Preface

This book gives an exposition of the foundations of modern measure theory and offers three levels of presentation: a standard university graduate course, an advanced study containing some complements to the basic course (the material of this level corresponds to a variety of special courses), and, finally, more specialized topics partly covered by more than 850 exercises. The target readership includes graduate students interested in deeper knowledge of measure theory, instructors of courses in measure and integration theory, and researchers in all fields of mathematics. The book may serve as a source for many advanced courses or as a reference.

Volume 1 (Chapters 1–5) is devoted to the classical theory of measure and integral, created chiefly by H. Lebesgue and developed by many other mathematicians, in particular, by E. Borel, G. Vitali, W. Young, F. Riesz, D. Egoroff, N. Lusin, J. Radon, M. Fréchet, H. Hahn, C. Carathéodory, and O. Nikodym, whose results are presented in these chapters. Almost all the results in Chapters 1–5 were already known in the first third of the 20th century, but the methods of presentation, certainly, take into account later developments. The basic material designed for graduate students and oriented towards beginners covers approximately 100 pages in the first five chapters (i.e., less than 1/4 of those chapters) and includes the following sections: §1.1–1.7, §2.1–2.11, §3.2–3.4, §3.9, §4.1, §4.3, and some fragments of §5.1–5.4. It corresponds to a one-semester university course of real analysis (measure and integration theory) taught by the author at the Department of Mechanics and Mathematics at the Lomonosov Moscow University. The curriculum of this course is found at the end of the Bibliographical and Historical Comments. The required background includes only the basics of calculus (convergence of sequences and series, continuity of functions, open and closed sets in the real line, the Riemann integral) and linear algebra. Although knowledge of the Riemann integral is not formally assumed, I am convinced that the Riemann approach should be a starting point of the study of integration; acquaintance with the basics of the Riemann theory enables one to appreciate the depth and beauty of Lebesgue's creation. Some additional notions needed in particular sections are explained in the appropriate places. Naturally, the classical basic material of the first five chapters (without supplements) does not differ much from what is contained in many well-known textbooks on measure and integration or probability theory, e.g., Bauer [**70**], Halmos [**404**], Kolmogorov, Fomin [**536**], Loève [**617**], Natanson [**707**], Neveu [**713**], Parthasarathy [**739**], Royden [**829**], Shiryaev [**868**], and other books. An important feature of our exposition is that the listed sections contain only minimal material covered in real lectures. In particular, less attention than usual is given to measures on semirings etc. In general, the technical set-theoretic ingredients are considerably shortened. However, the corresponding material is not completely excluded: it is just transferred to supplements and exercises. In this way, one can substantially ease the first acquaintance with the subject when the abundance of definitions and set-theoretical constructions often make obstacles for understanding the principal ideas. Other sections of the main body of the book, supplements and exercises contain many things that are very useful in applications but seldom included in textbooks. There are two reasons why the standard course is included in full detail (rather than just mentioned in prerequisites): it makes the book completely self-contained and available to a much broader audience, in addition, many topics in the advanced material continue our discussion started in the basic course; it would be unnatural to give a continuation of a discussion without its beginning and origins. It should be noted that brevity of exposition has not been my priority; moreover, due to the described structure of the book, certain results are first presented in more special cases and only later are given in more general form. For example, our discussion of measures and integrals starts from finite measures, since the consideration of infinite values does not require new ideas, but for the beginner may overshadow the essence by rather artificial troubles with infinities. The organization of the book does not suggest reading from cover to cover; in particular, almost all sections in the supplements are independent of each other and are directly linked only to specific sections of the main part. A detailed table of contents is given. Here are brief comments on the structure of chapters.

In Chapter 1, the principal objects are countably additive measures on algebras and σ-algebras, and the main theorems are concerned with constructions and extensions of measures.

Chapter 2 is devoted to the construction of the Lebesgue integral, for which measurable functions are introduced first. The main theorems in this chapter are concerned with passage to the limit under the integral sign. The Lebesgue integral — one of the basic objects in this book — is not the most general type of integral. Apparently, its role in modern mathematics is explained by two factors: it possesses a sufficient and reasonable generality combined with aesthetic attractiveness.

In Chapter 3, we consider the most important operations on measures and functions: the Hahn–Jordan decomposition of signed measures, product measures, multiplication of measures by functions, convolutions of functions and measures, transformations of measures and change of variables. We discuss in detail finite and infinite products of measures. Fundamental theorems due to Radon&Nikodym and Fubini are presented.

Chapter 4 is devoted to spaces of integrable functions and spaces of measures. We discuss the geometric properties of the space L^p, study the uniform integrability, and prove several important theorems on convergence and boundedness of sequences of measures. Considerable attention is given to weak convergence and the weak topology in L^1. Finally, the structure properties of spaces of functions and measures are discussed.

In Chapter 5, we investigate connections between integration and differentiation and prove the classical theorems on the differentiability of functions of bounded variation and absolutely continuous functions and integration by parts. Covering theorems and the maximal function are discussed. The Henstock–Kurzweil integral is introduced and briefly studied.

Whereas the first volume presents the ideas that go back mainly to Lebesgue, the second volume (Chapters 6–10) is to a large extent the result of the development of ideas generated in 1930–1960 by a number of mathematicians, among which primarily one should mention A.N. Kolmogorov, J. von Neumann, and A.D. Alexandroff; other chief contributors are mentioned in the comments. The central subjects in Volume 2 are: transformations of measures, conditional measures, and weak convergence of measures. These three themes are closely interwoven and form the heart of modern measure theory. Typical measure spaces here are infinite dimensional: e.g., it is often convenient to consider a measure on the interval as a measure on the space $\{0,1\}^\infty$ of all sequences of zeros and ones. The point is that in spite of the fact that any reasonable measure space is isomorphic to an interval, a significant role is played by diverse additional structures on measure spaces: algebraic, topological, and differential. This is partly explained by the fact that many problems of modern measure theory grew under the influence of probability theory, the theory of dynamical systems, information theory, the theory of representations of groups, nonlinear analysis, and mathematical physics. All these fields brought into measure theory not only problems, methods, and terminology, but also inherent ways of thinking. Note also that the most fruitful directions in measure theory now border with other branches of mathematics.

Unlike the first volume, a considerable portion of material in Chapters 6–10 has not been presented in such detail in textbooks. Chapters 6–10 require also a deeper background. In addition to knowledge of the basic course, it is necessary to be familiar with the standard university course of functional analysis including elements of general topology (e.g., the textbook by Kolmogorov and Fomin covers the prerequisites). In some sections it is desirable to be familiar with fundamentals of probability theory (for this purpose, a concise book, Lamperti [**566**], can be recommended). In the second volume many themes touched on in the first volume find their natural development (for example, transformations of measures, convergence of measures, Souslin sets, connections between measure and topology).

Chapter 6 plays an important technical role: here we study various properties of Borel and Souslin sets in topological spaces and Borel mappings of

Souslin sets, in particular, several measurable selection and implicit function theorems are proved here. The birth of this direction is due to a great extent to the works of N. Lusin and M. Souslin. The exposition in this chapter has a clear set-theoretic and topological character with almost no measures. The principal results are very elegant, but are difficult in parts in the technical sense, and I decided not to hide these difficulties in exercises. However, this chapter can be viewed as a compendium of results to which one should resort in case of need in the subsequent chapters.

In Chapter 7, we discuss measures on topological spaces, their regularity properties, and extensions of measures, and examine the connections between measures and the associated functionals on function spaces. The branch of measure theory discussed here grew from the classical works of J. Radon and A.D. Alexandroff, and was strongly influenced (and still is) by general topology and descriptive set theory. The central object of the chapter is Radon measures. We also study in detail perfect and τ-additive measures. A separate section is devoted to the Daniell–Stone method. This method could have been explained already in Chapter 2, but it is more natural to place it close to the Riesz representation theorem in the topological framework. There is also a brief discussion of measures on locally convex spaces and their characteristic functionals (Fourier transforms).

In Chapter 8, directly linked only to Chapter 7, the theory of weak convergence of measures is presented. We prove several fundamental results due to A.D. Alexandroff, Yu.V. Prohorov and A.V. Skorohod, study the weak topology on spaces of measures and consider weak compactness. The topological properties of spaces of measures on topological spaces equipped with the weak topology are discussed. The concept of weak convergence of measures plays an important role in many applications, including stochastic analysis, mathematical statistics, and mathematical physics. Among many complementary results in this chapter one can mention a thorough discussion of convergence of measures on open sets and a proof of the Fichtenholz–Dieudonné–Grothendieck theorem.

Chapter 9 is devoted to transformations of measures. We discuss the properties of images of measures under mappings, the existence of preimages, various types of isomorphisms of measure spaces (for example, point, metric, topological), the absolute continuity of transformed measures, in particular, Lusin's (N)-property, transformations of measures by flows generated by vector fields, Haar measures on locally compact groups, the existence of invariant measures of transformations, and many other questions important for applications. The "nonlinear measure theory" discussed here originated in the 1930s in the works of G.D. Birkhoff, J. von Neumann, N.N. Bogolubov, N.M. Krylov, E. Hopf and other researchers in the theory of dynamical systems, and was also considerably influenced by other fields such as the integration on topological groups developed by A. Haar, A. Weil, and others. A separate section is devoted to the theory of Lebesgue spaces elaborated by V. Rohlin (such spaces are called here Lebesgue–Rohlin spaces).

Chapter 10 is close to Chapter 9 in its spirit. The principal ideas of this chapter go back to the works of A.N. Kolmogorov, J. von Neumann, J. Doob, and P. Lévy. It is concerned with conditional measures — the object that plays an exceptional role in measure theory as well as in numerous applications. We describe in detail connections between conditional measures and conditional expectations, prove the main theorems on convergence of conditional expectations, establish the existence of conditional measures under broad assumptions and clarify their relation to liftings. In addition, a concise introduction to the theory of martingales is given with views towards applications in measure theory. A separate section is devoted to ergodic theory — a fruitful field at the border of measure theory, probability theory, and mathematical physics. Finally, in this chapter we continue our study of Lebesgue–Rohlin spaces, and in particular, discuss measurable partitions.

Extensive complementary material is presented in the final sections of all chapters, where there are also a lot of exercises supplied with complete solutions or hints and references. Some exercises are merely theorems from the cited sources printed in a smaller font and are placed there to save space (so that the absence of hints means that I have no solutions different from the ones found in the cited works). The symbol ° marks exercises recommendable for graduate courses or self-study. Note also that many solutions have been borrowed from the cited works, but sometimes solutions simpler than the original ones are presented (this fact, however, is not indicated). It should be emphasized that many exercises given without references are either taken from the textbooks listed in the bibliographical comments or belong to the mathematical folklore. In such exercises, I omitted the sources (which appear in hints, though), since they are mostly secondary. It is possible that some exercises are new, but this is never claimed for the obvious reason that a seemingly new assertion could have been read in one of hundreds papers from the list of references or even heard from colleagues and later recalled.

The book contains an extensive bibliography and the bibliographical and historical comments. The comments are made separately on each volume, the bibliography in Volume 1 contains the works cited only in that volume, and Volume 2 contains the cumulative bibliography, where the works cited only in Volume 1 are marked with an asterisk. For each item in the list of references we indicate all pages where it is cited. The comments, in addition to remarks of a historical or bibliographical character, give references to works on many special aspects of measure theory, which could not be covered in a book of this size, but the information about which may be useful for the reader. A detailed subject index completes the book (Volume 1 contains only the index for that volume, and Volume 2 contains the cumulative index).

For all assertions and formulas we use the triple enumeration: the chapter number, section number, and assertion number (all assertions are numbered independently of their type within each section); numbers of formulas are given in brackets.

This book is intended as a complement to the existing large literature of advanced graduate-text type and provides the reader with a lot of material from many parts of measure theory which does not belong to the standard course but is necessary in order to read research literature in many areas. Modern measure theory is so vast that it cannot be adequately presented in one book. Moreover, even if one attempts to cover all the directions in a universal treatise, possibly in many volumes, due depth of presentation will not be achieved because of the excessive amount of required information from other fields. It appears that for an in-depth study not so voluminous expositions of specialized directions are more suitable. Such expositions already exist in a several directions (for example, the geometric measure theory, Hausdorff measures, probability distributions on Banach spaces, measures on groups, ergodic theory, Gaussian measures). Here a discussion of such directions is reduced to a minimum, in many cases just to mentioning their existence.

This book grew from my lectures at the Lomonosov Moscow University, and many related problems have been discussed in lectures, seminar talks and conversations with colleagues at many other universities and mathematical institutes in Moscow, St.-Petersburg, Kiev, Berlin, Bielefeld, Bonn, Oberwolfach, Paris, Strasburg, Cambridge, Warwick, Rome, Pisa, Vienna, Stockholm, Copenhagen, Zürich, Barcelona, Lisbon, Athens, Edmonton, Berkeley, Boston, Minneapolis, Santiago, Haifa, Kyoto, Beijing, Sydney, and many other places. Opportunities to work in the libraries of these institutions have been especially valuable. Through the years of work on this book I received from many individuals the considerable help in the form of remarks, corrections, additional references, historical comments etc. Not being able to mention here all those to whom I owe gratitude, I particularly thank H. Airault, E.A. Alekhno, E. Behrends, P.A. Borodin, G. Da Prato, D. Elworthy, V.V. Fedorchuk, M.I. Gordin, M.M. Gordina, V.P. Havin, N.V. Krylov, P. Lescot, G. Letta, A.A. Lodkin, E. Mayer-Wolf, P. Malliavin, P.-A. Meyer, L. Mejlbro, E. Priola, V.I. Ponomarev, Yu.V. Prohorov, M. Röckner, V.V. Sazonov, B. Schmuland, A.N. Shiryaev, A.V. Skorohod, O.G. Smolyanov, A.M. Stepin, V.N. Sudakov, V.I. Tarieladze, S.A. Telyakovskii, A.N. Tikhomirov, F. Topsøe, V.V. Ulyanov, H. von Weizsäcker, and M. Zakai. The character of presentation was considerably influenced by discussions with my colleagues at the chair of theory of functions and functional analysis at the Department of Mechanics and Mathematics of the Lomonosov Moscow University headed by the member of the Russian Academy of Science P.L. Ulyanov. For checking several preliminary versions of the book, numerous corrections, improvements and other related help I am very grateful to A.V. Kolesnikov, E.P. Krugova, K.V. Medvedev, O.V. Pugachev, T.S. Rybnikova, N.A. Tolmachev, R.A. Troupianskii, Yu.A. Zhereb'ev, and V.S. Zhuravlev. The book took its final form after Z. Lipecki read the manuscript and sent his corrections, comments, and certain materials that were not available to me. I thank J. Boys for careful copyediting and the editorial staff at Springer-Verlag for cooperation.

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