)$, $A_i, B_i \in \mathcal{S}$, $B_i \subset A_i$, $n \in \mathbb{N}$, is the ring generated by $\mathcal{S}$.

HINT: verify that $(A\backslash B)\backslash(C\backslash D) = \bigl(A\backslash\bigl(B\cup(A\cap C)\bigr)\bigr)\cup\bigl((A\cap D)\backslash(B\cap D)\bigr)$ if $B\subset A$, $D\subset C$; next verify that the class of the indicated unions is closed with respect to intersections.

1.12.52.° Let m be an additive set function on a ring of sets $\mathcal{R}$. Prove the following Poincaré formula for all $A_1,\ldots,A_n\in\mathcal{R}$:

$$m\Bigl(\bigcup_{i=1}^n A_i\Bigr) = \sum_{i=1}^n m(A_i) - \sum_{1\le i<j\le n} m(A_i\cap A_j) + \sum_{1\le i<j<k\le n} m(A_i\cap A_j\cap A_k) - \cdots + (-1)^{n+1} m\Bigl(\bigcap_{i=1}^n A_i\Bigr).$$

1.12.53.° Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two semirings of sets. Prove that

$$\mathcal{R}_1\times\mathcal{R}_2 = \{R_1\times R_2\colon\ R_1\in\mathcal{R}_1, R_2\in\mathcal{R}_2\}$$

is a semiring. Show that $\mathcal{R}_1\times\mathcal{R}_2$ may not be a ring even if $\mathcal{R}_1$ and $\mathcal{R}_2$ are algebras.

1.12.54.° Let $\mathcal{F}$ be some collection of sets in a space X. Prove that every set A in the σ-algebra $\sigma(\mathcal{F})$ generated by $\mathcal{F}$ is contained in the σ-algebra generated by an at most countable subcollection $\{F_n\}\subset\mathcal{F}$.

HINT: verify that the union of all σ-algebras $\sigma(\{F_n\})$ generated by at most countable subcollections $\{F_n\}\subset\mathcal{F}$ is a σ-algebra.

1.12.55.° (Brown, Freilich [**134**]) The aim of this exercise is to show that Proposition 1.2.6 may be false if a σ-algebra is defined in the broader sense mentioned in §1.2. Suppose we are given a set X and a collection $\mathcal{S}$ of its subsets such that the union of all sets in $\mathcal{S}$ is $Y\subset X$. Prove that the following conditions are equivalent: (i) Y is an at most countable union of sets in $\mathcal{S}$; (ii) there exists a smallest family of sets $\mathcal{A}$ with the following properties: $\mathcal{A}$ is a σ-algebra on some subset $Z\subset X$ (i.e., Z is the unit of this σ-algebra) and $\mathcal{S}\subset\mathcal{A}$, where a smallest family is a family that is contained in every other family with the stated properties. Consider the example where $X=[0,1]$, $Y=[0,1/2]$, $\mathcal{S}$ is the class of all at most countable subsets of Y.

HINT: if Y is not the countable union of elements in $\mathcal{S}$, then Y does not belong to the class $\mathcal{P}$ of all sets $A\subset Y$ such that $A\subset\bigcup_{n=1}^\infty S_n$, where $S_n\in\mathcal{S}$. Let us fix $z\in X\backslash Y$ and consider the class $\mathcal{E}$ of all sets $E\subset Y\cup\{z\}$ such that either $E\in\mathcal{P}$ or $(Y\cup\{z\})\backslash E\in\mathcal{P}$. It is readily verified that $\mathcal{E}$ is a σ-algebra. One has $Y\notin\mathcal{E}$. If there exists a smallest family of sets $\mathcal{A}$ with the properties indicated in (ii), then the corresponding set Z cannot be smaller than Y, i.e., $Z=Y$ and hence $Y\in\mathcal{A}$. Therefore, $\mathcal{A}$ does not belong to $\mathcal{E}$, which gives a contradiction.

1.12.56. (Broughton, Huff [**132**]) Suppose we are given a sequence of σ-algebras $\mathcal{A}_n$ in a space X such that $\mathcal{A}_n$ is strictly contained in $\mathcal{A}_{n+1}$ for each n. Prove that $\bigcup_{n=1}^\infty\mathcal{A}_n$ is not a σ-algebra.

HINT: we may assume that there is a nonempty set $B\in\mathcal{A}_1$ not equal to X. If, for some n, we have $B\cap\mathcal{A}_{n+1}=B\cap\mathcal{A}_n$ and the same is true for $X\backslash B$, then $\mathcal{A}_{n+1}=\mathcal{A}_n$, which is a contradiction. Hence one can find $E\in\mathcal{A}_1$ and infinitely many p_k with $p_{k+1}>p_k$ such that $(E\cap\mathcal{A}_{p_k+1})\backslash(E\cap\mathcal{A}_{p_k})\neq\varnothing$. Then the classes $E\cap\mathcal{A}_{p_k}$ are strictly increasing σ-algebras on E. By induction, we construct a subsequence $\mathcal{A}_{j_1}, \mathcal{A}_{j_2}, \ldots$, where $j_{k+1} > j_k$, and sets $E_1 \supset E_2 \supset \ldots$ with $E_k \in \mathcal{A}_{j_k}$ and $E_{k+1} \in (E_k \cap \mathcal{A}_{j_{k+1}}) \backslash (E_k \cap \mathcal{A}_{j_k})$. We obtain disjoint sets $F_k := E_k \backslash E_{k+1}$, $F_k \in \mathcal{A}_{j_{k+1}} \backslash \mathcal{A}_{j_k}$. We may assume that $X = \bigcup_{k=1}^{\infty} F_k$. Let $\pi\colon X \to \mathbb{N}$, $\pi(F_k) = k$ and let $\mathcal{A}'_n := \{A\colon \pi^{-1}(A) \in \mathcal{A}_n\}$. It is easily verified that, for every n, there is the smallest set $B_n \in \mathcal{A}'_n$ with $n \in B_n$. Then $B_n \subset \{k \geq n\}$, $B_n \neq \{n\}$. If $m \in B_n$, then $B_m \subset B_n$, since $B_m \cap B_n \in \mathcal{A}'_m$. Let $n_1 := 1$. We find by induction $n_{k+1} \in B_{n_k}$, $n_{k+1} > n_k$. Then $B_{n_1} \supset B_{n_2} \supset \ldots$ Let $E := \{n_2, n_4, n_6, \ldots\}$. If $\pi^{-1}(E) \in \mathcal{A}_n$, i.e., $E \in \mathcal{A}'_n$, then $E \in \mathcal{A}'_{n_{2k}}$ for some k, whence one has $\{n_{2k}, n_{2k+2}, \ldots\} \in \mathcal{A}'_{n_{2k}}$ and $B_{n_{2k}} \subset \{n_{2k}, n_{2k+2}, \ldots\}$, contrary to the inclusion $n_{2k+1} \in B_{n_{2k}}$.

1.12.57.° Show that every set of positive Lebesgue measure contains a nonmeasurable subset.

1.12.58. Prove that there exists a sequence of sets $A_n \subset [0,1]$ such that for all n one has $A_{n+1} \subset A_n$, $\bigcap_{n=1}^{\infty} A_n = \varnothing$ and $\lambda^*(A_n) = 1$, where λ is Lebesgue measure.

HINT: let $\{r_n\}$ be some enumeration of the rational numbers and let $E \subset [0,1]$ be the nonmeasurable set from Vitali's example. Show that the sets

$$E_n := \bigl(E \cup (E + r_1) \cup \cdots \cup (E + r_n)\bigr) \cap [0,1]$$

have inner measure zero and take $A_n := [0,1] \backslash E_n$.

1.12.59. Show that every nonempty perfect set contains a nonempty perfect subset of Lebesgue measure zero. In particular, every set of positive Lebesgue measure contains a measure zero compact set of cardinality of the continuum.

HINT: it suffices to consider a compact set K of positive measure without isolated points; then, similarly to the construction of the classical Cantor set, delete from K the countable union of sets $J_n \cap K$, where J_n are disjoint intervals, in such a way that the remaining set is perfect, nonempty and has measure zero.

1.12.60.° Let C be the Cantor set in $[0,1]$. Show that

$$C + C := \{c_1 + c_2\colon\ c_1, c_2 \in C\} = [0,2], \quad C - C := \{c_1 - c_2\colon\ c_1, c_2 \in C\} = [-1,1].$$

HINT: the sets $C + C$ and $C - C$ are compact, hence it suffices to verify that they contain certain everywhere dense subsets in the indicated intervals, which can be done by using the description of C in terms of the ternary expansion.

1.12.61.° Give an example of two closed sets $A, B \subset \mathbb{R}$ of Lebesgue measure zero such that the set $A + B := \{a + b\colon\ a \in A, b \in B\}$ is $\mathbb{R}$.

HINT: take for A the Cantor set and for B the union of translations of A to all integer numbers.

1.12.62.° (Steinhaus [**910**]) Let A be a set of positive Lebesgue measure on the real line. Show that the set $A - A := \{a_1 - a_2\colon\ a_1, a_2 \in A\}$ contains some interval. Prove an analogous assertion for $\mathbb{R}^n$ (obtained in Rademacher [**775**]).

HINT: there is a compact set $K \subset A$ with $\lambda(K) > 0$; take an open set U with $K \subset U$ and $\lambda(U) < 2\lambda(K) = \lambda(K) + \lambda(K + h)$ and observe that there exists $\varepsilon > 0$ such that $K + h \subset U$ whenever $|h| < \varepsilon$; then $\lambda\bigl(K \cup (K + h)\bigr) \leq \lambda(U)$ for such h, whence $K \cap (K + h) \neq \varnothing$.

1.12.63. (P.L. Ulyanov, see Bary [**66**, Appendix, §23]) Let $E \subset [0,1]$ be a measurable set of positive measure. (i) Prove that for every sequence $\{h_n\}$ converging to zero and every $\varepsilon > 0$, there exist a measurable set $E_\varepsilon \subset E$ and a subsequence $\{h_{n_k}\}$

such that $\lambda(E_\varepsilon) > \lambda(E) - \varepsilon$ and for all $x \in E_\varepsilon$ we have $x + h_{n_k} \in E$, $x - h_{n_k} \in E$ for all k.

(ii) Prove that there exist a measurable set $E_0 \subset E$ and a sequence of numbers $h_n > 0$ converging to zero such that $\lambda(E_0) = \lambda(E)$ and for every $x \in E_0$, we have $x + h_n \in E$ for all $n \geq n(x)$.

Hint: (i) choose numbers n_k such that

$$\lambda\big(E \bigtriangleup (E + h_{n_k})\big) \leq \varepsilon 8^{-k},\ \lambda\big(E \bigtriangleup (E - h_{n_k})\big) \leq \varepsilon 8^{-k},$$

and take $E_\varepsilon = \bigcap_{k=1}^\infty \big((E + h_{n_k}) \cap (E - h_{n_k})\big)$. (ii) For $\{2^{-n}\}$ and $\varepsilon_1 = 1/2$, take the set $E_{1/2}$ according to (i) and proceed by induction: if for some n we have chosen a set $E_{2^{-n}}$ according to (i) and a subsequence $\{h_k^{(n)}\}$ in $\{2^{-n}\}$, then when choosing $E_{2^{-n-1}}$ for the number $n+1$, we take a subsequence in $\{h_k^{(n)}\}$. Let $E_0 = \bigcup_{n=1}^\infty E_{2^{-n}}$ and $h_n := h_n^{(n)}$.

1.12.64. Let A be a set of positive Lebesgue measure in $\mathbb{R}^n$ and let $k \in \mathbb{N}$. Prove that there exist a set B of positive Lebesgue measure and a number $\delta > 0$ such that the sets $B_{i_1,\ldots,i_n} := B + \delta(i_1, \ldots, i_n)$, where $i_j \in \{1, \ldots, k\}$, are disjoint and are contained in A.

1.12.65. (Jones [**469**]) In this exercise, by a Hamel basis we mean a Hamel basis of the space $\mathbb{R}^1$ over the field of rational numbers.

(i) Let M be a set in $[0, 1]$ and let $\lambda_*(M - M) > 0$. Prove that M contains a Hamel basis. Deduce that the Cantor set contains a Hamel basis and that every set of positive measure contains a Hamel basis.

(ii) Prove that there exists a Hamel basis containing a nonempty perfect set.

(iii) Let H be a Hamel basis and $DE := \{e_1 - e_2, e_1, e_2 \in E, e_1 \geq e_2\}$ for any set E. Prove that $\lambda^*(D^n H) > 0$ for some n and $\lambda_*(D^n H) = 0$ for all n, where D^n is defined inductively.

(iv) Let H be a Hamel basis and $TE := \{e_1 + e_2 - e_3, e_1, e_2, e_3 \in E\}$ for any set E. Prove that $\lambda^*(T^n H) > 0$ for some n and $\lambda_*(T^n H) = 0$ for all n.

1.12.66. Prove the existence of a nonmeasurable (in the sense of Lebesgue) Hamel basis of $\mathbb{R}^1$ over $\mathbb{Q}$ without using the continuum hypothesis (see Example 1.12.21).

Hint: let $\omega_\mathfrak{c}$ be the smallest ordinal number corresponding to the cardinality of the continuum. The family of all compacts of positive measure has cardinality $\mathfrak{c}$ and hence can be put in some one-to-one correspondence $\alpha \mapsto K_\alpha$ with ordinal numbers $\alpha < \omega_\mathfrak{c}$. By means of transfinite induction we find a family of elements $h_\alpha \in K_\alpha$ linearly independent over $\mathbb{Q}$. Namely, if such elements h_β are already found for all $\beta < \alpha$, where $\alpha < \mathfrak{c}$, then the collection of all linear combinations of these elements with rational coefficients has cardinality less than that of the continuum. Hence K_α contains an element h_α that is not such a linear combination. Let us complement the constructed family $\{h_\alpha, \alpha < \mathfrak{c}\}$ to a Hamel basis. We obtain a nonmeasurable set, since if it were measurable, then, according to what we proved earlier, it would have measure zero, which is impossible because the constructed family meets every compact set in $[0, 1]$ of positive measure.

1.12.67. Prove that there exists a bounded set E of measure zero such that $E + E$ is nonmeasurable.

Hint: let $H = \{h_\alpha\}$ be a Hamel basis over $\mathbb{Q}$ of zero measure with $h_\alpha \in [0, 1]$, $A = \{rh\colon\ r \in \mathbb{Q} \cap [0, 1], h \in H\}$. Set $E_1 := A + A$; it is readily seen that E_1 has inner measure zero because otherwise $E_1 - E_1$ would contain an interval, which is impossible, since any point in $E_1 - E_1$ is a linear combination of four vectors in H. If E_1 is nonmeasurable, then we take $E = A$; otherwise we set $E_2 := E_1 + E_1$ and construct inductively $E_{n+1} := E_n + E_n$. In finitely many steps we obtain a desired set, since $E_n - E_n$ cannot contain an interval and the union of all E_n covers $[0, 1]$.

1.12.68. (Ciesielski, Fejzić, Freiling [**181**]) Show that every set $E \subset \mathbb{R}$ contains a subset A with $\lambda_*(A + A) = 0$ and $\lambda^*(A + A) = \lambda^*(E + E)$, where λ is Lebesgue measure.

1.12.69. (Sodnomov [**895**]) Let $E \subset \mathbb{R}^1$ be a set of positive Lebesgue measure. Then, there exists a perfect set P with $P + P \subset E$.

1.12.70. Let $\beta \in (0, 1)$. The operation $T(\beta)$ over a finite family of disjoint intervals $I_1, \ldots, I_n$ of nonzero length consists of deleting from every I_j the open interval with the same center as I_j and length $\beta\lambda(I_j)$. Given a sequence of numbers $\beta_n \in (0, 1)$, let us define inductively compacts K_n obtained by consequent application of the operations $T(\beta_1)$,...,$T(\beta_n)$, starting with the interval $I = [0, 1]$.

(i) Show that $\lambda\bigl(\bigcap_{n=1}^\infty K_n\bigr) = \lim\limits_{n\to\infty} \prod_{i=1}^n (1 - \beta_i)$. In particular, letting $\beta_n = 1 - \alpha^{\frac{1}{n(n+1)}}$, where $\alpha \in (0, 1)$, we have $\lambda\bigl(\bigcap_{n=1}^\infty K_n\bigr) = \alpha$.

(ii) Show that there exists a sequence of pairwise disjoint nowhere dense compact sets A_n with the following properties: $\lambda(A_n) = 2^{-n}$ and the intersection of A_{n+1} with each interval contiguous to the set $\bigcup_{j=1}^n A_j$ has a positive measure.

(iii) Show that the intersections of the set $A := \bigcup_{n=1}^\infty A_{2n-1}$ and its complement with every interval $I \subset [0, 1]$ have positive measures.

Hint: see George [**351**, p. 62, 63].

1.12.71.° Prove that Lebesgue measure of every measurable set $E \subset \mathbb{R}^n$ equals the infimum of the sums $\sum_{k=1}^\infty \lambda_n(U_k)$ over all sequences of open balls U_k covering E.

Hint: observe that it suffices to prove the claim for open E and in this case use the fact that one can inscribe in E a disjoint collection of open balls V_j such that the set $E\backslash\bigcup_{j=1}^\infty V_j$ has measure zero, and then cover this set with a sequence of balls W_i with the sum of measures majorized by a given $\varepsilon > 0$.

1.12.72. Suppose that μ is a countably additive measure with values in $[0, +\infty]$ on the σ-algebra of Borel sets in $\mathbb{R}^n$ and is finite on balls, and let W be a nonempty open set in $\mathbb{R}^n$. Prove that there exists an at most countable collection of disjoint open cubes Q_j in W with edges parallel to the coordinate axes such that $\mu\bigl(W\backslash\bigcup_{j=1}^\infty W_j\bigr) = 0$.

Hint: we may assume that W is contained in a cube I; in the proof of Lemma 1.7.2 one can choose all cubes in such a way that their boundaries have μ-measure zero; to this end, we observe that at most countably many affine hyperplanes parallel to the coordinate hyperplanes have positive μ-measure. In addition, given a countable set of points t_i on the real line, the set of points of the form $r + t_i$, where r is binary-rational (i.e., $r = m2^{-k}$ with integer m, k), is countable as well; therefore, one can find $\alpha \neq 0$ such that the required cubes have edges of length $m2^{-k}$, where $m \in \mathbb{Z}$, $k \in \mathbb{N}$, and centers with coordinates of the form $\alpha + m2^{-k}$.

1.12.73.° Show that a set $E \subset \mathbb{R}$ is Lebesgue measurable precisely when for every $\varepsilon > 0$, there exist open sets U and V such that $E \subset U$, $U\backslash E \subset V$ and $\lambda(V) < \varepsilon$.

1.12.74.° Let μ be a Borel probability measure on the cube $I = [0,1]^n$ such that $\mu(A) = \mu(B)$ for any Borel sets $A, B \subset I$ that are translations of one another. Show that μ coincides with Lebesgue measure λ_n.

HINT: observe that μ coincides with λ_n on all cubes in I with edges parallel to the axes and having binary-rational lengths (the boundaries of such cubes have measure zero with respect to μ by the countable additivity and the hypothesis). It follows that μ coincides with λ_n on the algebra generated by the indicated cubes.

1.12.75.° (i) Show that for any countably additive function $\mu\colon \mathfrak{R} \to [0,+\infty)$ on a semiring $\mathfrak{R}$ and any $A, A_n \in \mathfrak{R}$ such that A_n either increase or decrease to A, one has the equality $\mu(A) = \lim\limits_{n\to\infty} \mu(A_n)$.

(ii) Give an example showing that the properties indicated in (i) do not imply the countable additivity of a nonnegative additive set function on a semiring.

HINT: (ii) consider the semiring of sets of the form $\mathbb{Q}\cap(a,b)$, $\mathbb{Q}\cap(a,b]$, $\mathbb{Q}\cap[a,b)$, $\mathbb{Q}\cap[a,b]$, where $\mathbb{Q}$ is the set of rational numbers in $[0,1]$; on such sets let μ equal $b-a$.

1.12.76.° Give an example of a nonnegative additive set function μ on a semiring $\mathfrak{R}$ such that $\mu(A) = \lim\limits_{n\to\infty} \mu(A_n)$ whenever $A, A_n \in \mathfrak{R}$ and A_n either increase or decrease to A, but the additive extension of μ to the ring generated by $\mathfrak{R}$ does not possess this property.

HINT: see Exercise 1.12.75.

1.12.77.° (i) Show that a bounded set $E \subset \mathbb{R}^n$ is Jordan measurable (see Definition in §1.1) precisely when the boundary of E (the set of points each neighborhood of which contains points from the set E and from its complement) has measure zero. (ii) Show that the collection of all Jordan measurable sets in an interval or in a cube is a ring.

1.12.78.° Prove Proposition 1.6.5.

1.12.79.° Show that a bounded nonnegative measure μ on a σ-algebra $\mathcal{A}$ is complete precisely when $\mathcal{A} = \mathcal{A}_\mu$; In particular, the Lebesgue extension of any complete measure coincides with the initial measure.

1.12.80.° Give an example of a σ-finite measure on a σ-algebra that is not σ-finite on some sub-σ-algebra.

HINT: consider Lebesgue measure on $\mathbb{R}^1$ and the sub-σ-algebra of all sets that are either at most countable or have at most countable complements.

1.12.81.° Let A_n be subsets of a space X. Show that

$$\{x\colon\ x \in A_n \text{ for infinitely many } n\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

1.12.82.° Let μ be a probability measure and let $A_1, \ldots, A_n$ be measurable sets with $\sum_{i=1}^n \mu(A_i) > n-1$. Prove that $\mu\bigl(\bigcap_{i=1}^n A_i\bigr) > 0$.

HINT: observe that $\sum_{i=1}^n \mu(C_i) = \sum_{i=1}^n (1-\mu(A_i)) < 1$, where C_i is the complement of A_i.

1.12.83.° **(Baire category theorem)** Let M_j, $j \in \mathbb{N}$, be closed sets in $\mathbb{R}^d$ such that their union is a closed cube. Prove that at least one of the sets M_j has inner points. Generalize to the case where M_j are closed sets in a complete metric space X with $\bigcup_{j=1}^\infty M_j = X$. A set in a metric space is called nowhere dense if its closure has no interior; a countable union of nowhere dense sets is said to be a first category set. The above result can be formulated as follows: a complete nonempty metric space is not a first category set.

HINT: assuming the opposite, construct a sequence of decreasing closed balls U_j with radii $r_j \to 0$ such that $U_j \cap M_j = \varnothing$.

1.12.84. Prove that $\mathbb{R}^1$ cannot be written as the union of a family of pairwise disjoint nondegenerate closed intervals.

HINT: verify that such a family must be countable and that the family of all endpoints of the given intervals is closed and has no isolated points; apply the Baire theorem. One can also use that a closed set without isolated points is uncountable (see Proposition 6.1.17 in Chapter 6).

1.12.85. Show that $\mathbb{R}^n$ with $n > 1$ cannot be written as the union of a family of closed balls with pairwise disjoint interiors.

HINT: apply Exercise 1.12.84 to a straight line which passes through the origin, contains no points of tangency of the given balls and is not tangent to any of them.

1.12.86.° Show that the σ-algebra $\mathcal{B}(\mathbb{R}^1)$ of all Borel subsets of the real line is the smallest class of sets that contains all closed sets and admits countable intersections and countable unions.

HINT: use that the indicated smallest class is monotone and contains the algebra of finite unions of rays and intervals; another approach is to verify that the collection of all sets belonging to the above class along with their complements is a σ-algebra and contains all closed sets. A stronger assertion is found in Example 1.12.3.

1.12.87. (i) Prove that the union of an arbitrary family of nondegenerate closed intervals on the real line is measurable.

(ii) Prove that the union of an arbitrary family of nondegenerate rectangles in the plane is measurable.

(iii) Prove that the union of an arbitrary family of nondegenerate triangles in the plane is measurable.

HINT: (i) it suffices to verify that the union of the family of all intervals I_α of length not smaller than $1/k$ is measurable for each k; there exists an at most countable subfamily I_{α_n} such that the union of their interiors equals the union of the interiors of all I_α; the set $\bigcup_\alpha I_\alpha \backslash \bigcup_{n=1}^\infty I_{\alpha_n}$ is at most countable, since every point is isolated (such a point may be only an endpoint of some interval I_α, and an interval of length $1/k$ cannot contain three such points). (ii) Consider all rectangles E_α with the shorter side length at least $1/k$; take a countable subfamily E_{α_n} with the union of interiors equal to the union of the interiors of all E_α and observe that any circle of a sufficiently small radius can meet at most finitely many sides of those rectangles E_α that are not covered by the rectangles E_{α_n}. (iii) Modify the proof of (ii) for triangles, considering subfamilies of triangles with sides at least $1/k$ and angles belonging to $[1/k, \pi - 1/k]$. We note that these assertions follow by the Vitali covering theorem proven in Chapter 5 (Theorem 5.5.2).

1.12.88. (Nikodym [**716**]) For any sequence of sets E_n let

$$\limsup_{n\to\infty} E_n := \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k, \quad \liminf_{n\to\infty} E_n := \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty E_k.$$

Let $(X, \mathcal{A}, \mu)$ be a probability space. Prove that a sequence of sets $A_n \in \mathcal{A}$ converges to a set $A \in \mathcal{A}$ in the Fréchet–Nikodym metric $d(B_1, B_2) = \mu(B_1 \triangle B_2)$ precisely when every subsequence in $\{A_n\}$ contains a further subsequence $\{E_n\}$ such that

$$A = \limsup_{n\to\infty} E_n = \liminf_{n\to\infty} E_n$$

up to a measure zero set.

HINT: see Theorem 1.12.6; this also follows by Theorem 2.2.5 in Chapter 2.

1.12.89.$^\circ$ Let $(X, \mathcal{A}, \mu)$ be a space with a probability measure, let $A_n \in \mathcal{A}_\mu$, and let

$$B := \{x\colon\ x \in A_n \text{ for infinitely many } n\},$$

i.e., $B = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_n$ according to Exercise 1.12.81.

(i) **(Borel–Cantelli lemma)** Show that if $\sum_{n=1}^\infty \mu(A_n) < \infty$, then $\mu(B) = 0$.

(ii) Prove that if $\mu(A_n) \geq \varepsilon > 0$ for all n, then $\mu(B) \geq \varepsilon$.

(iii) (Pták [**772**]) Show that if $\mu(B) > 0$, then one can find a subsequence $\{n_k\}$ such that $\mu(\bigcap_{k=1}^m A_{n_k}) > 0$ for all m.

HINT: the sets $B_k := \bigcup_{n=k}^\infty A_n$ decrease and one has $\mu(B_k) \leq \sum_{n=k}^\infty \mu(A_n)$, $\mu(B_k) \geq \mu(A_k)$. If $\mu(B) > 0$, we find the first number n_1 with $\mu(B \cap A_{n_1}) > 0$, then we find $n_2 > n_1$ with $\mu(B \cap A_{n_1} \cap A_{n_2}) > 0$ and so on. See also Exercise 2.12.35.

1.12.90. (i) Construct a sequence of sets $E_n \subset [0, 1]$ of measure $\sigma > 0$ such that the intersection of each subsequence in this sequence has measure zero.

(ii) Let μ be a probability measure and let A_n be μ-measurable sets such that $\mu(A_n) \geq \varepsilon > 0$ for all $n \in \mathbb{N}$. Show that there exists a subsequence n_k such that $\bigcap_{k=1}^\infty A_{n_k}$ is nonempty.

(iii) (Erdős, Kestelman, Rogers [**270**]) Let A_n be Lebesgue measurable sets in $[0, 1]$ with $\lambda(A_n) \geq \varepsilon > 0$ for all $n \in \mathbb{N}$. Show that there exists a subsequence n_k such that $\bigcap_{k=1}^\infty A_{n_k}$ is uncountable (see a stronger assertion in Exercise 3.10.107).

HINT: (i) define E_n inductively: $E_1 = (0, 1/2)$, $E_2 = (0, 1/4) \cup (3/4, 1)$ and so on; the set E_{n+1} consists of 2^n intervals $J_{n,k}$ that are the left halves of the intervals $J_{n-1,k}$ and the left halves of the contiguous intervals to the intervals $J_{n-1,k}$. (ii) Follows by the previous exercise.

1.12.91. Let a function $\alpha\colon \mathbb{N} \to [0, +\infty)$ be such that $\sum_{k=1}^\infty \alpha(k) < \infty$. Prove that the set E of all $x \in (0, 1)$ such that, for infinitely many natural numbers q, there exists a natural number p such that p and q are relatively prime and $|x - p/q| < \alpha(q)/q$, has measure zero. In Exercise 10.10.57 in Chapter 10 see a converse assertion.

HINT: for fixed q, let E_q be the set of all $x \in (0, 1)$ such that, for some $p \in \mathbb{N}$, one has $|x - p/q| < \alpha(q)/q$. This set consists of the intervals of length $2\alpha(q)/q$ centered at the points p/q, $p = 1, \ldots, q$, whence $\lambda(E_q) \leq 2\alpha(q)$. By the Borel–Cantelli lemma, $\lambda(E) = 0$.

1.12.92. (Gillis [**354**], [**355**]) Let $E_k \subset [0, 1]$ be measurable sets and let $\lambda(E_k) \geq \alpha$ for all k, where $\alpha \in (0, 1)$. Prove that for all $p \in \mathbb{N}$ and $\varepsilon > 0$, there exist $k_1 < \cdots < k_p$ such that $\lambda(E_{k_1} \cap \cdots \cap E_{k_p}) > \alpha^p - \varepsilon$.

1.12.93. (i) Let $E \subset [0, 1]$ be a set of Lebesgue measure zero. Prove that there exists a convergent series with positive terms a_n such that, for any $\varepsilon > 0$, the set E can be covered by a sequence of intervals I_n of length at most εa_n. (ii) Show that there is no such series that would suit every measure zero set.

1.12.94. (Wesler [**1010**]; Mergelyan [**682**] for $n = 2$) Let U_k be disjoint open balls of radii r_k in the unit ball U in $\mathbb{R}^n$ such that $U \backslash \bigcup_{k=1}^{\infty} U_k$ has measure zero. Show that $\sum_{k=1}^{\infty} r_k^{n-1} = \infty$.

HINT: see Crittenden, Swanson [**192**], Larman [**569**], and Wesler [**1010**].

1.12.95. (i) Let $\alpha = n^{-1}$, where $n \in \mathbb{N}$. Prove that for any sets A and B in $[0,1]$ of positive Lebesgue measure, there exist points $x, y \in [0,1]$ such that $\lambda(A \cap [x,y]) = \alpha\lambda(A)$ and $\lambda(B \cap [x,y]) = \alpha\lambda(B)$. (ii) Show that if $\alpha \in (0,1)$ does not have the form n^{-1} with $n \in \mathbb{N}$, then assertion (i) is false.

HINT: see George [**351**, p. 59].

1.12.96. A set $S \subset \mathbb{R}^1$ is called a Sierpiński set if $S \cap Z$ is at most countable for every set Z of Lebesgue measure zero.

(i) Under the continuum hypothesis show the existence of a Sierpiński set.

(ii) Prove that no Sierpiński set is measurable.

HINT: see Kharazishvili [**511**].

1.12.97. Let A be a set in $\mathbb{R}^d$ of Lebesgue measure greater than 1. Prove that there exist two distinct points $x, y \in A$ such that the vector $x - y$ has integer coordinates.

1.12.98.$^{\circ}$ Prove that each convex set in $\mathbb{R}^d$ is Lebesgue measurable.

HINT: show that the boundary of a bounded convex set has measure zero.

1.12.99. Let A be a bounded convex set in $\mathbb{R}^d$ and let A^{ε} be the set of all points with the distance from A at most ε. Prove that $\lambda_d(A^{\varepsilon})$, where λ_d is Lebesgue measure, is a polynomial of degree d in ε.

HINT: verify the claim for convex polyhedra.

1.12.100.$^{\circ}$ Prove Theorem 1.12.1.

1.12.101.$^{\circ}$ Let $(X, \mathcal{A}, \mu)$ be a probability space, $\mathcal{B}$ a sub-σ-algebra in $\mathcal{A}$, and let $\mathcal{B}^{\mu}$ be the σ-algebra generated by $\mathcal{B}$ and all sets of measure zero in $\mathcal{A}_{\mu}$.

(i) Show that $E \in \mathcal{B}^{\mu}$ precisely when there exists a set $B \in \mathcal{B}$ such that $E \triangle B \in \mathcal{A}_{\mu}$ and $\mu(E \triangle B) = 0$.

(ii) Give an example demonstrating that $\mathcal{B}^{\mu}$ may be strictly larger than the σ-algebra $\mathcal{B}_{\mu}$ that is the completion of $\mathcal{B}$ with respect to the measure $\mu|_{\mathcal{B}}$.

HINT: (i) the sets of the indicated form belong to $\mathcal{B}^{\mu}$ and form a σ-algebra. (ii) Take Lebesgue measure λ on the σ-algebra of all measurable sets in $[0,1]$ and $\mathcal{B} = \{\varnothing, [0,1]\}$. Then $\mathcal{B}_{\lambda} = \mathcal{B}$.

1.12.102.$^{\circ}$ Let μ be a probability measure on a σ-algebra $\mathcal{A}$. Suppose that $\mathcal{A}$ is countably generated, i.e., is generated by an at most countable family of sets. Show that the measure μ is separable. Give an example showing that the converse is false.

HINT: if $\mathcal{A}$ is generated by sets A_n, then the algebra $\mathcal{A}_0$ generated by those sets is at most countable. It remains to use that, for any $A \in \mathcal{A}$ and $\varepsilon > 0$, there exists $A_0 \in \mathcal{A}_0$ such that $\mu(A \triangle A_0) < \varepsilon$. As an example of a separable measure on a σ-algebra that is not countably generated, one can take Lebesgue measure on the σ-algebra of Lebesgue measurable sets in an interval (see §6.5). Another example: Lebesgue measure on the σ-algebra of all sets in $[0,1]$ that are either at most countable or have at most countable complements.

1.12.103. Let $(X, \mathcal{A}, \mu)$ be a measure space with a finite nonnegative measure μ and let $\mathcal{A}/\mu$ be the corresponding metric Boolean algebra with the metric d

introduced in §1.12(iii). Prove that the mapping $A \mapsto X\backslash A$ from $\mathcal{A}/\mu$ to $\mathcal{A}/\mu$ and the mappings $(A, B) \mapsto A \cup B$, $(A, B) \mapsto A \cap B$ from $(\mathcal{A}/\mu)^2$ to $\mathcal{A}/\mu$ are continuous.

1.12.104. Let μ be a separable probability measure on a σ-algebra $\mathcal{A}$ and let $\{X_t\}_{t\in T}$ be an uncountable family of sets of positive measure. Show that there exists a countable subfamily $\{t_n\} \subset T$ such that $\mu\bigl(\bigcap_{n=1}^{\infty} X_{t_n}\bigr) > 0$.

HINT: in the separable measure algebra $\mathcal{A}/\mu$ the given family has a point of accumulation X' with $\mu(X') > 0$, since an uncountable set cannot have the only accumulation point corresponding to the equivalence class of measure zero sets; there exist indices t_n with $\mu(X' \bigtriangleup X_{t_n}) < \mu(X')2^{-n}$.

1.12.105.° Let $\mathcal{A}$ be the class of all subsets on the real line that are either at most countable or have at most countable complements. If the complement of a set $A \in \mathcal{A}$ is at most countable, then we set $\mu(A) = 1$, otherwise we set $\mu(A) = 0$. Then $\mathcal{A}$ is a σ-algebra and μ is a probability measure on $\mathcal{A}$, the collection $\mathcal{K}$ of all sets with at most countable complements is a compact class, approximating μ, but there is no class $\mathcal{K}' \subset \mathcal{A}$ approximating μ and having the property that every (not necessarily countable) collection in $\mathcal{K}'$ with empty intersection has a finite subcollection with empty intersection.

HINT: if such a class $\mathcal{K}'$ exists, then, for every $x \in \mathbb{R}^1$, there is a set $K_x \in \mathcal{K}'$ such that $K_x \subset \mathbb{R}^1\backslash\{x\}$ and $\mu(K_x) > 0$. Then $\mu(K_x) = 1$ and hence each finite intersection of such sets is nonempty, but the intersection of all K_x is empty.

1.12.106.° Let μ be an atomless probability measure on a measurable space $(X, \mathcal{A})$ and let $\mathcal{F} \subset \mathcal{A}$ be a countable family of sets of positive measure. Show that there exists a set $A \in \mathcal{A}$ such that $0 < \mu(A \cap F) < \mu(F)$ for all $F \in \mathcal{F}$.

HINT: let $\mathcal{F} = \{F_n\}$ and $\mathcal{F}_n = \{A \in \mathcal{A}\colon\ \mu(A\cap F_n) = 0 \text{ or } \mu(A\cap F_n) = \mu(F_n)\}$. Then $\mathcal{F}_n$ is closed in $\mathcal{A}/\mu$. Since μ is atomless, the sets $\mathcal{F}_n$ are nowhere dense in $\mathcal{A}/\mu$. By Baire's theorem the intersection of their complements is not empty.

1.12.107. Let $\mathbb{Q}$ be the set of all rational numbers equipped with the σ-algebra $2^{\mathbb{Q}}$ of all subsets and let the measure μ on $2^{\mathbb{Q}}$ with values in $[0, +\infty]$ be defined as the cardinality of a set. Let $\nu = 2\mu$. Show that the distinct measures μ and ν coincide on all open sets in $\mathbb{Q}$ (with the induced topology), and on all sets from the algebra that consists of finite disjoint unions of sets of the form $\mathbb{Q} \cap (a, b]$ and $\mathbb{Q} \cap (c, +\infty)$, where $a, b, c \in \mathbb{Q}$ or $c = -\infty$ (this algebra generates $2^{\mathbb{Q}}$).

HINT: nonempty sets of the above types are infinite.

1.12.108. Prove that there exists no countably additive measure defined on all subsets of the space $X = \{0, 1\}^{\infty}$ that assumes only two values 0 and 1 and vanishes on all singletons.

HINT: let $X_n = \{(x_i) \in X\colon\ x_n = 0\}$; if such a measure μ exists, then, for any n, either $\mu(X_n) = 1$ or $\mu(X_n) = 0$; denote by Y_n that of the two sets X_n and $X\backslash X_n$ which has measure 1; then $\bigcap_{n=1}^{\infty} Y_n$ has measure 1 as well and is a singleton.

1.12.109. Prove that for every Borel set $E \subset \mathbb{R}^n$, there exists a Borel set $\widehat{E}$ that differs from E in a measure zero set and has the following property: for every point x at the boundary $\partial\widehat{E}$ of the set $\widehat{E}$ and every $r > 0$, one has

$$0 < \lambda_n\bigl(\widehat{E} \cap B(x, r)\bigr) < \omega_n r^n,$$

where $B(x, r)$ is the ball centered at x with the radius r and ω_n is the measure of the unit ball.

HINT: let E_0 be the set of all x such that $\lambda_n\bigl(E \cap B(x,r)\bigr) = 0$ for some $r > 0$, and let E_1 be the set of all x such that $\lambda_n\bigl(E \cap B(x,r)\bigr) = \omega_n r^n$ for some $r > 0$. Consider $\widehat{E} = (E \cup E_1)\backslash E_0$ and use the fact that E_0 and E_1 are open.

1.12.110. Prove that every uncountable set $G \subset \mathbb{R}$ that is the intersection of a sequence of open sets contains a nowhere dense closed set Z of Lebesgue measure zero that can be continuously mapped onto $[0, 1]$.

HINT: see Oxtoby [**733**, Lemma 5.1] or Chapter 6.

1.12.111. Prove that every uncountable set $G \subset \mathbb{R}$ that is the intersection of a sequence of open sets has cardinality of the continuum.

HINT: apply the previous exercise (see also Chapter 6, §6.1).

1.12.112. (i) Prove that the class of all Souslin subsets of the real line is obtained by applying the A-operation to the collection of all open sets. (ii) Show that in (i) it suffices to take the collection of all intervals with rational endpoints.

HINT: (i) use that every closed set is the intersection of a countable sequence of open sets and that $S(\mathcal{E})$ is closed with respect to the A-operation.

1.12.113. Prove that the classes of all Souslin and all Borel sets on the real line (or in the space $\mathbb{R}^n$) have cardinality of the continuum.

1.12.114. Let $(X, \mathcal{A}, \mu)$ be a space with a finite nonnegative measure μ such that there exists a set E that is not μ-measurable. Prove that there exists $\varepsilon > 0$ with the following property: if A and B are measurable, $E \subset A$, $X\backslash E \subset B$, then $\mu(A \cap B) \geq \varepsilon$.

HINT: assuming the converse one can find measurable sets A_n and B_n with $E \subset A_n$, $X\backslash E \subset B_n$, $\mu(A_n \cap B_n) < n^{-1}$; let $A = \bigcap_{n=1}^{\infty} A_n$, $B = \bigcap_{n=1}^{\infty} B_n$; then $E \subset A$, $X\backslash E \subset B$, $\mu(A \cap B) = 0$, whence one has $\mu^*(E) + \mu^*(X\backslash E) \leq \mu(X)$ and hence we obtain the equality $\mu^*(E) + \mu^*(X\backslash E) = \mu(X)$.

1.12.115. Construct an example of a separable probability measure μ on a σ-algebra $\mathcal{A}$ such that, for every countably generated σ-algebra $\mathcal{E} \subset \mathcal{A}$, the completion of $\mathcal{E}$ with respect to μ is strictly smaller than $\mathcal{A}$.

HINT: see Example 9.8.1 in Chapter 9.

1.12.116. (Zink [**1052**]) Let (X, S, μ) be a measure space with a complete atomless separable probability measure μ and let $\mu^*(E) > 0$. Then, there exist nonmeasurable sets E_1 and E_2 such that $E_1 \cap E_2 = \varnothing$, $E_1 \cup E_2 = E$ and one has $\mu^*(E_1) = \mu^*(E_2) = \mu^*(E)$.

1.12.117.° Let $\mathfrak{m}$ be a Carathéodory outer measure on a space X. Prove that a set A is Carathéodory measurable precisely when for all $B \subset A$ and $C \subset X\backslash A$ one has $\mathfrak{m}(B \cup C) = \mathfrak{m}(B) + \mathfrak{m}(C)$.

HINT: if A is Carathéodory measurable, then in the definition of measurability one can take $E = B \cup C$; if one has the indicated property, then an arbitrary set E can be written in the form $E = B \cup C$, $B = E \cap A$, $C = E\backslash A$.

1.12.118.° Suppose that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are outer measures on a space X. Show that $\max(\mathfrak{m}_1, \mathfrak{m}_2)$ is an outer measure too.

1.12.119.° (Young [**1029**]) Let $(X, \mathcal{A}, \mu)$ be a measure space with a finite nonnegative measure μ. Prove that a set $A \subset X$ belongs to $\mathcal{A}_\mu$ precisely when for each set B disjoint with A one has the equality $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

HINT: for the proof of sufficiency take $B = X\backslash A$; the necessity follows by the previous exercise.

1.12.120° Let $\mathfrak{m}$ be a Carathéodory outer measure on a space X. Prove that for any $E \subset X$ the function $\mathfrak{m}_E(B) = \mathfrak{m}(B \cap E)$ is a Carathéodory outer measure and all $\mathfrak{m}$-measurable sets are $\mathfrak{m}_E$-measurable.

1.12.121. Let τ be an additive, but not countably additive nonnegative set function that is defined on the class of all subsets of $[0,1]$ and coincides with Lebesgue measure on all Lebesgue measurable sets (see Example 1.12.29). Show that the corresponding outer measure $\mathfrak{m}$ from Example 1.11.5 is identically zero under the continuum hypothesis.

HINT: Theorem 1.11.8 yields the $\mathfrak{m}$-measurability of all sets, $\mathfrak{m}$ is countably additive on $\mathfrak{M}_{\mathfrak{m}}$ and $\mathfrak{m}(\{x\}) = 0$ for each x.

1.12.122. Prove that if $\mathfrak{X} \subset \mathfrak{M}_{\mathfrak{m}}$, then Method I from Example 1.11.5 gives a regular outer measure.

1.12.123. Let $\mathcal{S}$ be a collection of subsets of a set X, closed with respect to finite unions and finite intersections and containing the empty set, i.e., a lattice of sets (e.g., the class of all closed sets or the class of all open sets in $[0,1]$).

(i) Suppose that on $\mathcal{S}$ we have a modular set function m, i.e., $m(\varnothing) = 0$ and $m(A \cup B) + m(A \cap B) = m(A) + m(B)$ for all $A, B \in \mathcal{S}$. Show that by the equality $m(A\backslash B) = m(A) - m(B)$, $A, B \in \mathcal{S}$, $B \subset A$, the function m uniquely extends to an additive set function (which, in particular, is well-defined) on the semiring formed by the differences of elements in $\mathcal{S}$ (see Exercise 1.12.51), and then uniquely extends to an additive set function on the ring generated by $\mathcal{S}$.

(ii) Give an example showing that in (i) one cannot replace the modularity by the additivity even if m is nonnegative, monotone and subadditive on $\mathcal{S}$.

HINT: (i) use Exercise 1.12.51 and Proposition 1.3.10; in order to verify that m is well-defined we observe that if $A_1\backslash A_1' = A_2\backslash A_2'$, where $A_i, A_i' \in \mathcal{S}$, $A_i' \subset A_i$, then $m(A_1) + m(A_2') = m(A_2) + m(A_1')$ because $A_1 \cup A_2' = A_2 \cup A_1'$, $A_1 \cap A_2' = A_1' \cap A_2$, which is easily verified; see the details in Kelley, Srinivasan [**502**, Chapter 2, p. 23, Theorem 2]. (ii) Take $X = \{0, 1, 2\}$ and $\mathcal{S}$ consisting of X, $\varnothing$, $\{0, 1\}$, $\{1, 2\}$, $\{1\}$ with $m(X) = 2$, $m(\varnothing) = 0$ and $m = 1$ on all other sets in $\mathcal{S}$.

1.12.124. Suppose that $\mathcal{F}$ is a family of subsets of a set X, $\varnothing \in \mathcal{F}$. Let $\tau\colon \mathcal{F} \to [0, +\infty]$ be a set function with $\tau(\varnothing) = 0$. Let us define τ_* on all sets $A \subset X$ by formula (1.12.8).

(i) Prove that if $A_1, \ldots, A_n \subset X$ are disjoint sets and $A_1 \cup \cdots \cup A_n \subset A$, then one has $\tau_*(A) \geq \sum_{j=1}^n \tau_*(A_j)$.

(ii) Prove that τ_* coincides with τ on $\mathcal{F}$ if and only if, for all pairwise disjoint sets $F_1, \ldots, F_n \in \mathcal{F}$ and all $F \in \mathcal{F}$ with $\bigcup_{j=1}^n F_j \subset F$, one has $\tau(F) \geq \sum_{j=1}^n \tau(F_j)$.

(iii) Prove that if τ satisfies the condition in (ii) and the class $\mathcal{F}$ is closed with respect to finite unions of disjoint sets, then

$$\tau_*(A) = \sup\{\tau(F),\ F \in \mathcal{F},\ F \subset A\}, \quad \forall\, A \subset X.$$

HINT: (i) Let $\tau_*(A) < \infty$ and $\varepsilon > 0$. For every i, there exist disjoint sets $F_{ij} \in \mathcal{F}$, $j \leq n(i)$, such that $\bigcup_{j=1}^{n(i)} F_{ij} \subset A_i$ and $\tau_*(A_i) \leq \varepsilon 2^{-i} + \sum_{j=1}^{n(i)} \tau(F_{ij})$. All sets F_{ij} are pairwise disjoint and are contained in A. Therefore,

$$\sum_{i=1}^{n} \tau_*(A_i) \le \sum_{i=1}^{n} \varepsilon 2^{-i} + \sum_{i=1}^{n} \sum_{j=1}^{n(i)} \tau(F_{ij}) \le \varepsilon + \tau_*(A),$$

whence we obtain the claim, since ε is arbitrary.

(ii) Let $F_j, F \in \mathcal{F}$, $F_j \subset F$, where the sets F_j are pairwise disjoint. Then the inequality $\tau(F) \ge \sum_{j=1}^{n} \tau(F_j)$ yields the inequality $\tau(F) \ge \tau_*(F)$. Since the reverse inequality is obvious from the definition, we obtain the equality $\tau_* = \tau$ on $\mathcal{F}$. On the other hand, this equality obviously implies the indicated inequality.

(iii) Let $F_1, \ldots, F_n \in \mathcal{F}$ be disjoint sets and let $E := \bigcup_{j=1}^{n} F_j \subset A$. Then, by hypothesis, we have $E \in \mathcal{F}$ and $\sum_{j=1}^{n} \tau(F_j) \le \tau(E) \le \sup\bigl\{\tau(F)\colon\ F \in \mathcal{F}, F \subset A\bigr\}$, whence $\tau_*(A) \le \sup\bigl\{\tau(F)\colon\ F \in \mathcal{F}, F \subset A\bigr\}$; the reverse inequality is trivial.

1.12.125. Let $\mathcal{F}$ and τ be the same as in the previous exercise. (i) Prove that the outer measure τ^* coincides with τ on $\mathcal{F}$ precisely when $\tau(F) \le \sum_{n=1}^{\infty} \tau(F_n)$ whenever $F, F_n \in \mathcal{F}$ and $F \subset \bigcup_{n=1}^{\infty} F_n$.

(ii) Prove that if the condition in (i) is fulfilled and the class $\mathcal{F}$ is closed with respect to countable unions, then

$$\tau^*(A) = \inf\{\tau(F),\ F \in \mathcal{F},\ A \subset F\}, \quad \forall\, A \subset X.$$

Hint: the proof is similar to the reasoning in the previous exercise.

1.12.126. Suppose that $\mathcal{F}$ is a class of subsets of a space X, $\varnothing \in \mathcal{F}$. Let $\tau\colon \mathcal{F} \to [0, +\infty]$ be a set function with $\tau(\varnothing) = 0$. Prove that the following conditions are equivalent:

(i) τ^* coincides with τ on $\mathcal{F}$ and $\mathcal{F} \subset \mathfrak{M}_{\tau^*}$;

(ii) $\tau(A) = \tau^*(A \cap B) + \tau^*(A \backslash B)$ for all $A, B \in \mathcal{F}$.

Hint: (i) implies (ii) by the additivity of τ^* on $\mathfrak{M}_{\tau^*}$. Let (ii) be fulfilled. Letting $B = \varnothing$, we get $\tau(A) = \tau^*(A)$, $A \in \mathcal{F}$. Suppose that $F \in \mathcal{F}$ and $E \subset X$. Let $F_j \in \mathcal{F}$ and $E \subset \bigcup_{j=1}^{\infty} F_j$. Then

$$\sum_{j=1}^{\infty} \tau(F_j) = \sum_{j=1}^{\infty} \tau^*(F_j \cap F) + \sum_{j=1}^{\infty} \tau(F_j \backslash F) \ge \tau^*(E \cap F) + \tau^*(E \backslash F).$$

Taking the infimum over $\{F_j\}$, we obtain $\tau^*(E) \ge \tau^*(E \cap F) + \tau^*(E \backslash F)$, i.e., we have $F \in \mathfrak{M}_{\tau^*}$.

1.12.127. Suppose that $\mathcal{F}$ is a class of subsets of a space X, $\varnothing \in \mathcal{F}$. Let $\tau\colon \mathcal{F} \to [0, +\infty]$ be a set function with $\tau(\varnothing) = 0$. Denote by τ_* the corresponding inner measure (see formula (1.12.8)). Prove that the following conditions are equivalent:

(i) τ_* coincides with τ on $\mathcal{F}$ and $\mathcal{F} \subset \mathfrak{M}_{\tau_*}$;

(ii) $\tau(A) = \tau_*(A \cap B) + \tau_*(A \backslash B)$, $\forall\, A, B \in \mathcal{F}$.

Hint: the proof is completely analogous to the previous exercise, one has only take finitely many disjoint $F_j \subset A$; see also Glazkov [**360**], Hoffmann-Jørgensen [**440**, 1.26].

1.12.128. (i) Show that if in the situation of the previous exercise we have one of the equivalent conditions (i) and (ii), then on the algebra $\mathcal{A}_{\mathcal{F}}$ generated by $\mathcal{F}$, there exists an additive set function τ_0 that coincides with τ on $\mathcal{F}$.

(ii) Show that if, in addition to the hypotheses in (i), it is known that

$$\tau_*(F) \le \sum_{n=1}^{\infty} \tau_*(F_n) \quad \text{whenever } F, F_n \in \mathcal{A}_{\mathcal{F}} \text{ and } F \subset \bigcup_{n=1}^{\infty} F_n,$$

then there exists a countably additive measure μ on $\sigma(\mathcal{F})$ that coincides with τ on $\mathcal{F}$.

HINT: according to Theorem 1.11.4, the function τ_* is additive on $\mathfrak{M}_{\tau_*}$ and $\mathfrak{M}_{\tau_*}$ is an algebra. Since the algebra $\mathfrak{M}_{\tau_*}$ contains $\mathcal{F}$ by hypothesis, it also contains the algebra generated by $\mathcal{F}$. The second claim follows by the cited theorem, too.

1.12.129. Let $(X, \mathcal{A}, \mu)$ be a measure space, where $\mathcal{A}$ is a σ-algebra and μ is a countably additive measure with values in $[0, +\infty]$. Denote by $\mathfrak{L}_\mu$ the class of all sets $E \subset X$ for each of which there exist two sets $A_1, A_2 \in \mathcal{A}$ with $A_1 \subset E \subset A_2$ and $\mu(A_2 \backslash A_1) = 0$.

(i) Show that $\mathfrak{L}_\mu$ is a σ-algebra, coincides with $\mathcal{A}_\mu$ and belongs to $\mathfrak{M}_{\mu^*}$.

(ii) Show that if the measure μ is σ-finite, then $\mathfrak{L}_\mu$ coincides with $\mathfrak{M}_{\mu^*}$.

(iii) Let $X = [0, 1]$, let $\mathcal{A}$ be the σ-algebra generated by all singletons, and let the measure μ with values in $[0, +\infty]$ be defined as follows: $\mu(A)$ is the cardinality of A, $A \in \mathcal{A}$. Show that $\mathfrak{M}_{\mu^*}$ contains all sets, but $[0, 1/2] \notin \mathfrak{L}_\mu$.

HINT: (iii) show that $\mu^*(A)$ is the cardinality of A and that $\mathfrak{L}_\mu = \mathcal{A}$, by using that nonempty sets have measure at least 1.

1.12.130. Let us consider the following modification of Example 1.11.5. Let $\mathfrak{X}$ be a family of subsets of a set X such that $\varnothing \in \mathfrak{X}$. Suppose that we are given a function $\tau\colon \mathfrak{X} \to [0, +\infty]$ with $\tau(\varnothing) = 0$. Set

$$\widetilde{\mathfrak{m}}(A) = \inf\Bigl\{\sum_{n=1}^{\infty} \tau(X_n)\colon\ X_n \in \mathfrak{X}, A \subset \bigcup_{n=1}^{\infty} X_n\Bigr\}$$

if such sets X_n exist and otherwise let $\widetilde{\mathfrak{m}}(A) = \sup \widetilde{\mathfrak{m}}(A')$, where sup is taken over all sets $A' \subset A$ that can be covered by a sequence of sets in $\mathfrak{X}$.

(i) Show that $\widetilde{\mathfrak{m}}$ is an outer measure.

(ii) Let $X = [0,1]\times[0,1]$, $\mathfrak{X} = \bigl\{[a,b)\times t, a, b, t \in [0,1], a \le b\bigr\}$, $\tau([a,b)\times t) = b - a$. Let $\mathfrak{m}$ be given by formula (1.11.5). Show that $\mathfrak{m}$ and $\widetilde{\mathfrak{m}}$ do not coincide and that there exists a set $E \in \mathfrak{M}_{\mathfrak{m}} \cap \mathfrak{M}_{\widetilde{\mathfrak{m}}}$ such that $\mathfrak{m}(E) \ne \widetilde{\mathfrak{m}}(E)$.

HINT: (i) is verified similarly to the case of $\mathfrak{m}$; (ii) for E take the diagonal in the square.

1.12.131. Let μ be a measure with values in $[0, +\infty]$ defined on a measurable space $(X, \mathcal{A})$. The measure μ is called decomposable if there exists a partition of X into pairwise disjoint sets $X_\alpha \in \mathcal{A}$ of finite measure (indexed by elements α of some set Λ) with the following properties: (a) if $E \cap X_\alpha \in \mathcal{A}$ for all α, then $E \in \mathcal{A}$, (b) $\mu(E) = \sum_\alpha \mu(E \cap X_\alpha)$ for each set $E \in \mathcal{A}$, where convergence of the series $\sum_\alpha c_\alpha$, $c_\alpha \ge 0$, to a finite number s means by definition that among the numbers c_α at most countably many are nonzero and the corresponding series converges to s, and the divergence of such a series to $+\infty$ means the divergence of some of its countable subseries.

(i) Give an example of a measure that is not decomposable.

(ii) Show that a measure μ is decomposable precisely when there exists a partition of X into disjoint sets X_α of positive measure having property (a) and property (b'): if $A \in \mathcal{A}$ and $\mu(A \cap X_\alpha) = 0$ for all α, then $\mu(A) = 0$.

1.12.132. Let μ be a measure with values in $[0, +\infty]$ defined on a measurable space $(X, \mathcal{A})$. The measure μ is called semifinite if every set of infinite measure has a subset of finite positive measure.

(i) Give an example of a measure with values in $[0, +\infty]$ that is not semifinite.

(ii) Give an example of a semifinite measure that is not σ-finite.

(iii) Prove that for any measure μ with values in $[0, +\infty]$, defined on a σ-algebra $\mathcal{A}$, the formula $\mu_0(A) := \sup\{\mu(B)\colon\ B \subset A, B \in \mathcal{A}, \mu(B) < \infty\}$ defines a semifinite measure with values in $[0, +\infty]$ and μ is semifinite precisely when $\mu = \mu_0$.

(iv) Show that every decomposable measure is semifinite.

(v) Give an example of a semifinite measure μ with values in $[0, +\infty]$ that is defined on an algebra $\mathcal{A}$ and has infinitely many semifinite extensions to $\sigma(\mathcal{A})$.

Hint: (v) let $X = \mathbb{R}^1$, let $\mathcal{A}$ be the class of all finite sets and their complements, and let $\mu(A)$ be the cardinality (denoted Card) of $A \cap \mathbb{Q}$. For any $s \geq 0$ and $A \in \sigma(\mathcal{A})$, let $\mu_s(A) = \mathrm{Card}(A \cap \mathbb{Q})$ if $A \cap (\mathbb{R}^1 \backslash \mathbb{Q})$ is at most countable, $\mu_s(A) = s + \mathrm{Card}(A \cap \mathbb{Q})$ if $(\mathbb{R}^1 \backslash A) \cap (\mathbb{R}^1 \backslash \mathbb{Q})$ is at most countable.

1.12.133. Let μ be a measure μ with values in $[0, +\infty]$ defined on a measurable space $(X, \mathcal{A})$. A set E is called locally measurable if $E \cap A \in \mathcal{A}$ for every $A \in \mathcal{A}$ with $\mu(A) < \infty$. The measure μ is called saturated if every locally measurable set belongs to $\mathcal{A}$.

(i) Let $X = \mathbb{R}$, $\mathcal{A} = \{\mathbb{R}, \varnothing\}$, $\mu(\mathbb{R}) = +\infty$, $\mu(\varnothing) = 0$. Show that μ is a complete measure with values in $[0, +\infty]$ that is not saturated.

(ii) Show that every σ-finite measure is saturated.

(iii) Show that locally measurable sets form a σ-algebra.

(iv) Show that every measure with values in $[0, +\infty]$ can be extended to a saturated measure on the σ-algebra $\mathcal{L}$ of all locally measurable sets by the formula $\overline{\mu}(E) = \mu(E)$ if $E \in \mathcal{A}$, $\overline{\mu}(E) = +\infty$ if $E \notin \mathcal{A}$.

(v) Construct an example showing that $\overline{\mu}$ may not be a unique saturated extension of μ to the σ-algebra $\mathcal{L}$.

Hint: (i) observe that every set in X is locally measurable with respect to μ; (iii) use that $(X \backslash E) \cap A = A \backslash (A \cap E)$; (v) let $\mu_0(A) = 0$ if A is countable and $\mu_0(A) = \infty$ if A is uncountable; observe that μ_0 is saturated.

1.12.134. Let $(X, \mathcal{A}, \mu)$ be a measure space, where μ takes values in $[0, +\infty]$. The measure μ is called Maharam (or localizable) if μ is semifinite and each collection $\mathcal{M} \subset \mathcal{A}$ has the essential supremum in the following sense: there exists a set $E \in \mathcal{A}$ such that all sets $M \backslash E$, where $M \in \mathcal{M}$, have measure zero and if $E' \in \mathcal{A}$ is another set with such a property, then $E \backslash E'$ is a measure zero set.

(i) Prove that every decomposable measure is Maharam.

(ii) Give an example of a complete Maharam measure that is not decomposable.

Hint: (i) let the sets X_α, $\alpha \in \Lambda$, give a decomposition of the measure space $(X, \mathcal{A}, \mu)$ and $\mathcal{M} \subset \mathcal{A}$. Denote by $\mathcal{F}$ the family of all sets $F \in \mathcal{A}$ with $\mu(F \cap M) = 0$ for all $M \in \mathcal{M}$. It is clear that $\mathcal{F}$ contains the empty set and admits countable unions. For every α, let $c_\alpha := \sup\{\mu(F \cap X_\alpha), F \in \mathcal{F}\}$ and choose $F_{\alpha,n} \in \mathcal{F}$ such that $\lim\limits_{n\to\infty} \mu(F_{\alpha,n} \cap X_\alpha) = c_\alpha$. Let $F_\alpha := \bigcup_{n=1}^\infty F_{\alpha,n}$ and $\Psi := \bigcup_{\alpha\in\Lambda} (F_\alpha \cap X_\alpha)$. Then $\Psi \cap X_\alpha = F_\alpha$ and hence $\Psi \in \mathcal{A}$. Therefore, $E := X \backslash \Psi \in \mathcal{A}$. For any $M \in \mathcal{M}$ we have

$$\mu(M \backslash E) = \mu(M \cap \Psi) = \sum_\alpha \mu(M \cap \Psi \cap X_\alpha) = \sum_\alpha \mu(M \cap F_\alpha \cap X_\alpha) = 0$$

by the definition of $\mathcal{F}$. If E' is another set with such a property, then $X\backslash E' \in \mathcal{F}$ and $\Psi' := \Psi \cup (X\backslash E') \in \mathcal{F}$. Now it is readily shown that $\mu(\Psi \cap X_\alpha) = \mu(\Psi' \cap X_\alpha)$ for all α, whence $\mu\bigl((\Psi'\backslash\Psi) \cap X_\alpha\bigr) = 0$, i.e., $\mu(\Psi'\backslash\Psi) = 0$ and $\mu(E\backslash E') = 0$. (ii) Examples with various additional properties can be found in Fremlin [**327**, §216].

1.12.135. A measure with values in $[0, +\infty]$ is called locally determined if it is semifinite and saturated. Let μ be a measure with values in $[0, +\infty]$ defined on a measurable space $(X, \mathcal{A})$. Let $\mathcal{L}_\mu$ be the σ-algebra of locally $\mathcal{A}_\mu$-measurable sets, i.e., all sets L such that $L \cap A \in \mathcal{A}_\mu$ for all $A \in \mathcal{A}_\mu$ with $\mu(A) < \infty$. Let

$$\widetilde{\mu}(L) = \sup\{\mu(L \cap A)\colon\ A \in \mathcal{A}_\mu, \mu(A) < \infty\}, \quad L \in \mathcal{L}_\mu.$$

(i) Show that the measure $\widetilde{\mu}$ is locally determined and complete and that one has $\widetilde{\mu}(A) = \mu(A)$ whenever $A \in \mathcal{A}_\mu$ and $\mu(A) < \infty$.

(ii) Show that if μ is decomposable, then so is $\widetilde{\mu}$ and in this case $\widetilde{\mu}$ coincides with the completion of μ.

(iii) Show that if μ is Maharam, then so is $\widetilde{\mu}$.

(iv) Show that the measure μ is complete and locally determined precisely when one has $\mu = \widetilde{\mu}$.

Hint: the detailed verification of these simple assertions can be found, e.g., in Fremlin [**327**].

1.12.136. Let $(X, \mathcal{A})$ be a measurable space and let a measure μ on $\mathcal{A}$ with values in $[0, +\infty]$ be complete and locally determined. Suppose that there exists a family $\mathcal{D}$ of pairwise disjoint sets of finite measure in $\mathcal{A}$ such that if $E \in \mathcal{A}$ and $\mu(E \cap D) = 0$ for all $D \in \mathcal{D}$, then $\mu(E) = 0$. Prove that the measure μ is decomposable.

Hint: see Fremlin [**327**, §213O].

1.12.137. Let X be a set of cardinality of the continuum and let Y be a set of cardinality greater than that of the continuum. For every $E \subset X \times Y$, the sets $\{(a, y) \in E\}$ with fixed $a \in X$ will be called vertical sections of E, and the sets $\{(x, b) \in E\}$ with fixed $b \in Y$ will be called horizontal sections of E. Denote by $\mathcal{A}$ the class of all sets $A \subset X \times Y$ such that all their horizontal and vertical sections are either at most countable or have at most countable complements in the corresponding sections of $X \times Y$. Let $\gamma(A)$ be the number of those horizontal sections of the complement of A that are at most countable. Similarly, by means of vertical sections we define the function $v(A)$. Let $\mu(A) = \gamma(A) + v(A)$.

(i) Prove that $\mathcal{A}$ is a σ-algebra and that γ, v, and μ are countably additive measures with values in $[0, +\infty]$.

(ii) Prove that μ is semifinite in the sense of Exercise 1.12.132.

(iii) Prove that μ is not decomposable in the sense of Exercise 1.12.131.

Hint: (ii) if $(X \times Y)\backslash A$ has infinite number of finite or countable horizontal sections, then, given $N \in \mathbb{N}$, one can take points $y_1, \ldots, y_N \in Y$, giving such sections; let us take the set B such that the horizontal sections of its complement at the points y_i coincide with the corresponding sections of the complement of A, and all other sections of the complement of B coincide with $X \times y$; then $B \subset A$ and $\gamma(B) = N$, $v(B) = 0$. (iii) If sets E_α give a partition of $X \times Y$ and $\mu(E_\alpha) < \infty$, then the cardinality of this family of sets cannot be smaller than that of Y. Indeed, otherwise, since E_α is contained in a finite union of sets of the form $a \times Y$ and $X \times b$, one would find a set $X \times y$ whose intersection with every E_α is a set with the uncountable complement in $X \times y$, whence $\mu\bigl((X \times y) \cap E_\alpha\bigr) = 0$ for all α, but we have $\mu(X\times y)=1$. On the other hand, for every $x\in X$, there is a unique set E_{α_x} with $\mu\bigl((x\times Y)\cap E_{\alpha_x}\bigr)=1$, and since the complement of $(x\times Y)\cap E_{\alpha_x}$ in $x\times Y$ is at most countable, the set $x\times Y$ meets at most countably many sets E_α. Hence the cardinality of the family $\{E_\alpha\}$ is that of the continuum, which is a contradiction.

1.12.138. Let $X=[0,1]\times\{0,1\}$ and let $\mathcal{A}$ be the class of all sets $E\subset X$ such that the sections $E_x:=\{y\colon (x,y)\in E\}$ are either empty or coincide with $\{0,1\}$ for all x, excepting possibly the points of an at most countable set. Show that $\mathcal{A}$ is a σ-algebra and the function μ that to every set E assigns the cardinality of the intersection of E with the first coordinate axis, is a complete and semifinite countably additive measure with values in $[0,+\infty]$, but the measure generated by the outer measure μ^* is not semifinite.

1.12.139. (Luther [**639**]) Let μ be a measure with values in $[0,+\infty]$ defined on a ring $\mathcal{R}$, let $\overline{\mu}$ be the restriction of μ^* to the σ-ring $\mathcal{S}$ generated by $\mathcal{R}$, and let $\mathcal{R}_0$ and $\mathcal{S}_0$ be the subclasses in $\mathcal{R}$ and $\mathcal{S}$ consisting of all sets of finite measure. Set

$$\widetilde{\mu}(E)=\limsup\{\overline{\mu}(P\cap E), P\in\mathcal{R}_0\},\ \ E\in\mathcal{S}.$$

(i) Prove that the following conditions are equivalent: (a) μ is semifinite, (b) $\widetilde{\mu}$ is an extension of μ to $\mathcal{S}$, (c) any measure ν on $\mathcal{S}$ with values in $[0,+\infty]$ that agrees with μ on $\mathcal{R}_0$ coincides with μ on $\mathcal{R}$.

(ii) Show that any measure ν on $\mathcal{S}$ with values in $[0,+\infty]$ that agrees with μ on $\mathcal{R}_0$, coincides with $\widetilde{\mu}$ and $\overline{\mu}$ on $\mathcal{S}_0$, and that $\widetilde{\mu}\le\nu\le\overline{\mu}$ on $\mathcal{S}$.

(iii) Prove that the following conditions are equivalent: (a) $\overline{\mu}$ is semifinite, (b) μ is semifinite and has a unique extension to $\mathcal{S}$, (c) $\widetilde{\mu}=\overline{\mu}$, (d) for all $E\in\mathcal{S}$ one has $\overline{\mu}(E)=\limsup\{\overline{\mu}(P\cap E), P\in\mathcal{R}_0\}$.

(iv) Prove that if the measure $\overline{\mu}$ is σ-finite, then μ has a unique extension to $\mathcal{S}$.

(v) Give an example showing that in (iv) it is not sufficient to require the existence of some σ-finite extension of μ.

1.12.140. (Luther [**640**]) Let μ be a measure with values in $[0,+\infty]$ defined on a σ-ring $\mathcal{R}$. Prove that $\mu=\mu_1+\mu_2$, where μ_1 is a semifinite measure on $\mathcal{R}$, the measure μ_2 can assume only the values 0 and ∞, and in every set $R\in\mathcal{R}$ there exists a subset $R'\in\mathcal{R}$ such that $\mu_1(R')=\mu_1(R)$ and $\mu_2(R')=0$.

1.12.141. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two algebras of subsets of Ω and let μ_1,μ_2 be two additive real functions on $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively (or μ_1,μ_2 take values in the extended real line and vanish at $\varnothing$). (a) Show that the equality $\mu_1(E)=\mu_2(E)$ for all $E\in\mathcal{E}_1\cap\mathcal{E}_2$ is necessary and sufficient for the existence of an additive function μ that extends μ_1 and μ_2 to some algebra $\mathcal{F}$ containing $\mathcal{E}_1$ and $\mathcal{E}_2$. (b) Show that if $\mu_1,\mu_2\ge 0$, then the existence of a common nonnegative extension μ is equivalent to the following relations: $\mu_1(C)\ge\mu_2(D)$ for all $C\in\mathcal{E}_1$, $D\in\mathcal{E}_2$ with $D\subset C$ and $\mu_1(E)\le\mu_2(F)$ for all $E\in\mathcal{E}_1$, $F\in\mathcal{E}_2$ with $E\subset F$.

Hint: see Rao, Rao [**786**, §3.6, p. 82].

1.12.142. Let $(X,\mathcal{A},\mu)$ be a probability space and let μ^* be the corresponding outer measure. For a set $E\subset X$, we denote by $\mathfrak{m}_E$ the restriction of μ^* to the class of all subsets of E. Show that $\mathfrak{m}_E$ coincides with the outer measure on the space E generated by the restriction μ_E of μ to E in the sense of Definition 1.12.11. In particular, $\mathfrak{m}_E$ is a regular Carathéodory outer measure.

Hint: let $\widetilde{E}$ be a measurable envelope of E; for any set $B\subset E$ one has

$$\mathfrak{m}_E(B)=\inf\{\mu(A)\colon\ A\in\mathcal{A}, B\subset A\}.$$

By the definition of μ_E we have

$$\mu_E^*(B) = \inf\{\mu_E(C)\colon\ C \in \mathcal{A}_E, B \subset C\} = \inf\{\mu(A \cap \widetilde{E})\colon\ A \in \mathcal{A}, B \subset A \cap E\}.$$

Clearly, one has $\mathfrak{m}_E(B) \geq \mu_E^*(B)$. On the other hand, given $\varepsilon > 0$, we find a set $A_\varepsilon \in \mathcal{A}$ such that $\mu(A_\varepsilon \cap \widetilde{E}) < \mu_E^*(B) + \varepsilon$. Hence $\mu(A_\varepsilon) < \mu_E^*(B) + \varepsilon$ and $B \subset A_\varepsilon$, which yields the estimate $\mathfrak{m}_E(B) \leq \mu_E^*(B) + \varepsilon$. Hence $\mathfrak{m}_E(B) \leq \mu_E^*(B)$.

1.12.143. Suppose that μ is a measure with values in $[0, +\infty]$ on a measurable space $(X, \mathcal{A})$. Let μ^* and μ_* be the corresponding outer and inner measures and let $\mathfrak{m} := (\mu^* + \mu_*)/2$.

(i) (Carathéodory [**164**, p. 693]) Show that $\mathfrak{m}$ is a Carathéodory outer measure. Denote by ν the measure generated by $\mathfrak{m}$.

(ii) Let $X = \{0, 1\}$, $\mathcal{A} = \{X, \varnothing\}$, $\mu(X) = 1$. Show that $\mu \neq \nu$.

(iii) (Fremlin [**324**]) Prove that if μ is Lebesgue measure on $[0, 1]$, then $\mu = \nu$.

1.12.144. Let $\mathfrak{m}$ be a Carathéodory outer measure on a space X and let $\varphi\colon\ [0, +\infty] \to [0, +\infty)$ be a bounded concave function such that $\varphi(0) = 0$ and $\varphi(t) > 0$ if $t \neq 0$. Let $d(A, B) = \varphi\bigl(\mathfrak{m}(A \bigtriangleup B)\bigr)$, $A, B \in \mathfrak{M}_{\mathfrak{m}}$. Denote by $\widetilde{\mathfrak{M}}_\mu$ the factor-space of the space $\mathfrak{M}_{\mathfrak{m}}$ by the ring of $\mathfrak{m}$-zero sets. Show that $(\widetilde{\mathfrak{M}}_\mu, d)$ is a complete metric space.

1.12.145. (Steinhaus [**910**]) Let E be a set of positive measure on the real line. Prove that, for every finite set F, the set E contains a subset similar to F, i.e., having the form $c + tF$, where $t \neq 0$.

1.12.146. (i) Let μ be an atomless probability measure on a measurable space $(X, \mathcal{A})$. Show that every point $x \in X$ belongs to $\mathcal{A}_\mu$ and has μ-measure zero.

(ii) (Marczewski [**651**]) Prove that if a probability measure μ on a measurable space $(X, \mathcal{A})$ is atomless, then there exist nonempty sets of μ-measure zero.

Hint: (i) let us fix a point $x \in X$ and take its measurable envelope E. Then $\mu(E) = 0$. Indeed, if $c = \mu(E) > 0$, we find a set $A \in \mathcal{A}$ such that $A \subset E$ and $\mu(A) = c/2$, which is possible since μ is atomless. Then either $x \in A$ or $x \in E \backslash A$ and $\mu(A) = \mu(E \backslash A) = c/2$, which contradicts the fact that E is a measurable envelope of x. Alternatively, one can use the following fact that will be established in §9.1 of Chapter 9: there exists a function f from X to $[0, 1]$ such that for every $t \in [0, 1]$ one has $\mu(x\colon\ f(x) < t) = t$. It follows that for every $t \in [0, 1]$ the set $f^{-1}(t)$ has μ-measure zero. Assertion (ii) easily follows. Moreover, by the second proof, there exists an uncountable set of μ-measure zero.

1.12.147. (Kindler [**517**]) Let $\mathcal{S}$ be a family of subsets of a set Ω with $\varnothing \in \mathcal{S}$ and let $\alpha, \beta\colon\ \mathcal{S} \to (-\infty, +\infty]$ be two set functions vanishing at $\varnothing$. Prove that the following conditions are equivalent:

(i) there exists an additive set function μ on the set of all subsets of Ω taking values in $(-\infty, +\infty]$ and satisfying the condition $\alpha \leq \mu|_{\mathcal{S}} \leq \beta$;

(ii) if $A_i, B_j \in \mathcal{S}$ and $\sum_{i=1}^n I_{A_i} = \sum_{j=1}^m I_{B_j}$, then $\sum_{i=1}^n \alpha(A_i) \leq \sum_{j=1}^m \beta(B_j)$.

1.12.148. Prove Proposition 1.12.36. Moreover, show that there is a non-negative additive function α on the set of all subsets of X with $\alpha|_{\mathfrak{R}} \leq \beta$ and $\alpha(X) = \beta(X)$.

Hint: (a) by induction on n we prove the following fact: if $R_1, \ldots, R_n \in \mathfrak{R}$, then there are $R_1', \ldots, R_n' \in \mathfrak{R}$ such that $R_1' \subset R_2' \subset \ldots \subset R_n'$, $\sum_{i=1}^n I_{R_i} = \sum_{i=1}^n I_{R_i'}$ and $\sum_{i=1}^n \beta(R_i) \geq \sum_{i=1}^n \beta(R_i')$. For the inductive step to $n+1$, given $R_1, \ldots, R_{n+1} \in \mathfrak{R}$, set $S_{n+1} = R_{n+1}$ and use the inductive hypothesis to find $S_1, \ldots, S_n \in \mathfrak{R}$ such that $S_1 \subset \ldots \subset S_n$, $\sum_{i=1}^n I_{R_i} = \sum_{i=1}^n I_{S_i}$ and $\sum_{i=1}^n \beta(R_i) \geq \sum_{i=1}^n \beta(S_i)$. Now set $S_n' = S_{n+1} \cap S_n$, $S_i' = S_i$ for $i < n$. There are $R_1', \ldots, R_n' \in \mathfrak{R}$ such that $R_1' \subset R_2' \subset \ldots \subset R_n'$, $\sum_{i=1}^n I_{S_i'} = \sum_{i=1}^n I_{R_i'}$ and $\sum_{i=1}^n \beta(S_i') \geq \sum_{i=1}^n \beta(R_i')$. Let $R_{n+1}' = S_{n+1}' = S_n \cup S_{n+1}$. Then $S_i, S_i', R_i' \in \mathfrak{R}$. As $I_{R_n'} \leq \sum_{i=1}^n I_{S_i'}$, one has $R_n' \subset \bigcup_{i=1}^n S_i' \subset S_n \subset R_{n+1}'$. In addition,

$$\sum_{i=1}^{n+1} I_{R_i'} = \sum_{i=1}^{n} I_{S_i'} + I_{S_{n+1}'} = \sum_{i=1}^{n-1} I_{S_i} + I_{S_n \cap S_{n+1}} + I_{S_n \cup S_{n+1}} = \sum_{i=1}^{n+1} I_{S_i} = \sum_{i=1}^{n+1} I_{R_i}.$$

Finally,

$$\begin{aligned}\sum_{i=1}^{n+1} \beta(R_i') &\leq \sum_{i=1}^{n} \beta(S_i') + \beta(S_{n+1}') = \sum_{i=1}^{n-1} \beta(S_i) + \beta(S_n \cap S_{n+1}) + \beta(S_n \cup S_{n+1}) \\ &\leq \sum_{i=1}^{n-1} \beta(S_i) + \beta(S_n) + \beta(S_{n+1}) = \sum_{i=1}^{n} \beta(S_i) + \beta(S_{n+1}) \leq \sum_{i=1}^{n+1} \beta(R_i).\end{aligned}$$

(b) We may assume that $\beta(X) = 1$. Let us show that if $R_1, \ldots, R_n \in \mathfrak{R}$ are such that $\sum_{i=1}^n I_{R_i}(x) \geq m$ for all x, where $m \in \mathbb{N}$, then $\sum_{i=1}^n \beta(R_i) \geq m$. Let R_i' be as in (a). It suffices to verify our claim for the sets R_i'. As $R_i' \subset R_{i+1}'$, one has $R_n' = \cdots = R_{n-m+1}' = X$. Hence $\beta(R_j') = 1$ for $j \geq n+m-1$.

(c) On the linear space L of finitely valued functions on X we set

$$p(f) = \inf\Bigl\{\sum_{i=1}^n \alpha_i \beta(R_i)\colon\ R_i \in \mathfrak{R}, \alpha_i \geq 0, f \leq \sum_{i=1}^n \alpha_i I_{R_i}\Bigr\}.$$

It is readily verified that $p(f+g) \leq p(f) + p(g)$ and $p(\alpha f) = \alpha p(f)$ for all $f, g \in L$, $\alpha \geq 0$. In addition, $p(1) \geq 1$. Indeed, otherwise we can find $R_i \in \mathfrak{R}$ and $\alpha_i \geq 0$, $i = 1, \ldots, n$, of the form $\alpha_i = n_i/m$, where $n_i, m \in \mathbb{N}$, such that $\sum_{i=1}^n \alpha_i \beta(R_i) < 1$. Set $M := \{(i,j)\colon\ 1 \leq i \leq n, 1 \leq j \leq n_i\}$ and $R_{ij} = R_i$ if $(i,j) \in M$. Then

$$\sum_{(i,j)\in M} I_{R_{ij}} = \sum_{i=1}^n n_i I_{R_i} = m \sum_{i=1}^n \alpha_i I_{R_i} \geq m,$$

but

$$\sum_{(i,j)\in M} \beta(R_{ij}) = \sum_{i=1}^n n_i \beta(R_i) = m \sum_{i=1}^n \alpha_i \beta(R_i) < m,$$

which contradicts (b). By the Hahn–Banach theorem, there is a linear functional λ on L such that $\lambda(1) = p(1) \geq 1$ and $\lambda \leq p$. Let $\nu(E) := \lambda(I_E)$, $E \subset X$. Then $\nu(E) \leq \beta(R)$ if $E \subset R \in \mathfrak{R}$. Let $\alpha(E) := \nu^+(E) := \sup_{A\subset E} \nu(E)$. Then α is nonnegative and additive (see Proposition 3.10.16 in Ch. 3) and $\alpha(R) \leq \beta(R)$ if $R \in \mathfrak{R}$. Finally, $1 \leq \nu(X) \leq \alpha(X) \leq \beta(X) = 1$.

1.12.149. Let $(X, \mathcal{A}, \mu)$ be a probability space and let $\mathcal{S}$ be a family of subsets in X such that $\mu_*\bigl(\bigcup_{n=1}^\infty S_n\bigr) = 0$ for every countable collection $\{S_n\} \subset \mathcal{S}$. Prove that there exists a probability measure $\widetilde{\mu}$ defined on some σ-algebra $\widetilde{\mathcal{A}}$ such that $\mathcal{A}, \mathcal{S} \subset \widetilde{\mathcal{A}}$, $\widetilde{\mu}$ extends μ and vanishes on $\mathcal{S}$, and for each $A \in \widetilde{\mathcal{A}}$ there exists $A' \in \mathcal{A}$ with $\widetilde{\mu}(A \triangle A') = 0$.

HINT: let $\mathcal{Z}$ be the class of all subsets in X that can be covered by an at most countable subfamily in $\mathcal{S}$. It is clear that $\mu_*(Z) = 0$ if $Z \in \mathcal{Z}$. Let

$$\widetilde{\mathcal{A}} := \{A \bigtriangleup Z, A \in \mathcal{A}, Z \in \mathcal{Z}\}.$$

It is easily seen that $\widetilde{\mathcal{A}}$ is a σ-algebra and contains $\mathcal{A}$ and $\mathcal{S}$. Set $\widetilde{\mu}(A \bigtriangleup Z) := \mu(A)$ for $A \in \mathcal{A}$ and $Z \in \mathcal{Z}$. The definition is unambiguous because if $A \bigtriangleup Z = A' \bigtriangleup Z'$, $A, A' \in \mathcal{A}$, $Z, Z' \in \mathcal{Z}$, then $A \bigtriangleup A' = Z \bigtriangleup Z'$, whence $\mu(A \bigtriangleup A') = \mu_*(Z \bigtriangleup Z') = 0$, since $Z \bigtriangleup Z' \in \mathcal{Z}$. Note that $\widetilde{\mu}(Z) = 0$ for $Z \in \mathcal{Z}$, since one can take $A = \varnothing$. The countable additivity of $\widetilde{\mu}$ is easily verified.

1.12.150.° Let μ be a bounded nonnegative measure on a σ-algebra $\mathcal{A}$ in a space X. Denote by $\mathcal{E}$ the class of all sets $E \subset X$ such that

$$\mu^*(E) = \mu^*(E\backslash A) + \mu^*(E \cap A) \quad \text{for all } A \in \mathcal{A}.$$

Is it true that the function μ^* is additive on $\mathcal{E}$?

HINT: no. Let us consider the following example due to O.V. Pugachev. Let $X = \{1, -1, i, -i\}$. We define a measure μ on a σ-algebra $\mathcal{A}$ consisting of eight sets as follows:

$$\mu(\varnothing) = 0, \quad \mu(X) = 3,$$

$$\mu(1) = \mu(-1) = \mu(\{i, -i\}) = 1, \quad \mu(\{1, -1\}) = \mu(\{1, i, -i\}) = \mu(\{-1, i, -i\}) = 2.$$

Clearly, the domain of definition of μ is indeed a σ-algebra. It is easily seen that μ is additive, hence countably additive. For every $E \subset X$, we have

$$\mu^*(E) = \mu^*(E\backslash A) + \mu^*(E \cap A)$$

for all $A \in \mathcal{A}$, but μ^* is not additive on the algebra of all subsets in X.

1.12.151. (Radó, Reichelderfer [**777**, p. 260]) Let Φ be a finite nonnegative set function defined on the family $\mathcal{U}$ of all open sets in $(0, 1)$ such that:

(i) $\Phi\bigl(\bigcup_{n=1}^\infty U_n\bigr) = \sum_{n=1}^\infty \Phi(U_n)$ for every countable family of pairwise disjoint sets $U_n \in \mathcal{U}$,

(ii) $\Phi(U_1) \le \Phi(U_2)$ whenever $U_1, U_2 \in \mathcal{U}$ and $U_1 \subset U_2$,

(iii) $\Phi(U) = \lim\limits_{\varepsilon\to 0} \Phi(U_\varepsilon)$ for every $U \in \mathcal{U}$, where U_ε is the set of all points in U with distance more than ε from the boundary of U.

Is it true that Φ has a countably additive extension to the Borel σ-algebra of $(0, 1)$?

HINT: no; let $\Phi(U) = 1$ if $[1/4, 1/2] \subset U$ and $\Phi(U) = 0$ otherwise.

1.12.152. Let μ be a nonnegative σ-finite measure on a measurable space $(X, \mathcal{A})$ and let M_0 be the class of all sets of finite μ-measure. Let

$\sigma_\mu(A, B) = \mu(A \bigtriangleup B)/\mu(A \cup B)$ if $\mu(A \cup B) > 0$, $\sigma_\mu(A, B) = 0$ if $\mu(A \cup B) = 0$.

(i) (Marczewski, Steinhaus [**653**]) (a) Show that σ_μ is a metric on the space of equivalence classes in M_0, where $A \sim B$ whenever $\mu(A \bigtriangleup B) = 0$.

(b) Show that if $A_n, A \in M_0$ and $\sigma_\mu(A_n, A) \to 0$, then $\mu(A_n \bigtriangleup A) \to 0$.

(c) Show that if $\mu(A_n \bigtriangleup A) \to 0$ and $\mu(A) > 0$, then $\sigma_\mu(A_n, A) \to 0$.

(d) Observe that $\sigma_\mu(\varnothing, B) = 1$ if $\mu(B) > 0$ and deduce that in the case of Lebesgue measure on $[0, 1]$, the identity mapping $(M_0, d) \to (M_0, \sigma_0)$, where d is the Fréchet–Nikodym metric, is discontinuous at the point corresponding to $\varnothing$.

(ii) (Gładysz, Marczewski, Ryll-Nardzewski [**359**]) For all $A_1, \ldots, A_n \in M_0$ let

$$\sigma_\mu(A_1, \ldots, A_n) = \frac{\mu\bigl((A_1 \cup \cdots \cup A_n)\backslash(A_1 \cap \cdots \cap A_n)\bigr)}{\mu(A_1 \cup \cdots \cup A_n)}$$

if $\mu(A_1\cup\cdots\cup A_n)>0$ and $\sigma_\mu(A_1,\ldots,A_n)=0$ if $\mu(A_1\cup\cdots\cup A_n)=0$. Prove the inequality

$$\sigma_\mu(A_1,\ldots,A_n)\le\frac{1}{n-1}\sum_{i<j}\sigma_\mu(A_i,A_j).$$

Deduce that if $\sigma_\mu(A_i,A_j)<2/n$ for all $1\le i<j\le n$, then $\mu(A_1\cap\cdots\cap A_n)>0$.

1.12.153.° Let $A_1,\ldots,A_n$ be measurable sets in a probability space $(\Omega,\mathcal{A},P)$. Prove that

$$0\le\sum_{i=1}^n P(A_i)-P\Bigl(\bigcup_{i=1}^n A_i\Bigr)\le\sum_{1\le i<j\le n}P(A_i\cap A_j).$$

HINT: by using induction on n and the easily verified fact that A_n is the union of the disjoint sets $B_1:=\bigl(\bigcup_{i=1}^n A_i\bigr)\backslash\bigl(\bigcup_{i=1}^{n-1} A_i\bigr)$ and $B_2:=\bigcup_{i=1}^{n-1}(A_i\cap A_n)$ we obtain

$$\begin{aligned}\sum_{i=1}^n P(A_i)-P\Bigl(\bigcup_{i=1}^n A_i\Bigr)&=\sum_{i=1}^{n-1} P(A_i)-P\Bigl(\bigcup_{i=1}^{n-1} A_i\Bigr)+P(A_n)-P(B_1)\\&\le\sum_{1\le i<j\le n-1}P(A_i\cap A_j)+P(B_2).\end{aligned}$$

It remains to observe that $P(B_2)\le\sum_{i=1}^{n-1}P(A_i\cap A_n)$. More general inequalities of this type are considered in Galambos, Simonelli [**336**].

1.12.154. (Darji, Evans [**203**]) Let A be a measurable set in the unit cube I of $\mathbb{R}^n$, let $F\subset I\backslash A$ be a finite set, and let $\varepsilon>0$. Show that there exists a finite set $S\subset A$ with the following property: for every partition $\mathcal{P}$ of the cube I into finitely many parallelepipeds of the form $[a_i,b_i]\times\cdots\times[a_n,b_n]$ with pairwise disjoint interiors, letting $B:=\bigcup\{P\in\mathcal{P}\colon\ P\cap F\ne\varnothing, P\cap S=\varnothing\}$ we have $\lambda_n(A\cap B)<\varepsilon$.

1.12.155. (Kahane [**479**]) Let E be the set of all points in $[0,1]$ of the form $x=3\sum_{n=1}^\infty\varepsilon_n4^{-n}$, $\varepsilon_n\in\{0,1\}$. Show that $E+\dfrac{1}{2}E=[0,3/2]$, but for almost all real λ, the set $E+\lambda E$ has measure zero.

1.12.156. Multivariate distribution functions admit the following characterization. For any vectors $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$ let

$$[x,y):=[x_1,y_1)\times\cdots\times[x_n,y_n).$$

Given a function F on $\mathbb{R}^n$ let $F[x,y):=\sum_u s(u)F(u)$, where the summation is taken over all corner points u of the set $[x,y)$ and $s(u)$ equals $+1$ or -1 depending on whether the number of indices k with $u_k=y_k$ is even or odd. Prove that the function F on $\mathbb{R}^n$ is the distribution function of some probability measure precisely when the following conditions are fulfilled: 1) $F[x,y)\ge0$ whenever $x<y$ coordinatewise, 2) $F(x^j)\to F(x)$ whenever the vectors x^j increase to x, 3) $F(x)\to0$ as $\max_k x_k\to-\infty$ and $F(x)\to1$ as $\min_k x_k\to+\infty$.

HINT: see Vestrup [**976**, §2.3, 2.4].

1.12.157. Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{N}$. Show that $\mathcal{A}$ is generated by some finite or countable partition of $\mathbb{N}$ into disjoint sets, so that every element of $\mathcal{A}$ is an at most countable union of elements of this partition.

HINT: let $n\sim m$ if n and m cannot be separated by a set from $\mathcal{A}$. It is readily verified that we obtain an equivalence relation. Every equivalence class K is an element of $\mathcal{A}$. Indeed, let us fix some $k\in K$. For every $n\in\mathbb{N}\backslash K$, there is a set

$A_n \in \mathcal{A}$ such that $k \in A_n$, $n \notin A_n$. Then $K = \bigcap_{n=1}^{\infty} A_n$. Indeed, $\bigcap_{n=1}^{\infty} A_n \subset K$ by construction. On the other hand, if $l \in K$ and $l \notin \bigcap_{n=1}^{\infty} A_n$, then k is separated from l by the set $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$. Hence we obtain an at most countable family of disjoint sets $M_n \in \mathcal{A}$ with union $\mathbb{N}$ such that every element of $\mathcal{A}$ is a finite or countable union of some of these sets.

1.12.158. (i) Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{N}$ and let μ be a probability measure on $\mathcal{A}$. Show that μ extends to a probability measure on the class of all subsets of $\mathbb{N}$.

(ii) Let $\mathcal{A}$ be the σ-algebra generated by singletons of a set X and let $\mathcal{A}_0$ be its sub-σ-algebra. Show that any measure μ on $\mathcal{A}_0$ extends to a measure on $\mathcal{A}$.

Hint: (i) apply Exercise 1.12.157 (cf. Hanisch, Hirsch, Renyi [**406**]; the result also follows as a special case of extension of measures on Souslin spaces, which is considered in Volume 2). (ii) Observe that μ is concentrated at countably many atoms, and any atom is either countable or has a countable complement.

1.12.159. Let μ be a countably additive measure with values in $[0, +\infty]$ on a ring $\mathfrak{X}$ of subsets of a space X.

(i) Suppose that μ is σ-finite, i.e., $X = \bigcup_{n=1}^{\infty} X_n$, where one has $X_n \in \mathfrak{X}$ and $\mu(X_n) < \infty$. Show that μ has a unique countably additive extension to the σ-ring $\Sigma(\mathfrak{X})$ generated by $\mathfrak{X}$.

(ii) Suppose that the measure $\mathfrak{m} := \mu^*$ is σ-finite on $\mathfrak{X}_{\mathfrak{m}}$. Show that it is a unique extension of μ to $\sigma(\mathfrak{X})$.

Hint: (i) according to Corollary 1.11.9, μ^* is a countably additive extension of μ to $\Sigma(\mathfrak{X})$ (even to $\sigma(\mathfrak{X})$). Let ν be another countably additive extension of μ to $\Sigma(\mathfrak{X})$. We show that $\mu^* = \nu$ on $\Sigma(\mathfrak{X})$. Let $E \in \Sigma(\mathfrak{X})$. We may assume that $X_n \subset X_{n+1}$. It suffices to show that $\mu^*(E \cap X_n) = \nu(E \cap X_n)$ for every n. This follows by the uniqueness result in the case of algebras because it is readily seen that the set $E \cap X_n$ belongs to the σ-algebra generated by the intersections of sets in $\mathfrak{X}$ with X_n. (ii) See Vulikh [**1000**, Ch. IV, §5].

1.12.160. Two sets A and B on the real line are called metrically separated if, for every $\varepsilon > 0$, there exist open sets A_ε and B_ε such that $A \subset A_\varepsilon$ and $B \subset B_\varepsilon$ with $\lambda(A_\varepsilon \cap B_\varepsilon) < \varepsilon$, where λ is Lebesgue measure.

(i) Show that if sets A and B are metrically separated, then there exist Borel sets A_0 and B_0 such that $A \subset A_0$ and $B \subset B_0$ with $\lambda(A_0 \cap B_0) = 0$.

(ii) Let A be a Lebesgue measurable set on the real line and let $A = A_1 \cup A_2$, where the sets A_1 and A_2 are metrically separated. Show that A_1 and A_2 are Lebesgue measurable.

Hint: (i) let A_n and B_n be open sets such that $A \subset A_n$, $B \subset B_n$, and $\lambda(A_n \cap B_n) < n^{-1}$. Take the sets $A_0 := \bigcap_{n=1}^{\infty} A_n$ and $B_0 := \bigcap_{n=1}^{\infty} B_n$. (ii) According to (i) there exist Borel sets B_1 and B_2 with $A_1 \subset B_1$, $A_2 \subset B_2$, and $\lambda(B_1 \cap B_2) = 0$. Let $E := A \cap (B_1 \backslash A_1)$. It is readily verified that $E \subset B_1 \cap B_2$. Hence $\lambda(E) = 0$, which shows that A_1 is Lebesgue measurable.

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