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My Generalization of Sobolev's Embedding Theorem

The original method of proof of Sergei SOBOLEV consisted in writing

$$u = \sum_j \frac{\partial u}{\partial x_j} \star \frac{\partial E}{\partial x_j} \text{ for an elementary solution } E \text{ of } \Delta, \tag{31.1}$$

and it is not adapted to the case where the derivatives are in different spaces.

A different proof, by Louis NIRENBERG, and also by Emilio GAGLIARDO, can be used for the case where

$$\frac{\partial u}{\partial x_j} \in L^{p_j}(R^N) \text{ for } j = 1, \ldots, N; \tag{31.2}$$

In the late 1970s, I had heard a talk about this question by Alois KUFNER, then I was told that it had been noticed earlier by TROISI.

The case where the derivatives are in the same Lorentz space $L^{p,q}(R^N)$ with $1 < p < N$ can be treated with the theory of interpolation, as was done by Jaak PEETRE, but the limiting case where

$$\frac{\partial u}{\partial x_j} \in L^{N,p}(R^N) \text{ for } j = 1, \ldots, N, \tag{31.3}$$

was treated by Haïm BREZIS and Stephen WAINGER by analyzing a formula of O'NEIL about the nonincreasing rearrangement of a convolution product. The case $p = 1$ in (31.3) gives $u \in C_0(R^N)$, by noticing that $C_c(R^N)$ is dense in $L^{N,1}(R^N)$, whose dual is $L^{N',\infty}(R^N)$, which contains the derivatives $\frac{\partial E}{\partial x_j}$. The case $p = \infty$ in (31.3) gives $e^{\varepsilon\,|u|} \in L^1_{loc}(R^N)$, and u actually belongs to $BMO(R^N)$.

As far as I know, these classical methods do not permit one to treat the case where the derivatives are in different Lorentz spaces; of course, *this question is quite academic*, but serves as a training ground for situations which often occur where one has different information in different directions, for example because some coordinates represent *space* and another one represents *time* (and one has simplified the physical reality so that the model used has a *fake velocity of light* equal to $+\infty$).

First, it is useful to observe that Sobolev's embedding theorem for $p = 1$ is related to the *isoperimetric inequality*. The classical isoperimetric inequality says that among measurable sets A of R^N with a given volume, the $(N-1)$-dimensional measure of the boundary ∂A is minimum when A is a sphere; equivalently, for a given measure of the boundary, the volume is maximum for a sphere. Analytically it means that

$$meas(A) \leq C_0\big(meas(\partial A)\big)^{N/(N-1)}, \tag{31.4}$$

and it tells what the best constant C_0 is, while Sobolev's embedding theorem for $W^{1,1}(R^N)$ gives

$$\int_{R^N} |u|^{1^*}\,dx \leq C_1||grad(u)||_1^{N/(N-1)}, \tag{31.5}$$

but does not insist on identifying what the best constant C_1 is. The relation between the two inequalities is that one can apply the last inequality to $u = \chi_A$, the characteristic function of A, assuming that A has a finite perimeter; of course, χ_A does not belong to $W^{1,1}(R^N)$, but as its partial derivatives $\frac{\partial \chi_A}{\partial x_j}$ are Radon measures, one may apply the inequality to $\chi_A \star \varrho_\varepsilon$ and then let ε tend to 0; in this way one learns that $C_0 \leq C_1$. Conversely, knowing the isoperimetric inequality, one can approach a function u by a sum of characteristic functions, using $A_n = \{x \mid n\varepsilon \leq u(x) < (n+1)\varepsilon\}$ and deduce Sobolev's embedding theorem, so that $C_1 \leq C_0$ and the two inequalities are then essentially the same. However, the proof of the last part involves the technical study of functions of bounded variation (denoted by BV), which is classical in one dimension, but is indebted to the work of Ennio DE GIORGI[1], FEDERER[2] and Wendell FLEMING[3] for the development of the N-dimensional case.

As I observed, starting from Sobolev's embedding theorem $W^{1,1}(R^N) \subset L^{1^*}(R^N)$ (proven by Louis NIRENBERG), one can easily derive all the results already obtained, except for the question of identifying the best constants. For that, one uses the functions φ_n adapted to u, writes

$$||\varphi_n(u)||_{1^*} \leq C_0||\varphi_n'(u)grad(u)||_1 \leq ||\varphi_n'(u)grad(u)||_p e^{n/p'}, \tag{31.6}$$

by Hölder's inequality, and deduces the same inequality as before, $|a_{n-1} - a_n|e^{n/p^*} \in l^p(Z)$.

[1] Ennio DE GIORGI, Italian mathematician, 1928–1996. He received the Wolf Prize in 1990. He worked at Scuola Normale Superiore, Pisa, Italy.

[2] Herbert FEDERER, Austrian-born mathematician, born in 1920. He worked at Brown University, Providence, RI.

[3] Wendell Helms FLEMING, American mathematician, born in 1928. He works at Brown University, Providence, RI.

However, for the case of derivatives in (different) Lorentz spaces, I could only prove it by using a multiplicative variant of the isoperimetric inequality/Sobolev's embedding theorem.

Lemma 31.1. *The Sobolev's embedding theorem $W^{1,1}(R^N) \subset L^{1^*}(R^N)$ in its additive version*

$$||u||_{1^*} \le A \sum_{j=1}^{N} \Big|\Big|\frac{\partial u}{\partial x_j}\Big|\Big|_1 \text{ for all } u \in W^{1,1}(R^N), \tag{31.7}$$

is equivalent to the multiplicative version

$$||u||_{1^*} \le N\, A \left(\prod_{j=1}^{N} \Big|\Big|\frac{\partial u}{\partial x_j}\Big|\Big|_1\right)^{1/N} \text{ for all } u \in W^{1,1}(R^N). \tag{31.8}$$

Proof: One rescales with a different scaling in different directions, i.e., one applies the additive version to v defined by

$$v(x_1,\ldots,x_N) = u\left(\frac{x_1}{\lambda_1},\ldots,\frac{x_N}{\lambda_N}\right), \tag{31.9}$$

and one obtains

$$(\lambda_1\ldots\lambda_N)^{1/1^*}||u||_{1^*} \le A\,\lambda_1\ldots\lambda_N \sum_j \frac{1}{\lambda_j}\Big|\Big|\frac{\partial u}{\partial x_j}\Big|\Big|_1. \tag{31.10}$$

Then one notices that

$$\begin{aligned}&\text{if } \lambda_1\ldots\lambda_N = \mu > 0, \text{ the minimum of } \textstyle\sum_j \frac{\alpha_j}{\lambda_j} \text{ is attained when}\\ &\quad\lambda_j = \beta\,\alpha_j \text{ for all } j \text{ and the Lagrange multiplier } \beta \text{ satisfies}\\ &\quad\beta^N\alpha_1\ldots\alpha_N = \mu, \text{ so the minimum is } \tfrac{N}{\mu}(\alpha_1\ldots\alpha_N)^{1/N}.\end{aligned} \tag{31.11}$$

One applies (31.11) to the case $\alpha_j = \big|\big|\frac{\partial u}{\partial x_j}\big|\big|_1$ and one finds the multiplicative version, as the powers of μ are identical on both sides of the inequality (because the inequality is already invariant when one rescales all the coordinates in the same way). The multiplicative version implies the additive version by the *geometric-arithmetic inequality*

$$(a_1\ldots a_N)^{1/N} \le \frac{a_1+\ldots+a_N}{N} \text{ for all } a_1,\ldots,a_N > 0, \tag{31.12}$$

which, putting $a_j = e^{z_j}$, is but the convexity property of the exponential function. ☐

Lemma 31.2. *Let u satisfy*

$$\frac{\partial u}{\partial x_j} \in L^{p_j,q_j}(R^N), \text{ with } 1 < p_j < \infty \text{ and } 1 \le q_j \le \infty, \text{ for } j = 1,\ldots,N. \tag{31.13}$$

Let p_{eff}, p^*_{eff} and q_{eff} be defined by

$$\begin{aligned} \frac{1}{p_{eff}} &= \frac{1}{N}\sum_j \frac{1}{p_j} \\ \frac{1}{p^*_{eff}} &= \frac{1}{p_{eff}} - \frac{1}{N} \\ \frac{1}{q_{eff}} &= \frac{1}{N}\sum_j \frac{1}{q_j}. \end{aligned} \tag{31.14}$$

Then one has

$$|a_{n-1} - a_n| e^{n/p^*_{eff}} \in l^{q_{eff}}(Z). \tag{31.15}$$

One may allow some p_j to be 1 or ∞, but using only $q_j = +\infty$ in that case.

Proof: Let $f_j = \frac{\partial u}{\partial x_j}$ for $j = 1, \ldots, N$. One applies the multiplicative version to $\varphi_n(u)$, and one has to estimate $||\varphi'_n(u) f_j||_1$. A classical result of HARDY and LITTLEWOOD states that for all $f \in L^1(\Omega) + L^\infty(\Omega)$ and all measurable subsets $\omega \subset \Omega$ one has

$$\int_\omega |f(x)|\, dx \le \int_0^{meas(\omega)} f^*(s)\, ds, \tag{31.16}$$

and as the measure of the points where $\varphi'_n(u) \ne 0$ is at most e^n, one deduces that

$$||\varphi'_n(u) f_j||_1 \le K(e^n; f_j) \text{ for } j = 1, \ldots, N, \tag{31.17}$$

and then, using Hölder's inequality,

$$\begin{aligned} & e^{-n\theta_j} K(e^n; f_j) \in l^{q_j}(Z) \text{ with } \theta_j = \tfrac{1}{p'_j} \text{ for } j = 1, \ldots, N, \text{ imply} \\ & \quad e^{-n/p'_{eff}} ||\varphi_n(u)||_{1^*} \le N\, A \Big(\prod_j e^{-n\theta_j} K(e^n; f_j)\Big)^{1/N} \in l^{q_{eff}}(Z), \end{aligned} \tag{31.18}$$

which gives (31.14). In the case where $p_j = 1$, one has (31.18) with $\theta_j = 0$ and $q_j = +\infty$, and in the case where $p_j = \infty$, one has (31.18) with $\theta_j = 1$ and $q_j = +\infty$. ☐

For interpreting Lemma 31.2, one assumes that

$$\text{for all } \lambda > 0, \text{ one has } meas\{x \mid |u(x)| > \lambda\} < +\infty, \tag{31.19}$$

which is a way to impose that u tends to 0 at ∞.

If $p_{eff} < N$ then it means $u \in L^{p^*_{eff}, q_{eff}}(R^N)$.

If $p_{eff} = N$ and $q_{eff} = 1$, which means that $q_j = 1$ for $j = 1, \ldots, N$, then one has $|a_{n-1} - a_n| \in l^1(Z)$, so that one deduces a bound for a_n, i.e., $u \in L^\infty(R^N)$; using the density of $C_c^\infty(R^N)$ in $L^{p_j,1}(R^N)$, one deduces that $u \in C_0(R^N)$.

If $p_{eff} = N$ and $1 < q_{eff} < \infty$, then for every $\kappa > 0$ one has $e^{\kappa |u|^{q'_{eff}}} \in L^1_{loc}(R^N)$.

If $p_{eff} = N$ and $q_{eff} = \infty$, which means that $q_j = \infty$ for $j = 1, \ldots, N$, one deduces that $|a_n| \le \alpha |n| + \beta$, so that there exists $\varepsilon_0 > 0$ such that

$e^{\varepsilon_0 |u|} \in L^1_{loc}(R^N)$. This is the case when all the derivatives belong to $L^{N,\infty}(R^N)$, and because $\log(|x|)$ is such a function, it is not always true that $e^{\kappa\,|u|} \in L^1_{loc}(R^N)$ for all $\kappa > 0$. For that particular space of functions, u belongs to $BMO(R^N)$, which Fritz JOHN and Louis NIRENBERG had introduced for studying the case of $W^{1,N}(R^N)$, and they proved that for every function in $BMO(R^N)$ there exists $\varepsilon_0 > 0$, depending upon the semi-norm of u in $BMO(R^N)$, such that $e^{\varepsilon_0 |u|} \in L^1_{loc}(R^N)$.

If $p_{eff} > N$ then one has $u \in L^\infty(R^N)$. By considering $(u-\alpha)_+$ or $(u+\alpha)_-$ for $\alpha > 0$ (and letting then α tend to 0), one may assume that $u \in L^1(R^N)$, and in applying the usual rescaling argument one starts from a bound $||u||_\infty \le C\big(||u||_{r,s} + \sum_j ||\partial_j u||_{p_j,q_j}\big)$, where $||\cdot||_p$ denotes the norm in L^p and $||\cdot||_{p,q}$ denotes the norm in $L^{p,q}$. Applying this inequality to $u\big(\frac{x_1}{\lambda_1},\ldots,\frac{x_N}{\lambda_N}\big)$, and writing $\mu = \lambda_1 \cdots \lambda_N$, one obtains $||u||_\infty \le C\big(\mu^{1/r}\,||u||_{r,s} + \sum_j \mu^{1/p_j}\lambda_j^{-1}||\partial_j u||_{p_j,q_j}\big)$; the inequality between the arithmetic mean and the geometric mean implies $\sum_j \mu^{1/p_j}\lambda_j^{-1}||\partial_j u||_{p_j,q_j} \ge N\,\mu^{(1/p_{eff})-(1/N)}\big(\prod_j ||\partial_j u||_{p_j,q_j}\big)^{1/N}$, with equality if all $\mu^{1/p_j}\lambda_j^{-1}||\partial_j u||_{p_j,q_j}$ are equal, so that $||u||_\infty \le C\big(\mu^{1/r}\,||u||_{r,s} + N\,\mu^{1/p_{eff}}\mu^{-1/N}\big(\prod_j ||\partial_j u||_{p_j,q_j}\big)^{1/N}\big)$; because minimizing $\mu^a A + \mu^{-b}B$ for $\mu > 0$ gives $a\,\mu^{a-1}A - b\,\mu^{-b-1}B = 0$ and $\mu = (b\,B/a\,A)^{1/(a+b)}$, so that the minimum is $C' A^\eta B^{1-\eta}$ with $\eta = \frac{b}{a+b}$, one deduces that $||u||_\infty \le C''||u||^\theta_{r,s}\big(\prod_j ||\partial_j u||_{p_j,q_j}\big)^{(1-\theta)/N}$, with $\theta = \frac{(1/N)-(1/p_{eff})}{(1/r)+(1/N)-(1/p_{eff})}$. Choosing $r = p_i, s = q_i$, and applying the preceding inequality to the case where u is replaced by $\tau_{t\,e_i}u - u$, one deduces that u is Hölder-continuous of order γ_i in its ith variable, with $\gamma_i = \frac{(1/N)-(1/p_{eff})}{(1/p_i)+(1/N)-(1/p_{eff})}$.

Having different information on derivatives in different directions is usual for parabolic equations like the heat equation. For example, letting Ω be an open set of R^N, given $u_0 \in L^2(\Omega)$, one can show that there exists a unique solution u of $\frac{\partial u}{\partial t} - \Delta\,u = 0$ in $\Omega\times(0,T)$ satisfying the initial condition $u(x,0) = u_0(x)$ in Ω and the homogeneous Dirichlet boundary condition $\gamma_0 u = 0$ on $\partial\Omega\times(0,T)$, in the sense that $u \in C\big([0,T];L^2(\Omega)\big)$, $u \in L^2\big((0,T);H^1_0(\Omega)\big)$ and $\frac{\partial u}{\partial t} \in L^2\big((0,T);H^{-1}(\Omega)\big)$.

If $u_0 \in H^1_0(\Omega)$ then the solution also satisfies $u \in C^0\big([0,T];H^1_0(\Omega)\big)$, $\Delta\,u, \frac{\partial u}{\partial t} \in L^2\big((0,T);L^2(\Omega)\big) = L^2\big(\Omega\times(0,T)\big)$; if the boundary is of class C^1 or if the open set Ω is convex (or if an inequality holds for the total curvature of the boundary), then one has $u \in L^2\big((0,T);H^2(\Omega)\big)$.

If u_0 belongs to an interpolation space between $H^1_0(\Omega)$ and $L^2(\Omega)$ then one has intermediate results, but this requires enough smoothness for the boundary.

As an example, consider a function

$$u(x,t) \text{ defined on } R^N\times R \text{ and satisfying } u, \frac{\partial u}{\partial t}, \Delta\,u \in L^2(R^{N+1}), \quad (31.20)$$

and this implies that $\frac{\partial u}{\partial x_j} \in L^2(R^{N+1})$ for $j = 1, \ldots, N$ (by using Fourier transform, for example). Denoting the dual variables by (ξ, τ), the information is equivalent to $\mathcal{F}u, \tau\,\mathcal{F}u, |\xi|^2\mathcal{F}u \in L^2(R^{N+1})$ (and therefore $\xi_j\mathcal{F}u \in L^2(R^{N+1})$ for $j = 1, \ldots, N$). One has

$$\begin{aligned}&(1+|\tau|+|\xi|^2)\mathcal{F}u \in L^2(R^{N+1}), \text{ and if one shows } \tfrac{1}{1+|\tau|+|\xi|^2} \in L^{p,\infty}(R^{N+1})\\ &\quad \text{for some } p \in (2,\infty), \text{ then } \mathcal{F}u \in L^{q,2}(R^{N+1}) \text{ with } \tfrac{1}{q} = \tfrac{1}{2} + \tfrac{1}{p} \text{ and}\\ &\quad u \in L^{q',2}(R^{N+1}),\end{aligned} \tag{31.21}$$

the last property being due to the fact that $1 < q < 2$ and $\overline{\mathcal{F}}$ maps $L^1(R^{N+1})$ into $L^\infty(R^{N+1})$ and $L^2(R^{N+1})$ into itself, and by interpolation it maps $L^{q,2}(R^{N+1})$ into $L^{q',2}(R^{N+1})$. One has $\frac{1}{1+|\tau|+|\xi|^2} \in L^\infty(R^{N+1})$, and it is the behavior at ∞ that is interesting for obtaining the smallest value of p, so that one checks for what value of p one has $\frac{1}{|\tau|+|\xi|^2} \in L^{p,\infty}(R^{N+1})$ and one obtains the same information for the smaller function $\frac{1}{1+|\tau|+|\xi|^2}$. One uses the homogeneity properties of the function, and for $\lambda > 0$ one computes

$$\begin{gathered}meas\left\{(\xi,\tau) \mid \tfrac{1}{|\tau|+|\xi|^2} \ge \lambda\right\} = C\,\lambda^{-1-(N/2)}\\ \text{with } C = meas\left\{(\xi,\tau) \mid \tfrac{1}{|\tau|+|\xi|^2} \ge 1\right\},\end{gathered} \tag{31.22}$$

by using the change of coordinates $\tau = \lambda^{-1}\tau'$ and $\xi = \lambda^{-1/2}\xi'$. This corresponds to $p = 1 + \frac{N}{2} = \frac{N+2}{2}$, which gives $q = \frac{2(N+2)}{N+6}$ and $q' = \frac{2(N+2)}{N-2}$ if $N \ge 3$, so one has

$$\begin{aligned}&\text{for } N \ge 3, \text{ one has } u \in L^{2(N+2)/(N-2),2}(R^{N+1}) \cap L^2(R^{N+1}),\\ &\text{for } N = 2, \text{ one has } u \in L^r(R^3) \text{ for all } r \in [2,\infty),\\ &\text{for } N = 1, \text{ one has } u \in L^\infty(R^2) \cap L^2(R^2).\end{aligned} \tag{31.23}$$

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