

1 BASIC VARIATIONAL PRINCIPLES OF STATICS AND GEOMETRY IN STRUCTURAL MECHANICS

We have the right as well as are obliged to subject all our definitions to critical analysis from the standpoint of their application and revise them (fundamentally, if need be) if they do not work for us.

Young L (1969) Lectures on the calculus of variations and optimal control theory. W.B. Saunders company, Philadelphia London Toronto

1.1 Preliminaries

Let an area Ω with a piecewise smooth boundary Γ be defined in a k -dimensional space. Structural mechanics deals with one, two, and three-dimensional problems only, therefore we take the case of $k \leq 3$.

Consider a linear set $\mathcal{M}$ with elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \dots$ which are functions of points $x \in \Omega$. The elements of the set $\mathcal{M}$ will be assumed to exist as scalar, vector or tensor functions. We assume also that for every couple of elements, $\mathbf{a}$ and $\mathbf{b}$, from the set $\mathcal{M}$, two bilinear functionals, $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}, \mathbf{b})_\Gamma$, can be defined which acquire finite values. The functionals are specified by these formulas:

$$(\mathbf{a}, \mathbf{b}) = \int_{\Omega} \mathbf{a} \cdot \mathbf{b} \, d\Omega, \quad (1.1)$$

$$(\mathbf{a}, \mathbf{b})_\Gamma = \int_{\Gamma} \mathbf{a} \cdot \mathbf{b} \, d\Gamma, \quad (1.2)$$

where $\mathbf{a} \cdot \mathbf{b}$ means a scalar expression calculated according to the rule

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} ab & \text{– for scalar quantities,} \\ a^i b_i & \text{– for vector quantities,} \\ a^{ij} b_{ij} & \text{– for tensor quantities.} \end{cases} \quad (1.3)$$

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Here and further we use a common rule: the same indices on different levels are used for summation.

The one-dimensional version of the area Ω is an interval, $[x_1, x_2]$, over which an independent variable, x , can vary, so that

$$(a, b) = \int_{x_1}^{x_2} a \cdot b \, dx, \quad (a, b)_\Gamma = (a \cdot b)_{x=x_2} + (a \cdot b)_{x=x_1}. \quad (1.4)$$

The latter relationship can be rewritten also as

$$(a, b)_\Gamma = (+1a \cdot b)_{x=x_2} - (-1a \cdot b)_{x=x_1} = [na \cdot b]_{x_1}^{x_2}. \quad (1.5)$$

Here n is the cosine of the angle between the x axis and the external normal to the interval $[x_1, x_2]$ in its end points, so that

$$n = \begin{cases} 1 & \text{at } x = x_2, \\ -1 & \text{at } x = x_1. \end{cases}$$

Note that the bilinear functional (a, b) can be treated as a scalar product on the linear set $\mathcal{M}$, while the functional $(a, b)_\Gamma$ generates a scalar product on the linear set $\mathcal{M}_\Gamma$ which consists of functions defined on the boundary Γ . Indeed, both functionals are linear with respect to either argument and symmetric (insensitive to the order of their arguments). Also, the result is nonnegative when the arguments are equal:

$$(a, a) \geq 0, \quad (a, a)_\Gamma \geq 0.$$

Moreover, the condition $(a, a) = 0$ implies the equality $a = \mathbf{0}$ where $\mathbf{0}$ is a zero element of the $\mathcal{M}$ set, that is, all components of a are zero functions on Ω . Similarly, $(a, a)_\Gamma = 0$ implies $a = \mathbf{0}$ where $\mathbf{0}$ is a zero element of the set $\mathcal{M}_\Gamma$, i.e. all components of a are functions with zero values on Γ .

1.1.1 Formally conjugate differential operators

Now we consider two linear sets, $\mathcal{N}$ and $\mathcal{M}$, such that for any element $a \in \mathcal{N}$ a differential operation A is defined with its range of values in $\mathcal{M}$, and for any element $b \in \mathcal{M}$ a differential operation B is defined with its range of values in $\mathcal{N}$; in other words,

$$a \in \mathcal{N} \Rightarrow Aa \in \mathcal{M} \quad \text{and} \quad b \in \mathcal{M} \Rightarrow Bb \in \mathcal{N}. \quad (1.6)$$

Further on we will follow the terminology common in mechanics and use the words *differential operators* with objects like A and B thus treating

them as symbolic differentiation operations, although in mathematics [3] an operator is a bigger notion than a simple differentiation expression.

The differential operators $\mathbf{A}$ and $\mathbf{B}$ are called *formally conjugate* (sometimes Lagrange-conjugate) if they satisfy (1.6) and the following relationship holds:

$$(\mathbf{A}a, b) = (a, \mathbf{B}b) + (\mathbf{A}_\gamma a, \mathbf{B}_\gamma b)_\Gamma \quad (1.7)$$

where $\mathbf{A}_\gamma$ and $\mathbf{B}_\gamma$ are certain linear differential operations such that the bilinear functional $(\mathbf{A}_\gamma a, \mathbf{B}_\gamma b)_\Gamma$ makes sense for all elements a from $\mathcal{N}$ and all elements b from $\mathcal{M}$.

Further we will use the word *conjugate* for this relationship between the operators, always assuming the Lagrange-type conjugation. Also, an operator, $\mathbf{B}$, conjugate to the operator $\mathbf{A}$ will be denoted as $\mathbf{A}^\top$, that is, $\mathbf{B} = \mathbf{A}^\top$, if the relationship (1.7) holds¹.

Let's give an example of a matrix differential operation which conforms to a certain operator $\mathbf{A}$ and its conjugate operator $\mathbf{A}^\top$

$$\mathbf{A} = \left[\begin{array}{cc} \frac{d(\cdot)}{dx} & \frac{(\cdot)}{\rho} \\ \frac{d}{dx} \left(\frac{(\cdot)}{\rho} \right) & -\frac{d^2(\cdot)}{dx^2} \end{array} \right], \quad \mathbf{A}^\top = \left[\begin{array}{cc} -\frac{d(\cdot)}{dx} & -\frac{1}{\rho} \frac{d(\cdot)}{dx} \\ \frac{(\cdot)}{\rho} & -\frac{d^2(\cdot)}{dx^2} \end{array} \right] \quad (1.8)$$

where $\rho = \rho(x)$ is a sufficiently smooth function of the independent variable².

The linear sets $\mathcal{N}$ and $\mathcal{M}$ both will be a set of vectors of the type $\mathbf{a} = [a_1(x), a_2(x)]^\top$ with their components being sufficiently smooth functions defined on the interval $[0, l]$. The following holds for two vectors $\mathbf{a}$ and $\mathbf{b}$:

¹ We do not use a common mathematical notation for conjugate operators using $*$; instead, we choose $^\top$, not just because the asterisk $*$ is reserved in our book for marking quantities related to an exact solution of a problem. The matter is that the conjugation operation is a generalization of a matrix transposition which is usually denoted by the symbol $^\top$. There is no confusion; just remember that this symbol applied to a differential operator means something bigger than the mere transposition of a matrix.

² As we will see, this example gives the operators of geometry and equilibrium in the problem of bending of a planar curvilinear beam with its curvature radius, $\rho(x)$, variable along the arc coordinate x – see Section 4.7.

$$\mathbf{A}\mathbf{a} = \begin{bmatrix} a_1' + \frac{a_2}{\rho} \\ \left(\frac{a_1}{\rho}\right)' - a_2'' \end{bmatrix}, \quad \mathbf{A}^\top \mathbf{b} = \begin{bmatrix} -b_1' - \frac{b_2'}{\rho} \\ \frac{b_1}{\rho} - b_2'' \end{bmatrix} \quad (1.9)$$

where the stroke means the differentiation with respect to x . The scalar products of our interest can be represented now as

$$(\mathbf{A}\mathbf{a}, \mathbf{b}) = \int_0^l \left\{ \left(a_1' + \frac{a_2}{\rho} \right) b_1 + \left[\left(\frac{a_1}{\rho} \right)' - a_2'' \right] b_2 \right\} dx,$$

$$(\mathbf{a}, \mathbf{A}^\top \mathbf{b}) = \int_0^l \left\{ \left(-b_1' - \frac{b_2'}{\rho} \right) a_1 + \left(\frac{b_1}{\rho} - b_2'' \right) a_2 \right\} dx.$$

Integrating these products by parts will transform them into the following:

$$(\mathbf{A}\mathbf{a}, \mathbf{b}) = \int_0^l \left(-a_1 b_1' + \frac{a_2 b_1}{\rho} - \frac{a_1 b_2'}{\rho} + a_2' b_2' \right) dx + \left[a_1 b_1 + \frac{a_1 b_2}{\rho} - a_2' b_2 \right]_0^l,$$

$$(\mathbf{a}, \mathbf{A}^\top \mathbf{b}) = \int_0^l \left(-a_1 b_1' + \frac{a_2 b_1}{\rho} - \frac{a_1 b_2'}{\rho} + a_2' b_2' \right) dx + [-a_2 b_2']_0^l$$

which makes it clear that $(\mathbf{A}\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}, \mathbf{A}^\top \mathbf{b})$ are different only in non-integral terms; this is where the mutual conjugation of the $\mathbf{A}$ and $\mathbf{A}^\top$ operators can be seen.

Confining ourselves to this only example³, now we give general rules how to construct a conjugate operator, $\mathbf{A}^\top$, for a given original operator $\mathbf{A}$. The rules are simple [4] and consist of the following two operations:

- the matrix $\mathbf{A}$ of symbolic differentiations is transposed;
- every term of the type $F_\alpha(x)D^\alpha(\cdot)$ in the matrix obtained by the transposition is replaced by its conjugate term in the form $(-1)^\alpha D^\alpha [F_\alpha(x)(\cdot)]$ where D^α represents a symbolic form of the α -order differentiation, that is,

³ L. Young [13] gives a convincing reference which proves that one example is enough for every rule. This is an experience of “...a nine-year old girl, a lady indeed, who solved only one of summation exercises from her homework and wrote in her writing-book that the others can be solved similarly”.

$$D^{\alpha}(\cdot) = \frac{\partial^{|\alpha|}(\cdot)}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}}, \quad |\alpha| = \alpha_1 + \dots + \alpha_k.$$

With one dimension ($k = 1$) we can use the integration by parts, while with two or three dimensions we can use the Gauss–Ostrogradsky formula to check that these rules really produce the conjugate operator.

To complete this section, we note two simple but important properties of the conjugation operation. First, the definition (1.7) implies directly that the conjugation is mutual, that is, the conjugate operator of a conjugate operator is identical to the original operator:

$$(\mathbf{A}^{\top})^{\top} = \mathbf{A}. \quad (1.10)$$

Second, the conjugate of the product of two operators is equal to the product of the operators conjugate of the original cofactors, placed in the reverse order, that is,

$$(\mathbf{AB})^{\top} = \mathbf{B}^{\top} \mathbf{A}^{\top}. \quad (1.11)$$

Both properties are quite similar to the properties of the usual matrix transposition.

1.2 Basic integral identity

Let $\boldsymbol{\sigma}$ be a somehow ordered set of functions which determine the stress state of a mechanical system (a full set of stresses or internal forces).

For example, in three-dimensional elasticity this $\boldsymbol{\sigma}$ will be a stress tensor with its components σ^{ij} referred to a particular system of axes, (x_1, x_2, x_3) , which will be treated as a rectangular right-oriented Cartesian coordinate system if not stated otherwise. For further applications it will be convenient to use $\boldsymbol{\sigma}$ as a six-dimensional column vector, $\boldsymbol{\sigma} = [\sigma^{11}, \sigma^{22}, \sigma^{33}, \sigma^{12}, \sigma^{23}, \sigma^{31}]^{\top}$, remembering its symmetry and its tensor nature. Actually, the tensor nature of the stresses is relevant only when doing a transformation of the coordinate system.

Another example is a Timoshenko beam in bending. Here $\boldsymbol{\sigma}$ will be a two-component column vector, $\boldsymbol{\sigma} = [Q, M]^{\top}$, Q being the shear force and M the bending moment in a cross-section of the beam. When discussing and posing any particular problem of structural mechanics, we will assume that the ordering rules for the components of the internal stresses/forces in $\boldsymbol{\sigma}$ are defined and known.

Let $\boldsymbol{\varepsilon}$ be a set of components of a strain tensor (vector) which is in energy reciprocity with $\boldsymbol{\sigma}$. The energy reciprocity means that the usual

scalar product $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}$ in the sense of (1.3) will yield an expression of the elastic deformation energy.

For example, in three-dimensional elasticity $\boldsymbol{\varepsilon}$ means the strain tensor ε_{ij} or the column vector $\boldsymbol{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{12}, \gamma_{23}, \gamma_{31}]^T$ where $\gamma_{ij} = 2\varepsilon_{ij}$ at $i \neq j$. It is easy to notice that the vector notation for the set of the components of the stress and strain tensors will produce the following representation of the scalar product:

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}. \quad (2.1)$$

In the Timoshenko beam bending, $\boldsymbol{\varepsilon} = [\gamma, \chi]^T$ where γ is a shear strain, χ is a bending strain (the derivative of the slope of the beam's cross-sections).

A set of governing equations for the analysis of a mechanical system under static loading will be presented in the following form:

$$\mathbf{B}\boldsymbol{\sigma} + \mathbf{K}\mathbf{u} = \bar{\mathbf{X}} \quad \text{equilibrium equations} \quad (2.2-a)$$

$$\mathbf{A}\mathbf{u} = \boldsymbol{\varepsilon} \quad \text{geometric equations} \quad (2.2-b)$$

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} \text{ or } \boldsymbol{\varepsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma} \quad \text{physical equations} \quad (2.2-c)$$

Here

- $\mathbf{u} = [u_i]$ is a displacement vector;
- $\bar{\mathbf{X}} = [\bar{X}^i]$ is a vector of given external forces per unit of volume of an elastic body;
- $\mathbf{K} = \{k_{ij}\}$ is a tensor of elasticity coefficients of a medium in which the deformable solid in question is put;
- $\mathbf{C} = \{C^{ijkl}\}$ and $\mathbf{C}^{-1} = \{D_{ijkl}\}$ are algebraic, mutually inverse operators which represent the respective tensors of coefficients of elasticity and compliance for the material of the deformable system.

The operators $\mathbf{C}$, $\mathbf{C}^{-1}$ and $\mathbf{K}$ are symmetric, and $\mathbf{C}$ and $\mathbf{C}^{-1}$ are also positive definite, while operator $\mathbf{K}$ is nonnegative (which is sometimes referred to as 'positive semi-definite') in every point of the area Ω . The conditions thus formulated can be reduced to the following requirements:

$$\begin{aligned} C^{ijkl} &= C^{jikl} = C^{klij}, & M_c a^{ij} a_{ij} &\geq C^{ijkl} a_{ij} a_{kl} \geq m_c a^{ij} a_{ij} \quad (m_c > 0), \\ D_{ijkl} &= D_{klij} = D_{jikl}, & M_d a^{ij} a_{ij} &\geq D_{ijkl} a^{ij} a^{kl} \geq m_d a^{ij} a_{ij} \quad (m_d > 0), \\ k^{ij} &= k^{ji}, & k^{ij} b_i b_j &\geq m_k b_i b^i \quad (m_k \geq 0), \end{aligned} \quad (2.3)$$

where

- $\mathbf{a} = \{a^{ij}\}$ is an arbitrary symmetric tensor of second rank;
- $\mathbf{b} = \{b^i\}$ is a vector;
- $M_c = 1/m_d$, $M_d = 1/m_c$, m_k are scalars independent of coordinates, tensor $\mathbf{a}$, and vector $\mathbf{b}$.

Further we will refer to the operator $\mathbf{B}$ as an *operator of equilibrium* and to the operator $\mathbf{A}$, which is purely geometric and relates the strains with the displacements, as an *operator of geometry*.

The system of governing equations (2.2) should be supplemented with boundary conditions; in the operator-based form they can be written as

$$\mathbf{E}_p(\mathbf{H}_\sigma \boldsymbol{\sigma} - \bar{\mathbf{p}}) = \mathbf{0} \quad (\text{static boundary conditions}) \quad (2.4-a)$$

$$\mathbf{E}_u(\mathbf{H}_u \mathbf{u} - \bar{\mathbf{u}}) = \mathbf{0} \quad (\text{kinematic boundary conditions}) \quad (2.4-b)$$

and should be specified on the boundary Γ of the area Ω .

The formulas (2.4) use the notation:

- $\bar{\mathbf{p}}$ for a vector of given external boundary forces;
- $\bar{\mathbf{u}}$ for a vector of given external boundary displacements.

We will assume that the components of these two vectors are represented simultaneously either in a coordinate system global for the whole structure, (x_1, x_2, x_3) or as decompositions by axes of a local basis in each point of the boundary Γ . The only local coordinate basis that we will use will be a right-oriented orthonormalized triple of vectors $(\mathbf{n}, \mathbf{t}, \mathbf{b})$ where $\mathbf{n}$ is a unit vector of the exterior (with respect to the area Ω) normal to the boundary Γ ; $\mathbf{t}$ is a unit vector tangential to Γ . The condition of orthogonality and the right orientation of the triple $(\mathbf{n}, \mathbf{t}, \mathbf{b})$ define the third unit vector $\mathbf{b}$ as a vector product, $\mathbf{b} = \mathbf{n} \times \mathbf{t}$. In two-dimensional elasticity, the global coordinate system consists of the couple of axes, (x_1, x_2) , and the local basis is made up by the unit vectors $(\mathbf{n}, \mathbf{t})$ where the positive direction of the vector $\mathbf{t}$ tangential to the boundary Γ is defined in such way that the condition $\mathbf{n} \times \mathbf{t} = \mathbf{i}_1 \times \mathbf{i}_2$ holds, where $\mathbf{i}_1$ and $\mathbf{i}_2$ are unit vectors of the respective axes x_1 and x_2 .

Algebraic operators $\mathbf{E}_p$ and $\mathbf{E}_u$ take the boundary conditions in each point of the boundary Γ and extract those of them which are actually specified in a particular problem. The operators are symmetric; they make up a decomposition of the identity operator $\mathbf{I}$ into an orthogonal sum, that is,

$$\mathbf{E}_p + \mathbf{E}_u = \mathbf{I}, \quad \mathbf{E}_p \mathbf{E}_u = \mathbf{E}_u \mathbf{E}_p = \mathbf{O}, \quad (2.5)$$

where $\mathbf{O}$ is a zero (annihilating) operator.

The relation (2.5) implies the *idempotency* of the boundary condition extraction operators:

$$\mathbf{E}_p \mathbf{E}_p = \mathbf{E}_p, \quad \mathbf{E}_u \mathbf{E}_u = \mathbf{E}_u. \quad (2.6)$$

It should be obvious that the operators are defined on the whole boundary Γ , but they may have different values on different pieces of the boundary.

If we introduce the designation $\mathbf{p}$ for the boundary force vector and $\mathbf{u}$ for the boundary displacement vector, i.e. if we assume

$$\mathbf{p} = \mathbf{H}_\sigma \boldsymbol{\sigma}, \quad \mathbf{u} = \mathbf{H}_u \mathbf{u} \in \Gamma \quad (2.7)$$

then the conditions (2.5) show that one and only one boundary condition can be specified for each component of the two vectors: a static one (for the $\mathbf{p}$ vector components) or a kinematic one (for the $\mathbf{u}$ vector components).

It should be clear also that the couple of the predefined vectors, $\bar{\mathbf{p}}$ and $\bar{\mathbf{u}}$, and the calculated couple of vectors, $\mathbf{p}$ and $\mathbf{u}$, are represented by their decompositions by the coordinate axes of the same basis, either global or local, and only in this case the respective components of the vectors can be compared according to (2.4).

Let's explain this by an example of planar elasticity. Suppose that a part of the boundary Γ is subject to external forces $\bar{p}'$ tangential to the contour Γ and displacements $\bar{u}_n$ in the direction of the external normal to Γ (Fig. 1.1).

This means not all components of the vectors are specified in the local coordinates. In this particular case the decomposition of the vectors $\bar{\mathbf{p}}$ and $\bar{\mathbf{u}}$ by the axes of the local basis will give

$$\bar{\mathbf{p}} = \begin{bmatrix} \otimes \\ \bar{p}' \end{bmatrix}, \quad \bar{\mathbf{u}} = \begin{bmatrix} \bar{u}_n \\ \otimes \end{bmatrix} \quad (2.8)$$

where the symbol $\otimes$ designates values of components not known beforehand. Obviously, on this piece of the boundary Γ the algebraic operators $\mathbf{E}_p$ and $\mathbf{E}_u$ are described by the matrices of the following kind:

$$\mathbf{E}_p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{E}_u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (2.9)$$

to exclude the undefined quantities $\otimes$ from the boundary conditions (2.4).

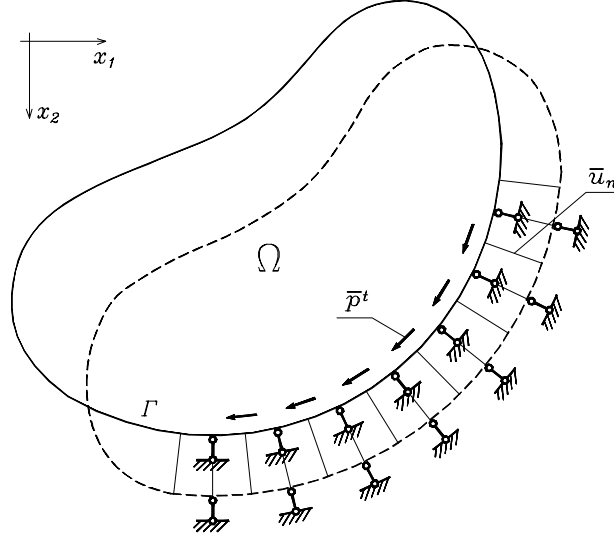


Fig. 1.1. Mixed boundary conditions on a piece of the boundary Γ of the area Ω

If $\bar{p}$ and $\bar{u}$ are specified in a local coordinate system, and if the components of the boundary force vector, $\mathbf{p}$, and of the boundary displacement vector, $\mathbf{u}$, are transformed by formal calculation according to (2.7) into quantities expressed in the global coordinate system,

$$\mathbf{p} = [p^1, p^2]^\top, \quad \mathbf{u} = [u_1, u_2]^\top,$$

then, before using the vectors $\mathbf{p}$ and $\mathbf{u}$ in the boundary conditions

$$\mathbf{E}_p(\mathbf{p} - \bar{\mathbf{p}}) = \mathbf{0}, \quad \mathbf{E}_u(\mathbf{u} - \bar{\mathbf{u}}) = \mathbf{0} \quad \in \Gamma,$$

they should be converted to the local coordinate system following common rules. The reason for this is the sensitivity of the boundary condition extraction operators, $\mathbf{E}_p$ and $\mathbf{E}_u$, to the coordinate system; they are constructed in such way that they are not invariant to the coordinates, instead they keep track of a coordinate system in which the predefined boundary forces and displacements are specified.

If there are no mixed boundary conditions anywhere on the contour Γ , then the latter can be divided into two parts, Γ_p and Γ_u , so that

$$\Gamma = \Gamma_p \cup \Gamma_u, \quad \Gamma_p \cap \Gamma_u = \emptyset \quad (2.10)$$

and

$$\mathbf{E}_p = \mathbf{I}, \mathbf{E}_u = \mathbf{O} \in \Gamma_p, \quad \mathbf{E}_p = \mathbf{O}, \mathbf{E}_u = \mathbf{I} \in \Gamma_u. \quad (2.11)$$

When (2.10) and (2.11) hold, the boundary conditions (2.4) can be written as

$$\mathbf{H}_\sigma \boldsymbol{\sigma} - \bar{\mathbf{p}} = \mathbf{0} \quad \in \Gamma_p \quad (\text{static boundary conditions}) \quad (2.12-a)$$

$$\mathbf{H}_u \mathbf{u} - \bar{\mathbf{u}} = \mathbf{0} \quad \in \Gamma_u \quad (\text{kinematic boundary conditions}) \quad (2.12-b)$$

In problems where the equilibrium operator $\mathbf{B}$ and the geometry operator $\mathbf{A}$ contain derivatives of orders higher than first, the vector of the boundary displacements, $\mathbf{u}$, exceeds the vector of internal displacements, $\mathbf{u}$, by the number of its components: the kinematic boundary conditions on the boundary Γ may contain both values of the vector function $\mathbf{u}$ itself and some derivatives of the components. The matrix differential operator $\mathbf{H}_u$ transforms the vector of internal displacements $\mathbf{u}$, to a space of vectors of the boundary displacements $\mathbf{u}$. When needed, this operator can perform a transformation of the boundary displacement vector to a local coordinate system.

The matrix differential operator $\mathbf{H}_\sigma$ works in a similar way. It transforms an internal force vector, $\boldsymbol{\sigma}$, to the space of boundary force vectors $\mathbf{p}$. In a particular case when the equilibrium and geometry operators contain differential operations of at most first order and the local coordinate system coincides with the global one, $\mathbf{H}_u$ is an identity operator, that is,

$$\mathbf{H}_u = \mathbf{I}. \quad (2.13)$$

In the same situation, the algebraic operator $\mathbf{H}_\sigma$ contains components of the unit vector $\mathbf{n}$ of an exterior normal to the boundary Γ with respect to the global coordinate system, i.e. the cosines of angles between the vector $\mathbf{n}$ and the direction of the respective axis x_i , in other words, the quantities

$$n_i = \cos(\mathbf{n}, x_i) = (\mathbf{i}_i, \mathbf{n}) \quad (2.14)$$

where $\mathbf{i}_i$ is the unit vector of the x_i axis.

As it will be seen from formulations of various types of structural mechanics problems, in all cases the operators of equilibrium and geometry at the given boundary conditions (2.4) satisfy a so-called *basic integral identity*

$$(\mathbf{A}\mathbf{u}, \boldsymbol{\sigma}) = (\mathbf{u}, \mathbf{B}\boldsymbol{\sigma}) + (\mathbf{H}_\sigma \boldsymbol{\sigma}, \mathbf{H}_u \mathbf{u})_\Gamma. \quad (2.15)$$

This equation can be validated directly by constructing the appropriate equilibrium and geometry operators. However, as we will see further, the basic integral identity (2.15) is actually true and fundamental for all linear

problems of structural mechanics. It implies all basic theorems given further below. In particular, comparing (2.15) and (1.7) will give an immediate result that the equilibrium operator and the geometry operator are mutually conjugate:

$$\mathbf{B} = \mathbf{A}^\top. \quad (2.16)$$

The latter relationship allows us to designate the equilibrium operator further as $\mathbf{A}^\top$ without noticing this fact additionally⁴.

Taking into account the designations from (2.7) will convert the basic integral identity into the following:

$$(\mathbf{A}\mathbf{u}, \boldsymbol{\sigma}) = (\mathbf{u}, \mathbf{A}^\top \boldsymbol{\sigma}) + (\mathbf{p}, \mathbf{u})_\Gamma. \quad (2.17)$$

Now, we can take the properties (2.5) and (2.6) of the boundary condition extraction operators into account and use a chain of obvious transformations

$$(\mathbf{p}, \mathbf{u})_\Gamma = ((\mathbf{E}_p + \mathbf{E}_u)\mathbf{p}, (\mathbf{E}_p + \mathbf{E}_u)\mathbf{u})_\Gamma = (\mathbf{E}_p \mathbf{p}, \mathbf{E}_p \mathbf{u})_\Gamma + (\mathbf{E}_u \mathbf{p}, \mathbf{E}_u \mathbf{u})_\Gamma \quad (2.18)$$

to represent the same integral identity as

$$(\mathbf{A}\mathbf{u}, \boldsymbol{\sigma}) = (\mathbf{u}, \mathbf{A}^\top \boldsymbol{\sigma}) + (\mathbf{E}_p \mathbf{p}, \mathbf{E}_p \mathbf{u})_\Gamma + (\mathbf{E}_u \mathbf{p}, \mathbf{E}_u \mathbf{u})_\Gamma. \quad (2.19)$$

This is the form of the basic integral identity which we will use most often throughout the book.

1.3 Various types of stress and strain fields

We will say that a whole set of stresses $\boldsymbol{\sigma}$, strains $\boldsymbol{\varepsilon}$, and displacements $\mathbf{u}$ determine a stress-and-strain state (SSS) of a mechanical system. As the stresses, strains, and displacements of the mechanical system may vary from point to point, another proper term will be a field or a distribution of the stresses, the strains, or the displacements. All three fields together will be called a *stress-and-strain field*. Further below, whenever the nature of a field is not specified, we will understand that the field is actually a stress-and-strain field. An arbitrary field of this kind will be denoted as F , and an expression like $F = \{\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \mathbf{u}\}$ should be read as a *field F consisting of stresses $\boldsymbol{\sigma}$, strains $\boldsymbol{\varepsilon}$, and displacements $\mathbf{u}$* . In their turn, the stresses $\boldsymbol{\sigma}$, the strains $\boldsymbol{\varepsilon}$, and the displacements $\mathbf{u}$ will be referred to as *elements of the*

⁴ Nearly everywhere. Rare exceptions will be specified.

field F . It should be emphasized that the elements of a field are not supposed to relate to one another anyhow in the most general case.

Now, let's introduce more notions and definitions which will be useful for further presentment.

We will say that external forces X , distributed over the volume of a body Ω , and contour forces p are in static conformance to a field $F = \{\sigma, \varepsilon, u\}$ if the following holds:

$$X = A^T \sigma + K u \in \Omega, \quad E_p p = E_p H_\sigma \sigma \in \Gamma, \quad (3.1)$$

or, in another words, if the forces X and p obey the equations of equilibrium inside the body Ω and the static boundary conditions on the surface Γ .

A field $F_s = \{\sigma_s, \varepsilon_s, u_s\}$ will be called statically admissible if the given external forces $\bar{X}$ and $\bar{p}$ are in static conformance to the stresses σ_s and displacements u_s , or, in other words,

$$\bar{X} = A^T \sigma_s + K u_s \in \Omega, \quad E_p \bar{p} = E_p H_\sigma \sigma_s \in \Gamma. \quad (3.2)$$

The “s” subscript emphasizes the static admissibility of the field and its elements. This definition implies that the property of static admissibility of a field does not depend on the strains ε_s .

A field $F_{so} = \{\sigma_{so}, \varepsilon_{so}, u_{so}\}$ will be called homogeneously statically admissible if internal forces in Ω created by this field are self-balanced while on the boundary Γ homogeneous static boundary conditions hold in such locations and in such directions where the original problem has the respective static boundary conditions. To put it otherwise, the elements of the field F_{so} satisfy the relationships

$$A^T \sigma_{so} + K u_{so} = 0 \in \Omega, \quad E_p H_\sigma \sigma_{so} = 0 \in \Gamma. \quad (3.3)$$

The additional subscript “o” emphasizes the homogeneous static admissibility of the field and its elements.

A field $F_k = \{\sigma_k, \varepsilon_k, u_k\}$ will be called kinematically admissible if the displacements u_k and the strains ε_k inside the area Ω satisfy the geometric equations and the kinematic boundary conditions on the boundary Γ . In other words, the elements of the field F_k satisfy the following relationships:

$$\varepsilon_k = A u_k \in \Omega, \quad E_u (H_u u_k - \bar{u}) = 0 \in \Gamma. \quad (3.4)$$

The subscript “k” emphasizes the kinematic admissibility of the field and its elements.

A field $F_{ko} = \{\boldsymbol{\sigma}_{ko}, \boldsymbol{\varepsilon}_{ko}, \mathbf{u}_{ko}\}$ will be called *homogeneously kinematically admissible* if the displacements $\mathbf{u}_{ko}$ and the strains $\boldsymbol{\varepsilon}_{ko}$ satisfy the geometric equations inside the area Ω and the homogeneous kinematic boundary conditions on the boundary Γ . In other words, the elements of the field F_{ko} meet the conditions

$$\boldsymbol{\varepsilon}_{ko} = \mathbf{A}\mathbf{u}_{ko} \in \Omega, \quad \mathbf{E}_u \mathbf{H}_u \mathbf{u}_{ko} = \mathbf{0} \in \Gamma. \quad (3.5)$$

A field $F_{s/2} = \{\boldsymbol{\sigma}_{s/2}, \boldsymbol{\varepsilon}_{s/2}, \mathbf{u}_{s/2}\}$ will be called *statically semi-admissible* if the elements of the field satisfy the static boundary conditions

$$\mathbf{E}_p \bar{\mathbf{p}} = \mathbf{E}_p \mathbf{H}_\sigma \boldsymbol{\sigma}_{s/2} \in \Gamma. \quad (3.6)$$

The equations of equilibrium in the volume of the body, Ω , are not required to hold with such a field.

A field $F_{k/2} = \{\boldsymbol{\sigma}_{k/2}, \boldsymbol{\varepsilon}_{k/2}, \mathbf{u}_{k/2}\}$ will be called *kinematically semi-admissible* if the displacements $\mathbf{u}_{k/2}$ and the strains $\boldsymbol{\varepsilon}_{k/2}$ satisfy the geometric equations inside the area Ω ,

$$\boldsymbol{\varepsilon}_{k/2} = \mathbf{A}\mathbf{u}_{k/2} \in \Omega. \quad (3.7)$$

There are no requirements to the values of the displacements $\mathbf{u}_{k/2}$ or their derivatives on the boundary Γ .

It will be useful to introduce, together with the definitions of stress-and-strain fields, the notion of a *field of external actions*, $V = \{\bar{\mathbf{X}}, \bar{\mathbf{p}}, \bar{\mathbf{u}}\}$, which will mean a set of given external forces, $\bar{\mathbf{X}}$, distributed over the area Ω , contour forces $\bar{\mathbf{p}}$ specified on the boundary Γ , and given boundary displacements $\bar{\mathbf{u}}$.

1.4 The general principle of statics and geometry

Consider two states of a mechanical system; one of those will be named the state 1 and the other the state 2. Let the state 1 be defined by the field $F_1 = \{\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \mathbf{u}_1\}$ and the state 2 by the field $F_2 = \{\boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \mathbf{u}_2\}$. For now, let's think both fields are absolutely independent.

Assuming $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1$ and $\mathbf{u} = \mathbf{u}_2$ in the basic integral identity (2.19) and taking into account the conjugation between the geometry and equilibrium operators will give

$$(\boldsymbol{\sigma}_1, \mathbf{A}\mathbf{u}_2) = (\mathbf{A}^T \boldsymbol{\sigma}_1, \mathbf{u}_2) + (\mathbf{E}_p \mathbf{p}_1, \mathbf{E}_p \mathbf{u}_2)_\Gamma + (\mathbf{E}_u \mathbf{p}_1, \mathbf{E}_u \mathbf{u}_2)_\Gamma \quad (4.1)$$

where the contour stresses, $\mathbf{p}_1 = \mathbf{H}_\sigma \boldsymbol{\sigma}_1$, are taken from the state 1 and the contour displacements, $\mathbf{u}_2 = \mathbf{H}_u \mathbf{u}_2$, from the state 2.

By adding external volumetric forces, $\mathbf{X}_1 = \mathbf{A}^\top \boldsymbol{\sigma}_1 + \mathbf{K} \mathbf{u}_1$, which are in static conformance to the field F_1 , we make the relationship (4.1) look like

$$(\mathbf{X}_1, \mathbf{u}_2) + (\mathbf{E}_p \mathbf{p}_1, \mathbf{E}_p \mathbf{u}_2)_\Gamma + (\mathbf{E}_u \mathbf{p}_1, \mathbf{E}_u \mathbf{u}_2)_\Gamma = (\boldsymbol{\sigma}_1, \mathbf{A} \mathbf{u}_2) + (\mathbf{K} \mathbf{u}_1, \mathbf{u}_2). \quad (4.2)$$

Actually, (4.2) is an equivalent of the basic integral identity (2.19). Now, by giving the states 1 and 2 a certain sense we can derive various versions and mechanical treatments of the basic integral identity in the form (4.2).

First of all, note that the left part in (4.2) is a *virtual work* $\mathbf{A}_{12}$ of the *external forces* in static conformance with the state 1 on the displacements of the state 2 corresponding to those forces,

$$\mathbf{A}_{12} = (\mathbf{X}_1, \mathbf{u}_2) + (\mathbf{E}_p \mathbf{p}_1, \mathbf{E}_p \mathbf{u}_2)_\Gamma + (\mathbf{E}_u \mathbf{p}_1, \mathbf{E}_u \mathbf{u}_2)_\Gamma. \quad (4.3)$$

We should add some explanation to the definition of the virtual work of external forces according to (4.3). The matter is that the external contour forces, $\mathbf{p}_1$, include the full set of the contour forces in the state 1, together with the contour forces $\mathbf{E}_u \mathbf{p}_1$ in such locations and in such directions where the kinematic boundary conditions are specified. In other words, the expression of $\mathbf{A}_{12}$ from (4.3) includes the work of the contour reactive forces of the state 1 in the form $(\mathbf{E}_u \mathbf{p}_1, \mathbf{E}_u \mathbf{u}_2)_\Gamma$. This part of the work can be different from zero because the field of displacements, $\mathbf{u}_2$, is not required to be homogeneously kinematically admissible, thus it does not have to satisfy homogeneous kinematic boundary conditions.

Finally, it should be noted that the expression (4.3) is not an actual work of the external forces of the state 1 but a *virtual work* which would be done by the forces hypothetically if these forces could be somehow maintained in the body when the displacements $\mathbf{u}_2$ appeared in our deformable body.

Next, let's define the right part of (4.2) as a *virtual work* $\mathbf{B}_{12}$ of the *internal forces (stresses)* of the state 1 on the respective displacements of the state 2, taken with the opposite sign.

$$\mathbf{B}_{12} = -(\boldsymbol{\sigma}_1, \mathbf{A} \mathbf{u}_2) - (\mathbf{K} \mathbf{u}_1, \mathbf{u}_2). \quad (4.4)$$

Note that the internal forces in the mechanical system in question are of a dual nature. On one hand, we can treat the stresses $\boldsymbol{\sigma}$ as internal forces which appear in the material of an elastic body. Another source of the internal forces is related to the forces

$$\mathbf{r} = \mathbf{K} \mathbf{u} \quad (4.5)$$

created by an elastic medium in which the deformable solid is placed. The forces of this kind are *reactions* of the elastic medium sometimes called a *response of an elastic foundation*.

As one can see from (4.4), the expression of the virtual work of the internal forces, B_{12} , consists of two components where the first term is interpreted as a virtual work of the stresses σ_1 and the second term is treated as a virtual work of the internal forces of the elastic medium. Note also that the assumed independence of the three elements of the field F_2 makes it impossible, in the general case, to identify the virtual work of the stresses σ_1 in the form $-(\sigma_1, A u_2)$ with the scalar product, $-(\sigma_1, \varepsilon_2)$, of the stresses of the field F_1 and the strains of the field F_2 . These two can be identified with each other only when the field F_2 is at least semi-admissible kinematically.

Now the relationships (4.3) and (4.4) turn (4.2) into

$$A_{12} + B_{12} = 0, \quad (4.6)$$

and this form of it is called the *general principle of statics and geometry* [8]. This entitlement for (4.6) is totally justified because, first, the relationship can be proved without any physical state equations and it does not have anything in common with any thermodynamic considerations being of pure static and geometric nature; second, it is formulated on the basis of two arbitrarily chosen states of the mechanical system.

An inverse proposition can be formulated just as easily: if (4.6) holds for a certain state 1, then this relationship implies the equilibrium equations for the state both in the volume of the body Ω and on its boundary Γ . To see this, note that (4.6) is a brief form of the relationship (4.2) which we assume to hold for a certain set of external forces X_1 and p_1 not subject to any additional conditions so far. Transforming (4.2) with help of the basic integral identity in the form (2.15) and regrouping its terms will give

$$(A^T \sigma_1 + K u_1 - X_1, u_2) + (H_\sigma \sigma_1 - p_1, u_2)_\Gamma = 0. \quad (4.7)$$

Assuming that the relationship (4.7) holds for any fields of displacements u_2 , we conclude that the scalar products from (4.7) should be equal to zero separately as well. The properties of the scalar products imply that the field $F_1 = \{\sigma_1, \varepsilon_1, u_1\}$ is balanced by the volumetric forces X_1 and the contour forces p_1 .

Thus, in terms of mechanics the general principle of statics and geometry (GPSG) can be read as follows:

It is necessary and sufficient for a linearly deformable mechanical system to remain in equilibrium in a certain state that the overall virtual work of all external and internal forces of this state of the system be equal to zero on any displacements.

It should be emphasized once more that we speak about the *virtual* work, and the two states of the system which participate in the GPSG formulation are not related to each other in any way.

This formulation of the general principle of the statics and geometry operates the following three categories:

- an expression of the work of internal forces (WIF);
- equilibrium equations (EE);
- geometric equations (GE).

The fact that the general principle of statics and geometry contains implicitly the geometric equations becomes obvious after we represent the virtual work of the stresses $\boldsymbol{\sigma}_1$ as $-(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_2)$; this can be done only if the geometric equations $\boldsymbol{\varepsilon}_2 = \mathbf{A}\mathbf{u}_2$ hold.

There is a tradition in courses of structural mechanics and strength of materials: the definition of the virtual work of internal forces is not introduced in axiomatic manner on the basis of an expression like (4.4); instead, certain direct mechanical considerations are used. Let's connect the considerations of that kind with our general case of the geometrically linear structural analysis and do the following reasoning.

Divide mentally a deformable solid body into a set of elementary bodies of small volume, Ω_i , so that $\sum \Omega_i = \Omega$ where the sum is taken over the whole set of the elementary bodies. Let Γ_i be a boundary surface of i -th elementary body. Next, replace the action of adjacent bodies on a certain elementary body by a vector of forces, $\mathbf{p}_{1i}$, resulted from the stresses $\boldsymbol{\sigma}_1$,

$$\mathbf{p}_{1i} = \mathbf{H}_\sigma \boldsymbol{\sigma}_{1i} \quad (4.8)$$

and acting in every point of the boundary surface Γ_i . Now, for the i -th elementary body the set of the forces $\mathbf{p}_{1i}$ becomes a set of external forces (with respect to this i -th body only) which perform their work on the displacements $\mathbf{u}_{2i} = \mathbf{H}_u \mathbf{u}_{2i}$. It is assumed implicitly that the internal stresses are self-balanced inside each of the elementary bodies in the sense that $\mathbf{A}^T \boldsymbol{\sigma}_{1i} + \mathbf{K} \mathbf{u}_{1i} = \mathbf{0} \in \Omega_i$. Then the virtual work $d\tilde{\mathbf{B}}_{12}$ of the forces $\mathbf{p}_{1i}$ on the displacements of the body, $\mathbf{u}_{2i}$, is defined by the integral

$$d\tilde{\mathbf{B}}_{12} = \int_{\Gamma_i} \mathbf{p}_{1i} \cdot \mathbf{u}_{2i} d\Gamma. \quad (4.9)$$

Now we use the basic integral identity in the form (2.17) in application to the elementary body; let's transform the integral into the form

$$d\tilde{B}_{12} = \int_{\Omega_i} \mathbf{A}\mathbf{u}_{2i} \cdot \boldsymbol{\sigma}_{1i} d\Omega - \int_{\Omega_i} \mathbf{u}_{2i} \cdot \mathbf{A}^T \boldsymbol{\sigma}_{1i} d\Omega = \int_{\Omega_i} \mathbf{A}\mathbf{u}_{2i} \cdot \boldsymbol{\sigma}_{1i} d\Omega + \int_{\Omega_i} \mathbf{K}\mathbf{u}_{1i} \cdot \mathbf{u}_{2i} d\Omega.$$

This allows for the above assumption $\mathbf{A}^T \boldsymbol{\sigma}_{1i} = -\mathbf{K}\mathbf{u}_{1i} \in \Omega_i$.

Now we sum the expression of the work thus obtained over all elementary bodies and derive this:

$$\tilde{B}_{12} = (\boldsymbol{\sigma}_1, \mathbf{A}\mathbf{u}_2) + (\mathbf{K}\mathbf{u}_1, \mathbf{u}_2). \quad (4.10)$$

The quantity $\tilde{B}_{12}$ is defined here as a virtual work of internal forces, but it is calculated with the assumption that the internal forces have been formally converted to external ones. If we compare (4.10) and (4.4), we will have to admit that B_{12} and $\tilde{B}_{12}$ can be matched only by assuming $B_{12} = -\tilde{B}_{12}$. This logical transformation is usually based on the statement that the internal forces resist to deformations in the body and thus perform a negative work⁵. This reasoning is logically vulnerable, which is obvious to anybody who ever tried to explain to an inquisitive student why it is necessary (!?) to alter the sign in (4.10) when deriving the expression of the virtual work of the internal forces.

Actually, the four categories introduced above are related to one another in such way that any three of them can be postulated, and then the fourth one will result from the three. This circumstance was noticed by L.A. Rozin; he composed the following table which we borrow from [8]:

GPSG	EE	WIF	GE
Implied	Postulated	Postulated	Postulated
Postulated	Implied	Postulated	Postulated
Postulated	Postulated	Implied	Postulated
Postulated	Postulated	Postulated	Implied

First two rows of the table conform to the verbal formulation of the general principle of statics and geometry given above; it is assumed that the virtual work of the internal forces is expressed by the relationship (4.4) while the geometric equations are as in (2.2-b). The statements contained

⁵ Here's a characteristic quote on this subject [2]: “*The work done by internal forces is negative because the internal forces tend to resist to displacements caused by the external forces; the internal forces are directed oppositely to the displacements of their application points*”. The essential point is that the subject being discussed is actually a virtual work of the internal forces.

in the third and fourth rows of the table can be proved to be true, too. We will not dwell on this; instead, we recommend [8] to the reader who wants details.

The notion of the virtual work can belong to the category of constructible concepts; one of the confirmations of this fact is that some publications (see [12], or [9], or [10], for example) treat the principle of virtual displacements in the statics of the deformable solid as an equality between the virtual work of the external forces and the virtual work of the internal forces, rather than an equality of the sum of the works to zero. Of course, this requires redefining the notion of the virtual work of the internal forces by introducing the expression (4.4) with the opposite sign. As it could be expected, no further logical contradiction comes from this redefinition of the virtual work of internal forces. Therefore it should be clear that one cannot really justify the necessity of minus in the right part of (4.4) by any logical reasoning.

1.4.1 The principle of virtual displacements as an implication of the general principle of statics and geometry

It is obvious that while the relationship (4.2) holds for arbitrary displacements $\mathbf{u}_2$ as long as the forces $\mathbf{X}_1$ and $\mathbf{p}_1$ balance the internal stresses in the system in its state 1, it is going to hold also for a particular case when the state 2 of the deformable system is assumed to be a homogeneously kinematically admissible state, so that $\mathbf{u}_2 = \mathbf{u}_{ko}$, which gives us immediately

$$(\mathbf{X}_1, \mathbf{u}_{ko}) + (\mathbf{E}_p \mathbf{p}_1, \mathbf{E}_p \mathbf{u}_{ko})_\Gamma = (\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_{ko}) + (\mathbf{K} \mathbf{u}_1, \mathbf{u}_{ko}) . \quad (4.11)$$

This relationship takes into account that a homogeneously kinematically admissible field meets the condition $\mathbf{E}_u \mathbf{u}_{ko} = \mathbf{0} \in \Gamma$.

Now, let's show that the equality (4.11) being held is enough to keep the equilibrium in the volume of the body and on a part of its surface where static boundary conditions are specified. To see this, substitute (4.1) and regroup the terms:

$$(\mathbf{A}^\top \boldsymbol{\sigma}_1 + \mathbf{K} \mathbf{u}_1 - \mathbf{X}_1, \mathbf{u}_{ko}) + (\mathbf{E}_p \mathbf{H}_\sigma \boldsymbol{\sigma}_1 - \mathbf{E}_p \mathbf{p}_1, \mathbf{E}_p \mathbf{u}_{ko})_\Gamma = 0 . \quad (4.12)$$

We use the same reasoning as before and come to the principle of virtual displacements:

It is necessary and sufficient for a linearly deformable mechanical system to remain in equilibrium that the overall virtual work of all external and internal forces on any homogeneously kinematically admissible displacements be equal to zero.

Some courses of structural mechanics advance another requirement when formulating the principle of virtual displacements: the displacements on which the virtual work is calculated must be infinitesimal. This is relevant when we deal with a geometric nonlinearity (second-order analysis). The requirement that homogeneously kinematically admissible displacements should be infinitesimal is not essential in first-order analysis and can be omitted. Even though this is obvious, some authors were confused and suggested various justifications. However, in the linear analysis this principle is totally independent from any particular cases of the nonlinear analysis, therefore no justification is required.

It should be said that the direct application of the principle of virtual displacements sometimes produces a quicker and nicer solution than the use of equilibrium equations. Here is an example to illustrate this statement. Consider a problem shown in Fig. 1.2.

The bars in intermediate nodes of the system are connected as in scissors; two couples of bars come to one node, so that two bars that form a couple and belong to one straight line are attached stiffly while the couples are joined by hinges.

We are required to find the stress in the bar that connects nodes 0 and 1 in this simple, statically determinate system. We could use the equilibrium equations in nodes and move from the system's right end to its left end to find the stress (which we denote as S_{01}); however, the direct application of the principle of virtual displacements will bring us to our goal much faster.

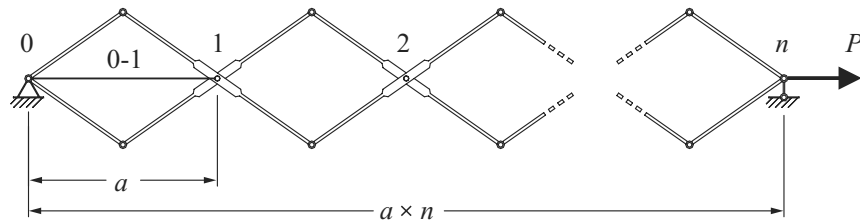


Fig. 1.2. An example of application of the principle of virtual displacements

The system is statically determinate, therefore its internal forces (stresses) in its elements do not depend on physical properties of the elements. That's why we can assume, without limiting the generality, that all bars except for 0-1 are perfectly rigid, that is, their strains are zero – as

is the virtual work of the internal forces which appear in the bars under external forces. As a result, the virtual work of all internal forces in the system can be expressed as $-S_{01}\Delta$ where Δ is a virtual (homogeneously kinematically admissible) displacement of the node 1 along the x -axis. It is easy to see that this displacement of the node 1 causes all cells of the system to distort their size equally, so the difference between the displacements of any two adjacent nodes will be equal to Δ , and the last node, n , will acquire the displacement $n\Delta$. Thus, the virtual work of the external forces will be equal to $Pn\Delta$. The principle of virtual displacements gives

$$Pn\Delta - S_{01}\Delta = 0$$

and, accordingly, $S_{01} = nP$.

1.4.2 The principle of virtual stress increments as an implication of the general principle of statics and geometry

A principle reciprocal with the principle of virtual displacements will be derived again from the general principle of statics and geometry, but this time the state 1 will be a homogeneously statically admissible state; this assumption allows us to rewrite (4.2) in the form

$$(E_u p_{so}, E_u u_2)_\Gamma = (\sigma_{so}, A u_2) + (K u_{so}, u_2). \quad (4.13)$$

We choose only the class of kinematically admissible states from all possible kinds of the state 2 of the system. Then, according to (3.4), the relationship (4.13) will become

$$(E_u p_{so}, E_u \bar{u})_\Gamma = (\sigma_{so}, \epsilon_2) + (K u_{so}, u_2). \quad (4.14)$$

The scalar product in the left part of (4.14) can be interpreted as a virtual work, A_{12} , of all external forces of the state 1 on real displacements of the system (there are no active external forces in the state 1, hence the zero virtual work done by them). If we postulate the expression for the virtual work of the internal forces of the state 1 on the displacements of the state 2 directly via the strains,

$$B_{12} = -(\sigma_{so}, \epsilon_2) - (K u_{so}, u_2), \quad (4.15)$$

we will come to a formulation of the principle of virtual stress increments:

It is necessary and sufficient for a certain state 2 of a linearly deformable mechanical system to be kinematically admissible that the sum of the virtual work, A_{12} , of all external forces of any homogeneously statically admissible state 1 on the real displacements of the system and the virtual work, B_{12} , of the internal forces of the same state on the displacements of the state 2 be equal to zero.

The necessity is obvious because this is just a verbal presentation of the proved equality (4.14), therefore we start with proving the sufficiency.

So, let (4.14) hold for a certain, arbitrary so far, state 2 with any field $F_{so} = \{\sigma_{so}, \varepsilon_{so}, u_{so}\}$ which satisfies the conditions (3.3). Let's prove that the state 2 is kinematically admissible. To do it, we take a scalar product of the first condition in (3.3) and an arbitrary vector λ and then subtract the obtained expression, equal to zero, from the right part of (4.14). The result is

$$(\sigma_{so}, \varepsilon_2) + (Ku_{so}, u_2) - (\lambda, A^T \sigma_{so}) - (\lambda, Ku_{so}) - (E_u p_{so}, E_u \bar{u})_\Gamma = 0. \quad (4.16)$$

With this identity we can already assume that the field $\{\sigma_{so}, \varepsilon_{so}, u_{so}\}$ does not satisfy the first condition in (3.3) because the condition itself is a corollary from (4.16) as the vector λ is an arbitrary one. In particular, we can assume $\lambda = u_2$ and use the basic integral formula (2.19) to derive

$$(\lambda, A^T \sigma_{so}) = (u_2, A^T \sigma_{so}) = (\sigma_{so}, Au_2) - (p_{so}, H_u u_2)_\Gamma,$$

which will yield the following after substituting to (4.16):

$$(\sigma_{so}, \varepsilon_2) - (\sigma_{so}, Au_2) + (p_{so}, H_u u_2)_\Gamma - (E_u p_{so}, E_u \bar{u})_\Gamma = 0.$$

Now we use the second condition in (3.3) and the fact that $E_p + E_u = I$ to transform the latter identity into the following:

$$(\sigma_{so}, \varepsilon_2 - Au_2) - (E_u p_{so}, E_u (\bar{u} - u_2))_\Gamma = 0. \quad (4.17)$$

As (4.17) should hold for any homogeneously statically admissible field $\{\sigma_{so}, \varepsilon_{so}, u_{so}\}$, we can conclude that either scalar product from the left part of (4.17) should be equal to zero, thus

$$\varepsilon_2 - Au_2 = 0 \quad \in \Omega, \quad E_u (\bar{u} - u_2) = 0 \quad \in \Gamma, \quad (4.18)$$

which are the same requirements (3.4) of the kinematic admissibility of the state 2.

1.4.3 Theorem of field orthogonality

We can derive the following statement from the formula (4.13) as a particular case:

The virtual work of internal stresses of a homogeneously statically admissible state (field) on the displacements of a homogeneously kinematically admissible state (field) is equal to zero.

To put it another way,

$$(\boldsymbol{\sigma}_{so}, \boldsymbol{\varepsilon}_{ko}) + (\mathbf{K}\mathbf{u}_{so}, \mathbf{u}_{ko}) = 0. \quad (4.19)$$

To see this, we use the formula (4.13) and take a homogeneously kinematically admissible state as the state 2. Seeing that $\mathbf{E}_u \mathbf{u}_{ko} = \mathbf{0} \in \Gamma$, we have (4.19).

The formulation of the field orthogonality theorem can be expressed in another way:

If the virtual work of the internal stresses of a certain state 1 of a system on the displacements of any homogeneously kinematically admissible state of the same system is equal to zero, then the state 1 is homogeneously statically admissible.

In other words, the equality

$$(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_{ko}) + (\mathbf{K}\mathbf{u}_1, \mathbf{u}_{ko}) = 0, \quad (4.20)$$

which is true for an arbitrary homogeneously kinematically admissible field $F_{ko} = \{\boldsymbol{\sigma}_{ko}, \boldsymbol{\varepsilon}_{ko}, \mathbf{u}_{ko}\}$, implies that the state 1 is homogeneously statically admissible. It is clear that this formulation of the orthogonality theorem follows from the principle of virtual displacements.

Finally,

If the virtual work of the internal stresses of any homogeneously statically admissible state on the displacements of a certain state 2 of the system is equal to zero, then the state 2 is homogeneously kinematically admissible.

In other words, the equality

$$(\boldsymbol{\sigma}_{so}, \boldsymbol{\varepsilon}_2) + (\mathbf{K}\mathbf{u}_{so}, \mathbf{u}_2) = 0, \quad (4.21)$$

which is true for an arbitrary homogeneously statically admissible state $F_{so} = \{\boldsymbol{\sigma}_{so}, \boldsymbol{\varepsilon}_{so}, \mathbf{u}_{so}\}$, implies that the state 2 is homogeneously kinematically admissible. It is clear also that the latter formulation is a corollary from the principle of virtual stress increments.

All three framed statements above, taken together, will be called a *theorem of field orthogonality*. Further on, after we introduce the notion of an energy metrics, the fundamental relationship (4.19) will be treated also as a condition of orthogonality between a homogeneously statically admissible field, F_{so} , and a homogeneously kinematically admissible field, F_{ko} , in this metrics. This is why we have the right to name the theorem in the way we do in the title of this subsection.

Let's give a tiny illustration of the field orthogonality theorem using a three-bar truss as an example (Fig. 1.3).

In this case, the “stress” vector, $\boldsymbol{\sigma} = [N_1, N_2, N_3]^T$, consists of longitudinal forces/stresses in the bars of the truss which are assumed positive when in tension. The “strain” vector corresponding to the $\boldsymbol{\sigma}$ vector, $\boldsymbol{\varepsilon} = [\Delta_1, \Delta_2, \Delta_3]^T$, consists of the elongations of the same bars. If we denote as U and V the displacements of the only free node of the truss in the respective directions x and y , we find that the homogeneously kinematically admissible strains, $\boldsymbol{\varepsilon}_{ko}$, can be calculated as

$$\boldsymbol{\varepsilon}_{ko} = [U \sin \alpha + V \cos \alpha, V, -U \sin \alpha + V \cos \alpha]^T.$$

The condition of equilibrium of the free node in its projections onto the axes x and y in the absence of external loads will give the homogeneously statically admissible stresses in the bars:

$$\boldsymbol{\sigma}_{so} = [A, -2A \cos \alpha, A]^T$$

where A is an arbitrary constant.

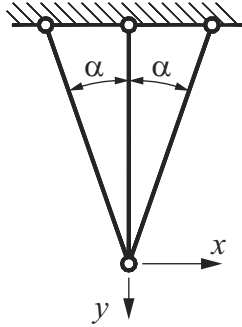


Fig. 1.3. An illustration to the theorem of field orthogonality

Direct substitution helps verify that the equality $(\boldsymbol{\sigma}_{so}, \boldsymbol{\varepsilon}_{ko}) = \boldsymbol{\sigma}_{so}^T \boldsymbol{\varepsilon}_{ko} = 0$ holds; this corresponds to (4.19) because $\mathbf{K} = \mathbf{O}$ in this case.

1.4.4 Integral identity by Papkovich

P.F. Papkovich [6] derived an integral identity in the theory of elasticity, which he called a *general expression of the work of external forces*. Papkovich used this identity to derive various relationships and theorems applicable to the variational formulations in elasticity. Now we will show how this identity looks for a general elastic body all equations of which are written in the operator form that we use in this chapter.

Consider the following four independent states of a mechanical system. Let us assume that in the state 1 we know all external forces (active and reactive), such as: a vector of external volumetric forces, $\mathbf{X}_1$, and a vector of surface forces, $\mathbf{p}_1$. The forces are supposed to be known on the whole boundary surface of the body, including places where the kinematic conditions are specified in the original problem. Only a displacement field, $\mathbf{u}_2$, is supposed to be known in the state 2. The state 3 of the system is defined by a stress field, $\boldsymbol{\sigma}_3$, only. Finally, we know only a strain field, $\boldsymbol{\varepsilon}_4$, in the state 4. The four listed states of the system are in no relation with one another, that is, they are arbitrary.

Following the expression (4.3), we define a virtual work, A_{12} , of the external forces of the state 1 on the respective displacements of the state 2 as

$$A_{12} = (\mathbf{X}_1, \mathbf{u}_2) + (\mathbf{E}_p \mathbf{p}_1, \mathbf{E}_p \mathbf{u}_2)_\Gamma + (\mathbf{E}_u \mathbf{p}_1, \mathbf{E}_u \mathbf{u}_2)_\Gamma. \quad (4.22)$$

Note once more that the term $\mathbf{E}_u \mathbf{p}_1$ means the reactive forces in the state 1 which appear in the exact locations of the contour Γ and in the exact directions where the kinematic conditions are defined.

According to Papkovich, the expression of A_{12} can be represented also in the form

$$\begin{aligned} A_{12} = & (\mathbf{X}_1 - \mathbf{A}^\top \boldsymbol{\sigma}_3, \mathbf{u}_2) + (\boldsymbol{\sigma}_3, \mathbf{A} \mathbf{u}_2 - \boldsymbol{\varepsilon}_4) - (\mathbf{E}_p \mathbf{H}_\sigma \boldsymbol{\sigma}_3 - \mathbf{E}_p \mathbf{p}_1, \mathbf{E}_p \mathbf{u}_2)_\Gamma - \\ & - (\mathbf{E}_u \mathbf{H}_\sigma \boldsymbol{\sigma}_3 - \mathbf{E}_u \mathbf{p}_1, \mathbf{E}_u \mathbf{u}_2)_\Gamma + (\boldsymbol{\sigma}_3, \boldsymbol{\varepsilon}_4). \end{aligned} \quad (4.23)$$

The equivalence of the expressions (4.22) and (4.23) can be checked directly by using the basic integral identity (2.19) with the second and third states of the system, that is, in the form

$$(\mathbf{A}^\top \boldsymbol{\sigma}_3, \mathbf{u}_2) = (\boldsymbol{\sigma}_3, \mathbf{A} \mathbf{u}_2) - (\mathbf{E}_p \mathbf{H}_\sigma \boldsymbol{\sigma}_3, \mathbf{E}_p \mathbf{u}_2)_\Gamma - (\mathbf{E}_u \mathbf{H}_\sigma \boldsymbol{\sigma}_3, \mathbf{E}_u \mathbf{u}_2)_\Gamma. \quad (4.24)$$

As can be seen, substituting (4.24) to the right part of (4.23) will annihilate all terms related to the third and fourth states, and the rest will coincide with (4.22).

Further, taking mechanical states with particular properties as each of the four independent states of the system can turn (4.23) into various

relationships and theorems related to the variational formulations of structural mechanics [6].

1.5 Final comments to Chapter 1

First of all, we have to explicate our usage of the term *principle*. The literature on the structural mechanics uses the words *principle* and *law* sometimes as mutual equivalents. One can occasionally run into an expression like “law of virtual displacements”. In principle⁶, this should not make any confusion if the context makes it clear. Nevertheless, in order to avoid ambiguity in terms we had better adhere to the linguistic basis once established. We prefer the terminology advocated by L.A. Rozin. A quote from [8] helps explain what we mean:

“Variational formulations of problems can be represented either as integral identities which mechanics treats as variational principles and refers to as variational equations, or as requirements that certain functionals should be stationary, which mechanics treats as variational laws”.

As the definitions of the static and kinematic semi-admissibility of stress and strain fields are new for the reader⁷, we should emphasize the differences between the notions once again. The criterion for the static semi-admissibility is that the static boundary conditions should be met while the equilibrium equations in the volume of a body are neglected. On the contrary, the kinematic semi-admissibility uses the criterion of the geometric equations satisfied inside the body while the boundary conditions of the geometric type may be violated.

Further, as we use the different notions of stress and strain fields, it would be appropriate for us to name the two basic principles of statics and geometry as the *principle of homogeneously kinematically admissible displacements* and the *principle of homogeneously statically admissible stresses*, to emphasize the reciprocity of the principles by their very names. However, there is a long-living historical tradition of naming these principles otherwise (the principle of virtual displacements and the principle of virtual increments of the stress state) which is widely used in literature, so any deviations from this tradition are undesirable. The *virtual* displacements in mechanics are displacements which do not conflict with

⁶ Another use of the term *principle*, and of a totally different meaning, too.

⁷ Strictly speaking, one of the notions, the static semi-admissibility of a stress field, was introduced by us earlier in the paper [11].

constraints of a particular mechanical system. In our situation they are equivalent to displacements from a homogeneously kinematically admissible field.

It should be said that the notion of the homogeneously kinematically admissible displacements (as well as just kinematically admissible displacements) actually needs a more accurate definition. Formal mathematics advances two requirements relevant to this notion:

- the displacements should satisfy the respective kinematic boundary conditions;
- the displacements should be sufficiently smooth functions.

Different definitions of a problem's solution (a classic, strong, weak, semi-weak solution [1]) require a different smoothness of the displacements, therefore the very set of the kinematically admissible displacements will change together with the definition of the solution. In the context that we use, we require the smoothness corresponding to the semi-weak solution definition. This admits the $\mathcal{A}\mathbf{u}$ operation, at the least.

The theorem of field orthogonality was obtained here as a corollary of the principle of virtual displacements and the principle of virtual variations of the stress state. The reader can make sure (we do not dwell on it here) that the inverse relationship is also true: the theorem of field orthogonality implies the two basic versions of the variational principles. In that sense the theorem of field orthogonality can be treated as a fundamental static-geometric statement.

And the final note. As we have said before, the general principle of statics and geometry written as an integral identity (4.2) is a good starting point for obtaining various partial but in a certain way important versions of the basic variational principles. All of those are derived from (4.2) by giving particular meanings to the states 1 and 2 of a mechanical system, which states were quite arbitrary from the beginning. This is the way they derive a whole bunch of partial variational principles in [8] where the interested reader can find the details. We confine ourselves intentionally to only two: the principle of virtual displacements and the principle of virtual variations of the stress state. These two principles can be called 'basic' due to the following reasons. First, these two were the first to appear, and many known works on elasticity and/or structural mechanics discuss nothing more in their presentment of variational principles of statics and geometry – refer to [5] for an example. Second, as we will show in the next chapter, these two principles are in tight connection with two classic variational laws: the Lagrange variational principle and the Castigliano variational principle, which we also call basic ones and which have an important property of extremality. Third, all the other variational principles

and laws (which we call *supplementary*) can be derived from the basic ones by means of various known transformations such as: the Lagrange transformation, the Legendre transformation, the Friedrichs transformation. More details on the use of these transformations can be found in the book by L.A. Rozin [7].

Finally, we should mention a so-called Vorobiov identity [5] formulated for a general nonlinear analysis. In the particular case of linear elasticity this identity becomes

$$(\mathbf{A}\mathbf{u}_k, \boldsymbol{\sigma}_s) = (\mathbf{u}_k, \bar{\mathbf{X}}) + (\mathbf{E}_p \bar{\mathbf{p}}, \mathbf{E}_p \mathbf{u}_k)_\Gamma + (\mathbf{E}_u \mathbf{H}_\sigma \boldsymbol{\sigma}_s, \mathbf{E}_u \bar{\mathbf{u}})_\Gamma.$$

Taking into account the static admissibility of the stresses $\boldsymbol{\sigma}_s$ and the kinematic admissibility of the displacements $\mathbf{u}_k$ will transform this identity into the following:

$$(\mathbf{A}\mathbf{u}_k, \boldsymbol{\sigma}_s) = (\mathbf{u}_k, \mathbf{A}^\top \boldsymbol{\sigma}_s) + (\mathbf{E}_p \mathbf{H}_\sigma \boldsymbol{\sigma}_s, \mathbf{E}_p \mathbf{u}_k)_\Gamma + (\mathbf{E}_u \mathbf{H}_\sigma \boldsymbol{\sigma}_s, \mathbf{E}_u \mathbf{u}_k)_\Gamma.$$

Now it is easy to see that the Vorobiov identity in linear elasticity is just another form of the same basic integral identity (2.19) in application to statically admissible stresses and kinematically admissible displacements.

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