# 1. The Introduction of Quarks

About 70 years ago, only a small number of “elementary particles”,[1] thought to be the basic building blocks of matter, were known: the proton, the electron, and the photon as the quantum of radiation. All these particles are stable (the neutron is stable only in nuclear matter, the free neutron decays by beta decay: $\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}$). Owing to the availability of large accelerators, this picture of a few elementary particles has profoundly changed: today, the standard reference *Review of Particle Properties*[2] lists more than 100 particles. The number is still growing as the energies and luminosities of accelerators are increased.

## 1.1 The Hadron Spectrum

The symmetries known from classical and quantum mechanics can be utilized to classify the “elementary-particle zoo”. The simplest baryons are p and n; the simplest leptons $\mathrm{e}^-$ and $\mu^-$. Obviously there are many other particles that must be classified as baryons or leptons.

The symmetries are linked to conserved quantum numbers such as the baryon number $B$, isospin $T$ with $z$ component $T_3$, strangeness $S$, hypercharge $Y = B + S$, charge $Q = T_3 + Y/2$, spin $I$ with $z$ component $I_z$, parity $\pi$, and charge conjugation parity $\pi_c$. Conservation laws for such quantum numbers manifest themselves by the absence of certain processes. For example, the hydrogen atom does not decay into two photons: $\mathrm{e}^- + \mathrm{p} \to \gamma + \gamma$, although this process is not forbidden either by energy–momentum conservation or by charge conservation. Since our world is built mainly out of hydrogen, we know from our existence that there must be at least one other conservation law that is as fundamental as charge conservation. The nonexistence of the decays $\mathrm{n} \to \mathrm{p} + \mathrm{e}^-$ and $\mathrm{n} \to \gamma + \gamma$ also indicates the presence of a new quantum number. The proton and neutron are given a baryonic charge $B = 1$, the electron $B = 0$. Similarly the electron is assigned leptonic charge $L = 1$, the nucleons $L = 0$. From the principle of simplicity it appears very unsatisfactory to regard all observed particles

[1] For a detailed discussion of the content of this chapter see W. Greiner and B. Müller: *Symmetries* (Springer, Berlin, Heidelberg 1994).

[2] See the *Review of Particle Physics* by W.-M. Yao et al., Journal of Physics G **33** (2006) 1, and information available online at http://pdg.lbl.gov/

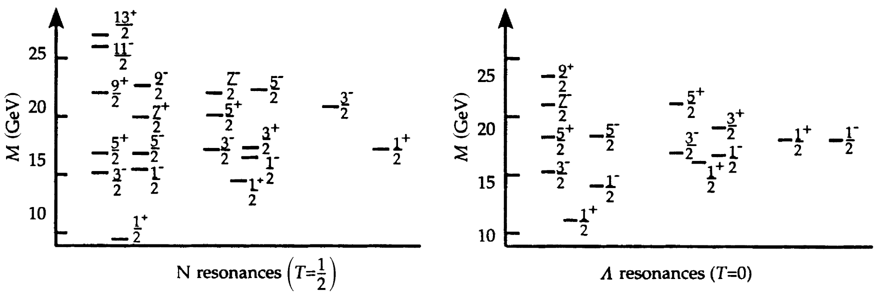


**Fig. 1.1.** The mass spectra of baryons. Plotted are the average masses of the multiplets. For example, the state $N_{5/2^+}$ at 1.68 MeV stands for two particles, one protonlike and one neutronlike, both with spin 5/2 and positive internal parity. The figure contains 140 particle states in total

as elementary. To give an impression of the huge number of hadrons known today, we have collected together the baryon resonances in Fig. 1.1. The data are taken from the "Review of Particle Properties". Particles for which there is only weak evidence or for which the spin $I$ and internal parity $P$ have not been determined have been left out. Note that each state represents a full multiplet. The number of members in a multiplet is $N = 2T + 1$ with isospin $T$. Thus the 13 $\Delta$ resonances shown correspond to a total of 52 different baryons.

When looking at these particle spectra, one immediately recognizes the similarity to atomic or nuclear spectra. One would like, for example, to classify the nucleon resonances (N resonances) in analogy to the levels of a hydrogen atom. The $\frac{1}{2}^+$ ground state (i.e., the ordinary proton and neutron) would then correspond to the $1s_{\frac{1}{2}}$ state, the states $\frac{3}{2}^-$, $\frac{1}{2}^-$, and $\frac{1}{2}^+$ at approximately 1.5 GeV to the hydrogen levels $2p_{\frac{3}{2}}$, $2p_{\frac{1}{2}}$, and $2s_{\frac{1}{2}}$, the states $\frac{5}{2}^+$, $\frac{3}{2}^+$, $\frac{3}{2}^-$, $\frac{1}{2}^-$, $\frac{1}{2}^+$ to the sublevels of the third main shell $3d_{\frac{5}{2}}$, $3d_{\frac{3}{2}}$, $3p_{\frac{3}{2}}$, $3p_{\frac{1}{2}}$, $3s_{\frac{1}{2}}$, and so on.

Although one should not take this analogy too seriously, it clearly shows that a model in which the baryons are built from spin-$\frac{1}{2}$ particles almost automatically leads to the states depicted in Fig. 1.1. The quality of any such model is measured by its ability to predict the correct energies. We shall discuss specific models in Sect. 3.1.

We therefore interpret the particle spectra in Fig. 1.1 as strong evidence that the baryons are composed of several more fundamental particles and that most of the observable baryon resonances are excitations of a few ground states. In this way the excited states $\frac{3}{2}^-$ and $\frac{1}{2}^-$ are reached from the nucleon ground state N(938 MeV) $\frac{1}{2}^+$ by increasing the angular momentum of one postulated component particle by one: $\frac{1}{2}^+$ can be coupled with $1^-$ to give $\frac{1}{2}^-$ or $\frac{3}{2}^-$. As the energy of the baryon resonances increases with higher spin (i.e., total angular momentum of all component particles), one can deduce that all relative orbital angular momenta vanish in the ground states.

To investigate this idea further, one must solve a purely combinatorial problem: How many component particles (called *quarks* in the following) are needed, and what properties are required for them to correctly describe the ground states of the hadron spectrum? It turns out that the existence of several quarks must be postulated. The quantum numbers given in Table 1.1 must be given to them.

**Table 1.1.** Quark charge ($Q$), isospin ($T$, $T_3$), and strangeness ($S$)

| | $Q$ | $T$ | $T_3$ | $S$ |
|---|---|---|---|---|
| u | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| d | $-\frac{1}{3}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 |
| s | $-\frac{1}{3}$ | 0 | 0 | $-1$ |
| c | $\frac{2}{3}$ | 0 | 0 | 0 |
| t | $\frac{2}{3}$ | 0 | 0 | 0 |
| b | $-\frac{1}{3}$ | 0 | 0 | 0 |

The three light quarks u, d, s can be identified with the three states in the fundamental representation of SU(3). This is initially a purely formal act. It gains importance only as one shows that the *branching ratios* of particle reactions and the *mass differences* between stable baryons show – at least approximately – the same symmetries. This means that the so-called *flavor* SU(3) can be interpreted as the symmetry group of a more fundamental interaction.

Hadrons are therefore constructed as flavor SU(3) states. As the spin of the quarks must also be taken into account, the total symmetry group becomes SU(3) × SU(2). As an example we give the decomposition of the neutron into quark states[3]:

$$\begin{aligned}|\mathrm{n}\uparrow\rangle = \frac{1}{\sqrt{18}}\Big(& 2\,|\mathrm{d}\uparrow\rangle\,|\mathrm{d}\uparrow\rangle\,|\mathrm{u}\downarrow\rangle - |\mathrm{d}\uparrow\rangle\,|\mathrm{d}\downarrow\rangle\,|\mathrm{u}\uparrow\rangle - |\mathrm{d}\downarrow\rangle\,|\mathrm{d}\uparrow\rangle\,|\mathrm{u}\uparrow\rangle \\ & - |\mathrm{d}\uparrow\rangle\,|\mathrm{u}\uparrow\rangle\,|\mathrm{d}\downarrow\rangle + 2\,|\mathrm{d}\uparrow\rangle\,|\mathrm{u}\downarrow\rangle\,|\mathrm{d}\uparrow\rangle - |\mathrm{d}\downarrow\rangle\,|\mathrm{u}\uparrow\rangle\,|\mathrm{d}\uparrow\rangle \\ & - |\mathrm{u}\uparrow\rangle\,|\mathrm{d}\uparrow\rangle\,|\mathrm{d}\downarrow\rangle - |\mathrm{u}\uparrow\rangle\,|\mathrm{d}\downarrow\rangle\,|\mathrm{d}\uparrow\rangle + 2\,|\mathrm{u}\downarrow\rangle\,|\mathrm{d}\uparrow\rangle\,|\mathrm{d}\uparrow\rangle \Big) \ . \end{aligned} \tag{1.1}$$

Particularly interesting for the topic of this volume are the corresponding decompositions of the states $\Omega^-$, $\Delta^{++}$, and $\Delta^-$ (see [3]):

$$\begin{aligned} \left|\Omega^-\right\rangle &= |\mathrm{s}\uparrow\rangle\,|\mathrm{s}\uparrow\rangle\,|\mathrm{s}\uparrow\rangle \ , \\ \left|\Delta^{++}\right\rangle &= |\mathrm{u}\uparrow\rangle\,|\mathrm{u}\uparrow\rangle\,|\mathrm{u}\uparrow\rangle \ , \\ \left|\Delta^-\right\rangle &= |\mathrm{d}\uparrow\rangle\,|\mathrm{d}\uparrow\rangle\,|\mathrm{d}\uparrow\rangle \ . \end{aligned} \tag{1.2}$$

To obtain the spin quantum numbers of hadrons, one must assume that the quarks have spin $\frac{1}{2}$. This poses a problem: spin-$\frac{1}{2}$ particles should obey Fermi statistics, i.e., no two quarks can occupy the same state. So the three quarks in $\Omega^-$, $\Delta^{++}$, and $\Delta^-$ must differ in at least one quantum number, as we shall discuss in Chapt. 4. Before proceeding to the composition of baryons from quarks, we shall first repeat the most important properties of the symmetry groups SU(2) and SU(3).

SU(2) and SU(3) are special cases of the group SU($N$) the special unitary group in $N$ dimensions. Any unitary square matrix $\hat{U}$ with $N$ rows and $N$ columns can be written as (for more details see [3])

$$\hat{U} = \mathrm{e}^{\mathrm{i}\hat{H}} \ , \tag{1.3}$$

where $\hat{H}$ is a Hermitian matrix. The matrices $\hat{U}$ form the group SU($N$) of unitary matrices in $N$ dimensions. $\hat{H}$ is Hermitian, i.e.,

$$\hat{H}^*_{ij} = \hat{H}_{ji} \ . \tag{1.4}$$

Of the $N^2$ complex parameters (elements of the matrices), $N^2$ real parameters for $\hat{H}$ and hence for $\hat{U}$ remain, owing to the auxiliary conditions (1.4). Since $\hat{U}$

[3] W. Greiner and B. Müller: *Quantum Mechanics: Symmetries* (Springer, Berlin, Heidelberg, 1994).

is unitary, i.e. $\hat{U}^\dagger \hat{U} = 1$, $\det \hat{U}^\dagger \det \hat{U} = (\det \hat{U})^* \det \hat{U} = 1$ and thus

$$\left|\det \hat{U}\right| = 1 \ . \tag{1.5}$$

Owing to (1.4), $\mathrm{tr}\left\{\hat{H}\right\} = \alpha$ ($\alpha$ real) and

$$\det \hat{U} = \det\left(\mathrm{e}^{\mathrm{i}\hat{H}}\right) = \mathrm{e}^{\mathrm{i}\,\mathrm{tr}\hat{H}} = \mathrm{e}^{\mathrm{i}\alpha} \ . \tag{1.6}$$

If we additionally demand the condition

$$\det \hat{U} = +1 \ , \tag{1.7}$$

i.e., $\alpha = 0 \mod 2\pi$, only $N^2 - 1$ parameters remain. This group is called the *special unitary group* in $N$ dimensions ($\mathrm{SU}(N)$).

Let us now consider a group element $\hat{U}$ of $\mathrm{U}(N)$ as a function of $N^2$ parameters $\phi_\mu$ ($\mu = 1, \dots, n$). To this end, we write (1.3) as

$$\hat{U}(\phi_1, \dots, \phi_n) = \exp\left(-\mathrm{i}\sum_\mu \phi_\mu \hat{L}_\mu\right) \ , \tag{1.8}$$

where $\hat{L}_\mu$ are for the time being unknown operators:

$$-\mathrm{i}\hat{L}_\mu = \left.\frac{\partial \hat{U}(\boldsymbol{\phi})}{\partial \phi_\mu}\right|_{\boldsymbol{\phi}=0} \tag{1.9}$$

($\boldsymbol{\phi} = (\phi_1, \dots, \phi_n)$). For small $\phi_\mu (\delta\phi_\mu)$ we can expand $\hat{U}$ in a series ($\mathbb{1}$ is the $N \times N$ unit matrix):

$$\hat{U}(\boldsymbol{\phi}) \approx \mathbb{1} - \mathrm{i}\sum_{\mu=1}^{n} \delta\phi_\mu \hat{L}_\mu - \frac{1}{2}\sum_{\mu,\nu} \delta\phi_\mu \delta\phi_\nu \hat{L}_\mu \hat{L}_\mu + \dots \ . \tag{1.10}$$

Boundary conditions (1.4) and (1.5) imply after some calculation that the operators $\hat{L}_i$ must satisfy the commutation relations

$$\left[\hat{L}_i, \hat{L}_j\right] = c_{ijk}\hat{L}_k \ . \tag{1.11}$$

Equation (1.11) defines an algebra, the *Lie algebra* of the group $\mathrm{U}(N)$.

The operators $\hat{L}_i$ generate the group by means of (1.10) and are thus called *generators*. Obviously there are as many generators as the group has parameters, i.e., the group $\mathrm{U}(N)$ has $N^2$ generators and the group $\mathrm{SU}(N)$ has $N^2 - 1$. The quantities $c_{ijk}$ are called *structure constants* of the group. They contain all the information about the group. In the Lie algebra of the group (i.e., the $\hat{L}_k$), there is a maximal number $R$ of commutating elements $\hat{L}_i$ ($i = 1, \dots, R$)

$$\left[\hat{L}_i, \hat{L}_j\right] = 0 \quad (i = 1, \dots, R) \ . \tag{1.12}$$

$R$ is called the *rank* of the group. The eigenvalues of the $\hat{L}_i$ are, as we shall see, used to classify elementary-particle spectra. We shall now discuss the concepts introduced here using the actual examples of the spin and isospin group SU(2) and the group SU(3).

**SU(2)**. U(2) is the group of lineary independent Hermitian $2 \times 2$ matrices. A well-known representation of it is given by the Pauli matrices and the unit matrix

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \hat{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \; \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.13)$$

These span the space of Hermitian $2 \times 2$ matrices, i.e., they are linearly independent. SU(2) has only three generators; the unit matrix is not used. From (1.3) we can write a general group element of the group SU(2) as

$$\hat{U}(\boldsymbol{\phi}) = \exp\left(-\mathrm{i}\sum_{i=1}^{3}\phi_i\hat{\sigma}_i\right) \quad (1.14)$$

(or, using the summation convention, $\exp(-\mathrm{i}\phi_i\hat{\sigma}_i)$). Here $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ is a shorthand for the parameter of the transformation. The Pauli matrices satisfy the commutation relations

$$\left[\hat{\sigma}_i, \hat{\sigma}_j\right] = 2\mathrm{i}\varepsilon_{ijk}\hat{\sigma}_k \;, \quad (1.15)$$

with

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{for two equal indices,} \\ 1 & \text{for even permutations of the indices,} \\ -1 & \text{for odd permutations of the indices.} \end{cases}$$

Usually, instead of $\hat{\sigma}_i$, the $\hat{S}_i = \frac{1}{2}\,\hat{\sigma}_i$ are used as generators, i.e.

$$\left[\hat{S}_i, \hat{S}_j\right] = \mathrm{i}\varepsilon_{ijk}\hat{S}_k \;.$$

According to (1.11), $\mathrm{i}\varepsilon_{ijk}$ are the structure constants of SU(2). Equation (1.15) shows that no generator commutes with any other, i.e., the rank of SU(2) is 1. According to the *Racah theorem*, the rank of a group is equal to the number of Casimir operators (i.e., those operators are polynomials in the generators and commute with all generators). Thus there is one Casimir operator for SU(2), namely the square of the well-known angular momentum (spin) operator:

$$\hat{C}_{\mathrm{SU(2)}} = \sum_{i=1}^{3}\hat{S}_i^2 \;. \quad (1.16)$$

The representation of SU(2) given in (1.13) (and generally of SU($N$)) by $2 \times 2$ matrices (generally $N \times N$ matrices) is called the *fundamental representation* of SU(2) (SU($N$)). It is the smallest nontrivial representation of SU(2) (SU($N$)). It is a $2 \times 2$ representation for SU(2), a $3 \times 3$ representation for SU(3), and so on. From Schur's first lemma the Casimir operators in the fundamental representation are multiples of the unit matrix (see Exercise 1.1):

$$\hat{C}_{\mathrm{SU}(2)} = \sum_{i=1}^{3} \left(\frac{\hat{\sigma}_i}{2}\right)^2 = \frac{3}{4}\mathbb{1} \ . \tag{1.17}$$

**SU(3)**. The special unitary group in three dimensions has $3^2-1=8$ generators. In the fundamental representation they can be expressed by the Gell-Mann matrices $\hat{\lambda}_1, \ldots, \hat{\lambda}_8$:

$$\hat{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \hat{\lambda}_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \hat{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\hat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} , \quad \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ,$$

$$\hat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} , \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \ . \tag{1.18}$$

**Table 1.2.** The nonvanishing, completely antisymmetric structure constants $f_{ijk}$ and the symmetric constants $d_{ijk}$

| $ijk$ | $f_{ijk}$ | $ijk$ | $d_{ijk}$ |
|---|---|---|---|
| 123 | 1 | 118 | $\frac{1}{\sqrt{3}}$ |
| 147 | 1/2 | 146 | 1/2 |
| 156 | −1/2 | 157 | 1/2 |
| 246 | 1/2 | 228 | $\frac{1}{\sqrt{3}}$ |
| 257 | 1/2 | 247 | −1/2 |
| 345 | 1/2 | 256 | 1/2 |
| 367 | −1/2 | 338 | $\frac{1}{\sqrt{3}}$ |
| 458 | $\frac{\sqrt{3}}{2}$ | 344 | 1/2 |
| 678 | $\frac{\sqrt{3}}{2}$ | 355 | 1/2 |
| | | 366 | −1/2 |
| | | 377 | −1/2 |
| | | 448 | $-\frac{1}{2\sqrt{3}}$ |
| | | 558 | $-\frac{1}{2\sqrt{3}}$ |
| | | 668 | $-\frac{1}{2\sqrt{3}}$ |
| | | 778 | $-\frac{1}{2\sqrt{3}}$ |
| | | 888 | $-\frac{1}{\sqrt{3}}$ |

The Gell-Mann matrices are Hermitian,

$$\hat{\lambda}_i^\dagger = \hat{\lambda}_i \ , \tag{1.19}$$

and their trace vanishes,

$$\mathrm{tr}\left\{\hat{\lambda}_i\right\} = 0 \ . \tag{1.20}$$

They define the Lie algebra of SU(3) by the commutation relations

$$\left[\hat{\lambda}_i, \hat{\lambda}_j\right] = 2\mathrm{i} f_{ijk}\hat{\lambda}_k \ , \tag{1.21}$$

where the structure constants $f_{ijk}$ are, like the $\varepsilon_{ijk}$ in SU(2), completely antisymmetric, i.e.,

$$f_{ijk} = -f_{jik} = -f_{ikj} \ . \tag{1.22}$$

The anticommutation relations of the $\hat{\lambda}_i$ are written as

$$\left\{\hat{\lambda}_i, \hat{\lambda}_j\right\} = \frac{4}{3}\delta_{ij}\mathbb{1} + 2d_{ijk}\hat{\lambda}_k \ . \tag{1.23}$$

The constants $d_{ijk}$ are completely symmetric:

$$d_{ijk} = d_{jik} = d_{ikj} \ . \tag{1.24}$$

The nonvanishing structure constants are given in Table 1.2.

As in SU(2), generators $\hat{F}_i = \frac{1}{2}\hat{\lambda}_i$ ("hyperspin") are used instead of the $\hat{\lambda}_i$ with the commutation relations

$$\left[\hat{F}_i, \hat{F}_j\right] = \mathrm{i}\, f_{ijk} \hat{F}_k \ . \tag{1.25}$$

One can easily check that among the $\hat{F}_i$ only the commutators $\left[\hat{F}_1, \hat{F}_8\right] = \left[\hat{F}_2, \hat{F}_8\right] = \left[\hat{F}_3, \hat{F}_8\right] = 0$ vanish. As the $\hat{F}_i$, $i = 1, 2, 3$, do not commute with each other, there are at most two commuting generators, i.e., SU(3) has rank two (in general SU($N$) has rank $N-1$), and hence two Casimir operators, one of which is simply

$$\hat{C}_1 = \sum_{i=1}^{8} \hat{F}_i^2 = -\frac{2\mathrm{i}}{3} \sum_{i,j,k} f_{ijk}\, \hat{F}_i\, \hat{F}_j\, \hat{F}_k \ . \tag{1.26}$$

In the fundamental representation

$$\left(\hat{C}_1\right)_{j\ell} = \frac{1}{4} \sum_{i=1}^{8} \sum_{k=1}^{3} \left(\hat{\lambda}_i\right)_{jk} \left(\hat{\lambda}_i\right)_{k\ell} = \frac{4}{3}\delta_{j\ell} \ . \tag{1.27}$$

From the structure constants $f_{ijk}$, new matrices $\hat{U}_i$ can be constructed according to

$$\left(\hat{U}_i\right)_{jk} = -\mathrm{i}\, f_{ijk} \ , \tag{1.28}$$

which also satisfy the commutation relations

$$\left[\hat{U}_i, \hat{U}_j\right] = \mathrm{i}\, f_{ijk} \hat{U}_k \ . \tag{1.29}$$

This representation of the Lie algebra of SU(3) is called *adjoint* (or *regular*). In it (see Exercise 1.2)

$$\begin{aligned}
(\hat{C}_l)_{kl} &= \sum_{i=1}^{8} (\hat{U}_i^2)_{kl} = \sum_{i,j} (\hat{U}_i)_{kj} (\hat{U}_i)_{jl} \\
&= -\sum_i \sum_j f_{ikj} f_{ijl} = \sum_{i,j} f_{ijk} f_{ijl} \\
&= 3\delta_{kl} \ .
\end{aligned} \tag{1.30}$$

A form of the complete SU(3) group element according to (1.3) is ($\hat{U}(0)$ designates in contrast to $\hat{U}_i$ the transformation matrix from (1.3))

$$\hat{U}(\theta) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}\cdot\hat{\boldsymbol{F}}} \ , \tag{1.31}$$

where $\hat{\boldsymbol{F}}$ is the vector of eight generators and $\boldsymbol{\theta}$ the vector of eight parameters.

After this short digression into the group structure of SU(2) and SU(3), we return to the classification of elementary particles. As indicated above, the eigenvalues of commuting generators of the group serve to classify the hadrons. For SU(2) there is only one such operator among the $\hat{T}_i$ $(i = 1, 2, 3)$, usually chosen to be $\hat{T}_3$ (the $z$ component). The structure of SU(2) multiplets is thus one-dimensional and characterized by a number $T_3$. In the framework of QCD the most important application of SU(2) is the isospin group (with generators $\hat{T}_i$) and the angular momentum group with the spin operator $\hat{S}_i$. The small mass difference between neutron and proton (0.14% of the total mass) leads to the thought that both can be treated as states of a single particle, the nucleon. According to the matrix representation

$$\hat{T}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\hat{\tau}_3 \ , \tag{1.32}$$

one assigns the isospin vector $\Psi_\mathrm{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to the proton and $\Psi_\mathrm{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the neutron, so that the isospin eigenvalues $T_3 = \pm\frac{1}{2}$ are assigned to the nucleons:

$$\hat{T}_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \tag{1.33}$$

$$\hat{T}_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{1.34}$$

Analogously one introduces

$$\hat{\tau}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \hat{\tau}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{1.35}$$

such that the

$$\hat{T}_k = \frac{1}{2}\,\hat{\tau}_k \qquad (k = 1, 2, 3) \tag{1.36}$$

satisfy the same commutation relations as the spin operators. One can check by direct calculation that raising and lowering operators can be constructed from the $\hat{\tau}_i$:

$$\hat{\tau}_+ = \frac{1}{2}(\hat{\tau}_1 + \mathrm{i}\hat{\tau}_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ ,$$

$$\hat{\tau}_- = \frac{1}{2}(\hat{\tau}_1 - \mathrm{i}\hat{\tau}_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ . \tag{1.37}$$

They have the following well-known action on nucleon states:

$$\begin{aligned} &\hat{\tau}_+\Psi_\mathrm{p} = 0 \ , \qquad &&\hat{\tau}_+\Psi_\mathrm{n} = \Psi_\mathrm{p} \ , \\ &\hat{\tau}_-\Psi_\mathrm{p} = \Psi_\mathrm{n} \ , \qquad &&\hat{\tau}_-\Psi_\mathrm{n} = 0 \ , \end{aligned} \tag{1.38}$$

i.e., the operators change nucleon states into each other (they are also called *ladder operators*). From (1.14) and (1.31), we can give the general transformation in the abstract three-dimensional isospin space

$$\hat{U}(\boldsymbol{\phi}) = \hat{U}(\phi_1, \phi_2, \phi_3) = \mathrm{e}^{-\mathrm{i}\phi_\mu \hat{T}_\mu} \quad , \tag{1.39}$$

where the $\phi_\mu$ represent the rotation angles in isospin space. The Casimir operator of isospin SU(2) is

$$\hat{T}^2 = \hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2 \qquad . \tag{1.40}$$

We can now describe each particle state by an abstract vector $|TT_3\rangle$ (analogously to the spin, as the isospin SU(2) is isomorphic to the spin SU(2)), where the following relations hold:

$$\hat{T}^2 \,|TT_3\rangle = T(T+1)\,|TT_3\rangle \quad , \tag{1.41}$$

$$\hat{T}_3 \,|TT_3\rangle = T_3 \,|TT_3\rangle \quad . \tag{1.42}$$

Thus the nucleons represent an isodoublet with $T = \frac{1}{2}$ and $T_3 = \pm\frac{1}{2}$. The pions ($\pi^\pm$, $\pi^0$) (masses $m(\pi^0) = 135\,\mathrm{MeV}/c^2$ and $m(\pi^\pm) = 139.6\,\mathrm{MeV}/c^2$, i.e. a mass difference of $4.6\,\mathrm{MeV}/c^2$) constitute an isotriplet with $T = 1$ and $T_3 = -1, 0, 1$. Obviously there is a relation between isospin and the electric charge of a particle. For the nucleons the charge operator is immediately obvious:

$$\hat{Q} = \hat{T}_3 + \frac{1}{2}\mathbb{1} \tag{1.43}$$

in units of the elementary charge $e$, while one finds in a similarly simple way for the pions

$$\hat{Q} = \hat{T}_3 \quad . \tag{1.44}$$

To unify both relations, one can introduce an additional quantum number $Y$ (the so-called hypercharge) and describe any state by $T_3$ and $Y$:

$$\hat{Y}\,|YT_3\rangle = Y\,|YT_3\rangle \quad , \tag{1.45}$$

$$\hat{T}_3\,|YT_3\rangle = T_3\,|YT_3\rangle \quad . \tag{1.46}$$

In this way the nucleon is assigned $Y = 1$ and the pion $Y = 0$, so that (1.42) and (1.43) can be written as

$$\hat{Q} = \frac{1}{2}\hat{Y} + \hat{T}_3 \quad . \tag{1.47}$$

Relation (1.45) is the *Gell-Mann–Nishijima relation*. The hypercharge characterizes the center of a charge multiplet. It is often customary to express $Y$ by the strangeness $S$ and the baryon number $B$ using $Y = B + S$. Here $B = +1$ for all baryons, $B = -1$ for antibaryons, and $B = 0$ otherwise (in particular for mesons). Thus $Y = S$ for mesons. To classify elementary particles in the framework of SU(3), it is customary to display them in a $T_3$–$Y$ diagram (see [3]). The baryons with spin $\frac{1}{2}$ constitute an octet in this diagram (see Fig. 1.2).

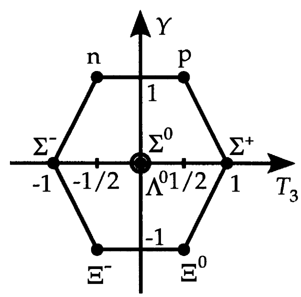


**Fig. 1.2.** An octet of spin-$\frac{1}{2}$ baryons

The spectrum of antiparticles is obtained from this by reflecting the expression with respect to the $Y$ and $T_3$ axes. The heavier baryons and the mesons can be classified analogously. We introduced the hypercharge by means of the charge and have thus added another quantum number. SU(2) has rank 1, i.e., it provides only one such quantum number. SU(3), however, has rank 2 and thus two commuting generators, $\hat{F}_3$ and $\hat{F}_8$. We can therefore make the identification $\hat{T}_3 = \hat{F}_3$ and $\hat{Y} = 2/\sqrt{3}\hat{F}_8$ and interpret the multiplets as SU(3) multiplets. The SU(3)-multiplet classification was introduced by M. Gell-Mann and is initially purely schematic. There are no small nontrivial representations among these multiplets (with the exception of the singlet, interpreted as the $\Lambda^*$ hyperon with mass 1405 MeV/$c^2$ and spin $\frac{1}{2}$). The smallest nontrivial representation of SU(3) is the triplet. This reasoning led Gell-Mann and others to the assumption that physical particles are connected to this triplet, the quarks (from James Joyce's Finnegan's Wake: "Three quarks for Muster Mark"). Today we know that there are six quarks. They are called up, down, strange, charm, bottom, and top quarks. The sixth quark, the top quark, has only recently been discovered[4] and has a large mass[5] $m_{\mathrm{top}} = 178.0 \pm 4.3\,\mathrm{GeV}/c^2$. The different kinds of quarks are called "flavors". The original SU(3) flavor symmetry is therefore only important for low energies, where c, b, and t quarks do not play a role owing to their large mass. It is, also, still relevant for hadronic ground-state properties.

All particles physically observed at this time are combinations of three quarks (baryons) or a quark and an antiquark (mesons) plus, in each case, an arbitrary number of quark–antiquark pairs and gluons. This requires that quarks have

(1) baryon number $\frac{1}{3}$

(2) electric charges in multiples of $\pm\frac{1}{3}$.

Uneven multiples of charge $\frac{1}{3}$ have never been conclusively observed in nature, and there, therefore, seems to exist some principle assuring that quarks can exist in bound states in elementary particles but never free. This is the problem of quark confinement, which we shall discuss later. Up to now, we have considered the SU(3) symmetry connected with the flavor of elementary particles. Until the early 1970s it was commonly believed that this symmetry was the basis of the strong interaction. Today the true strong interaction is widely acknowledged to be connected with another quark quantum number, the *color*. The dynamics of color (chromodynamics) determines the interaction of the quarks (which is, as we shall see, flavor-blind).

Quantum electrodynamics is reviewed in the following chapter. Readers familiar with it are advised to continue on page 77 with Chap. 3.

[4] CDF collaboration (F. Abe *et al.* – 397 authors): Phys. Rev. Lett. **73**, 225 (1994); Phys. Rev. D**50**, 2966 (1994); Phys. Rev. Lett. **74**, 2626 (1995).

[5] DØ collaboration (V. M. Abazov *et al.*): Nature **429**, 638 (10 June 2004); the preprint hep-ex/0608032 by the CDF and DØ collaborations gives a mass of $m_{\mathrm{top}} = 171.4 \pm 2.1\,\mathrm{GeV}/c^2$, resulting from a combined analysis of all data available in 2006.

## EXERCISE

### 1.1 The Fundamental Representation of a Lie Algebra

**Problem.** (a) What are the fundamental representations of the group SU($N$)? (b) Show that according to Schur's lemma the Casimir operators in these fundamental representations are multiples of the unit matrix.

**Solution.** (a) The fundamental representations are those nontrivial representations of a group that have the lowest dimension. All higher-dimensional representations can be constructed from them. We shall demonstrate this using the special unitary groups SU($N$).

**SU(2)**. As we have learned, its representation is characterized by the angular-momentum quantum number $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and states are classified by $(j) \equiv |jm\rangle$, $m = -j, \ldots, +j$. The scalar representation is $j = 0$. The lowest-dimensional representation with $j \neq 0$ would then be $j = \frac{1}{2}$. From it we can construct all others by simply coupling one to another:

$$\left[\frac{1}{2}\right] \times \left[\frac{1}{2}\right] = \left[\frac{1}{2}\right]^2 = [1] + [0] \quad , \tag{1a}$$

$$\left[\frac{1}{2}\right] \times \left[\frac{1}{2}\right] \times \left[\frac{1}{2}\right] = \left[\frac{1}{2}\right]^3 = \left[\frac{3}{2}\right] + \left[\frac{1}{2}\right] + \left[\frac{1}{2}\right] \quad . \tag{1b}$$

"$\times$" indicates the direct product, "+" the direct sum. The first two $j = \frac{1}{2}$ representations can be coupled to $j = 0, 1$. Adding another $j = \frac{1}{2}$, it couples with $j = 1$ to give $j = \frac{3}{2}, \frac{1}{2}$ and with $j = 0$ to give only $j = \frac{1}{2}$. In total, $\left[\frac{1}{2}\right]^3$ contains the representations $\left[\frac{3}{2}\right]$, $\left[\frac{1}{2}\right]$, $\left[\frac{1}{2}\right]$. Figure 1.3 depicts this angular momentum coupling graphically. It must be noted that a representation can appear more than once, e.g., $\left[\frac{1}{2}\right]$ appears twice in $\left[\frac{1}{2}\right]^3$ and [1] thrice in $\left[\frac{1}{2}\right]^4$.

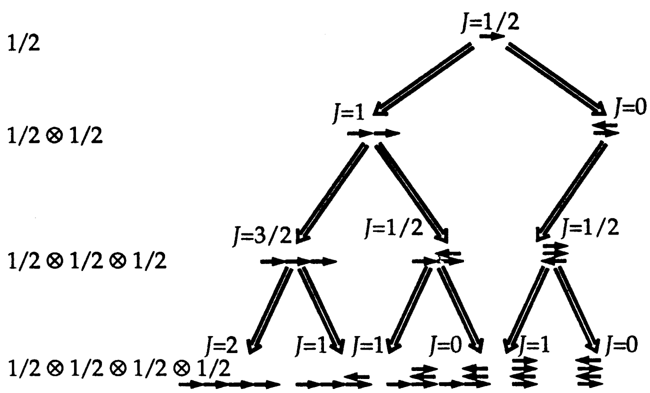


**Fig. 1.3.** Multiple coupling of spins $\frac{1}{2}$ to various total spins $J$

*Exercise 1.1*

In the next example, an alternative representation according to "maximal weight" is of interest. For this, all operators in the algebra that commute with each other are considered (*Cartan subalgebra*). Their eigenvalues classify states in a representation. In the case of SU(2) there is only one operator commuting with itself. This can be chosen to be any of the $j_i$, usualy one takes $j_3$, the third component of the angular momentum vector. Its eigenvalues are $m = -j, \ldots, +j$. The "maximal weight" is $m_{\max} = j$. In direct products $\left[\frac{1}{2}\right]^n$ the maximal weight is $m_{\max} = \frac{n}{2}$, which is the "maximal weight" of the "straight coupling" (see Fig. 1.3).

**SU(3)**. Its representations (multiplets) are classified by the eigenvalues of the Casimir operators. These give us, in the case of SU(3), two numbers $[p, q]$. These are in turn connected to the rank of the algebra, i.e., the number of commuting generators in the algebra. In general, the representations of SU($N$) are characterized by $N-1$ numbers. Another possibility would be to classify representations by their "maximal weight". As is known, each state in a representation of SU(3) (a multiplet) is labeled by the eigenvalues of the third component of isospin $\hat{T}_3$ and hypercharge $\hat{Y}$. The weight is given by the tuple $(T_3, Y)$. A weight $(T_3, Y)$ is higher than $(T_3', Y')$ if

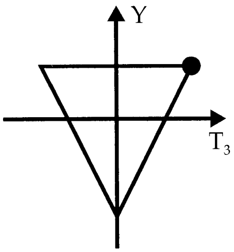


**Fig. 1.4.** The quark weight diagram

$$T_3 > T_3' \quad \text{or} \quad T_3 = T_3' \quad \text{and} \quad Y > Y' \ . \tag{2}$$

The highest weight in a representation is given by the maximal value of $T_3$, and, if there is more than one, by the maximal value of $Y$. This is demonstrated by the following examples:

(1) $[p, q] = [1, 0]$ .

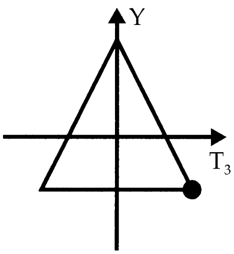


**Fig. 1.5.** The antiquark weight diagram

This is the representation whose "weight diagram" is depicted in Fig. 1.4. The states carry the weights

$$(T_3, Y) = \left(\frac{1}{2}, \frac{1}{3}\right), \left(-\frac{1}{2}, \frac{1}{3}\right), \left(0, -\frac{2}{3}\right) \ .$$

The tuple $\left(\frac{1}{2}, \frac{1}{3}\right)$ is the maximal weight.

(2) $[p, q] = [0, 1]$ .

This is the representation of antiquarks with the "weight diagram" in Fig. 1.5. The states carry the weights

$$(T_3, Y) = \left(0, \frac{2}{3}\right), \left(\frac{1}{2}, -\frac{1}{3}\right), \left(-\frac{1}{2}, -\frac{1}{3}\right) \ .$$

The state of maximal weight is $\left(\frac{1}{2}, -\frac{1}{3}\right)$.

In the case of SU(3), the trivial (scalar) representation is $[p, q] = [0, 0]$. The first nontrivial representations are $[1, 0]$ and $[0, 1]$ of the same lowest dimension. Mathematically, one of these representations, either $[1, 0]$ or $[0, 1]$, is sufficient to construct all higher SU(3) multiplets by multiple coupling (see [3]). Nevertheless, physically, one prefers to treat both representations $[1, 0]$ and $[0, 1]$

*Exercise 1.1*

equivalently side by side. In this way, the quark [1, 0] and antiquark [0, 1] character of the multiplet states can be better revealed (see again [3] for more details). Thus, by definition we have two fundamental representations. All others can be constructed from these two representations! To do so, we must construct the direct product of states

$$\begin{aligned}&(1,0)^p(0,1)^q \to\\&|T_3(1)Y(1)\rangle\,|T_3(2)Y(2)\rangle\cdots\\&|T_3(p)Y(p)\rangle\,\left|\overline{T}_3(1)\overline{Y}(1)\right\rangle\left|\overline{T}_3(2)\overline{Y}(2)\right\rangle\cdots\left|\overline{T}_3(q)\overline{Y}(q)\right\rangle\ .\end{aligned}\tag{3}$$

Here, $(T_3, Y)$ describe the quark and $(\overline{T}_3, \overline{Y})$ the antiquark quantum numbers, respectively. Owing to the additivity of the isospin component $\hat{T}_3$ and the hypercharge $\hat{Y}$, it holds that

$$\hat{T}_3 = \sum_i \hat{T}_3(i)\ ,\tag{4a}$$

$$\hat{Y} = \sum_i \hat{Y}(i)\ .\tag{4b}$$

Thus many-quark states have $T_3$ and $Y$ eigenvalues

$$(T_3, Y) = \left(\sum_{i=1}^{p} T_3(i) + \sum_{i=1}^{q} \overline{T}_3(i),\ \sum_{i=1}^{p} Y(i) + \sum_{i=1}^{q} \overline{Y}(i)\right)\ .\tag{5}$$

In these, there is one state of maximal weight, namely the one that is composed of $p$ quarks of maximal weight $\left(\frac{1}{2}, \frac{1}{3}\right)$ and $q$ antiquarks of maximal weight $\left(\frac{1}{2}, -\frac{1}{3}\right)$, i.e.,

$$(T_3)_{\max} = \frac{p+q}{2}\ ,\quad (Y)_{\max} = \frac{p-q}{3}\ .\tag{6}$$

It characterizes a representation contained in (5). If we subtract it, there is a remainder. Within this there is another state (or several states) of maximal weight. They are analogously given tuples $[p, q]$, i.e., a multiplet. We repeat the above steps until nothing is left, i.e., the direct product is completely reduced. In this way we can construct all SU(3) decompositions (for more details, see [3]).

We consider $[p_1, q_1] \times [p_2, q_2] = [1, 0] \times [0, 1]$ and first add the two weight diagrams, i.e., at each point of the one diagram, we add the other diagram (see Fig. 1.6).

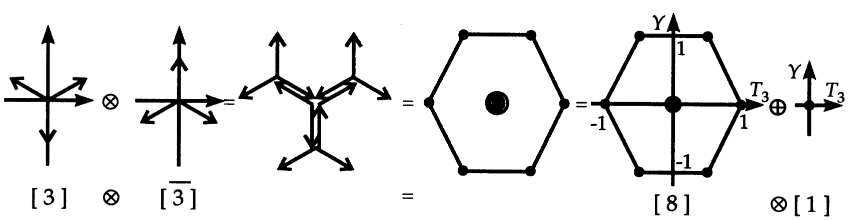


**Fig. 1.6.** Adding [1, 0] and [0, 1] weight diagrams

*Exercise 1.1*

We are thus led to a weight diagram whose center is occupied three times! The maximal weight appearing there is

$$(T_3, Y)_{\mathrm{max}} = (1, 0) \; . \tag{7}$$

For $[p, q]$, it follows from (6) that

$$[p, q] = [1, 1] \; , \tag{8}$$

corresponding to an octet with dimension 8. On subtracting the octet which is twice degenerate at the center, only the singlet remains

$$(T_3, Y)_{\mathrm{max}} = (0, 0) \; , \tag{9a}$$

that is,

$$[p, q] = [0, 0] \; . \tag{9b}$$

We thus obtain the following result:

$$[1, 0] \times [0, 1] = [1, 1] + [0, 0] \; . \tag{10}$$

*Note*: Constructing $[1, 0] \times [1, 0]$ with this method, we obtain

$$[1, 0] \times [1, 0] = [2, 0] + [0, 1] \; . \tag{11}$$

On the right-hand side, [0, 1] appears. This obviously means that mathematically, we can construct [0, 1] from [1, 0]. Thus one is inclined to call only [1, 0] the fundamental representation. Physically, however, the right-hand of equation (11) describes two-quark states and not, as [0, 1] does, antiquark states. In other words, in order to keep the quark-antiquark structure side by side, we keep both [1, 0] and [0, 1] as elementary multiplets.

**SU**($N$). Its multiplet states are classified by $N-1$ numbers:

$$[h_1, \cdots, h_{N-1}] \; . \tag{12}$$

Analogously to SU(3), there is the scalar (trivial) representation

$$[0, \cdots, 0] \tag{13}$$

and $N-1$ fundamental representations

$$\begin{aligned} &[1, 0, \cdots, 0] \; , \\ &[0, 1, \cdots, 0] \; , \\ &\qquad \vdots \\ &[0, \cdots, 0, 1] \; . \end{aligned} \tag{14}$$

From these, all other multiplets in (12) can be constructed by direct products.

*Exercise 1.1*

**Solution.** (b) Schur's lemma indicates that any operator $\hat{H}$ commuting with all operators $\hat{U}(\boldsymbol{\alpha})$ (the components of $\boldsymbol{\alpha}$ denote the group parameters), in particular with the generators $\hat{L}_i$,

$$\left[\hat{H}, \hat{U}(\boldsymbol{\alpha})\right] = 0 \Leftrightarrow \quad \left[\hat{H}, \hat{L}_i\right] = 0 \quad \Rightarrow \quad \left[\hat{H}, \hat{C}(\lambda)\right] = 0 \ ,$$

has the property that every state in a multiplet of the group is an eigenvector and that all states in a multiplet are degenerate. $\hat{C}(\lambda)$ is a Casimir operator of the group in the irreducible representation $\lambda$.

Since $\hat{C}(\lambda)$ commutes with $\hat{H}$, $\hat{C}(\lambda)$ and $\hat{H}$ can be simultaneously diagonalized, i.e., $\hat{C}(\lambda)$, too, is diagonal with respect to any state of the irreducible representation (multiplet) of the group. Calling $C(\lambda)$ the eigenvalues of $\hat{C}(\lambda)$, $\hat{C}(\lambda)$ has the following form with respect to the irreducible representation of the group:

$$\hat{C}(\lambda) = C(\lambda)\mathbb{1}(\lambda) \ , \tag{15}$$

where $\mathbb{1}_\lambda$ is the unit matrix with the multiplet's dimension. As the fundamental representation is by construction irreducible, (15) holds. In matrix representation, the Casimir operator has the following form:

$$\begin{pmatrix} C(\lambda_1)\mathbb{1}(\lambda_1) & 0 & 0 & \cdots \\ 0 & C(\lambda_2)\mathbb{1}(\lambda_2) & 0 & \cdots \\ 0 & 0 & C(\lambda_3)\mathbb{1}(\lambda_3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

Each diagonal submatrix appearing in it is of the form $C(\lambda)\mathbb{1}(\lambda)$ and characterizes a representation (multiplet) of the same dimension as this multiplet.

## EXERCISE

### 1.2 Casimir Operators of SU(3)

**Problem.** The regular (adjoint) representation of SU(3) is given by the eight generators $\hat{U}_i$, $i = 1, \ldots, 8$ with

$$(\hat{U}_i)_{jk} = -\mathrm{i} f_{ijk} \tag{1}$$

($\hat{U}_i$ are $8 \times 8$ matrices). Show that for $\hat{C}_1$, one of the two Casimir operators of SU(3) in the regular representation, it holds that

$$\hat{C}_1 = \sum_{i=1}^{8} \hat{U}_i^2 = 3\mathbb{1}_{8\times 8} \ . \tag{2}$$

*Exercise 1.2*

**Table 1.3.** The eigenvalues of the Casimir operator $\hat{C}_1$ for the regular representation

| $m$ | $\sum_{ij} f_{ijm}^2$ |
|---|---|
| 1 | $2f_{123}^2 + 2f_{147}^2 + 2f_{156}^2 = 3$ |
| 2 | $2f_{123}^2 + 2f_{246}^2 + 2f_{257}^2 = 3$ |
| 3 | $2f_{123}^2 + 2f_{345}^2 + 2f_{367}^2 = 3$ |
| 4 | $2f_{246}^2 + 2f_{345}^2 + 2f_{147}^2 + 2f_{458} = 3$ |
| 5 | $2f_{156}^2 + 2f_{257}^2 + 2f_{345}^2 + 2f_{458} = 3$ |
| 6 | $2f_{156}^2 + 2f_{246}^2 + 2f_{367}^2 + 2f_{678} = 3$ |
| 7 | $2f_{147}^2 + 2f_{257}^2 + 2f_{367}^2 + 2f_{678} = 3$ |
| 8 | $2f_{458}^2 + 2f_{678}^2 = 3$ |

**Solution.** Each irreducible representation of SU(3) is uniquely determined by the eigenvalues of its Casimir operators. Each state in a multiplet has the same eignvalues with respect to $\hat{C}_1$. Thus this operator must be proportional to the unit matrix. This is checked here using an example. Using (1) it follows for $\hat{C}_1$ that

$$(\hat{C}_1)_{lm} = -\sum_{i,j} f_{ilj} f_{ijm} \quad . \tag{3}$$

From the Table 1.2 on page 6 of the $f_{ijk}$, one recognizes that $f_{ilj} \neq 0$ and $f_{ijm} \neq 0$, which implies that $l = m$:

$$(\hat{C}_1)_{lm} = +\sum_{i,j} f_{ijm}^2 \delta_{lm} = 3\delta_{lm} \quad . \tag{4}$$

This proves (2).

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