5

The Fundamental Solution for a Sub-Laplacian and Applications

In this chapter, we enter the core of the study of the sub-Laplacians $\mathcal{L}$ on the homogeneous Carnot groups (and hence on the stratified Lie groups) of homogeneous dimension $Q \geq 3$, by showing that they possess a *fundamental solution* Γ resemblant to the fundamental solution $c_N \, |x|^{2-N}$ of the usual Laplace operator Δ on $\mathbb{R}^N$, $N \geq 3$. This property is one of the most striking analogies between $\mathcal{L}$ and the classical Laplace operator. Indeed, we shall see that it holds

$$\Gamma = d^{2-Q},$$

where Q is the homogeneous dimension of $\mathbb{G}$ and d is a *symmetric homogeneous norm* on $\mathbb{G}$, smooth out of the origin (the relevant definitions will be given in Section 5.1). We shall also call d an *$\mathcal{L}$-gauge*.

To do this, we first fix some results on homogeneous norms and the Carnot–Carathéodory distance. Then, in Section 5.3, we define the fundamental solution Γ, whose existence follows from the hypoellipticity and the homogeneity properties of $\mathcal{L}$. We then collect many of its remarkable properties.

As a first application, we provide *mean value formulas* for $\mathcal{L}$, generalizing to the sub-Laplacian setting the Gauss theorem for classical harmonic functions. These formulas will play a central rôle throughout the book and are proved by only using integration by parts and the coarea theorem (see Theorems 5.5.4 and 5.6.1).

From the mean value formulas, we derive *Harnack-type inequalities* for $\mathcal{L}$ and the *Brelot convergence property* for monotone sequences of $\mathcal{L}$-harmonic functions (see Theorem 5.7.10). Furthermore, as an application of the Harnack theorem, in Section 5.8 we derive several *Liouville-type theorems* for $\mathcal{L}$. As another application of the properties of the fundamental solution of $\mathcal{L}$, we prove the *Sobolev–Stein embedding theorem* in the stratified group setting (see Section 5.9).

To end with the applications of Γ, we provide three sections devoted to the following topics: some remarks on the *analytic-hypoelliptic* sub-Laplacians, $\mathcal{L}$-harmonic approximations, and finally an integral representation formula for the fundamental solution by R. Beals, B. Gaveau and P. Greiner [BGG96].

Finally, three appendices close the chapter. The first is devoted to *the weak and the strong maximum principles* for $\mathcal{L}$. The second one provides an improved version of the pseudo-triangle inequality. In the third appendix, we prove in details the existence of geodesics on Carnot groups. As a direct application of the maximum principles, we give a decomposition theorem for $\mathcal{L}$-harmonic functions, resemblant to the decomposition of a holomorphic function on an annulus of $\mathbb{C}$ into the sum of the regular and singular parts from its Laurent expansion.

Convention. Throughout this chapter, we fix a stratified group $\mathbb{H}$ of step r and m generators. Q denotes the homogeneous dimension of $\mathbb{H}$.

Together with $\mathbb{H}$*,*
a stratification $V = (V_1, \dots, V_r)$ *of the algebra of* $\mathbb{H}$ *will be fixed.*

Moreover, $\mathcal{L}$ will be any sub-Laplacian on $\mathbb{H}$ related to the given stratification. We recall that any stratification of the algebra of $\mathbb{H}$ brings along a homogeneous Carnot group on $\mathbb{R}^N$ isomorphic to $\mathbb{H}$. Hence, together with the couple $(\mathbb{H}, V)$, we fix $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$, a homogeneous Carnot group isomorphic to $\mathbb{H}$, as described in Proposition 2.2.22. We let $\Psi : \mathbb{G} \to \mathbb{H}$ be the Lie-group isomorphism, as in the cited proposition. We still denote by $\mathcal{L}$ the sub-Laplacian on $\mathbb{G}$ which is Ψ-related to the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}$ (see (2.68), page 147).

Obviously, *the "homogeneous version"* $\mathbb{G}$ *of* $\mathbb{H}$ *depends upon the stratification* V *but not on the sub-Laplacian* $\mathcal{L}$.

Thus, any definition and result given henceforward for homogeneous Carnot groups has its counterpart (and is actually intended) for *any couple* $(\mathbb{H}, V)$, where $\mathbb{H}$ is an abstract stratified group and V is a stratification for $\mathbb{G}$.

Notation. We introduce the notation for the homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$. Its dilations $\{\delta_\lambda\}_{\lambda>0}$ are denoted by

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \dots, \lambda^r x^{(r)}), \quad x^{(i)} \in \mathbb{R}^{N_i}, \quad 1 \le i \le r.$$

We denote by $m := N_1$ the number of generators of $\mathbb{G}$ and assume that the homogeneous dimension

$$Q = N_1 + 2\,N_2 + \dots + r\,N_r \ge 3.$$

As we showed in Chapter 1, Section 1.4 (page 56), the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ can be written as follows (see (1.90a))

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 = \operatorname{div}(A(x)\nabla^T),$$

where $\{X_1, \dots, X_m\}$ is a family of vector fields that form a linear basis of the first layer of $\mathfrak{g}$, the Lie algebra of $\mathbb{G}$. The matrix A is given by

$$A(x) = \begin{pmatrix} X_1 I(x) & \cdots & X_m I(x) \end{pmatrix} \cdot \begin{pmatrix} (X_1 I(x))^T \\ \vdots \\ (X_m I(x))^T \end{pmatrix}$$

and takes the following block form (see (1.91))

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix},$$

where $A_{1,1}$ is a strictly positive definite constant $m \times m$ matrix.

The characteristic form of $\mathcal{L}$

$$q_{\mathcal{L}}(x,\xi) := \langle A(x)\xi, \xi\rangle = \sum_{j=1}^{N_1} \langle X_j I(x), \xi\rangle^2, \quad x,\ \xi \in \mathbb{R}^N, \tag{5.1a}$$

is non-negative definite and, for every fixed $x \in \mathbb{R}^N$, the set

$$\{\xi \in \mathbb{R}^N \mid q_{\mathcal{L}}(x,\xi) = 0\}$$

is a linear space of dimension $N - m$. The vector-valued operator

$$\nabla_{\mathcal{L}} := (X_1, \ldots, X_m) \tag{5.1b}$$

is called the $\mathcal{L}$-gradient operator in $\mathbb{G}$. Due to identity (5.1a), we have

$$|\nabla_{\mathcal{L}} u|^2 = \sum_{j=1}^{m} |X_j u|^2 = \langle A\nabla^T u, \nabla^T u\rangle, \quad u \in C^1(\mathbb{R}^N, \mathbb{R}^N). \tag{5.1c}$$

5.1 Homogeneous Norms

Definition 5.1.1. *We call* homogeneous norm on (*the homogeneous Carnot group*) $\mathbb{G}$, *every* continuous[1] *function* $d : \mathbb{G} \to [0, \infty)$ *such that*:

1. $d(\delta_\lambda(x)) = \lambda\, d(x)$ *for every* $\lambda > 0$ *and* $x \in \mathbb{G}$;
2. $d(x) > 0$ *iff* $x \neq 0$.

   Moreover, we say that d is symmetric *if*
3. $d(x^{-1}) = d(x)$ *for every* $x \in \mathbb{G}$.

Example 5.1.2. Define

$$|x|_{\mathbb{G}} := \left(\sum_{j=1}^{r} |x^{(j)}|^{\frac{2r!}{j}} \right)^{\frac{1}{2r!}}, \quad x = (x^{(1)}, \ldots, x^{(r)}) \in \mathbb{G}, \tag{5.2}$$

where $|x^{(j)}|$ denotes the Euclidean norm on $\mathbb{R}^{N_j}$. Then $|\cdot|_{\mathbb{G}}$ is a homogeneous norm on $\mathbb{G}$ smooth out of the origin. It is symmetric if $x^{-1} = -x$ for any $x \in \mathbb{G}$. In general, if $\mathbb{G}$ is any Carnot group (with inverse x^{-1} not necessarily equal to $-x$) the map $x \mapsto |\mathrm{Log}\,(x)|_{\mathbb{G}}$ is a symmetric homogeneous norm on $\mathbb{G}$ smooth out of the origin. This follows from the facts that $\mathrm{Log}\,(\delta_\lambda(x)) = \delta_\lambda(\mathrm{Log}\,(x))$ and $\mathrm{Log}\,(x^{-1}) = -\mathrm{Log}\,(x)$.

[1] With respect to the Euclidean topology.

Example 5.1.3 (Control norm). Let d be the control distance related to a system of generators of $\mathbb{G}$ (see Section 5.2). Define

$$d_0(x) := d(x, 0), \quad x \in \mathbb{G}.$$

Then (see Theorem 5.2.8 in Section 5.2) we shall see that d_0 is a symmetric homogeneous norm on $\mathbb{G}$.

From the next (elementary) proposition it will follow that the homogeneous norms on $\mathbb{G}$ are all equivalent.

Proposition 5.1.4 (Equivalence of the homogeneous norms). *Let d be a homogeneous norm on $\mathbb{G}$. Then there exists a constant* $\mathbf{c} > 0$ *such that*

$$\mathbf{c}^{-1}\,|x|_{\mathbb{G}} \le d(x) \le \mathbf{c}\,|x|_{\mathbb{G}} \quad \forall\, x \in \mathbb{G}, \tag{5.3}$$

where $|\cdot|_{\mathbb{G}}$ *has been defined in* (5.2).

Proof. Due to the δ_λ-homogeneity of d and $|\cdot|_{\mathbb{G}}$, inequalities (5.3) hold taking $\mathbf{c} := \max\{H, 1/h\}$, where

$$H := \sup\{d(x) : |x|_{\mathbb{G}} = 1\}, \quad h := \inf\{d(x) : |x|_{\mathbb{G}} = 1\}.$$

We explicitly remark that $H < \infty$ and $h > 0$, since the set

$$\{x : |x|_{\mathbb{G}} = 1\}$$

is a compact subset of $\mathbb{G}$ not containing the origin and d is a continuous function strictly positive in $\mathbb{G} \setminus \{0\}$. □

Corollary 5.1.5. *For every fixed (non-necessarily symmetric) homogeneous norm d on $\mathbb{G}$, there exists a constant* $\mathbf{c} > 0$ *such that*

$$\mathbf{c}^{-1}\,d(x) \le d(x^{-1}) \le \mathbf{c}\,d(x) \quad \forall\, x \in \mathbb{G}. \tag{5.4}$$

Proof. The function $x \mapsto d(x^{-1})$ is a homogeneous norm on $\mathbb{G}$. Indeed, recall that δ_λ is an automorphism of $\mathbb{G}$, whence $\delta_\lambda(x^{-1}) = (\delta_\lambda(x))^{-1}$. Then the assertion follows from Proposition 5.1.4. □

Any homogeneous norm turns out to be locally Hölder continuous with respect to the Euclidean metric in the following sense.

Proposition 5.1.6. *Let d be a homogeneous norm on $\mathbb{G}$. Then, for every compact set $K \subset \mathbb{R}^N$, there exists a constant* $\mathbf{c}_K > 0$ *such that*

$$d(y^{-1} \circ x) \le \mathbf{c}_K\,|x - y|^{1/r} \quad \forall\, x,\ y \in K, \tag{5.5}$$

where r is the step of $\mathbb{G}$.

(See also Proposition 5.15.1 (page 309) in Appendix C for an estimate from below of $d(y^{-1} \circ x)$.)

Proof. Let $K \subset \mathbb{R}^N$ be a compact set. It is easy to see that there exists a constant $\mathbf{c} = \mathbf{c}(K) > 0$ such that $|x|_{\mathbb{G}} \leq \mathbf{c}\,|x|^{1/r}$ for every $x \in K$, where $|\cdot|_{\mathbb{G}}$ has been defined in (5.2). We now use Proposition 5.1.4, and we obtain that there exists a constant $\mathbf{c} = \mathbf{c}(K) > 0$ such that $d(x) \leq \mathbf{c}\,|x|^{1/r}$ for every $x \in K$. Hence, (5.5) will follow if we prove that there exists a constant $\mathbf{c} = \mathbf{c}(K) > 0$ such that $|y^{-1} \circ x| \leq \mathbf{c}\,|x - y|$ for every $x, y \in K$. If we apply the mean value theorem to the function $F(x, y) := y^{-1} \circ x$, we obtain

$$\begin{aligned}|y^{-1} \circ x| &= |F(x, y) - F(x, x)| \\ &\leq \max_{t\in[0,1]} \|\mathcal{J}_F(x, t\,x + (1-t)y)\|\,|x - y| \leq \mathbf{c}\,|x - y|,\end{aligned}$$

and the assertion is proved. □

Any homogeneous norm satisfies a kind of *pseudo-triangle inequality*.

Proposition 5.1.7 (Pseudo-triangle inequalities. I). *Let d be a homogeneous norm on $\mathbb{G}$. Then there exists a constant $\mathbf{c} > 0$ such that:*

1) $d(x \circ y) \leq \mathbf{c}(d(x) + d(y))$,
2) $d(x \circ y) \geq \frac{1}{\mathbf{c}}\,d(x) - d(y^{-1})$,
3) $d(x \circ y) \geq \frac{1}{\mathbf{c}}\,d(x) - \mathbf{c}\,d(y)$

for every $x, y \in \mathbb{G}$.

Proof. Due to the δ_λ-homogeneity of d, inequality 1) is equivalent to the following one

$$d(x \circ y) \leq \mathbf{c} \quad \text{if } d(x) + d(y) = 1.$$

This inequality holds true taking

$$\mathbf{c} := \max\{d(x \circ y)\ :\ d(x) + d(y) = 1\}.$$

Obviously, $1 \leq \mathbf{c} < \infty$. From inequality 1) we now obtain

$$d(x) = d((x \circ y) \circ y^{-1}) \leq \mathbf{c}(d(x \circ y) + d(y^{-1})),$$

whence 2). Now, 3) follows from 2) and Corollary 5.1.5 with a suitable change of the constant $\mathbf{c}$. □

Given a homogeneous norm d_0 on $\mathbb{G}$, the function

$$\mathbb{G} \times \mathbb{G} \ni (x, y) \mapsto d(x, y) := d_0(y^{-1} \circ x)$$

is a *pseudometric* on $\mathbb{G}$. Indeed, we have the following proposition.

Proposition 5.1.8 (Pseudo-triangle inequalities. II). *With the above notation, there exists a positive constant $\mathbf{c} > 0$ such that:*

1) $d(x, y) \le \mathbf{c}\, d(y, x)$ *for every* $x, y \in \mathbb{G}$ (*here* $\mathbf{c}$ *can be taken* $= 1$ *iff* d_0 *is also symmetric*),
2) $d(x, y) \le \mathbf{c}(d(x, z) + d(z, y))$ *for every* $x, y, z \in \mathbb{G}$ (the pseudo-triangle inequality for d),
3) $d(x, y) = 0$ *iff* $x = y$.

Proof. It immediately follows from Corollary 5.1.5 and Proposition 5.1.7. □

5.2 Control Distances or Carnot–Carathéodory Distances

We begin with some important definitions.

Definition 5.2.1 (X-subunit path). *Let* $X = \{X_1, \dots, X_m\}$ *be any family of vector fields in* $\mathbb{R}^N$. *A piece-wise regular path* $\gamma : [0, T] \to \mathbb{R}^N$ *is said to be* X-subunit *if*

$$\langle \dot\gamma(t), \xi\rangle^2 \le \sum_{j=1}^m \langle X_j I(\gamma(t)), \xi\rangle^2 \quad \forall\, \xi \in \mathbb{R}^N,$$

almost everywhere in $[0, T]$. *We shall denote by* $\mathcal{S}(X)$ *the set of all* X*-subunit paths, and we put*

$$l(\gamma) = T$$

if $[0, T]$ *is the domain of* $\gamma \in \mathcal{S}(X)$.

We explicitly remark that every integral curve of $\pm X_j$ $(j \in \{1, \dots, m\})$ is X-subunit.

Convention. We assume $\mathbb{R}^N$ is X*-connected* in the following sense (a proof of this fact in the case of stratified vector fields will be given in Theorem 19.1.3 on page 716):

For every $x, y \in \mathbb{R}^N$, *there exists*

$$\gamma \in \mathcal{S}(X), \quad \gamma : [0, T] \to \mathbb{R}^N \quad \text{such that } \gamma(0) = x \text{ and } \gamma(T) = y.$$

Then the following definition makes sense.

Definition 5.2.2 (X-Carnot–Carathéodory distance). *Suppose* $\mathbb{R}^N$ *is* X*-connected. Then, for every* $x, y \in \mathbb{R}^N$, *we set*

$$d_X(x, y) := \inf \big\{ l(\gamma) : \ \gamma \in \mathcal{S}(X),\ \gamma(0) = x,\ \gamma(T) = y \big\}. \tag{5.6}$$

Under suitable hypotheses, the above inf is actually a minimum. For example, in Appendix C, we show that this occurs if $\{X_1, \dots, X_m\}$ are generators of the first layer of the stratified algebra of a homogeneous Carnot group (see also [HK00]).

Proposition 5.2.3 (d_X is a metric). *If* $\mathbb{R}^N$ *is* X*-connected, then the function* $(x, y) \mapsto d_X(x, y)$ *is a metric on* $\mathbb{R}^N$, *called the* X-control distance *or the* Carnot–Carathéodory distance *related to* X.

In what follows, when there is no risk of confusion, we shall simply write d instead of d_X.

Proof. It is quite easy to see that d is non-negative, symmetric and satisfies the triangle inequality. To prove positivity, i.e.

$$\big(d(x, y) = 0\big) \quad \Longrightarrow \quad (x = y), \tag{5.7}$$

we compare d with the Euclidean metric (which has an interest in its own). Given $x \in \mathbb{R}^N$ and $r > 0$, define

$$M(x, r) := \sup \left\{ \sum_{j=1}^{n} |X_j I(z)| \ : \ |z - x| \le r \right\}, \tag{5.8a}$$

where $|\cdot|$ denotes the Euclidean norm. We next show the following inequality

$$M(x, |x - y|)\, d(x, y) \ge |x - y| \quad \forall\, x, y \in \mathbb{R}^N. \tag{5.8b}$$

This will obviously imply (5.7). By contradiction, assume (5.8b) is false for some x and y in $\mathbb{R}^N$. Then there exists $\gamma \in \mathcal{S}(X)$, $\gamma : [0, T] \to \mathbb{R}^N$, such that $\gamma(0) = x$, $\gamma(T) = y$ and $M(x, |x - y|)\, T < |x - y|$. As a consequence, if we put

$$t^* := \sup \big\{ t \in [0, T] \ : \ |\gamma(s) - x| < |x - y| \text{ for } 0 \le s \le t \big\},$$

we have

$$\begin{aligned} |\gamma(t^*) - x| &= \left| \int_0^{t^*} \dot{\gamma}(s)\, \mathrm{d}s \right| \le \sum_{j=1}^{m} \int_0^{t^*} |X_j I(\gamma(s))|\, \mathrm{d}s \\ &\le M(x, |x - y|)\, t^* \le M(x, |x - y|)\, T < |x - y|. \end{aligned}$$

It follows that $t^* = T$ and $|y - x| = |\gamma(T) - x| < |x - y|$. This contradiction proves (5.8b). $\square$

If the vector fields $X_1, \dots, X_m$ are left invariant w.r.t. the translations on a Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$, then the $X = \{X_1, \dots, X_m\}$-control distance has the same property. Indeed, we have the following proposition.

Proposition 5.2.4 (Control distance of a left-invariant family). *Let* $\mathbb{G} = (\mathbb{R}^N, \circ)$ *be a Lie group on* $\mathbb{R}^N$*, and let* d *be the control distance related to a family of left invariant vector fields* $X = \{X_1, \dots, X_m\}$ *on* $\mathbb{G}$*. Then*

$$d(x, y) = d(y^{-1} \circ x, 0) \quad \forall\, x, y \in \mathbb{G}, \tag{5.9a}$$

and

$$d(x^{-1}, 0) = d(x, 0) \quad \forall\, x \in \mathbb{G}. \tag{5.9b}$$

The proof of this proposition will easily follow from the next lemma.

Lemma 5.2.5. *In the hypotheses of Proposition* 5.2.4, *let* $\gamma : [0, T] \to \mathbb{R}^N$ *be a* X*-subunit curve. Then* $\alpha \circ \gamma$ *is* X*-subunit for every* $\alpha \in \mathbb{G}$.

Proof. If we denote by Γ the path $\alpha \circ \gamma$, we have

$$\dot{\Gamma}(s) = \mathcal{J}_{\tau_\alpha}(\gamma(s)) \cdot \dot{\gamma}(s) \text{ almost everywhere in } [0, T].$$

Then, for every $\xi \in \mathbb{R}^N$,

$$\begin{aligned}
\langle \dot{\Gamma}(s), \xi \rangle^2 &= \left\langle \dot{\gamma}(s), \left(\mathcal{J}_{\tau_\alpha}(\gamma(s))\right)^T \xi \right\rangle^2 \quad \text{(since } \gamma \text{ is } X\text{-subunit)} \\
&\le \sum_{j=1}^m \left\langle X_j I(\gamma(s)), \left(\mathcal{J}_{\tau_\alpha}(\gamma(s))\right)^T \xi \right\rangle^2 \\
&= \sum_{j=1}^m \left\langle \mathcal{J}_{\tau_\alpha}(\gamma(s)) \cdot X_j I(\gamma(s)), \xi \right\rangle^2 \quad \text{(Proposition 1.2.3, page 14)} \\
&= \sum_{j=1}^m \left\langle X_j I(\alpha \circ \gamma(s)), \xi \right\rangle^2 = \sum_{j=1}^m \left\langle X_j I(\Gamma(s)), \xi \right\rangle^2.
\end{aligned}$$

This proves that Γ is X-subunit. □

Proof (of Proposition 5.2.4). Let $x, y \in \mathbb{G}$, and let γ be a X-subunit path connecting x and y. For every $\alpha \in \mathbb{G}$, by the previous lemma, $\alpha \circ \gamma$ is a X-subunit path connecting $\alpha \circ x$ and $\alpha \circ y$. Then $d(\alpha \circ x, \alpha \circ y) \le d(x, y)$. Since x, y, α are arbitrary, this inequality obviously also implies $d(x, y) \le d(\alpha \circ x, \alpha \circ y)$. Thus, we have proved

$$d(\alpha \circ x, \alpha \circ y) = d(x, y) \quad \forall\, x,\ y,\ \alpha \in \mathbb{G}. \tag{5.10}$$

Choosing in (5.10) $\alpha = y^{-1}$, we obtain (5.9a).

Putting $x = 0$ in (5.9a), we obtain $d(y^{-1}, 0) = d(0, y) = d(y, 0)$, which is (5.9b). This completes the proof. □

The control distance related to homogeneous vector fields is homogeneous. More precisely, the following assertion holds.

Proposition 5.2.6 (Control distance of a homogeneous family). *Let* d *be the control distance related to a family* $X = \{X_1, \ldots, X_m\}$ *of smooth vector fields in* $\mathbb{R}^N$. *Assume the* X_j*'s are* ϱ_λ*-homogeneous of degree one with respect to the "dilations"*

$$\varrho_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \varrho_\lambda(x_1, \ldots, x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N),$$

where $\sigma_1, \ldots, \sigma_N$ *are positive real numbers. Then*

$$d(\varrho_\lambda(x), \varrho_\lambda(y)) = \lambda\, d(x, y) \quad \forall\, x, y \in \mathbb{R}^N\ \forall\, \lambda > 0. \tag{5.11}$$

For the proof of this proposition we need the following lemma.

Lemma 5.2.7. *In the hypotheses of Proposition* 5.2.6, *let* $\gamma : [0, T] \to \mathbb{R}^N$ *be a* X*-subunit curve. Then, for every* $\lambda > 0$, *the curve*

$$\Gamma : [0, \lambda T] \to \mathbb{R}^N, \quad \Gamma(s) := \varrho_\lambda(\gamma(s/\lambda))$$

is a X*-subunit path.*

Proof. For every $\xi \in \mathbb{R}^N$, we have (note that ϱ_λ is a linear and symmetric map for it is represented by a diagonal matrix)

$$\begin{aligned} \langle \dot{\Gamma}(s), \xi \rangle^2 &= \lambda^{-2} \big\langle \varrho_\lambda(\dot{\gamma}(s/\lambda)), \xi \big\rangle^2 = \lambda^{-2} \big\langle \dot{\gamma}(s/\lambda), \varrho_\lambda(\xi) \big\rangle^2 \\ \text{(since } \gamma \text{ is } X\text{-subunit)} &\le \lambda^{-2} \sum_{j=1}^m \big\langle X_j I(\gamma(s/\lambda)), \varrho_\lambda(\xi) \big\rangle^2 \\ &= \sum_{j=1}^m \big\langle \lambda^{-1} \varrho_\lambda(X_j I(\gamma(s/\lambda))), \xi \big\rangle^2 \\ \text{(Corollary 1.3.6, page 35)} &= \sum_{j=1}^m \big\langle X_j I(\varrho_\lambda(\gamma(s/\lambda))), \xi \big\rangle^2. \end{aligned}$$

Since $\varrho_\lambda(\gamma(s/\lambda)) = \Gamma(s)$, this proves the lemma. □

Proof (of Proposition 5.2.6). Let $x, y \in \mathbb{R}^N$, and let $\gamma : [0, T] \to \mathbb{R}^N$ be a X-subunit curve connecting x and y. By the previous lemma, $\Gamma(s) = \varrho_\lambda(\gamma(s/\lambda))$ $(0 \le s \le \lambda t)$ is a X-subunit curve, so that, since Γ connects $\varrho_\lambda(x)$ and $\varrho_\lambda(y)$,

$$d(\varrho_\lambda(x), \varrho_\lambda(y)) \le l(\Gamma) = \lambda T = \lambda\, l(\gamma).$$

Then, being γ an arbitrary X-subunit curve connecting x and y,

$$d(\varrho_\lambda(x), \varrho_\lambda(y)) \le \lambda\, d(x, y) \quad \forall\, x, y \in \mathbb{R}^N\ \forall\, \lambda > 0. \tag{5.12}$$

This inequality obviously implies (replace x and y with $\varrho_{1/\lambda}(\widetilde{x})$ and $\varrho_{1/\lambda}(\widetilde{y})$, respectively, and then λ with $1/\widetilde{\lambda}$; then remove "$\sim$")

$$d(x, y) \le \frac{1}{\lambda}\, d(\varrho_\lambda(x), \varrho_\lambda(y)) \quad \forall\, x, y \in \mathbb{R}^N\ \forall\, \lambda > 0.$$

Then (5.12) holds with the equality sign and the proposition is proved. □

Theorem 5.2.8 (Control distance of a homogeneous Carnot group). *Let* $\mathbb{G}$ *be a homogeneous Carnot group on* $\mathbb{R}^N$, *and let* d *be the control distance related to any family of generators for* $\mathbb{G}$. *Then*

$$\mathbb{G} \ni x \mapsto d_0(x) := d(x, 0) \tag{5.13}$$

is a symmetric homogeneous norm on $\mathbb{G}$.

Proof. By means of Propositions 5.2.3, 5.2.4 and 5.2.6, we are only left to prove that d_0 is continuous. For the proof of this fact, we refer to Theorem 19.1.3 on page 716. □

Remark 5.2.9. The homogeneous norm (5.13), in general, is not smooth.

Corollary 5.2.10. *Let $\mathbb{G}$ be a Carnot group. Denote by d the control distance related to a family of generators for $\mathbb{G}$. Then, for every compact subset K of $\mathbb{G}$, there exists a positive constant $C(K)$ such that*

$$d(x, y) \le C(K)\,|x - y|^{1/r} \quad \forall\, x, y \in K,$$

where r denotes the step of $\mathbb{G}$.

(See also Proposition 5.15.1 (page 309) in Appendix C for an estimate from below of $d(x, y)$.)

Proof. It follows from Propositions 5.1.6, 5.2.4 and Theorem 5.2.8. □

5.3 The Fundamental Solution

Throughout the sequel, we shall make use of some maximum principles for sub-Laplacians, which (for the reader's convenience) we postpone to Appendix A of the present chapter (see Section 5.13).

For our purposes, it is convenient to give the definition of fundamental solution of a sub-Laplacian $\mathcal{L}$ on a homogeneous Carnot group as follows.

Definition 5.3.1 (Fundamental solution). *Let $\mathbb{G}$ be a homogeneous Carnot group on $\mathbb{R}^N$. Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. A function $\Gamma : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a* fundamental solution *for $\mathcal{L}$ if*:

(i) $\Gamma \in C^\infty(\mathbb{R}^N \setminus \{0\})$;
(ii) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *and* $\Gamma(x) \longrightarrow 0$ *when x tends to infinity*;
(iii) $\mathcal{L}\Gamma = -\mathrm{Dirac}_0$, *being* Dirac_0 *the Dirac measure supported at* $\{0\}$. *More explicitly (recall that $\mathcal{L}^* = \mathcal{L}$, being $\mathcal{L}^*$ the formal adjoint of $\mathcal{L}$),*

$$\int_{\mathbb{R}^N} \Gamma\, \mathcal{L}\varphi \,\mathrm{d}x = -\varphi(0) \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N). \tag{5.14}$$

Theorem 5.3.2 (Existence of the fundamental solution). *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$ (whose homogeneous dimension Q is > 2). Then there exists a fundamental solution Γ for $\mathcal{L}$.*

(*Note*. Such a fundamental solution is indeed *unique*, as it will be proved in Proposition 5.3.10.)

Proof. The existence of such a fundamental solution may be proved by means of very general arguments from the theory of distributions, based on the hypoellipticity of $\mathcal{L}$ and of its formal adjoint $\mathcal{L}^*$ $(= \mathcal{L})$, jointly with the well-behaved homogeneity properties of $\mathcal{L}$.

Indeed, from the hypoellipticity of $\mathcal{L}$ (see property (A0), page 63) we infer the existence of a "local" fundamental solution satisfying $\mathcal{L}\Gamma = -\mathrm{Dirac}_0$ on a neighborhood of the origin (see F. Trèves [Tre67, Theorems 52.1, 52.2]). Moreover, by using the homogeneity properties of $\mathcal{L}$, a "local-to-global" argument can be performed. It is out of our scopes here to give the details. The complete proof is due to G.B. Folland and can be found in [Fol75, Theorem 2.1] (see also L. Gallardo [Gal82] for some further properties of Γ obtained via probabilistic techniques). An alternative proof can be found in [BLU02, Theorem 3.9]. □

From the integral identity (5.14) and condition (i) in the above Definition 5.3.1 we immediately get the $\mathcal{L}$-harmonicity of Γ out of the origin. Indeed, if we replace in (5.14) a test function φ with support in $\mathbb{R}^N \setminus \{0\}$, by the smoothness of Γ out of the origin, we can integrate by parts obtaining

$$\int_{\mathbb{R}^N} (\mathcal{L}\Gamma)\, \varphi \,\mathrm{d}x = 0 \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}).$$

This obviously implies

$$\mathcal{L}\Gamma = 0 \text{ in } \mathbb{R}^N \setminus \{0\}. \tag{5.15}$$

A simple change of variable and the left-invariance of $\mathcal{L}$ w.r.t. the translations on $\mathbb{G}$ give the following theorem.

Theorem 5.3.3 (Γ left-inverts $\mathcal{L}$). *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. If Γ is a fundamental solution for $\mathcal{L}$, then*

$$\int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x)\, \mathcal{L}\varphi(x) \,\mathrm{d}x = -\varphi(y) \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N) \tag{5.16}$$

and every $y \in \mathbb{R}^N$.

Proof. The change of variable $z = y^{-1} \circ x$ gives

$$\int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x)\, \mathcal{L}\varphi(x) \,\mathrm{d}x = \int_{\mathbb{R}^N} \Gamma(z)\, (\mathcal{L}\varphi)(y \circ z) \,\mathrm{d}z. \tag{5.17}$$

On the other hand, since $\mathcal{L}$ is left-invariant on $\mathbb{G}$, then

$$(\mathcal{L}\varphi)(y \circ z) = \mathcal{L}(\varphi(y \circ z)).$$

Replacing this identity in (5.17) and using (5.14) with $\varphi(\cdot)$ replaced by $\varphi(y \circ \cdot)$, one gets the thesis. □

Remark 5.3.4. The integral identity (5.16) means that

$$\mathcal{L}(\Gamma(y^{-1}\circ\cdot)) = -\mathrm{Dirac}_y$$

in the weak sense of distributions. Here Dirac_y denotes the Dirac measure supported at $\{y\}$.

Due to identity (5.16), we can say that Γ is a left inverse of $\mathcal{L}$. We next prove that Γ is a right inverse too.

Theorem 5.3.5 (Γ right-inverts $\mathcal{L}$). *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. If Γ is a fundamental solution for $\mathcal{L}$, then, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, the function*

$$\mathbb{R}^N \ni y \mapsto u(y) := \int_{\mathbb{R}^N} \Gamma(y^{-1}\circ x)\,\varphi(x)\,\mathrm{d}x \tag{5.18a}$$

is smooth and satisfies the equation

$$\mathcal{L}u = -\varphi. \tag{5.18b}$$

The proof of this theorem requires some prerequisites. First of all, we note that a change of variable in the integral at the right-hand side of (5.18a) gives

$$u(y) = \int_{\mathbb{R}^N} \Gamma(z)\,\varphi(y\circ z)\,\mathrm{d}z.$$

Then we can differentiate under the integral sign and get the smoothness of u. Moreover, if $\mathrm{supp}(\varphi) \subseteq \{x : d(x) \le R\}$ (here d denotes a fixed homogeneous norm on $\mathbb{G}$), then

$$\begin{aligned} |u(y)| &\le \sup\{|\Gamma(z)| : d(y\circ z) \le R\} \int_{\mathbb{R}^N} |\varphi(z)|\,\mathrm{d}z \\ &=: C(y) \int_{\mathbb{R}^N} |\varphi(z)|\,\mathrm{d}z. \end{aligned} \tag{5.19}$$

On the other hand, by Corollary 5.1.5 and Proposition 5.1.7,

$$d(z) \ge \frac{1}{\mathbf{c}} d(y) - \mathbf{c} d(y\circ z),$$

for a suitable positive constant $\mathbf{c}$ independent of x, y, z. As a consequence,

$$C(y) \le \sup\left\{ |\Gamma(z)| : d(z) \ge \frac{1}{\mathbf{c}} d(y) - \mathbf{c}R \right\},$$

so that, since $\Gamma(z)$ vanishes as z goes to infinity, inequality (5.19) implies

$$\lim_{y\to\infty} u(y) = 0. \tag{5.20}$$

We then show a crucial property of the $(\varepsilon, \mathbb{G})$-*mollifiers*. The relevant definition is the following one.

Definition 5.3.6 (Mollifiers). *Let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be a homogeneous Carnot group on $\mathbb{R}^N$. Let O be a fixed non-empty open neighborhood of the origin* 0. *Let also be given a function $J \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$, $J \geq 0$, such that*

$$\operatorname{supp}(J) \subset O \text{ and } \int_{\mathbb{R}^N} J = 1.$$

For any $\varepsilon > 0$, we set $J_\varepsilon(x) := \varepsilon^{-Q} J(\delta_{1/\varepsilon}(x))$.

Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. We set, for every $x \in \mathbb{R}^N$,

$$\begin{aligned} u_\varepsilon(x) &:= (u *_{\mathbb{G}} J_\varepsilon)(x) := \int_{\mathbb{R}^N} u(y)\, J_\varepsilon(x \circ y^{-1})\, \mathrm{d}y \\ &= \int_{\delta_\varepsilon(O)} u(z^{-1} \circ x)\, J_\varepsilon(z)\, \mathrm{d}z. \end{aligned} \tag{5.21}$$

We call u_ε a mollifier of u *(or* $(\varepsilon, \mathbb{G})$-mollifier) related to the kernel J. Note that this mollifier depends only on $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ and J.

For the use of mollifiers in a context of subelliptic PDE's, see also [CDG97].

Example 5.3.7. Let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be a homogeneous Carnot group on $\mathbb{R}^N$. Let ϱ be a fixed homogeneous symmetric norm on $\mathbb{G}$. We set a notation which will be used throughout the book. For every $x \in \mathbb{G}$ and every $r > 0$, we set

$$B_\varrho(x, r) := \left\{ y \in \mathbb{G} \,:\, \varrho(x^{-1} \circ y) < r \right\}.$$

We say that $B_\varrho(x, r)$ is the *ϱ-ball with center x and radius r*. Also, fixed a point $x \in \mathbb{G}$ and a set $A \subset \mathbb{G}$, we let

$$\varrho\text{-dist}(x, A) := \inf \left\{ \varrho(x^{-1} \circ a) : a \in A \right\}.$$

We call ϱ-dist(x, A) the *ϱ-distance of x from A*. The notation $\mathrm{dist}_\varrho(x, A)$ will also be available.

Let now a function $J \in C_0^\infty(\mathbb{R}^N)$, $J \geq 0$ be given such that

$$\operatorname{supp}(J) \subset B_\varrho(0, 1) \text{ and } \int_{\mathbb{R}^N} J = 1.$$

For any $\varepsilon > 0$, we set $J_\varepsilon(x) := \varepsilon^{-Q} J(\delta_{1/\varepsilon}(x))$.

Let $u \in L^1_{\mathrm{loc}}(\Omega)$, $\Omega \subseteq \mathbb{R}^N$ open. For the open set

$$\Omega_\varepsilon := \{x \in \Omega \,:\, \varrho\text{-dist}(x, \partial\Omega) > \varepsilon\},$$

we define

$$\begin{aligned} u_\varepsilon(x) &:= (u *_{\mathbb{G}} J_\varepsilon)(x) := \int_{B_\varrho(x,\, \varepsilon)} u(y)\, J_\varepsilon(x \circ y^{-1})\, \mathrm{d}y \\ &= \int_{B_\varrho(0, \varepsilon)} u(y^{-1} \circ x)\, J_\varepsilon(y)\, \mathrm{d}y \end{aligned} \tag{5.22}$$

for every $x \in \Omega_\varepsilon$.

We call u_ε a *mollifier of* u (or $(\varepsilon, \mathbb{G})$-*mollifier*) *related to the homogeneous norm* ϱ. Note that this mollifier depends only on $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$, J and ϱ.

Remark 5.3.8. Let the notation in Definition 5.3.6 be fixed. Let $u \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then the following fact holds:

$$(\star) \quad u_\varepsilon \to u \quad \text{as} \quad \varepsilon \to 0 \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N).$$

Indeed, we have (perform the change of variable $y = \delta_{1/\varepsilon}(z)$)

$$u_\varepsilon(x) = \int_{\delta_\varepsilon(O)} u(z^{-1} \circ x)\, \varepsilon^{-Q} J(\delta_{1/\varepsilon}(z))\, \mathrm{d}z = \int_O u(\delta_\varepsilon(y^{-1}) \circ x) J(y)\, \mathrm{d}y.$$

As a consequence (recall that $\int_O J = 1$),

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_\varepsilon(x) - u(x)|\, \mathrm{d}x &= \int_{\mathbb{R}^N} \left| \int_O u(\delta_\varepsilon(y^{-1}) \circ x)\, J(y)\, \mathrm{d}y - u(x) \right| \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left| \int_O \big(u(\delta_\varepsilon(y^{-1}) \circ x) - u(x)\big)\, J(y)\, \mathrm{d}y \right| \mathrm{d}x \\
&\le \int_O \left\{ \int_{\mathbb{R}^N} \left| u(\delta_\varepsilon(y^{-1}) \circ x) - u(x) \right| \mathrm{d}x \right\} J(y)\, \mathrm{d}y.
\end{aligned}$$

Given $\sigma > 0$, by well-known results, there exists $\overline{\varepsilon} = \overline{\varepsilon}(\sigma, \mathbb{G}, O) > 0$ such that if $0 < \varepsilon < \overline{\varepsilon}$, then the integral in braces is $\le \sigma$ for every fixed $y \in O$. This proves that

$$\int_{\mathbb{R}^N} |u_\varepsilon(x) - u(x)|\, \mathrm{d}x \le \sigma \int_O J(y)\, \mathrm{d}y = \sigma \quad \forall\, 0 < \varepsilon < \overline{\varepsilon},$$

i.e. $(\star)$ holds. □

Proposition 5.3.9 ($\mathcal{L}$-harmonicity of the mollifier). *Let* $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ *be a homogeneous Carnot group on* $\mathbb{R}^N$. *Let* $\mathcal{L}$ *be a sub-Laplacian on* $\mathbb{G}$. *Let* $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ *be a weak solution to* $\mathcal{L}u = 0$ *in* $\mathbb{R}^N$, *i.e.*

$$\int_{\mathbb{R}^N} u\, \mathcal{L}\varphi\, \mathrm{d}y = 0 \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N). \tag{5.23}$$

Then, if u_ε *denotes the mollification on* $\mathbb{G}$ *w.r.t. any kernel* J (*as in Definition* 5.3.6), *we have*

$$\mathcal{L}u_\varepsilon = 0 \quad \textit{in } \mathbb{G} \quad \textit{for every } \varepsilon > 0. \tag{5.24}$$

Proof. First, note that, being $\operatorname{supp}(J_\varepsilon) \subset \delta_\varepsilon(O)$,

$$u_\varepsilon(x) = \int_{\mathbb{R}^N} u(y^{-1} \circ x)\, J_\varepsilon(y)\, \mathrm{d}y.$$

For every test function φ, we thus have (Fubini–Tonelli's theorem certainly applies)

$$\begin{aligned}\int_{\mathbb{R}^N} u_\varepsilon(x)\,\mathcal{L}\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^N} \mathcal{L}\varphi(x)\Big(\int_{\mathbb{R}^N} u(y^{-1}\circ x)\,J_\varepsilon(y)\,\mathrm{d}y\Big)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N} J_\varepsilon(y)\Big(\int_{\mathbb{R}^N} (\mathcal{L}\varphi)(x)u(y^{-1}\circ x)\,\mathrm{d}x\Big)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^N} J_\varepsilon(y)\Big(\int_{\mathbb{R}^N} (\mathcal{L}\varphi)(y\circ z)\,u(z)\,\mathrm{d}z\Big)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^N} J_\varepsilon(y)\Big(\int_{\mathbb{R}^N} \mathcal{L}\big(z\mapsto \varphi(y\circ z)\big)\,u(z)\,\mathrm{d}z\Big)\,\mathrm{d}y.\end{aligned}$$

The inner integral in the far right-hand side is equal to zero by the hypothesis (5.23). Then

$$\int_{\mathbb{R}^N} u_\varepsilon(x)\,\mathcal{L}\varphi(x)\,\mathrm{d}x = 0 \quad \forall\,\varphi\in C_0^\infty(\mathbb{R}^N),$$

so that the claimed (5.24) follows from the hypoellipticity of $\mathcal{L}$ and the fact that $\mathcal{L}^* = \mathcal{L}$. □

With Proposition 5.3.9 at hand, it is easy to prove the uniqueness of the fundamental solution.

Proposition 5.3.10 (Uniqueness of the fundamental solution). *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. The fundamental solution of $\mathcal{L}$ (whose existence is granted by Theorem 5.3.2) is unique.*

Proof. Let Γ and Γ' be fundamental solutions for $\mathcal{L}$. Then the function $u = \Gamma - \Gamma'$ has the following properties: $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $u(x)\to 0$ as $x\to\infty$ and

$$\int_{\mathbb{R}^N} u\,\mathcal{L}\varphi\,\mathrm{d}y = 0 \qquad \forall\,\varphi\in C_0^\infty(\mathbb{R}^N).$$

As a consequence, by Proposition 5.3.9, $\mathcal{L}u_\varepsilon = 0$ in $\mathbb{R}^N$ for every $\varepsilon > 0$.

Thus, since $u_\varepsilon(x)\to 0$ as $x\to\infty$ (argue as in (5.20)), the maximum principle in Section 5.13 implies $u_\varepsilon = 0$ in $\mathbb{R}^N$. On the other hand, $u_\varepsilon \to u$ as $\varepsilon\to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ (see Remark 5.3.8). Then $u = 0$ almost everywhere in $\mathbb{R}^N$, so that $\Gamma = \Gamma'$ in $\mathbb{R}^N\setminus\{0\}$. □

We now prove that Γ is "$\mathbb{G}$-symmetric" with respect to the origin. More precisely, the following assertion holds.

Proposition 5.3.11 (Symmetry of Γ). *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. Let Γ be the fundamental solution of $\mathcal{L}$. Then*

$$\Gamma(x^{-1}) = \Gamma(x) \quad \forall\,x\in\mathbb{G}\setminus\{0\}.$$

Proof. Given $\varphi\in C_0^\infty(\mathbb{R}^N)$, define

$$u(x) := -\int_{\mathbb{R}^N} \Gamma(y^{-1}\circ x)\,\mathcal{L}\varphi(y)\,\mathrm{d}y, \quad x\in\mathbb{G}.$$

The function u is smooth and vanishes at infinity (see (5.20)). Moreover, for every $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{L}u(x)\,\psi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^N} u(x)\,\mathcal{L}\psi(x)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^N} \mathcal{L}\varphi(y)\left(\int_{\mathbb{R}^N} \Gamma(y^{-1}\circ x)\,\mathcal{L}\psi(x)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \mathcal{L}\varphi(x)\,\psi(x)\,\mathrm{d}x \quad \text{(by Theorem 5.3.3).}\end{aligned}$$

This proves that $\mathcal{L}(u-\varphi)=0$ in $\mathbb{G}$. Thus, since $u-\varphi$ vanishes at infinity, by the maximum principle, $u=\varphi$ in $\mathbb{R}^N$. In particular,

$$\varphi(0) = u(0) = -\int_{\mathbb{R}^N} \Gamma(y^{-1}\circ x)\mathcal{L}\varphi(y)\,\mathrm{d}y \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^N),$$

so that $x \mapsto \Gamma(x^{-1})$ is a fundamental solution of $\mathcal{L}$ (see Definition 5.3.1). The uniqueness of the fundamental solution (Proposition 5.3.10) implies $\Gamma(x^{-1})=\Gamma(x)$ for every $x\in\mathbb{G}\setminus\{0\}$. □

Finally, we are in the position to prove Theorem 5.3.5.

Proof (of Theorem 5.3.5). Let u be the function defined in (5.18a). Then $u \in C^\infty(\mathbb{R}^N)$ and, for any test function $\psi \in C_0^\infty(\mathbb{R}^N)$, one has

$$\begin{aligned}\int_{\mathbb{R}^N} (\mathcal{L}u)(y)\,\psi(y)\,\mathrm{d}y &= \int_{\mathbb{R}^N} u(y)\,\mathcal{L}\psi(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \mathcal{L}\psi(y)\left(\int_{\mathbb{R}^N} \Gamma(y^{-1}\circ x)\,\varphi(x)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \varphi(x)\left(\int_{\mathbb{R}^N} \Gamma(y^{-1}\circ x)\mathcal{L}\psi(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \varphi(x)\left(\int_{\mathbb{R}^N} \Gamma(x^{-1}\circ y)\,\mathcal{L}\psi(y)\,\mathrm{d}y\right)\mathrm{d}x.\end{aligned}$$

Here we used Proposition 5.3.11 ensuring that $\Gamma(y^{-1}\circ x) = \Gamma((x^{-1}\circ y)^{-1}) = \Gamma(x^{-1}\circ y)$. Now, by identity (5.16), the inner integral at the far right-hand side is equal to $-\psi(x)$. Then

$$\int_{\mathbb{R}^N} (\mathcal{L}u)(y)\,\psi(y)\,\mathrm{d}y = -\int_{\mathbb{R}^N} \varphi(x)\,\psi(x)\,\mathrm{d}x \quad \forall\,\psi \in C_0^\infty(\mathbb{R}^N).$$

This gives identity (5.18b). □

From the uniqueness of Γ we easily obtain its δ_λ-homogeneity.

Proposition 5.3.12 (δ_λ-homogeneity of Γ). *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. Let Γ be the fundamental solution of $\mathcal{L}$. Then Γ is δ_λ-homogeneous of degree $2-Q$, i.e.*

$$\Gamma(\delta_\lambda(x)) = \lambda^{2-Q}\Gamma(x) \qquad \forall\,x\in\mathbb{G}\setminus\{0\} \quad \forall\,\lambda>0.$$

Proof. For any fixed $\lambda > 0$, define

$$\Gamma'(x) := \lambda^{Q-2}\Gamma(\delta_\lambda(x)) \quad \forall\, x \in \mathbb{G} \setminus \{0\}.$$

It is quite obvious that $\Gamma' \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $\Gamma'(x) \to 0$ as $x \to \infty$. Moreover, for every test function $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
\int_{\mathbb{R}^N} \Gamma'(x)\, \mathcal{L}\varphi(x)\, \mathrm{d}x &= \lambda^{Q-2} \int_{\mathbb{R}^N} \Gamma(\delta_\lambda(x))\, \mathcal{L}\varphi(x)\, \mathrm{d}x \\
&\qquad \text{(by using the change of variable } y = \delta_\lambda(x)) \\
&= \lambda^{-2} \int_{\mathbb{R}^N} \Gamma(y)\, (\mathcal{L}\varphi)(\delta_{1/\lambda}(y))\, \mathrm{d}y \\
&\qquad \text{(since } \mathcal{L} \text{ is } \delta_\lambda\text{-homogeneous of degree 2)} \\
&= \int_{\mathbb{R}^N} \Gamma(y) \mathcal{L}\big(\varphi(\delta_{1/\lambda}(y))\big)\, \mathrm{d}y = -\varphi(\delta_{1/\lambda}(0)) = -\varphi(0).
\end{aligned}$$

This proves that Γ' is a fundamental solution of $\mathcal{L}$. Then, by Proposition 5.3.10, $\Gamma = \Gamma'$ and the assertion is proved. $\square$

From the strong maximum principle in Theorem 5.13.8 we obtain another important property of Γ.

Proposition 5.3.13 (Positivity of Γ). *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. Let Γ be the fundamental solution of $\mathcal{L}$. Then*

$$\Gamma(x) > 0 \quad \forall\, x \in \mathbb{G} \setminus \{0\}.$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$. Define

$$u(y) := \int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x)\varphi(x)\, \mathrm{d}x, \qquad y \in \mathbb{G}.$$

The function u is smooth, vanishes at infinity and satisfies the equation $\mathcal{L}u = -\varphi$ (see (5.20) and Theorem 5.3.5). Then

$$\mathcal{L}u \leq 0 \text{ in } \mathbb{G} \text{ and } \lim_{y\to\infty} u(y) = 0.$$

By the maximum principle in Corollary 5.13.6, it follows that $u \geq 0$ in $\mathbb{G}$. Hence

$$\int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x)\, \varphi(x)\, \mathrm{d}x \geq 0 \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N),\ \varphi \geq 0.$$

Thus, $\Gamma \geq 0$, so that, since it is $\mathcal{L}$-harmonic in $\mathbb{G} \setminus \{0\}$, the strong maximum principle in Theorem 5.13.8 implies $\Gamma \equiv 0$ or $\Gamma(x) > 0$ for any $x \neq 0$. The first case would contradict identity (5.14). Then $\Gamma > 0$ at any point of $\mathbb{G} \setminus \{0\}$. This ends the proof. $\square$

Corollary 5.3.14 (Pole of Γ). *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. The fundamental solution Γ of $\mathcal{L}$ has a pole at* 0, *i.e.*

$$\lim_{x\to 0} \Gamma(x) = \infty. \tag{5.25}$$

Proof. Since Γ is smooth and strictly positive out the origin, we have

$$h := \min\{\Gamma(x) : d(x) = 1\} > 0.$$

Here d denotes any fixed homogeneous norm on $\mathbb{G}$. Then, by Proposition 5.3.12,

$$\Gamma(x) = d(x)^{2-Q}\,\Gamma(\delta_{1/d(x)}(x)) \geq h\, d^{2-Q}(x).$$

From this inequality (5.25) immediately follows. □

5.3.1 The Fundamental Solution in the Abstract Setting

The aim of this brief section is to show how to derive a "fundamental solution" for a sub-Laplacian $\mathcal{L}$ on an abstract stratified group $\mathbb{H}$, *starting from* the fundamental solution of the related sub-Laplacian $\widetilde{\mathcal{L}}$ on a homogeneous-Carnot-group copy $\mathbb{G}$ of $\mathbb{H}$.

Many other alternative "more intrinsic" definitions may be certainly provided, for example, by making use of the integration on an abstract Lie group. We considered more "in the spirit" of our exposition to pass through the homogeneous group $\mathbb{G}$. Besides, this also makes unnecessary to furnish the (lengthy) theory of integration on manifolds.

Let $\mathbb{H}$ be a stratified group with an algebra $\mathfrak{h}$. Let $\mathcal{L} = \sum_{j=1}^m X_j^2$ be a sub-Laplacian on $\mathbb{H}$, and let $V = (V_1, \ldots, V_r)$ be the stratification of $\mathfrak{h}$ related to $\mathcal{L}$, according to Definition 2.2.25, page 144. Let also $\mathcal{E}$ be a basis for $\mathfrak{h}$ adapted to the stratification V.

By Proposition 2.2.22 on page 139 (whose notation we presently follow), there exists a homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \diamond_{\mathcal{E}})$ which is isomorphic to $\mathbb{H}$. Let $\Psi : \mathbb{G} \to \mathbb{H}$ be the isomorphism as in Proposition 2.2.22-(1). With the therein notation, we have

$$\Psi = \operatorname{Exp} \circ \pi_{\mathcal{E}}^{-1},$$

where, for every $\xi \in \mathbb{R}^N$, $\pi_{\mathcal{E}}^{-1}(\xi)$ is the vector field in $\mathfrak{h}$ having ξ as N-tuple of the coordinates w.r.t. the basis $\mathcal{E}$.

Let $\widetilde{X}_1, \ldots, \widetilde{X}_m$ be the vector fields in $\mathfrak{g}$ (the algebra of $\mathbb{G}$) which are Ψ-related to $X_1, \ldots, X_m$, respectively, i.e.

$$\mathrm{d}\Psi(\widetilde{X}_j) = X_j \quad \text{for every } j = 1, \ldots, m.$$

We set

$$\widetilde{\mathcal{L}} := \sum_{j=1}^m (\widetilde{X}_j)^2,$$

and (since $\widetilde{\mathcal{L}}$ is a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$) we let $\widetilde{\Gamma}$ denote its (unique) fundamental solution.

A possible definition of a fundamental solution for $\mathcal{L}$ is

$$\Gamma := \widetilde{\Gamma} \circ \Psi^{-1}. \tag{5.26}$$

Our task here is to show how Γ depends on the (arbitrary) choice of the above basis $\mathcal{E}$. We show that, roughly speaking "up to a multiplicative factor", Γ in (5.26) is intrinsic (see below for the precise statement).

To this aim, let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two bases of $\mathfrak{h}$ adapted to the stratification V. With the above notation, let (for $i = 1, 2$) $\mathbb{G}_i = (\mathbb{R}^N, \diamond_{\mathcal{E}_i})$ and $\mathfrak{g}_i$ denotes the algebra of $\mathbb{G}_i$. Moreover, we set

$$\Psi_i := \operatorname{Exp} \circ \pi_{\mathcal{E}_i}^{-1}, \quad i = 1, 2. \tag{5.27}$$

Again for $i = 1, 2$, we also let

$$\widetilde{X}_1^{(i)}, \dots, \widetilde{X}_m^{(i)}$$

be the vector fields in $\mathfrak{g}_i$ which are Ψ_i-related to $X_1, \dots, X_m$, respectively, i.e.

$$\mathrm{d}\Psi_i(\widetilde{X}_j^{(i)}) = X_j \quad \text{for every } j = 1, \dots, m. \tag{5.28}$$

We set

$$\widetilde{\mathcal{L}}_i := \sum_{j=1}^{m} (\widetilde{X}_j^{(i)})^2, \quad i = 1, 2.$$

Finally, $\widetilde{\Gamma}_i$ denotes the fundamental solution of $\widetilde{\mathcal{L}}_i$. We claim that

$$\widetilde{\Gamma}_2 = \mathbf{c}_{1,2} \cdot \widetilde{\Gamma}_1 \circ (\Psi_1^{-1} \circ \Psi_2), \quad \text{where} \quad \mathbf{c}_{1,2} := \left| \det\left(\pi_{\mathcal{E}_1} \circ \pi_{\mathcal{E}_2}^{-1}\right)\right|. \tag{5.29}$$

This will give

$$\widetilde{\Gamma}_2 \circ \Psi_2^{-1} = \mathbf{c}_{1,2} \cdot \widetilde{\Gamma}_1 \circ \Psi_1^{-1}, \tag{5.30}$$

which proves the "almost" intrinsic nature of the definition (5.26) of Γ. We explicitly remark that $\pi_{\mathcal{E}_1} \circ \pi_{\mathcal{E}_2}^{-1}$ is a linear automorphism of $\mathbb{R}^N$, so that its determinant is well-posed.

(*Note*. We explicitly remark that the uniqueness of the fundamental solution of $\mathcal{L}$ *up to a multiplicative factor* is a matter of fact. Indeed, even in the homogeneous Carnot setting, the fundamental solution *depends* on the measure within the integral in (5.14) of Definition 5.3.1. In order to have a unique fundamental solution, the choice of the Lebesgue measure on $\mathbb{R}^N$ was quite natural (though arbitrary). Instead, in the context of an abstract Lie group $\mathbb{H}$, the Haar measure is unique only *up to a multiplicative factor*, and there is no effective way to prefer a Haar measure instead of another. These remarks show that the constant in (5.30) is perfectly justified.)

We now turn to the claimed (5.29). By invoking Proposition 5.3.10, set

$$\widetilde{\Gamma}_{1,2} := \mathbf{c}_{1,2} \cdot \widetilde{\Gamma}_1 \circ (\Psi_1^{-1} \circ \Psi_2),$$

then (5.29) will follow if we show that $\widetilde{\Gamma}_{1,2}$ satisfies (i), (ii) and (iii) of Definition 5.3.1 w.r.t. the sub-Laplacian $\widetilde{\mathcal{L}}_2$.

To begin with, observe that (thanks to (5.27))

$$\Psi_1^{-1} \circ \Psi_2 = \left(\operatorname{Exp} \circ \pi_{\mathcal{E}_1}^{-1}\right)^{-1} \circ \left(\operatorname{Exp} \circ \pi_{\mathcal{E}_2}^{-1}\right) = \pi_{\mathcal{E}_1} \circ \pi_{\mathcal{E}_2}^{-1}, \tag{5.31}$$

and this last map is a linear isomorphism of $\mathbb{R}^N$. As a consequence, since $\widetilde{\Gamma}_1 \in C^\infty(\mathbb{R}^N \setminus \{0\})$, we immediately infer that the same holds for $\widetilde{\Gamma}_{1,2}$. This proves (i). Moreover, since $\widetilde{\Gamma}_1 \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\widetilde{\Gamma}_1(x) \longrightarrow 0$ when x tends to infinity, the same holds for $\widetilde{\Gamma}_{1,2}$, again thanks to (5.31). This proves (ii).

Finally, we prove (iii). First, note that from (5.28) one gets

$$\mathrm{d}\Psi_1(\widetilde{X}_j^{(1)}) = X_j = \mathrm{d}\Psi_2(\widetilde{X}_j^{(2)}) \quad \text{for every } j = 1, \dots, m,$$

i.e. for every $j = 1, \dots, m$,

$$\widetilde{X}_j^{(1)} = \mathrm{d}(\Psi_1^{-1} \circ \Psi_2)(\widetilde{X}_j^{(2)}).$$

Set $\Psi_{1,2} := \Psi_1^{-1} \circ \Psi_2$. This gives

$$\widetilde{\mathcal{L}}_1 = \mathrm{d}\Psi_{1,2}(\widetilde{\mathcal{L}}_2),$$

i.e. it holds

$$(\widetilde{\mathcal{L}}_1 f) \circ \Psi_{1,2} = \widetilde{\mathcal{L}}_2(f \circ \Psi_{1,2}) \quad \forall\, f \in C^\infty(\mathbb{R}^N, \mathbb{R}). \tag{5.32}$$

Let now $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \widetilde{\Gamma}_{1,2}(\widetilde{\mathcal{L}}_2\varphi) &= \mathbf{c}_{1,2} \cdot \int_{\mathbb{R}^N} \widetilde{\Gamma}_1(\Psi_{1,2}(x))(\widetilde{\mathcal{L}}_2\varphi)(x)\,\mathrm{d}x \\
&\quad \text{(by the linear change of variable } y = \Psi_{1,2}(x) = \pi_{\mathcal{E}_1} \circ \pi_{\mathcal{E}_2}^{-1}\text{, see (5.31))} \\
&= \mathbf{c}_{1,2} \cdot \int_{\mathbb{R}^N} \widetilde{\Gamma}_1(y)(\widetilde{\mathcal{L}}_2\varphi)\big(\Psi_{1,2}^{-1}(y)\big)\,\frac{\mathrm{d}y}{\mathbf{c}_{1,2}} \qquad \text{(see (5.32))} \\
&= \int_{\mathbb{R}^N} \widetilde{\Gamma}_1(y)\widetilde{\mathcal{L}}_1\big(\varphi \circ \Psi_{1,2}^{-1}\big)(y)\,\mathrm{d}y \\
&\quad (\widetilde{\Gamma}_1 \text{ is the fundamental solution of } \widetilde{\mathcal{L}}_1 \text{ and } \varphi \circ \Psi_{1,2}^{-1} \in C_0^\infty(\mathbb{R}^N)) \\
&= -\big(\varphi \circ \Psi_{1,2}^{-1}\big)(0) = -\varphi(0).
\end{aligned}$$

This proves (iii), and the proof is complete. □

5.4 $\mathcal{L}$-gauges and $\mathcal{L}$-radial Functions

On every homogeneous Carnot group, there exist distinguished smooth symmetric homogeneous norms playing a fundamental rôle for the sub-Laplacians. We call these norms *gauges*, according to the following definition.

Definition 5.4.1 ($\mathcal{L}$-gauge). *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. We call $\mathcal{L}$-*gauge on $\mathbb{G}$ *a homogeneous symmetric norm d* smooth *out of the origin and satisfying*

$$\mathcal{L}(d^{2-Q}) = 0 \quad in\ \mathbb{G} \setminus \{0\}. \tag{5.33}$$

*An $\mathcal{L}$-*radial function on $\mathbb{G}$ *is a function $u : \mathbb{G} \setminus \{0\} \to \mathbb{R}$ such that*

$$u(x) = f(d(x)) \qquad \forall\, x \in \mathbb{G} \setminus \{0\}$$

for a suitable $f : (0, \infty) \to \mathbb{R}$ and a given $\mathcal{L}$-gauge d on $\mathbb{G}$.

The $\mathcal{L}$-gauges are deeply related to the fundamental solution of $\mathcal{L}$.

Proposition 5.4.2. *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. Let Γ be the fundamental solution of $\mathcal{L}$. Then*

$$d(x) := \begin{cases} (\Gamma(x))^{1/(2-Q)} & \text{if } x \in \mathbb{G} \setminus \{0\}, \\ 0 & \text{if } x = 0 \end{cases}$$

is an $\mathcal{L}$-gauge on $\mathbb{G}$.

Proof. The assertion follows from condition (i) in Definition 5.3.1, (5.15), Propositions 5.3.11, 5.3.12, 5.3.13 and Corollary 5.3.14. □

In the next section, we shall show the reverse part of Proposition 5.4.2 (see Theorem 5.5.6): *if d is an $\mathcal{L}$-gauge on $\mathbb{G}$, then there exists a positive constant β_d such that $\Gamma = \beta_d\, d^{2-Q}$ is the fundamental solution of $\mathcal{L}$.* As a consequence, by Proposition 5.3.10, the $\mathcal{L}$-gauge is unique up to a multiplicative constant.

In Section 9.8, we shall also prove the following fact: *if d is a homogeneous norm on $\mathbb{G}$, smooth out of the origin and such that*

$$\mathcal{L}(d^{\alpha}) = 0 \text{ in } \mathbb{G} \setminus \{0\}$$

for a suitable $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then $\alpha = 2 - Q$ and d is an $\mathcal{L}$-gauge on $\mathbb{G}$ (see Corollary 9.8.8, page 461, for the precise statement).

The sub-Laplacian of an $\mathcal{L}$-radial function takes a noteworthy form.

Proposition 5.4.3. *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. Let $f(d)$ be a smooth $\mathcal{L}$-radial function on $\mathbb{G} \setminus \{0\}$. Then*

$$\mathcal{L}(f(d)) = |\nabla_{\mathcal{L}} d|^2 \left(f''(d) + \frac{Q-1}{d}\, f'(d) \right), \tag{5.34}$$

where, if $\mathcal{L} = \sum_{j=1}^{m} X_j^2$, we set $\nabla_{\mathcal{L}} = (X_1, \ldots, X_m)$.

Proof. An easy computation gives (see also Ex. 6, Chapter 1)

$$\mathcal{L}(f(d)) = \sum_{j=1}^{m} X_j^2(f(d)) = f''(d)\sum_{j=1}^{m}(X_j d)^2 + f'(d)\sum_{j=1}^{m} X_j^2(d),$$

so that

$$\mathcal{L}(f(d)) = f''(d)\,|\nabla_{\mathcal{L}} d|^2 + f'(d)\,\mathcal{L}(d). \tag{5.35}$$

(Note that (5.35) holds with the only assumption that f and d are smooth functions on some open subsets of $\mathbb{G}$ and $\mathbb{R}$, respectively, and $f \circ d$ is defined.)

Applying this formula to the function $f(s) = s^{2-Q}$ and keeping in mind (5.33), we obtain

$$0 = (1 - Q)\,d^{-Q}\,|\nabla_{\mathcal{L}} d|^2 + d^{1-Q}\,\mathcal{L}(d),$$

hence

$$\mathcal{L}(d) = |\nabla_{\mathcal{L}} d|^2\,\frac{Q-1}{d}.$$

Identity (5.34) follows by replacing this last identity in (5.35). □

If ϱ is any (sufficiently smooth) homogeneous norm on $\mathbb{G}$, the integration of a ϱ-radial function over a "ϱ-radially-symmetric" domain reduces to an integration of a single variable function.

Proposition 5.4.4. *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. Let ϱ be any homogeneous norm on $\mathbb{G}$ smooth on $\mathbb{G} \setminus \{0\}$. Let $f(\varrho)$ be a function defined on the ϱ-ball $B_\varrho(0, r)$ of radius r centered at the origin,*

$$B_\varrho(0, r) := \{x \in \mathbb{G} : \varrho(x) < r\}.$$

Then, if $f(\varrho) \in L^1(B_\varrho(0, r))$, it holds

$$\int_{B_\varrho(0,r)} f(\varrho(x))\,\mathrm{d}x = Q\,\omega_\varrho \int_0^r s^{Q-1}\,f(s)\,\mathrm{d}s, \tag{5.36}$$

where ω_ϱ denotes the Lebesgue measure of $B_\varrho(0, 1)$,

$$\omega_\varrho := |B_\varrho(0, 1)|.$$

Proof. The coarea formula gives

$$\int_{B_\varrho(0,r)} f(\varrho(x))\,\mathrm{d}x = \int_0^r f(s)\left(\int_{\{\varrho=s\}} \frac{1}{|\nabla\varrho|}\,\mathrm{d}H^{N-1}\right)\mathrm{d}s. \tag{5.37}$$

On the other hand, by using the δ_λ-homogeneity of ϱ, we have

$$\int_0^r \left(\int_{\{\varrho=s\}} \frac{1}{|\nabla\varrho|}\,\mathrm{d}H^{N-1}\right)\mathrm{d}s = |B_\varrho(0, r)| = \omega_\varrho\, r^Q$$

for every $r > 0$. Differentiating this last identity with respect to r, we obtain

$$\int_{\{\varrho=r\}} \frac{1}{|\nabla\varrho|}\,\mathrm{d}H^{N-1} = Q\omega_\varrho\, r^{Q-1}. \tag{5.38}$$

By using this identity in (5.37), we immediately get (5.36). □

Corollary 5.4.5. *Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. Let ϱ be any homogeneous norm on $\mathbb{G}$. The function ϱ^α is locally integrable in $\mathbb{R}^N$ if and only if $\alpha > -Q$.*

Proof. If ϱ is also smooth on $\mathbb{G} \setminus \{0\}$, by Proposition 5.4.4,

$$\int_{B_\varrho(0,r)} \varrho^\alpha(x)\,\mathrm{d}x = Q\omega_\varrho \int_0^r s^{\alpha+Q-1}\,\mathrm{d}s,$$

and the assertion trivially follows.

If ϱ is only continuous, we argue as follows. If $\alpha \geq 0$, the assertion is trivial. If $\alpha < 0$, we have

$$\begin{aligned}
\int_{B_\varrho(0,r)} \varrho^\alpha(x)\,\mathrm{d}x &= \sum_{k=0}^N \int_{\{r/2^{k+1} \leq \varrho < r/2^k\}} \varrho^\alpha(x)\,\mathrm{d}x \\
&\leq (r/2)^\alpha \sum_{k=0}^N \frac{1}{2^{k\alpha}} \int_{\{r/2^{k+1} \leq \varrho(x) < r/2^k\}} \mathrm{d}x \quad (\text{by } x = \delta_{r/2^k}(y)) \\
&= (r/2)^\alpha \sum_{k=0}^N \frac{1}{2^{k\alpha}} \left(\frac{r}{2^k}\right)^Q \int_{\{1/2 \leq \varrho(y) < 1\}} \mathrm{d}y \\
&= \mathbf{c}_\varrho\, r^Q\, (r/2)^\alpha \sum_{k=0}^N 2^{-k(\alpha+Q)},
\end{aligned}$$

hence, if $\alpha > -Q$, then ϱ^α is integrable on $B_\varrho(0,r)$. The reverse assertion follows by the same computation as above, by taking the obvious bound from below. □

A couple of examples of explicit $\mathcal{L}$-gauges are in order. See also Example 5.10.3 and Chapter 18.

Example 5.4.6 (Δ-gauge). The classical Laplace operator in $\mathbb{R}^N$, $N \geq 3$,

$$\Delta := \sum_{j=1}^N \partial_{x_j}^2$$

is the canonical (sub-)Laplacian on the Euclidean group

$$\mathbb{E} := (\mathbb{R}^N, +, \delta_\lambda)$$

with $\delta_\lambda x = \lambda x$. The homogeneous dimension of $\mathbb{E}$ is N and a Δ-gauge function is the Euclidean norm

$$x \mapsto |x| := \left(\sum_{j=1}^N x_j^2\right)^{1/2}, \qquad x = (x_1, \ldots, x_N).$$

Indeed, $|\cdot|$ is smooth and strictly positive out of the origin, δ_λ-homogeneous of degree 1 and, as it is well known,

$$\Delta(|x|^{2-N}) = 0 \quad \forall x \neq 0. \quad \square$$

On every group of Heisenberg type, noteworthy *explicit* $\mathcal{L}$-gauges are known. They were discovered by A. Kaplan [Kap80].

Example 5.4.7 ($\Delta_{\mathbb{H}}$-gauges). Let $\mathbb{H} = (\mathbb{R}^{m+n}, \circ, \delta_\lambda)$ be a (prototype) group of Heisenberg type (see Section 3.6, page 169). Denoting by (x, t) the points of $\mathbb{H}$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, we know that the canonical sub-Laplacian $\Delta_{\mathbb{H}}$ on $\mathbb{H}$ takes the form (see (3.14) on page 171)

$$\Delta_{\mathbb{H}} = \Delta_x + \frac{1}{4}\,|x|^2 \Delta_t + \sum_{k=1}^{n} \langle B^{(k)} x, \nabla_x \rangle\, \partial_{t_k}. \tag{5.39a}$$

Here, the $B^{(k)}$'s are n skew-symmetric $m \times m$ orthogonal matrices satisfying the relation

$$B^{(i)}\, B^{(j)} = -B^{(j)}\, B^{(i)} \text{ for every } i,\ j \in \{1, \dots, n\} \text{ with } i \neq j.$$

The dilations $\{\delta_\lambda\}_{\lambda>0}$ are given by

$$\delta_\lambda(x, t) = (\lambda\, x, \lambda^2 t).$$

Then

$$Q = m + 2\,n \tag{5.39b}$$

is the homogeneous dimension of $\mathbb{H}$. We want to show that

$$d(x, t) := \big(|x|^4 + 16\,|t|^2\big)^{1/4} \tag{5.39c}$$

is a $\Delta_{\mathbb{H}}$-gauge. Here $|x|$ and $|t|$ denote respectively the Euclidean norm of $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$. First of all, we remark that d is strictly positive and smooth out of the origin, it is δ_λ-homogeneous of degree one and symmetric, since $(x, t)^{-1} = (-x, -t)$. Now, we aim to compute $\Delta_{\mathbb{H}}(d^{2-Q})$. To this end, it is convenient to fix the following notation:

$$v(r, s) := r^4 + 16\,s^2,\ r = |x|,\ s = |t|; \quad \alpha = (2 - Q)/4.$$

Then

$$G(x, t) := (d(x, t))^{2-Q} = (r^4 + 16\,s^2)^\alpha = v^\alpha(r, s).$$

Since $B^{(k)}$ is skew-symmetric, we have

$$\langle B^{(k)} x, \nabla_x G(x, t) \rangle = 4\alpha\, v^{\alpha-1} r^2\, \langle B^{(k)} x, x \rangle = 0$$

for every $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$ and $1 \le k \le n$. Then, by using (5.39a),

$$\begin{aligned} \Delta_{\mathbb{H}} G &= \Delta_x G + \frac{r^2}{4} \Delta_t G \\ &= \alpha v^{\alpha-1} \left(\Delta_x v + \frac{r^2}{4} \Delta_t v \right) \\ &\quad + \alpha(\alpha - 1)\, v^{\alpha-2} \left(|\nabla_x v|^2 + \frac{r^2}{4} |\nabla_t v|^2 \right). \end{aligned} \tag{5.39d}$$

Now, thanks to the radial symmetry of v with respect to x and t, we easily obtain

$$\Delta_x v = (8+4m)r^2, \quad \frac{r^2}{4}\,\Delta_t v = 8nr^2 \tag{5.39e}$$

and

$$|\nabla_x v|^2 = 16r^6, \quad \frac{r^2}{4}\,|\nabla_t v|^2 = (16\,rs)^2. \tag{5.39f}$$

Replacing (5.39e) and (5.39f) in (5.39d), we get

$$\begin{aligned}\Delta_{\mathbb{H}} G &= 4\alpha\, v^{\alpha-1} r^2(2+m+2n) + 16\alpha(\alpha-1)\, v^{\alpha-2} r^2 (r^4 + 16\,s^2)\\ &= 4\alpha\, v^{\alpha-1} r^2 (2+Q+4(\alpha-1))\\ &= 4\alpha\, v^{\alpha-1} r^2 (Q-2+4\alpha) = 0 \quad \text{(being } \alpha = (2-Q)/4).\end{aligned}$$

Then

$$\Delta_{\mathbb{H}}(d^{2-Q}) = 0 \text{ in } \mathbb{H}\setminus\{0\},$$

and d in (5.39c) is a $\Delta_{\mathbb{H}}$-gauge. $\square$

Remark 5.4.8 (Cylindrically-symmetric functions on $\mathbb{H}$*).* In the notation of the above Example 5.4.7, we shall call *cylindrically-symmetric* any function u on $\mathbb{H}$ such that

$$u(x,t) = v(|x|,t)$$

for some function $v = v(r,t)$, $r \in \mathbb{R}$, $r > 0$, $t \in \mathbb{R}^n$. Assume v is smooth. Then

$$\langle B^{(k)}x, \nabla_x u(x,t)\rangle = \langle B^{(k)}x, x/|x|\rangle\, \partial_r v(|x|,t) = 0,$$

since $\langle B^{(k)}x, x\rangle = 0$ for every $x \in \mathbb{R}^m$. As a consequence, keeping in mind (3.14) and (3.16), we obtain the following form of $\Delta_{\mathbb{H}}$ and $|\nabla_{\mathbb{H}}|$ for cylindrically-symmetric functions $u(x,t) = v(|x|,t)$:

$$\begin{aligned}\Delta_{\mathbb{H}} u &= v_{rr} + \frac{m-1}{r}\, v_r + \frac{1}{4}\, r^2\, \Delta_t,\\ |\nabla_{\mathbb{H}} u|^2 &= v_r^2 + \frac{1}{4}\, r^2\, |\nabla_t v|^2.\end{aligned} \tag{5.39g}$$

Here, $r = |x|$ and $v_r = \delta_r v$, $v_{rr} = \delta_{rr} v$.

5.5 Gauge Functions and Surface Mean Value Theorem

Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge. The aim of this section is to prove a mean value theorem on the boundary of the d-balls for the $\mathcal{L}$-harmonic functions. When $\mathcal{L}$ is the classical Laplace operator, our result will give back the Gauss theorem for classical harmonic functions. We shall obtain the mean value theorem as a byproduct of a representation formula for general

C^2 functions. *These representation formulas will play a major rôle throughout the book.*

For any $x \in \mathbb{G}$ and $r > 0$, we recall that we have already defined the d-ball of center x and radius r as follows:

$$B_d(x,r) := \{y \in \mathbb{G} : d(x^{-1} \circ y) < r\}. \tag{5.40}$$

Then

$$B_d(x,r) = x \circ B_d(0,r).$$

By using the translation invariance of the Lebesgue measure and the δ_λ-homogeneity of d, one also easily recognizes that

$$|B_d(x,r)| = r^Q \, |B_d(0,1)| =: \omega_d \, r^Q. \tag{5.41}$$

We explicitly remark that

$$\partial B_d(x,r) := \{y \in \mathbb{G} : d(x^{-1} \circ y) = r\}$$

is a smooth manifold of dimension $N-1$. Indeed, by Sard's lemma, this holds true for almost every $r > 0$. The assertion then follows for *every* $r > 0$, since $\partial B_d(x,r)$ is diffeomorphic to $\partial B_d(x,1)$ via the dilation δ_r. (Note that, so far, d may be any homogeneous norm smooth out of the origin.)

Definition 5.5.1 (The kernels of the mean value formulas). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge. We set, for $x \in \mathbb{G} \setminus \{0\}$,*

$$\Psi_{\mathcal{L}}(x) := |\nabla_{\mathcal{L}} d|^2(x).$$

Moreover, for every $x, y \in \mathbb{G}$ with $x \neq y$, we define the functions

$$\Psi_{\mathcal{L}}(x,y) := \Psi_{\mathcal{L}}(x^{-1} \circ y) \quad \textit{and} \quad \mathcal{K}_{\mathcal{L}}(x,y) := \frac{|\nabla_{\mathcal{L}} d|^2(x^{-1} \circ y)}{|\nabla(d(x^{-1} \circ \cdot))|(y)}. \tag{5.42}$$

Remark 5.5.2. We explicitly remark that

$$\Psi_{\mathcal{L}} \text{ is } \delta_\lambda\text{-homogeneous of degree zero},$$

a fact which will be used repeatedly. We would like to recall that $\Psi_{\mathcal{L}}$ appears in the "radial" form of $\mathcal{L}$, see (5.34).

We also explicitly remark that, while $\Psi_{\mathcal{L}}$ is translation-invariant (i.e. $\Psi_{\mathcal{L}}(\alpha \circ x, \alpha \circ y) = \Psi_{\mathcal{L}}(x,y)$), the function $\mathcal{K}_{\mathcal{L}}$ does not necessarily share the same property. Moreover, when $\mathcal{L} = \Delta$ is the classical Laplace operator, then $\Psi_{\mathcal{L}} = \mathcal{K}_{\mathcal{L}} = 1$. For a general sub-Laplacian $\mathcal{L}$, we shall prove the following fact (see Proposition 9.8.9, page 462):

The function $\Psi_{\mathcal{L}}$ is constant if and only if
$\mathbb{G}$ is the Euclidean group.

For instance, consider the following example.

Example 5.5.3. Let $\mathbb{H} = (\mathbb{R}^{m+n}, \circ, \delta_\lambda)$ be a group of Heisenberg type. Denoting by (x,t) the points of $\mathbb{H}$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, we proved in Example 5.4.7 that the "Folland function"

$$d(x,t) := (|x|^4 + 16\,|t|^2)^{1/4}$$

is a $\Delta_{\mathbb{H}}$-gauge. Using (5.39g), we obtain

$$\begin{aligned}\Psi_{\mathbb{H}}(x,t) &= |\nabla_{\mathbb{H}} d(x,t)|^2\\ &= \left(\frac{r}{d}\right)^6 + 16\left(\frac{r\,s}{d^3}\right)^2 = \left(\frac{r}{d}\right)^2\left(\frac{r^4}{d^4} + 16\,\frac{s^2}{d^4}\right),\end{aligned}$$

where $r = |x|$ and $s = |t|$. Then

$$\Psi_{\mathbb{H}}(x,t) = \frac{|x|^2}{\sqrt{|x|^4 + 16\,|t|^2}}, \qquad (x,t) \neq (0,0). \qquad \square$$

Let $\Omega \subseteq \mathbb{G}$ be an open set and $u, v \in C^2(\Omega)$. By using the divergence form of the sub-Laplacian $\mathcal{L}$ (see (1.90a) on page 64),

$$\mathcal{L} = \operatorname{div}(A(x) \cdot \nabla^T),$$

we easily get

$$v\,\mathcal{L}u - u\,\mathcal{L}v = \operatorname{div}(v\,A \cdot \nabla^T u) - \operatorname{div}(u\,A \cdot \nabla^T v). \tag{5.43a}$$

Let us now assume that Ω is bounded with boundary $\partial\Omega$ of class C^1 and exterior normal $\nu = \nu(y)$ at any point $y \in \partial\Omega$. Then, if u, v are of class C^2 in a neighborhood of $\overline{\Omega}$, integrating (5.43a) on Ω and using the divergence theorem, we obtain the Green identity[2]

$$\int_\Omega (v\,\mathcal{L}u - u\,\mathcal{L}v)\,\mathrm{d}H^N = \int_{\partial\Omega} \big(v\,\langle A \cdot \nabla^T u, \nu\rangle - u\,\langle A \cdot \nabla^T v, \nu\rangle\big)\,\mathrm{d}H^{N-1}. \tag{5.43b}$$

Hereafter, $\mathrm{d}H^N$ (respectively, $\mathrm{d}H^{N-1}$) stands for the N-dimensional (respectively, $(N-1)$-dimensional) Hausdorff measure in $\mathbb{R}^N$. If we choose $v \equiv 1$ in (5.43b), we

[2] Another way to write the Green identity is the following one: if $\mathcal{L} = \sum_{j=1}^m X_j^2$, we have

$$\int_\Omega (v\,\mathcal{L}u - u\,\mathcal{L}v)\,\mathrm{d}H^N = \int_{\partial\Omega}\left(v\sum_{j=1}^m X_j u\,\langle X_j I, \nu\rangle - u\sum_{j=1}^m X_j v\,\langle X_j I, \nu\rangle\right)\mathrm{d}H^{N-1}.$$

This follows from (5.43b), recalling that (see (1.90b), page 64) A is the $N \times N$ symmetric matrix $A(x) = \sigma(x)\,\sigma(x)^T$, where $\sigma(x)$ is the $N \times m$ matrix whose columns are $X_1 I(x), \dots, X_m I(x)$.

obtain[3]

$$\int_{\Omega} \mathcal{L}u \, \mathrm{d}H^N = \int_{\partial\Omega} \langle A \cdot \nabla^T u, \nu \rangle \, \mathrm{d}H^{N-1}, \tag{5.43c}$$

so that

$$\int_{\Omega} \mathcal{L}u \, \mathrm{d}H^N = 0 \quad \forall\, u \in C_0^\infty(\Omega). \tag{5.43d}$$

Let us now consider an arbitrary open set $O \subseteq \mathbb{R}^N$ such that $\overline{B_d(x,r)} \subset O$ for a suitable $r > 0$. For $0 < \varepsilon < r$, we define the "d-ring"

$$D_{\varepsilon,r} := B_d(x,r) \setminus \overline{B_d(x,\varepsilon)} = \{y \in \mathbb{R}^N : \varepsilon < d(x^{-1} \circ y) < r\}.$$

Given $u \in C^2(O)$, we apply the Green identity (5.43b) to the functions u and $v := d^{2-Q}(x^{-1} \circ \cdot)$ on the open set $D_{\varepsilon,r}$. Since v *is $\mathcal{L}$-harmonic in* $\mathbb{G} \setminus \{0\}$ (note that here we apply, for the first time and with crucial consequences, the fact d *is an $\mathcal{L}$-gauge*), we obtain

$$\int_{D_{\varepsilon,r}} v \, \mathcal{L}u = S_r(u) - S_\varepsilon(u) + T_\varepsilon(u) - T_r(u), \tag{5.43e}$$

where we have used the following notation

$$S_\rho(u) := \int_{\partial B_d(x,\rho)} v \, \langle A \cdot \nabla^T u, \nu \rangle \, \mathrm{d}H^{N-1},$$

$$T_\rho(u) := \int_{\partial B_d(x,\rho)} u \, \langle A \cdot \nabla^T v, \nu \rangle \, \mathrm{d}H^{N-1}.$$

Since v is constant on $\partial B_d(x,\rho)$, keeping in mind (5.43c), we have

$$S_\rho(u) = \rho^{2-Q} \int_{\partial B_d(x,\rho)} \langle A \cdot \nabla^T u, \nu \rangle \, \mathrm{d}H^{N-1} = \rho^{2-Q} \int_{B_d(x,\rho)} \mathcal{L}u \, \mathrm{d}H^N, \tag{5.43f}$$

so that, by means of (5.41),

$$S_\varepsilon(u) = \varepsilon^2 \, \mathcal{O}(|\mathcal{L}u|) \longrightarrow 0, \quad \text{as } \varepsilon \to 0. \tag{5.43g}$$

To evaluate $T_\rho(u)$, we first remark that on $\partial B_d(x,\rho)$ one has

$$\nu = \frac{\nabla(d(x^{-1} \circ \cdot))}{|\nabla(d(x^{-1} \circ \cdot))|},$$

and

$$\langle A \cdot \nabla^T v, \nu \rangle = (2-Q)\, d^{1-Q}(x^{-1} \circ \cdot)\, \frac{\langle A \cdot \nabla^T(d(x^{-1} \circ \cdot)), \nabla^T(d(x^{-1} \circ \cdot)) \rangle}{|\nabla(d(x^{-1} \circ \cdot))|}$$

$$\text{(see (5.1c) and (5.42))} = (2-Q)\, \rho^{1-Q}\, \frac{\Psi_{\mathcal{L}}(x^{-1} \circ \cdot)}{|\nabla(d(x^{-1} \circ \cdot))|}.$$

---

[3] Or, equivalently (see the previous note),

$$\int_{\Omega} \mathcal{L}u \, \mathrm{d}H^N = \int_{\partial\Omega} \sum_{j=1}^m X_j u \, \langle X_j I, \nu \rangle \, \mathrm{d}H^{N-1}.$$

Therefore, keeping in mind the second definition in (5.42),

$$T_\rho(u) = (2-Q)\,\rho^{1-Q} \int_{\partial B_d(x,\rho)} u(y)\,\mathcal{K}_{\mathcal{L}}(x,y)\,\mathrm{d}H^{N-1}(y), \tag{5.43h}$$

so that

$$T_\varepsilon(u) = (u(x) + o(1))\,T_\varepsilon(1), \quad \text{as } \varepsilon \to 0. \tag{5.43i}$$

To compute $T_\varepsilon(1)$, we observe that (5.43e) with $u = 1$ gives

$$T_\varepsilon(1) = T_r(1) \text{ for } 0 < \varepsilon < r < \infty. \tag{5.43j}$$

Hence, for every $\varepsilon > 0$,

$$T_\varepsilon(1) = T_1(1) = (2-Q) \int_{\partial B_d(x,1)} \mathcal{K}_{\mathcal{L}}(x,\cdot)\,\mathrm{d}H^{N-1}. \tag{5.43k}$$

Finally, since $v = d^{2-Q}(x,\cdot) \in L^1(B_d(x,r))$ (see Corollary 5.4.5)

$$\lim_{\varepsilon\to 0} \int_{D_{\varepsilon,r}} v\,\mathcal{L}u\,\mathrm{d}H^N = \int_{D_{0,r}} v\,\mathcal{L}u\,\mathrm{d}H^N.$$

Therefore, as $\varepsilon \to 0$, identity (5.43e) together with (5.43f)-(5.43k) give

$$\begin{aligned} \int_{B_d(x,r)} v\,\mathcal{L}u\,\mathrm{d}H^N &= r^{2-Q} \int_{B_d(x,r)} \mathcal{L}u\,\mathrm{d}H^N + T_1(1)\,u(x) \\ &\quad - (2-Q)\,r^{1-Q} \int_{\partial B_d(x,r)} u\,\mathcal{K}_{\mathcal{L}}(x,\cdot)\,\mathrm{d}H^{N-1}. \end{aligned} \tag{5.43l}$$

We now observe that

$$T_1(1) = (2-Q) \int_{\partial B_d(x,1)} \mathcal{K}_{\mathcal{L}}(x,\cdot)\,\mathrm{d}H^{N-1}$$

does not depend on x, i.e.

$$\begin{aligned} &(Q-2) \int_{\partial B_d(x,1)} \mathcal{K}_{\mathcal{L}}(x,\cdot)\,\mathrm{d}H^{N-1} \\ &\quad = (Q-2) \int_{\partial B_d(0,1)} \mathcal{K}_{\mathcal{L}}(0,\cdot)\,\mathrm{d}H^{N-1} =: (\beta_d)^{-1}. \end{aligned} \tag{5.43m}$$

Indeed, from (5.43j) we have

$$\begin{aligned} T_1(1)\,\frac{1}{Q} &= \int_0^1 T_r(1)\,r^{Q-1}\,\mathrm{d}r \\ &= (2-Q) \int_0^1 \int_{\partial B_d(x,r)} \mathcal{K}_{\mathcal{L}}(x,y)\,\mathrm{d}H^{N-1}(y)\,\mathrm{d}r \\ &\text{(by the coarea formula)} = (2-Q) \int_{B_d(x,1)} \Psi_{\mathcal{L}}(x,y)\,\mathrm{d}H^{N-1}(y) \\ &= (2-Q) \int_{B_d(0,1)} \Psi_{\mathcal{L}}\,\mathrm{d}H^N. \end{aligned}$$

Here we used the very definition (5.42) of the functions $\Psi_{\mathcal{L}}$ and $\mathcal{K}_{\mathcal{L}}$ *and the left invariance of* $\Psi_{\mathcal{L}}$. Incidentally, we have also proved that

$$(\beta_d)^{-1} = Q(Q-2)\int_{B_d(0,1)} \Psi_{\mathcal{L}}\,\mathrm{d}H^N. \tag{5.43n}$$

From identity (5.43l), moving terms around, we obtain

$$\begin{aligned} u(x) = {} & \frac{(Q-2)\,\beta_d}{r^{Q-1}}\int_{\partial B_d(x,r)} u\,\mathcal{K}_{\mathcal{L}}(x,\cdot)\,\mathrm{d}H^{N-1} \\ & - \beta_d \int_{B_d(x,r)} (d^{2-Q}(x^{-1}\circ\cdot) - r^{2-Q})\,\mathcal{L}u\,\mathrm{d}H^N. \end{aligned} \tag{5.44}$$

We have thus proved the following fundamental result.

Theorem 5.5.4 (Surface mean value theorem). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge on $\mathbb{G}$. Let O be an open subset of $\mathbb{G}$, and let $u \in C^2(O,\mathbb{R})$.*

Then, for every $x \in O$ and $r > 0$ such that $\overline{B_d(x,r)} \subset O$, we have

$$u(x) = \mathcal{M}_r(u)(x) - \mathcal{N}_r(\mathcal{L}u)(x), \tag{5.45}$$

where

$$\begin{aligned} \mathcal{M}_r(u)(x) &= \frac{(Q-2)\,\beta_d}{r^{Q-1}}\int_{\partial B_d(x,r)} \mathcal{K}_{\mathcal{L}}(x,z)\,u(z)\,\mathrm{d}H^{N-1}(z), \\ \mathcal{N}_r(w)(x) &= \beta_d \int_{B_d(x,r)} (d^{2-Q}(x^{-1}\circ z) - r^{2-Q})\,w(z)\,\mathrm{d}H^N(z), \end{aligned} \tag{5.46}$$

and β_d and $\mathcal{K}_{\mathcal{L}}$ are defined, respectively, in (5.43m) *and* (5.42).

In particular, if $\mathcal{L}u = 0$, i.e. u is $\mathcal{L}$-harmonic in O, we have

$$u(x) = \mathcal{M}_r(u)(x). \tag{5.47}$$

Remark 5.5.5. When $\mathcal{L} = \Delta$ is the classical Laplace operator in $\mathbb{R}^N$, $N \geq 3$, the kernel $\mathcal{K}_{\mathcal{L}}$ is constant ($\mathcal{K}_{\mathcal{L}} \equiv 1$). Then, in this case,

$$\mathcal{M}_r(u)(x) = \frac{1}{\sigma_N\, r^{N-1}}\int_{|x-y|=r} u(y)\,\mathrm{d}H^{N-1}(y) =: \fint_{|x-y|=r} u(y)\,\mathrm{d}H^{N-1}y$$

(see (5.43m) and (5.46)) and (5.47) gives back the Gauss theorem for classical harmonic functions. □

From Theorem 5.5.4, we straightforwardly also obtain the following result (see also Corollary 9.8.8 on page 461 for yet another improvement).

Theorem 5.5.6 ("Uniqueness" of the $\mathcal{L}$-gauges. I). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let d be an $\mathcal{L}$-gauge on $\mathbb{G}$, and let β_d be the positive constant defined in* (5.43m). *Then*

$$\Gamma = \beta_d\, d^{2-Q} \tag{5.48}$$

is the fundamental solution of $\mathcal{L}$.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and choose $r > 0$ such that $\operatorname{supp}(\varphi) \subset B_d(0,r)$. Then, by the mean value formula (5.45) (being $u \equiv 0$ on $\partial B_d(0,r)$),

$$\varphi(0) = -\beta_d \int_{B_d(0,r)} (d^{2-Q}(z) - r^{2-Q})\, \mathcal{L}\varphi(z)\, \mathrm{d}H^N(z).$$

On the other hand, by identity (5.43d),

$$\int_{B_d(0,r)} r^{2-Q}\, \mathcal{L}\varphi\, \mathrm{d}H^N = 0.$$

Then, if Γ is the function defined in (5.48), $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ thanks to Corollary 5.4.5 and

$$-\varphi(0) = \int_{\mathbb{R}^N} \Gamma(z)\, \mathcal{L}\varphi(z)\, \mathrm{d}H^N(z)$$

for every $\varphi \in C_0^\infty(\Omega)$. Moreover, Γ is smooth in $\mathbb{G} \setminus \{0\}$ and $\Gamma(z) \to 0$ as $z \to \infty$, since $Q - 2 > 0$. Thus, by Definition 5.3.1, Γ is the fundamental solution of $\mathcal{L}$. □

From Theorem 5.5.4 we obtain the following *asymptotic* formula for $\mathcal{L}$.

Theorem 5.5.7 (Asymptotic surface formula for $\mathcal{L}$). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge.*

Let $\Omega \subseteq \mathbb{G}$ be open, and let $u \in C^2(\Omega, \mathbb{R})$. Then, for every $x \in \Omega$, we have

$$\lim_{r \to 0^+} \frac{\mathcal{M}_r(u)(x) - u(x)}{r^2} = a_d\, \mathcal{L}u(x), \tag{5.49a}$$

where

$$a_d := \beta_d \int_{B_d(0,1)} (d^{2-Q}(y) - 1)\, \mathrm{d}y. \tag{5.49b}$$

Proof. The change of variable $z = x \circ \delta_r(y)$ in the integral defining $\mathcal{N}_r$ gives

$$\mathcal{N}_r(1)(x) = a_d\, r^2 \quad \text{for every } x \in \mathbb{G}.$$

As a consequence, if $u \in C^2(\Omega, \mathbb{R})$, from (5.45) we obtain

$$\begin{aligned} \mathcal{M}_r(u)(x) - u(x) &= \mathcal{N}_r(\mathcal{L}u)(x) = \mathcal{N}_r(\mathcal{L}u - \mathcal{L}u(x))(x) + \mathcal{N}_r(1)(x)\, \mathcal{L}u(x) \\ &= a_d\, r^2(\mathcal{L}u(x) + o(1)), \quad \text{as } r \to 0^+. \end{aligned}$$

This proves (5.49a). □

5.6 Superposition of Average Operators. Solid Mean Value Theorems. Koebe-type Theorems

As in the previous section, $\mathcal{L}$ and d will respectively denote a sub-Laplacian and an $\mathcal{L}$-gauge on the homogeneous Carnot group $\mathbb{G}$. We shall denote by $B_d(x,r)$ the

d-ball with center $x \in \mathbb{G}$ and radius $r \geq 0$ and by Q the homogeneous dimension of $\mathbb{G}$. We shall also assume, as usual, $Q \geq 3$.

Given an open set $O \subseteq \mathbb{R}^N$ and a real number $r > 0$, we let

$$O_r := \{x \in O \mid d\text{-dist}(x, \partial O) > r\},$$

where

$$d\text{-dist}(x, \partial O) := \inf_{y \in \partial O} d(y^{-1} \circ x).$$

Since the d-balls are connected, one easily recognizes that $B_d(x, \rho) \subseteq O$ for every $x \in O$ and $0 < \rho < d\text{-dist}(x, \partial O)$. It follows that

$$\overline{B_d(x, \rho)} \subseteq O \qquad \forall\, x \in O_r \text{ and } 0 < \rho \leq r.$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an L^1-function vanishing out of the interval $]0, 1[$ and such that $\int_{\mathbb{R}} \varphi = 1$. For $r > 0$, define

$$\varphi_r(t) := \frac{1}{r}\, \varphi\Big(\frac{t}{r}\Big), \quad t \in \mathbb{R}.$$

Let us now consider a function $u \in C^2(O)$. Then, if $x \in O_r$, from (5.45) we obtain

$$u(x) = \mathcal{M}_\rho(u)(x) - \mathcal{N}_\rho(\mathcal{L}u)(x) \qquad \text{for } 0 < \rho \leq r.$$

We now multiply both sides of this identity times $\varphi_r(\rho)$ and integrate with respect to ρ. We thus get

$$u(x) = \Phi_r(u)(x) - \Phi_r^*(\mathcal{L}u)(x), \quad x \in O_r, \tag{5.50a}$$

where

$$\Phi_r(u)(x) := \int_0^\infty \varphi_r(\rho)\, \mathcal{M}_\rho(u)(x)\, \mathrm{d}\rho \tag{5.50b}$$

and

$$\Phi_r^*(w)(x) := \int_0^\infty \varphi_r(\rho)\, \mathcal{N}_\rho(w)(x)\, \mathrm{d}\rho. \tag{5.50c}$$

By using the coarea formula, the average operator Φ_r can be written as follows (we agree to let $u = 0$ out of O)

$$\Phi_r(u)(x) = \int_{\mathbb{R}^N} u(z)\, \phi_r(x^{-1} \circ z)\, \mathrm{d}z \tag{5.50d}$$

with

$$\phi_r(z) := r^{-Q}\, \phi\big(\delta_{1/r}(z)\big)$$

and

$$\phi(z) := (Q-2)\beta_d\, \Psi_{\mathcal{L}}(z)\, \frac{\varphi(d(z))}{d(z)^{Q-1}}\,. \tag{5.50e}$$

We note that ϕ vanishes out of $B_d(0, 1)$ and

$$\int_{\mathbb{R}^N} \phi(z)\,\mathrm{d}z = (Q-2)\beta_d \int_0^\infty \frac{\varphi(\rho)}{\rho^{Q-1}} \left(\int_{d(z)=\rho} \frac{|\nabla_{\mathcal{L}} d(z)|^2}{|\nabla d(z)|}\,\mathrm{d}H^{N-1}(y) \right) \mathrm{d}\rho$$
$$= \int_0^\infty \varphi(\rho)\,\mathrm{d}\rho = 1.$$

Here we used (5.42), (5.43h), (5.43k) and (5.43m).

If the function φ is smooth and its support is contained in $]0,1[$, then $\phi \in C_0^\infty(\mathbb{R}^N)$, $\mathrm{supp}(\phi) \subseteq B_d(0,1)$ and

$$x \mapsto \Phi_r(u)(x)$$

is a smooth map in O_r, whenever u is just an $L^1_{\mathrm{loc}}(O)$-function. We would like to explicitly remark that in the integral (5.50d) the function

$$z \mapsto \phi_r(x^{-1} \circ z) \text{ vanishes out of } B_d(x,r).$$

As a consequence, if $x \in O_r$, that integral is performed on a compact set contained in O, since $\overline{B_d(x,r)} \subseteq O$.

If we choose

$$\varphi(t) = \begin{cases} Q\,t^{Q-1} & \text{if } 0 < t < 1, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\Phi_r(u) = \mathrm{M}_r(u) \text{ and } \Phi_r^*(w) = \mathrm{N}_r(w),$$

where

$$\mathrm{M}_r(u)(x) := \frac{m_d}{r^Q} \int_{B_d(x,r)} \Psi_{\mathcal{L}}(x^{-1} \circ y)\, u(y)\,\mathrm{d}y, \tag{5.50f}$$

and

$$\mathrm{N}_r(w)(x) := \frac{n_d}{r^Q} \int_0^r \rho^{Q-1} \left(\int_{B_d(x,\rho)} \left(d^{2-Q}(x^{-1} \circ y) - \rho^{2-Q} \right) w(y)\,\mathrm{d}y \right) \mathrm{d}\rho, \tag{5.50g}$$

being

$$m_d := Q(Q-2)\beta_d \text{ and } n_d := Q\,\beta_d. \tag{5.50h}$$

Hence, from (5.50a), we obtain the following theorem.

Theorem 5.6.1 (Solid mean value theorem). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge on $\mathbb{G}$. Let O be an open subset of $\mathbb{G}$, and let $u \in C^2(O, \mathbb{R})$.*

Then, for every $x \in O$ and $r > 0$ such that $\overline{B_d(x,r)} \subset O$, we have

$$u(x) = \mathrm{M}_r(u)(x) - \mathrm{N}_r(\mathcal{L}u)(x), \tag{5.51}$$

where

$$\mathrm{M}_r(u)(x) = \frac{m_d}{r^Q}\int_{B_d(x,r)} \Psi_{\mathcal{L}}(x^{-1}\circ y)\,u(y)\,\mathrm{d}H^N(y),$$
$$\mathrm{N}_r(w)(x) = \frac{n_d}{r^Q}\int_0^r \rho^{Q-1}\left(\int_{B_d(x,\rho)} \left(d^{2-Q}(x^{-1}\circ y) - \rho^{2-Q}\right)w(y)\,\mathrm{d}y\right)\mathrm{d}\rho,$$

and m_d, n_d *and* $\Psi_{\mathcal{L}}$ *are defined in* (5.50h) (*see also* (5.43m)) *and* (5.42).

In particular, if $\mathcal{L}u = 0$, *i.e.* u *is* $\mathcal{L}$*-harmonic in* O, *we have*

$$u(x) = \mathrm{M}_r(u)(x). \tag{5.52}$$

Remark 5.6.2. When $\mathcal{L} = \Delta$ is the classical Laplace operator in $\mathbb{R}^N$, $N \geq 3$, one has $\Psi_{\mathcal{L}} \equiv 1$. Then, in this case,

$$\mathrm{M}_r(u)(x) = \frac{1}{\omega_N\, r^N}\int_{|x-y|<r} u(y)\,\mathrm{d}y =: \fint_{|x-y|<r} u(y)\,\mathrm{d}H^N(y)$$

and (5.52) gives back the “Solid” Gauss theorem for classical harmonic functions. □

The mean value theorems given by identities (5.47) and (5.52) characterize the $\mathcal{L}$-harmonic functions. Indeed, the following Gauss–Koebe–Levi–Tonelli type theorem holds.

Theorem 5.6.3 (Gauss–Koebe–Levi–Tonelli type theorem). *Let* $\mathcal{L}$ *be a sub-Laplacian on the homogeneous Carnot group* $\mathbb{G}$, *and let* d *be an* $\mathcal{L}$*-gauge.*

Let O *be an open subset of* $\mathbb{G}$, *and let* $u : O \to \mathbb{R}$ *be a continuous function. Assume that one of the following conditions is satisfied*:

(i) $u(x) = \mathcal{M}_r(u)(x)$ *for every* $x \in O$ *and* $r > 0$ *such that* $\overline{B_d(x,r)} \subset O$,
(ii) $u(x) = \mathrm{M}_r(u)(x)$ *for every* $x \in O$ *and* $r > 0$ *such that* $\overline{B_d(x,r)} \subset O$.

Then $u \in C^\infty(O)$ *and*

$$\mathcal{L}u = 0 \quad \textit{in } O.$$

Proof. Assume condition (i) is satisfied. Then

$$u(x) = \mathrm{M}_\rho(u)(x) \quad \text{for } 0 < \rho \leq r. \tag{5.53}$$

Let $\varphi \in C_0^\infty(]0,1[,\mathbb{R})$ be such that $\int_{\mathbb{R}}\varphi = 1$. Multiply both sides of (5.53) times $\varphi_r(\rho) = \varphi(\rho/r)/r$. An integration with respect to ρ gives

$$u(x) = \Phi_r(u)(x) \quad \forall\, x \in O_r,$$

where $\Phi_r(u)$ is the integral operator (5.50b). From (5.50b) we get

$$u(x) = \int_O u(z)\,\phi_r(x^{-1}\circ z)\,\mathrm{d}z,$$

where $\phi_r(z) = r^{-Q}\,\phi(\delta_{1/r}(z))$ and ϕ is the smooth function defined by (5.50e). It follows that $u \in C^\infty(O)$. Condition (i) and identity (5.45) now give $\mathcal{N}_r(\mathcal{L}u)(x) = 0$

for every $x \in O$ and $r > 0$ such that $\overline{B_d(x,r)} \subset O$. Since the kernel appearing in the integral operator $\mathcal{N}_r$ is strictly positive, this implies $\mathcal{L}u = 0$ in O. Thus, the theorem is proved if condition (i) is fulfilled.

Let us now assume (ii). Since $\rho \mapsto \mathcal{M}_\rho(u)(x)$ is continuous on $]0, r]$ and

$$\mathrm{M}_r(u)(x) = \frac{Q}{r^Q} \int_0^r \rho^{Q-1} \mathcal{M}_\rho(u)(x)\,\mathrm{d}\rho,$$

from (ii) we get

$$\begin{aligned} Qr^{Q-1}u(x) &= \frac{\mathrm{d}}{\mathrm{d}r}(r^Q u(x)) = Q\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \rho^{Q-1}\mathcal{M}_\rho(u)(x)\,\mathrm{d}\rho \\ &= Qr^{Q-1}\,\mathcal{M}_r(u)(x). \end{aligned}$$

Hence $u(x) = \mathcal{M}_r(u)(x)$ for every $x \in O$ and $r > 0$ such that $\overline{B_d(x,r)} \subset O$. Then u satisfies condition (i), so that $u \in C^\infty(O)$ and $\mathcal{L}u = 0$ in O. $\square$

Remark 5.6.4 (Another Koebe-type result). In the hypotheses of Theorem 5.6.3, if $u : O \to \mathbb{R}$ is continuous and satisfies

$$u(x) = \Phi_r(u)(x) \qquad \forall\, r > 0\ \forall\, x \in O_r,$$

where Φ_r is the integral operator in (5.50d) related to a smooth function $\varphi \in C_0^\infty(]0,1[,\mathbb{R})$ (see also (5.50b)), then $u \in C^\infty(O)$. As a consequence, by identity (5.50a), $\Phi_r^*(\mathcal{L}u) = 0$ in O_r for every $r > 0$. It follows that

$$\mathcal{L}u = 0 \quad \text{in } O. \quad \square$$

From Theorem 5.6.1 we obtain another asymptotic formula for the sub-Laplacians.

Theorem 5.6.5 (Asymptotic solid formula for $\mathcal{L}$). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge.*

Let $\Omega \subseteq \mathbb{G}$ be open, and let $u \in C^2(\Omega,\mathbb{R})$. Then, for every $x \in \Omega$, we have

$$\lim_{r\to 0^+} \frac{\mathrm{M}_r(u)(x) - u(x)}{r^2} = \mathrm{a}_d\, \mathcal{L}u(x), \tag{5.54}$$

where $\mathrm{a}_d = Q\,a_d/(Q+2)$ (*and* a_d *is as in* (5.49b)).

Proof. From (5.50g) we obtain (by recalling the definition (5.49b) of a_d)

$$\mathrm{N}_r(1)(x) = \frac{Q\,\beta_d}{r^Q}\int_0^r \rho^{Q+1}\left(\int_{B_d(0,1)} \left(d^{2-Q}(y) - 1\right)\mathrm{d}y\right)\mathrm{d}\rho = r^2\,\mathrm{a}_d.$$

Then, by Theorem 5.6.1 (arguing as in the proof of Theorem 5.5.7), we get

$$\mathrm{M}_r(u)(x) - u(x) = \mathrm{a}_d\, r^2(\mathcal{L}u(x) + o(1)), \quad \text{as } r \to 0^+,$$

and (5.54) follows. $\square$

5.7 Harnack Inequalities for Sub-Laplacians

In this section, we shall prove some type of Harnack inequalities for sub-Laplacians. Our main tool will be the solid average operator M_r in (5.50f).

Let d be an $\mathcal{L}$-gauge on $\mathbb{G}$, let $\mathbf{c}_d$ be the positive constant of the pseudo-triangle inequality (see Proposition 5.1.8)

$$d(x^{-1} \circ y) \le \mathbf{c}_d \left(d(x^{-1} \circ z) + d(z^{-1} \circ y) \right) \qquad \forall\, x,\ y,\ z \in \mathbb{G}, \tag{5.55}$$

and let $\Psi_{\mathcal{L}} = |\nabla_{\mathcal{L}} d|^2$ be the kernel appearing in the average operator M_r in (5.50f).

The following lemma will play a fundamental rôle in this section: it will allow us to compare the average on different d-balls of a given non-negative function.

Lemma 5.7.1. *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge. For every $r > 0$, there exists a point $z_0 = z_0(r) \in \mathbb{G}$ satisfying the following conditions*:

(i) $d(z_0) = \lambda\, r$,
(ii) $\Psi_{\mathcal{L}}(x^{-1} \circ y) \ge \mu$ *and* $\Psi_{\mathcal{L}}(y^{-1} \circ x) \ge \mu$

for every $x \in B_d(z_0, r)$ and every $y \in B_d(0, 2\,\mathbf{c}_d\, r)$. Here $\lambda \ge 1$ and $\mu > 0$ are real constants independent of r (depending only on $\mathbb{G}$, d and $\mathcal{L}$).

Proof. We split the proof into three steps.

(I) The set

$$\{y \in \mathbb{G} \setminus \{0\} \mid \Psi_{\mathcal{L}}(y) = 0\}$$

has empty interior. Indeed, suppose by contradiction $\Psi_{\mathcal{L}}(y) = 0$ for every y in a neighborhood U of a suitable $y_0 \ne 0$. Then, since $\Psi_{\mathcal{L}} = |\nabla_{\mathcal{L}} d|^2$, this gives $\nabla_{\mathcal{L}} \Psi_{\mathcal{L}} \equiv 0$ on U, so that, by Proposition 1.5.6 (page 69), d is constant in an open set containing y_0. As a consequence, the function $r \mapsto d(\delta_r(y_0)) = r\, d(y_0)$ is constant near $r = 1$. This implies $d(y_0) = 0$, which is a contradiction with the assumption $y_0 \ne 0$.

(II) From step (I) and the homogeneity of $\Psi_{\mathcal{L}}$ we infer the existence of a point $y_0 \in \mathbb{G}$, $d(y_0) = 1$, such that

$$\Psi_{\mathcal{L}}(y_0) > 0 \quad \text{and} \quad \Psi_{\mathcal{L}}(y_0^{-1}) > 0.$$

Then, by the continuity of $\Psi_{\mathcal{L}}$ out of the origin, there exist two positive constants σ and μ (with $\sigma \le 1$) such that

$$\Psi_{\mathcal{L}}(\xi \circ y_0 \circ \eta) \ge \mu, \qquad \Psi_{\mathcal{L}}(\xi \circ y_0^{-1} \circ \eta) \ge \mu \tag{5.56}$$

for every $\xi, \eta \in \mathbb{G}$ satisfying the inequalities $d(\xi), d(\eta) \le 2\,\mathbf{c}_d\, \sigma$.

(III) For any fixed $r > 0$, we let $z_0 := \delta_{\lambda r}(y_0)$ with $\lambda = 1/\sigma$. Let $x \in B_d(z_0, r)$ and $y \in B_d(0, 2\,\mathbf{c}_d\, r)$. From the second inequality in (5.56) we obtain

$$\begin{aligned}\Psi_{\mathcal{L}}(x^{-1} \circ y) &= \Psi_{\mathcal{L}}\big((x^{-1} \circ z_0) \circ z_0^{-1} \circ y\big) \\ &\qquad \text{(by the homogeneity of } \Psi_{\mathcal{L}}) \\ &= \Psi_{\mathcal{L}}\big(\delta_{1/(\lambda r)}(x^{-1} \circ z_0) \circ y_0^{-1} \circ \delta_{1/(\lambda r)}(y)\big) \ge \mu,\end{aligned}$$

since $d(\delta_{1/(\lambda r)}(x^{-1} \circ z_0)) \le \sigma$ and $d(\delta_{1/(\lambda r)}(y)) \le 2\,\mathbf{c}_d\,\sigma$. Analogously, from the first inequality in (5.56) we obtain

$$\begin{aligned}\Psi_{\mathcal{L}}(y^{-1} \circ x) &= \Psi_{\mathcal{L}}\big(y^{-1} \circ z_0 \circ (z_0^{-1} \circ x)\big)\\ &= \Psi_{\mathcal{L}}\big(\delta_{1/(\lambda r)}(y^{-1}) \circ y_0 \circ \delta_{1/(\lambda r)}(z_0^{-1} \circ x)\big) \ge \mu,\end{aligned}$$

since $d(\delta_{1/(\lambda r)}(z_0^{-1} \circ x)) \le \sigma$ and $d(\delta_{1/(\lambda r)}(y^{-1})) \le 2\,\mathbf{c}_d\,\sigma$. □

To state the next theorem, it is convenient to introduce the following constants

$$\theta_0 := \mathbf{c}_d(1 + \mathbf{c}_d(1 + \lambda)), \qquad \theta := \mathbf{c}_d(\lambda + \theta_0). \tag{5.57}$$

Theorem 5.7.2 (Non-homogeneous Harnack inequality). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge. Let $\Omega \subseteq \mathbb{G}$ be open. Finally, let $x_0 \in \Omega$ and $r > 0$ be such that $\overline{B_d(x_0, \theta\, r)} \subset \Omega$ (here θ is as in* (5.57), *see also Lemma* 5.7.1 *and* (5.55)).

Then, for any $p \in (Q/2, \infty]$, we have

$$\sup_{B_d(x_0,r)} u \le \mathbf{c}\Big\{\inf_{B_d(x_0,\,r)} u + r^{2-Q/p}\ \| \mathcal{L}u \|_{L^p(B_d(x_0,\,\theta\, r))}\Big\} \tag{5.58}$$

for every non-negative smooth function $u : \Omega \to \mathbb{R}$. Here $\mathbf{c}$ is a positive constant depending only on $\mathbb{G}$, d, $\mathcal{L}$ and p and not depending on u, r, x_0 and Ω.

Proof. Since $\mathcal{L}$ is left invariant, we may assume $x_0 = 0$.

We split the proof in several steps. M_r will be the average operator in (5.50f) and $z_0 = z_0(r)$ the point of $\mathbb{G}$ given by Lemma 5.7.1.

(I) There exists an absolute constant[4] $\mathbf{c} > 0$ such that

$$\mathrm{M}_r(u)(x) \le \mathbf{c}\,\mathrm{M}_{\theta_0 r}(u)(z_0) \qquad \forall\, x \in B_d(0, r). \tag{5.59}$$

Indeed, for every $x \in B_d(0, r)$, we have

$$\overline{B_d(x, r)} \subseteq \overline{B_d(0, 2\,\mathbf{c}_d\, r)} \subseteq \overline{B_d(0, \theta_0\, r)} \subset \Omega,$$

whence (being u non-negative)

$$\begin{aligned}\mathrm{M}_r(u)(x) &\le \Big(\sup_{\mathbb{G}\setminus\{0\}} \Psi_{\mathcal{L}}\Big)\,\frac{m_d}{r^Q}\int_{B_d(x,r)} u(z)\,\mathrm{d}z\\ &\qquad\text{(by the first inequality in Lemma 5.7.1-(ii))}\\ &\le \frac{\mathbf{c}_1}{r^Q}\int_{B_d(x,r)} \Psi_{\mathcal{L}}(z_0^{-1} \circ z)\, u(z)\,\mathrm{d}z,\end{aligned}$$

where $\mathbf{c}_1 = m_d/\mu\ \sup_{\mathbb{G}\setminus\{0\}} \Psi_{\mathcal{L}}$. We remark that $\mathbf{c}_1 < \infty$, since $\Psi_{\mathcal{L}}$ is smooth out of the origin and δ_λ-homogeneous of degree zero. Then, since we also have

[4] Hereafter, we call *absolute constant* any positive real constant independent of u, r, x_0 and Ω.

$$\overline{B_d(x,r)} \subseteq \overline{B_d(z_0,\theta_0 r)} \subseteq \overline{B_d(0,\theta r)} \subset \Omega,$$

we get (again by the non-negativity of u)

$$\mathrm{M}_r(u)(x) \le \frac{\mathbf{c}_1}{r^Q} \int_{B_d(z_0,\theta_0 r)} \Psi_{\mathcal{L}}(z_0^{-1} \circ z)\, u(z)\, \mathrm{d}z = \frac{\mathbf{c}_1}{m_d} \mathrm{M}_{\theta_0 r}(u)(z_0).$$

This proves (5.59) with $\mathbf{c} = \mathbf{c}_1/m_d$.

(II) There exists an absolute constant $\mathbf{c} > 0$ such that

$$\mathrm{M}_r(u)(z_0) \le \mathbf{c}\, \mathrm{M}_{\theta_0 r}(u)(y) \qquad \forall\, y \in B_d(0,r). \tag{5.60}$$

This inequality can be proved just by proceeding as in the previous step, by using the second inequality in Lemma 5.7.1-(ii) and the inclusions

$$\overline{B_d(z_0,r)} \subseteq \overline{B_d(y,\theta_0 r)} \subseteq \overline{B_d(0,\theta r)} \subset \Omega.$$

(III) Let finally $x,\ y \in B_d(0,r)$. Then, by repeatedly using the solid mean value theorem 5.6.1, we have

$$\begin{aligned} u(x) &= \mathrm{M}_r(u)(x) - \mathrm{N}_r(\mathcal{L}u)(x) \qquad \text{(by (5.59))}\\ &\le \mathbf{c}\mathrm{M}_{\theta_0 r}(u)(z_0) - \mathrm{N}_r(\mathcal{L}u)(x)\\ &= \mathbf{c}\big(u(z_0) + \mathrm{N}_{\theta_0 r}(\mathcal{L}u)(z_0)\big) - \mathrm{N}_r(\mathcal{L}u)(x)\\ &= \mathbf{c}\big(\mathrm{M}_r(u)(z_0) - \mathrm{N}_r(\mathcal{L}u)(z_0) + \mathrm{N}_{\theta_0 r}(\mathcal{L}u)(z_0)\big) - N_r(\mathcal{L}u)(x). \end{aligned}$$

On the other hand, from (5.60),

$$\mathrm{M}_r(u)(z_0) \le \mathbf{c}\, \mathrm{M}_{\theta_0 r}(u)(y) = \mathbf{c}\, \big(u(y) - \mathrm{N}_{\theta_0 r}(\mathcal{L}u)(y)\big).$$

By using this last estimate in the previous one, we infer that $u(x)$ is bounded from above by a suitable absolute constant $\mathbf{c}$ times

$$u(y) + |\mathrm{N}_{\theta_0 r}(\mathcal{L}u)(y)| + |\mathrm{N}_r(\mathcal{L}u)(z_0)| + |\mathrm{N}_{\theta_0 r}(\mathcal{L}u)(z_0)| + |N_r(\mathcal{L}u)(x)|$$

for all $x,\ y \in B_d(0,r)$.

An elementary computation based on Hölder's inequality shows that

$$|\mathrm{N}_r(w)(x)| \le \mathbf{c}_p\, r^{2-Q/p}\ \parallel w \parallel_{L^{p'}(B_d(x,r))} \tag{5.61}$$

for $Q/2 < p \le \infty$, $p' = p/(p-1)$ and

$$\mathbf{c}_p := n_d \left(\int_{d(z)<1} (d(z)^{2-Q} - 1)\, \mathrm{d}z \right)^{1/p'}.$$

Thus, keeping in mind the inclusions

$$\overline{B_d(y,\theta_0 r)},\ \overline{B_d(z_0,r)},\ \overline{B_d(z_0,\theta_0 r)},\ \overline{B_d(x,r)} \subseteq \overline{B_d(0,\theta r)},$$

from the upper bound of $u(x)$ and from (5.61) we obtain (5.58) when $x_0 = 0$. This ends the proof. $\square$

Theorem 5.7.2 contains the following "homogeneous" Harnack inequality.

Corollary 5.7.3 (Homogeneous invariant Harnack inequality). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge. Let Ω be an open subset of $\mathbb{G}$, and let $u : \Omega \to \mathbb{R}$ be a non-negative smooth solution to $\mathcal{L}u = 0$. Then*

$$\sup_{B_d(x_0,r)} u \leq \mathbf{c} \inf_{B_d(x_0,r)} u \tag{5.62}$$

for every $x_0 \in \Omega$ and $r > 0$ such that $\overline{B_d(x_0, \theta\, r)} \subset \Omega$. The constant $\mathbf{c}$ depends only on $\mathbb{G}$, $\mathcal{L}$ and d and does not depend on u, r, x_0 and Ω.

By using a covering argument, from the non-homogeneous Harnack inequality of Theorem 5.7.2 one obtains the following theorem.

Theorem 5.7.4 (Non-homogeneous, non-invariant Harnack inequality). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge. Let Ω be an open subset of $\mathbb{G}$, and let K and K_0 be compact and connected subsets of Ω such that $K \subset \mathrm{int}(K_0)$.*

Then, for every $p \in (Q/2, \infty]$, there exists a positive constant $\mathbf{c} = \mathbf{c}(K, K_0, \Omega, \mathcal{L}, d, p, Q)$ such that

$$\sup_K u \leq \mathbf{c}\left\{\inf_K u + \| \mathcal{L}u \|_{L^p(K_0)}\right\} \tag{5.63}$$

for every $u \in C^\infty(\Omega, \mathbb{R})$, $u \geq 0$.

Proof. Let $\{D_j \mid j = 1, \ldots, q\}$ be a finite family of d-balls $D_j = B_d(x_j, r_j)$ such that (here θ is as in (5.57)):

(i) $K \subset \bigcup_{j=1}^q D_j$,
(ii) $\theta\, D_j := B_d(x_j, \theta\, r_j) \subset K_0$ for any $j = 1, \ldots, q$,
(iii) $D_j \cap D_{j+1} \neq \emptyset$ for every $j \in \{1, \ldots, q-1\}$.

We explicitly remark that such a covering exists, since K is compact, connected and contained in the interior of K_0.

By Theorem 5.7.2, we have

$$\sup_{D_j} u \leq \mathbf{c}\left\{\inf_{D_j} u + \| \mathcal{L}u \|_{L^p(\theta\, D_j)}\right\}, \quad j = 1, \ldots, q,$$

for every $u \in C^\infty(\Omega, \mathbb{R})$, $u \geq 0$. The constant $\mathbf{c}$ is independent of u.

Then, inequality (5.63) will follow by a repeated application of the following elementary lemma. $\square$

Lemma 5.7.5. *Let A_1 and A_2 be arbitrary sets such that $A_1 \cap A_2 \neq \emptyset$. Suppose $u : A_1 \cap A_2 \to \mathbb{R}$ is a non-negative function satisfying*

$$\sup_{A_i} u \leq \mathbf{c}\left\{\inf_{A_i} u + L_i\right\}, \quad i = 1, 2, \tag{5.64}$$

for suitable constants $\mathbf{c} \geq 1$ *and* $L_i \geq 0, i = 1, 2$. *Then*

$$\sup_{A_1 \cup A_2} u \leq \mathbf{c}^2 \Big\{ \inf_{A_1 \cup A_2} u + L_1 + L_2 \Big\}.$$

Proof. We have to show that

$$u(x) \leq \mathbf{c}\{u(y) + L_1 + L_2\} \tag{5.65}$$

for every $x, y \in A_1 \cup A_2$. Now, if $x, y \in A_1$ or $x, y \in A_2$, then inequality (5.65) directly follows from (5.64). Suppose $x \in A_1$ and $y \in A_2$ and choose a point $z \in A_1 \cap A_2$. By hypothesis (5.64),

$$u(x) \leq \mathbf{c}\,\{u(z) + L_1\} \quad \text{and} \quad u(z) \leq \mathbf{c}\,\{u(y) + L_2\}.$$

Hence

$$u(x) \leq \mathbf{c}\,\{\mathbf{c}\,(u(y) + L_2) + L_1\},$$

and (5.65) follows. □

From Theorem 5.7.4 one obtains the following improvement of Theorem 5.7.2.

Corollary 5.7.6 (Non-homogeneous invariant Harnack inequality). *Let* $\mathcal{L}$ *be a sub-Laplacian on the homogeneous Carnot group* $\mathbb{G}$, *and let* d *be an* $\mathcal{L}$*-gauge. Let* $\Omega \subseteq \mathbb{G}$ *be open, and let* r, R *and* R_0 *be real constants such that* $0 < r < R < R_0$. *Assume*

$$\frac{r}{R}, \frac{R}{R_0} \leq \rho \quad \textit{and} \quad \overline{B_d(x_0, R_0)} \subset \Omega$$

for suitable $\rho < 1$ *and* $x_0 \in \Omega$.

Then, for every $p \in (Q/2, \infty]$, *there exists a constant* $\mathbf{c} > 0$ *such that*

$$\sup_{B_d(x_0,r)} u \leq \mathbf{c}\Big\{ \inf_{B_d(x_0, r)} u + R^{2-Q/p} \parallel \mathcal{L}u \parallel_{L^p(B_d(x_0, R))} \Big\} \tag{5.66}$$

for every $u \in C^\infty(\Omega, \mathbb{R})$, $u \geq 0$. *The constant* $\mathbf{c}$ *depends only on* $\mathbb{G}$, $\mathcal{L}$, d, p *and* ρ *and does not depend on* u, r, R, R_0, Ω *and* x_0.

Proof. Since $\mathcal{L}$ is left invariant, we may assume $x_0 = 0$. Let us put

$$u_R(x) := u(\delta_R(x)), \quad x \in \delta_{1/R}(\Omega).$$

Then, by applying Theorem 5.7.4 to the function u_R, the compact sets $K = \overline{B_d(0, \rho)}$ and $K_0 = \overline{B_d(0, 1)}$ and the open set $B_d(0, 1/\rho)$ (which is contained in $B_d(0, R_0/R) \subseteq \delta_{1/R}(\Omega)$), we obtain

$$\begin{aligned} \sup_{B_d(0,\rho)} u_R &\leq \mathbf{c}\Big\{ \inf_{B_d(0,\rho)} u_R + \parallel \mathcal{L}u_R \parallel_{L^p(B_d(0,1))} \Big\} \\ &= \mathbf{c}\Big\{ \inf_{B_d(0,\rho)} u_R + R^{2-Q/p} \parallel \mathcal{L}u \parallel_{L^p(B_d(0,R))} \Big\}, \end{aligned}$$

with $\mathbf{c} > 0$ depending only on the parameters in the assertion of the corollary. Note that we have also used the δ_λ-homogeneity of $\mathcal{L}$ (of degree 2). From this inequality (5.66) follows, since

$$\sup_{B_d(0,\rho)} u_R = \sup_{B_d(0,R\,\rho)} u \geq \sup_{B_d(0,r)} u \quad \text{and} \quad \inf_{B_d(0,\rho)} u_R \leq \inf_{B_d(0,r)} u.$$

This ends the proof. □

Theorem 5.7.4 and the δ_λ-homogeneity of $\mathcal{L}$ easily imply the following Harnack inequality on rings.

Corollary 5.7.7 (Harnack inequality on rings). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge. Let u be a smooth non-negative function on the ring*

$$A_{R,c} := \big\{x \in \mathbb{G} \,|\, cR < d(x) < R/c\big\},$$

where $R > 0$ and $0 < c < 1$. Let $0 < a < b < c$. Then, for every $p \in (Q/2, \infty]$, there exists a constant $\mathbf{c} > 0$ such that

$$\sup_{A_{R,a}} u \leq \mathbf{c}\Big\{ \inf_{A_{R,a}} u + R^{2-Q/p} \;\| \mathcal{L}u \|_{L^p(A_{R,b})}\Big\}. \tag{5.67}$$

The constant $\mathbf{c}$ depends only on $\mathbb{G}$, $\mathcal{L}$, p, d, a, b and c and does not depend on u and R.

Proof. The change of variable $x \mapsto \delta_R(x)$ reduces (5.67) to the analogous inequality with $R = 1$. This last one follows from Theorem 5.7.4. □

Remark 5.7.8 (The Harnack inequality in the abstract setting). According the convention in the *incipit* of the chapter, given an abstract stratified group $\mathbb{H}$, via the isomorphism Ψ between $\mathbb{H}$ and a homogeneous Carnot group $\mathbb{G}$, all the Harnack inequalities of the present section do possess a counterpart in $\mathbb{H}$.

For example, it suffices to consider the results in Remark 2.2.28 in order to derive the following result from Theorem 5.7.4.

Theorem 5.7.9 (A Harnack inequality in the abstract setting). *Let $\mathbb{H}$ be an abstract stratified group, and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{H}$. Let Ω be an open subset of $\mathbb{H}$, and let K be a compact and connected subset of Ω.*

Then there exists a positive constant $\mathbf{c} = \mathbf{c}(\mathbb{H}, \mathcal{L}, \Omega, K)$ such that

$$\sup_K u \leq \mathbf{c} \inf_K u,$$

for every non-negative function $u \in C^\infty(\Omega, \mathbb{R})$ satisfying $\mathcal{L}u = 0$ in Ω.

We close this section by giving the following "monotone convergence" theorem.

Theorem 5.7.10 (The Brelot convergence property). *Let $\mathbb{H}$ be an abstract stratified group, and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{H}$. Let $\Omega \subseteq \mathbb{H}$ be open and connected. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{L}$-harmonic functions in Ω, i.e.*

$$u_n \in C^\infty(\Omega, \mathbb{R}) \text{ and } \mathcal{L}u_n = 0 \text{ in } \Omega \quad \text{for every } n \in \mathbb{N}.$$

Assume the sequence $\{u_n\}_{n\in\mathbb{N}}$ is monotone increasing and

$$\sup_{n\in\mathbb{N}}\{u_n(x_0)\} < \infty \tag{5.68}$$

at some point $x_0 \in \Omega$. Then there exists an $\mathcal{L}$-harmonic function u in Ω such that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly convergent on every compact subset of Ω to u.

Proof. By the results in Remark 2.2.28, it suffices[5] to consider the case when $\mathcal{L}$ is a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$.

Let K be a compact subset of Ω. Since Ω is connected, there exists a compact and connected set K^* such that

$$K \subseteq K^* \subset \Omega \text{ and } x_0 \in K_0.$$

Then, by Theorem 5.7.4,

$$\begin{aligned}\sup_K(u_n - u_m) &\le \sup_{K^*}(u_n - u_m) \le \mathbf{c}\inf_{K^*}(u_n - u_m)\\ &\le \mathbf{c}(u_n(x_0) - u_m(x_0)) \qquad \text{for every } n \ge m \ge 1.\end{aligned}$$

The constant $\mathbf{c}$ is independent of n and m. Then, by condition (5.68), $\{u_n\}_n$ is uniformly convergent on K. Since K is an arbitrary compact subset of Ω, $\{u_n\}_{n\in\mathbb{N}}$ is locally uniformly convergent to a continuous function $u : \Omega \to \mathbb{R}$. On the other hand, by the solid mean value Theorem 5.6.1, for every $x \in \Omega$ and $r > 0$ such that $\overline{B_d(x, r)} \subset \Omega$, we have

$$u_n(x) = \mathrm{M}_r(u_n)(x) \quad \forall\, n \in \mathbb{N}.$$

Letting n tend to infinity (by the uniform convergence $u_n \to u$), we get

$$u(x) = \mathrm{M}_r(u)(x) \qquad \forall\, x \in \Omega, r > 0 : \ \overline{B_d(x, r)} \subset \Omega,$$

and now the Koebe-type Theorem 5.6.3 implies

$$u \in C^\infty(\Omega, \mathbb{R}) \text{ and } \mathcal{L}u = 0 \text{ in } \Omega. \quad \square$$

[5] Indeed, following the notation in Remark 2.2.28 and in the assertion of the above theorem, the following facts hold: the (abstract) sub-Laplacian $\mathcal{L}$ is Ψ-related to the sub-Laplacian $\widetilde{\mathcal{L}}$ on $\mathbb{G}$; set $\widetilde{u}_n := u_n \circ \Psi$, $\widetilde{\Omega} := \Psi^{-1}(\Omega)$, $\widetilde{x}_0 := \Psi^{-1}(x_0)$, then $\widetilde{\Omega}$ is open and connected in $\mathbb{G}$ (recall that Ψ is a homeomorphism), the sequence $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ is monotone increasing on $\widetilde{\Omega}$, bounded in $\widetilde{x}_0$ and $\widetilde{\mathcal{L}}\widetilde{u}_n = 0$ on $\widetilde{\Omega}$. Finally, if K is a compact subset of $\mathbb{H}$, then $\widetilde{K} := \Psi^{-1}(K)$ is a compact subset of $\mathbb{G}$, and

$$\sup_K |u_n - u| = \sup_{\widetilde{K}} |\widetilde{u}_n - \widetilde{u}|.$$

The above Brelot convergence property implies the following strong minimum principle (see also Section 5.13 for a more exhaustive investigation of maximum–minimum principles).

Corollary 5.7.11 (Strong minimum principle). *Let $\mathcal{L}$ be a sub-Laplacian on an abstract stratified group $\mathbb{H}$. A non-negative solution to $\mathcal{L}u = 0$ on an open connected set $\Omega \subseteq \mathbb{H}$ vanishes identically iff it vanishes at a point.*

Proof. Apply the result of Theorem 5.7.10 to the sequence $\{n \cdot u \mid n \in \mathbb{N}\}$. □

5.8 Liouville-type Theorems

The classical Liouville theorem for entire harmonic functions also holds in the sub-Laplacian setting. Indeed, the Harnack inequality of Corollary 5.7.3 implies the following theorem:

Theorem 5.8.1 (Liouville theorem for sub-Laplacians). *Let $\mathbb{H}$ be an abstract stratified group. Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{H}$. Let $u \in C^\infty(\mathbb{H}, \mathbb{R})$ be a function satisfying*

$$u \geq 0 \text{ and } \mathcal{L}u = 0 \text{ in } \mathbb{H}.$$

Then u is constant.

Proof. By the results in Remark 2.2.28, it suffices[6] to consider the case when $\mathcal{L}$ is a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$. Define

$$m := \inf_{\mathbb{G}} u \quad \text{and} \quad v := u - m.$$

Then $v \geq 0$ and $\mathcal{L}v = 0$ in $\mathbb{G}$. From Harnack inequality (5.62) we obtain

$$\sup_{B_d(0,r)} v \leq \mathbf{c} \inf_{B_d(0,r)} v, \qquad \mathbf{c} \text{ independent of } r.$$

From this inequality, letting r tend to infinity, we obtain

$$0 \leq \sup_{\mathbb{G}} v \leq \mathbf{c} \inf_{\mathbb{G}} v = 0,$$

which implies $v \equiv 0$ and $u \equiv m$. □

Theorem 5.8.1 can be viewed as a particular case of the following stronger version of the Liouville property for $\mathcal{L}$.

[6] Indeed, the (abstract) sub-Laplacian $\mathcal{L}$ is Ψ-related to the sub-Laplacian $\widetilde{\mathcal{L}}$ on $\mathbb{G}$ and, set $\widetilde{u} := u \circ \Psi$, we have $\widetilde{u} \geq 0$ and $\widetilde{\mathcal{L}}\widetilde{u} = 0$ on $\mathbb{G}$. Once we have proved that $\widetilde{u}$ is constant on $\mathbb{G}$, the same follows for $u = \widetilde{u} \circ \Psi^{-1}$.

Theorem 5.8.2 (Liouville-type theorem-polynomial lower bound). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let u be an entire solution to $\mathcal{L}u = 0$, i.e. a function $u \in C^\infty(\mathbb{G}, \mathbb{R})$ such that $\mathcal{L}u = 0$ in $\mathbb{G}$. Assume there exists a polynomial function p on $\mathbb{G}$ such that*

$$u \geq p \quad \textit{in } \mathbb{G}.$$

Then[7] *u is a polynomial function and* $\deg_{\mathbb{G}} u \leq \deg_{\mathbb{G}} p$.

The inequality $\deg_{\mathbb{G}} u \leq \deg_{\mathbb{G}} p$ can be strict: take, for example, $u \equiv 0$, $p = -x_1^2$, so that $\deg_{\mathbb{G}} u = 0 < 2 = \deg_{\mathbb{G}} p$.

Remark 5.8.3. All the results in this section involving polynomial functions are obviously related to the fixed coordinates on a homogeneous Carnot group.

Abstract counterparts of these results are available in the obvious way. A polynomial function on the abstract stratified group $\mathbb{H}$ is a function $P : \mathbb{H} \to \mathbb{R}$ such that $p := P \circ \operatorname{Exp}$ is a polynomial function on the vector space $\mathfrak{h}$ (the Lie algebra of $\mathbb{H}$), i.e. p is a polynomial function when expressed in coordinates w.r.t. any (or equivalently, w.r.t. at least one) basis for $\mathfrak{h}$. All the details are left to the reader. □

Theorem 5.8.2 is an easy consequence of the following one.

Theorem 5.8.4 (Liouville-type theorem-polynomial sub-Laplacian). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let u be a smooth function on $\mathbb{G}$ satisfying*

$$u \geq 0 \quad \textit{and} \quad \mathcal{L}u = w \textit{ in } \mathbb{G},$$

where w is a polynomial function. Then u is a polynomial function and

$$\deg_{\mathbb{G}} u \leq 2 + \deg_{\mathbb{G}} w.$$

More precisely,

$$\deg_{\mathbb{G}} u = \begin{cases} \deg_{\mathbb{G}} w & \textit{if } \deg_{\mathbb{G}} u = 0, \\ 2 + \deg_{\mathbb{G}} w & \textit{if } \deg_{\mathbb{G}} u \geq 2. \end{cases} \tag{5.69}$$

The case $\deg_{\mathbb{G}} u = 1$ *cannot occur, since a polynomial of $\mathbb{G}$-degree 1 cannot be a non-negative function.*

The proof of this theorem will follow from a representation formula that we shall deduce from identity (5.50a) in Section 5.6.

We first show how Theorem 5.8.2 can be obtained from Theorem 5.8.4.

Proof (of Theorem 5.8.2). We first recall that $\mathcal{L}$ is a differential operator, δ_λ-homogeneous of degree two, and its coefficients are polynomial functions. Then, if $u \geq p$ and $\mathcal{L}u = 0$, we have

$$u - p \geq 0 \text{ and } \mathcal{L}(u - p) = w,$$

[7] See Definition 1.3.3 on page 33 for the definition of the $\mathbb{G}$-degree $\deg_{\mathbb{G}}(p)$ of a polynomial p and the $\mathbb{G}$-length $|\alpha|_{\mathbb{G}}$ of the multi-index α.

where $w := -\mathcal{L}p$ is a polynomial whose $\mathbb{G}$-degree does not exceed

$$\max\{0, \deg_{\mathbb{G}} p - 2\}.$$

From Theorem 5.8.4 it follows that $u - p$ is a polynomial and that

$$\deg_{\mathbb{G}}(u-p) \le 2 + \deg_{\mathbb{G}}(w) \le 2 + \max\{0, \deg_{\mathbb{G}} p - 2\} = \max\{2, \deg_{\mathbb{G}} p\}.$$

Hence, $u = (u-p) + p$ is a polynomial function and its $\mathbb{G}$-degree does not exceed $\max\{2, \deg_{\mathbb{G}} p\}$.

If $\deg_{\mathbb{G}} p \ge 2$, this gives the assertion of Theorem 5.8.2. It remains to consider the cases $\deg_{\mathbb{G}} p = 0$ and $\deg_{\mathbb{G}} p = 1$ (indeed, in these cases the above argument only proves that $\deg_{\mathbb{G}} u \le 2$, which is weaker than $\deg_{\mathbb{G}} u \le \deg_{\mathbb{G}} p$). In both cases, $\mathcal{L}p \equiv 0$, so that the hypotheses of Theorem 5.8.2 rewrites as

$$u - p \ge 0, \quad \mathcal{L}(u-p) = 0 \qquad \text{in } \mathbb{G}.$$

Hence, by Liouville Theorem 5.8.1, $u - p$ is constant, whence

$$u = (u-p) + p = p + \text{constant},$$

so that u is a polynomial function of the same $\mathbb{G}$-degree as p. $\square$

We next prove a representation formula having its own interest and useful for the proof of Theorem 5.8.4.

Proposition 5.8.5. *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $(\mathbb{G}, *)$, and let d be an $\mathcal{L}$-gauge on $\mathbb{G}$. Let $u \in C^\infty(\mathbb{G}, \mathbb{R})$ be such that*

$$\mathcal{L}u = w \quad \textit{in } \mathbb{G}, \tag{5.70a}$$

where w is a polynomial function. Then

$$u(x) = \Phi_r(u)(x) - \sum_{|\alpha|_{\mathbb{G}} \le m} C_Q^{(\alpha)}\, w^{(\alpha)}(x)\, r^{2+|\alpha|_{\mathbb{G}}} \tag{5.70b}$$

for any $x \in \mathbb{G}$ and $r > 0$. Φ_r denotes the integral operator (5.50b) *related to a smooth function $\varphi \in C_0^\infty((0,1), \mathbb{R})$. Moreover, m is the $\mathbb{G}$-degree of w,*

$$w^{(\alpha)}(x) := D_y^\alpha|_{y=0} w(x * y)$$

and, for any α with $|\alpha|_{\mathbb{G}} \le m$, $C_Q^{(\alpha)}$ is a positive constant depending only on α, Q and d. In particular, $w^{(\alpha)}$ is a polynomial function with $\mathbb{G}$-degree not exceeding $m - |\alpha|_{\mathbb{G}}$.

Proof. Identity (5.50a) in Section 5.6 and hypothesis (5.70a) give

$$u(x) = \Phi_r(u)(x) - \Phi_r^*(w)(x),$$

where $\Phi_r^*(w)(x) = \int_0^\infty \varphi_r(\rho)\,\mathcal{N}_\rho(w)(x)\,\mathrm{d}\rho$, and

$$\begin{aligned}\mathcal{N}_\rho(w)(x) &= \beta_d \int_{B_d(x,\rho)} (d^{2-Q}(x^{-1} * y) - \rho^{2-Q})\, w(y)\,\mathrm{d}H^N(y)\\ &= \beta_d \int_{B_d(0,\rho)} (d^{2-Q}(y) - \rho^{2-Q})\, w(x * y)\,\mathrm{d}H^N(y).\end{aligned}$$

We now claim that

$$w(x * y) = \sum_{|\alpha|_{\mathbb{G}} \le m} \frac{w^{(\alpha)}(x)}{\alpha!}\, y^\alpha, \tag{5.71}$$

where $w^{(\alpha)} = D_y^\alpha|_{y=0} w(x * y)$ is a polynomial function of $\mathbb{G}$-degree $\le m - |\alpha|_{\mathbb{G}}$. Taking this claim for granted for a moment, we have

$$\begin{aligned}&\Phi_r^*(w)(x)\\ &= \sum_{|\alpha|_{\mathbb{G}} \le m} \frac{w^{(\alpha)}(x)}{\alpha!}\, \beta_d \int_0^\infty \varphi\left(\frac{\rho}{r}\right)\left(\int_{B_d(x,\rho)} (d^{2-Q}(y) - \rho^{2-Q})\, y^\alpha\,\mathrm{d}H^N(y)\right)\frac{\mathrm{d}\rho}{r}\\ &\quad\text{(by using the change of variables } y = \delta_\rho(z) \text{ and } \rho = r\,\sigma)\\ &= \sum_{|\alpha|_{\mathbb{G}} \le m} C_Q^{(\alpha)}\, w^{(\alpha)}(x)\, r^{2+|\alpha|_{\mathbb{G}}},\end{aligned}$$

where

$$C_Q^{(\alpha)} = \frac{\beta_d}{\alpha!} \int_0^\infty \varphi(\sigma)\left(\int_{B_d(0,1)} (d^{2-Q}(z) - 1)\, z^\alpha\,\mathrm{d}H^N(z)\right)\mathrm{d}\sigma.$$

Then, we are left with the proof of (5.71). Since w is a polynomial function and $(x, y) \mapsto x * y$ has polynomial components too, one has

$$w(x * y) = \sum_{|\alpha|_{\mathbb{G}} + |\beta|_{\mathbb{G}} \le n} c_{\alpha,\beta}\, x^\alpha\, y^\beta$$

for a suitable positive integer n and real constants $c_{\alpha,\beta}$. We have to prove only that $n \le m$. Now, since $\deg_{\mathbb{G}} w \le m$,

$$w(z) = \sum_{|\gamma|_{\mathbb{G}} \le m} c_\gamma\, z^\gamma, \quad c_\gamma \in \mathbb{R} \text{ for any } \gamma.$$

Then, since δ_λ is an automorphism of the group $\mathbb{G}$,

$$\begin{aligned}\sum_{|\alpha|_{\mathbb{G}} + |\beta|_{\mathbb{G}} \le n} c_{\alpha,\beta}\, \lambda^{|\alpha|_{\mathbb{G}} + |\beta|_{\mathbb{G}}}\, x^\alpha\, y^\beta &= w(\delta_\lambda(x) * \delta_\lambda(y))\\ = w(\delta_\lambda(x * y)) &= \sum_{|\gamma|_{\mathbb{G}} \le m} c_\gamma\, \lambda^{|\gamma|_{\mathbb{G}}}\, (x * y)^\gamma\end{aligned}$$

for every $x,\ y \in \mathbb{G}$ and $\lambda > 0$. As a consequence,

$$\sum_{m<|\alpha|_{\mathbb{G}}+|\beta|_{\mathbb{G}}\leq n} c_{\alpha,\beta}\, x^{\alpha}\, y^{\beta} = 0 \quad \forall\, x,\ y \in \mathbb{G},$$

so that

$$w(x * y) = \sum_{|\alpha|_{\mathbb{G}}+|\beta|_{\mathbb{G}}\leq m} c_{\alpha,\beta}\, x^{\alpha}\, y^{\beta},$$

and the claim follows. This completes the proof. □

We are now in the position to prove Theorem 5.8.4.

Proof (of Theorem 5.8.4). In the Harnack inequality (5.58) of Theorem 5.7.2, take $x_0 = 0$, $r = 2\,|x|$ and $p = \infty$. We obtain

$$u(x) \leq \mathbf{c}\left\{u(0) + |x|^{m+2}\right\}, \tag{5.72}$$

where $\mathbf{c}$ is a positive absolute constant.

Let us recall the notation used in Corollary 1.5.5 (page 68). If $\mathcal{L}$ is the sum of squares of the vector fields $X_1, \ldots, X_{N_1}$ and if $\beta = (i_1, \ldots, i_k)$ is a multi-index with components in $\{1, \ldots, N_1\}$, we set

$$X^{\beta} := X_{i_1} \circ \cdots \circ X_{i_k} \text{ and } |\beta| = k.$$

Let us now use the representation formula of Proposition 5.8.5. Since $w^{(\alpha)}$ in (5.70b) has $\mathbb{G}$-degree $\leq m - |\alpha|$, for any non-negative multi-index β with $|\beta| > m$, we have

$$X^{\beta}u(x) = X^{\beta}\big(\Phi_r(u)(x)\big) \quad \forall\, x \in \mathbb{G}.$$

Then, since the X_j's are left-invariant on $(\mathbb{G}, *)$ and δ_λ-homogeneous of degree one, we have (see also (5.50d) and (5.50e))

$$\begin{aligned} X^{\beta}u(x) &= \int_{\mathbb{G}} u(y)\, X^{\beta}\big(\phi_r(x^{-1} * y)\big)\,\mathrm{d}y \\ &= r^{-|\beta|} \int_{\mathbb{R}^N} u(y)\,\big(X^{\beta}\widetilde{\phi}\big)_r(y^{-1} * x)\,\mathrm{d}y, \end{aligned}$$

where $\widetilde{\phi}(z) := \phi(z^{-1})$ and

$$\big(X^{\beta}\widetilde{\phi}\big)_r = r^{-Q}\,\big(X^{\beta}\widetilde{\phi}\big) * \delta_{1/r}.$$

Hence,

$$X^{\beta}u(x) = r^{-|\beta|} \int_{B_d(0,1)} u(x * \delta_r(z^{-1}))\,\big(X^{\beta}\widetilde{\phi}\big)(z)\,\mathrm{d}z. \tag{5.73}$$

Using inequality (5.72) in (5.73), we obtain

$$|X^{\beta}u(x)| \leq \mathbf{c}\, r^{-|\beta|}\,(1 + |x| + r)^{m+2}$$

for every $x \in \mathbb{G}$ and $r > 0$. The constant $\mathbf{c}$ depends on $u(0)$, but it is independent of x and r. Letting r tend to infinity, we obtain

$$X^{\beta}u(x) = 0 \qquad \forall\, x \in \mathbb{G}$$

and for every β with $|\beta| > m + 2$. By the cited Corollary 1.5.5, this implies that u is a polynomial function of $\mathbb{G}$-degree $\leq m + 2$. □

Using the same argument as above, we can prove the following improvement of Theorem 5.8.4.

Theorem 5.8.6 (Liouville: polynomial sub-Laplacian and bound). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let $u : \mathbb{G} \to \mathbb{R}$ be a smooth function satisfying*

$$u \geq p \quad \textit{and} \quad \mathcal{L}u = w \textit{ in } \mathbb{G},$$

where p and w are polynomial functions. Then u is a polynomial and

$$\deg_{\mathbb{G}} u \leq \max\{\deg_{\mathbb{G}} p, 2 + \deg_{\mathbb{G}} w\}.$$

Proof. Set $v = u - p$. We have

$$v \geq 0, \quad \mathcal{L}v = w - \mathcal{L}p \qquad \text{in } \mathbb{G}.$$

Since $\mathcal{L}$ has polynomial coefficients, we can apply Theorem 5.8.4 to derive

$$\begin{aligned}
\deg_{\mathbb{G}}(u-p) &\leq 2 + \deg_{\mathbb{G}}(w - \mathcal{L}p) \leq 2 + \max\{\deg_{\mathbb{G}} w, \deg_{\mathbb{G}} \mathcal{L}p\} \\
&\leq 2 + \begin{cases} \max\{\deg_{\mathbb{G}} w, \deg_{\mathbb{G}} p - 2\} & \text{if } \deg_{\mathbb{G}} p \geq 2, \\ \max\{\deg_{\mathbb{G}} w, 0\} & \text{if } \deg_{\mathbb{G}} p \leq 1 \end{cases} \\
&= \begin{cases} \max\{2 + \deg_{\mathbb{G}} w, \deg_{\mathbb{G}} p\} & \text{if } \deg_{\mathbb{G}} p \geq 2, \\ 2 + \deg_{\mathbb{G}} w & \text{if } \deg_{\mathbb{G}} p \leq 1 \end{cases} \\
&= \max\{\deg_{\mathbb{G}} p, 2 + \deg_{\mathbb{G}} w\}. \quad \square
\end{aligned}$$

Summing up all the above statements, we obtain the following Liouville theorem “of polynomial type”.

Theorem 5.8.7 (Liouville-type theorem). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let $u : \mathbb{G} \to \mathbb{R}$ be smooth and such that*

$$u \geq p \quad \textit{and} \quad \mathcal{L}u = w \quad \textit{in } \mathbb{G},$$

where p and w are polynomial functions. Then u is a polynomial, and

$$\deg_{\mathbb{G}} u \leq \begin{cases} \deg_{\mathbb{G}} p & \textit{if } w \equiv 0, \\ \max\{\deg_{\mathbb{G}} p, 2 + \deg_{\mathbb{G}} w\} & \textit{otherwise.} \end{cases}$$

5.8.1 Asymptotic Liouville-type Theorems

We close this section by giving some more “asymptotic” Liouville-type theorems, easy consequences of Theorem 5.8.2.

Theorem 5.8.8 (Asymptotic Liouville. I). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let u be an entire solution to $\mathcal{L}u = 0$. Assume there exists a real number $m \geq 0$ such that*

$$u(x) = \mathcal{O}(\varrho^m(x)), \quad \textit{as } x \to \infty. \tag{5.74}$$

Then u is a polynomial function and

$$\deg_{\mathbb{G}} u \le [m]. \tag{5.75}$$

Here ϱ is (any fixed) homogeneous norm on $\mathbb{G}$ and $[m]$ denotes the integer part of m, i.e. $[m] \in \mathbb{Z}$ and $[m] \le m < [m] + 1$.

Proof. By condition (5.74) and Proposition 5.1.4, we get

$$u(x) \ge p(x) \quad \forall\, x \in \mathbb{G},$$

where

$$p(x) = -\mathbf{c}\left(1 + \sum_{j=1}^{r} |x^{(j)}|^{2\,r!/j}\right)^{[m]+1},$$

and $\mathbf{c}$ is a suitable positive constant. Then, by Theorem 5.8.2, u is a polynomial function. Let $n := \deg_{\mathbb{G}} u$. Assume, by contradiction, $n \ge [m] + 1$. Writing

$$u(x) = \sum_{k=0}^{n} u_k(x),$$

where u_k is δ_λ-homogeneous of degree k, from condition (5.74) we get

$$\sum_{k=0}^{n} \varrho(x)^{k-n}\, u_k\big(\delta_{1/\varrho(x)}(x)\big) = \frac{u(x)}{\varrho(x)^n} \longrightarrow 0, \quad \text{as } \varrho(x) \to \infty.$$

Hence, $u_n(y) = 0$ for every $y \in \mathbb{G}$ such that $\varrho(y) = 1$, which implies $\deg_{\mathbb{G}} u \le n-1$, a contradiction. Then $n \le [m]$ and the proof is complete. □

Theorem 5.8.8 together with Theorem 5.6.1 give the following corollary.

Corollary 5.8.9 (Asymptotic Liouville. II). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge.*

Let u be an entire solution to $\mathcal{L}u = 0$. Assume there exists $x_0 \in \mathbb{G}$ and a real number $m \ge 0$ such that

$$⨍_{B_d(x_0,r)} |u(y)|\,\mathrm{d}y = \mathcal{O}(r^m), \quad \textit{as } r \to \infty. \tag{5.76}$$

Then u is a polynomial function and

$$\deg_{\mathbb{G}} u \le [m].$$

In (5.76) *we have set*

$$⨍_D := \frac{1}{|D|} \int_D .$$

Proof. Let $x \in \mathbb{G} \setminus \{0\}$ and put $r = \max\{d(x), d(x_0)\}$. By the pseudo-triangle inequality (5.55), $B_d(x,r) \subseteq B_d(x_0, 2\,\mathbf{c}_d\, r)$. Then, from the solid mean value Theorem 5.6.1, we obtain

$$\begin{aligned}|u(x)| &= |\mathrm{M}_r(u)(x)| \le \mathrm{M}_r(|u|)(x)\\ &\le \frac{m_d}{r^Q}\Big(\sup_{\mathbb{G}\setminus\{0\}} \Psi_{\mathcal{L}}\Big)\int_{B_d(x_0,2\mathbf{c}_d\, r)} |u(y)|\,\mathrm{d}y,\end{aligned}$$

so that

$$|u(x)| \le \mathbf{c}\, \fint_{B_d(x_0,2\mathbf{c}_d\, r)} |u(y)|\,\mathrm{d}y,$$

being $\mathbf{c} > 0$ independent of x and r. This inequality and (5.76) give

$$u(x) = \mathcal{O}(r^m) = \mathcal{O}(d(x)^m), \quad \text{as } x \to \infty.$$

Then, by Theorem 5.8.8, u is a polynomial function with the $\mathbb{G}$-degree $\le [m]$. □

Remark 5.8.10. By using Proposition 5.1.4, one easily recognizes that condition (5.76) in the previous corollary is equivalent to the following one

$$\int_{B_r} |u(y)|\,\mathrm{d}y = \mathcal{O}(r^{m+Q}), \quad \text{as } r \to \infty, \tag{5.77}$$

where $B_r := \{y \in \mathbb{G} : \varrho(y) < r\}$ is the ball of radius r centered at the origin, with respect to a homogeneous norm ϱ on $\mathbb{G}$.

Corollary 5.8.11 (Asymptotic Liouville. III). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let u be an entire solution to $\mathcal{L}u = 0$, and $1 \le p \le \infty$. Assume* (*with the notation of Remark* 5.8.10)

$$\| u \|_{L^p(B_r)} = \mathcal{O}(r^{m+Q/p}), \quad \textit{as } r \to \infty. \tag{5.78}$$

Then u is a polynomial function of the $\mathbb{G}$-degree not exceeding $[m]$.

Proof. The Hölder inequality gives

$$\int_{B_r} |u(y)|\,\mathrm{d}y \le \mathbf{c}\, r^{Q(1-1/p)} \;\| u \|_{L^p(B_r)} .$$

Then (5.78) implies (5.77), and the assertion follows from Remark 5.8.10 and Corollary 5.8.9. □

5.9 Some Results on $\mathbb{G}$-fractional Integrals and the Sobolev–Stein Embedding Inequality

Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$ of homogeneous dimension Q. Let $0 < \alpha < Q$. Given a function $f : \mathbb{G} \to \mathbb{R}$, we formally define

$$I_\alpha(f)(x) := \int_{\mathbb{G}} \frac{f(y)}{(d(x,y))^{Q-\alpha}}\,\mathrm{d}y,$$

where $d(x, y)$ stands for $d(y^{-1} \circ x)$ and $z \mapsto d(z)$ is an $\mathcal{L}$-gauge function on $\mathbb{G}$. By analogy with the Euclidean setting, we shall call I_α the $\mathbb{G}$-*fractional integral of order* α.

The following theorem, when $\mathbb{G}$ is the usual Euclidean group $(\mathbb{R}^N, +)$, gives back a celebrated theorem by Hardy, Littlewood and Sobolev.

Theorem 5.9.1 (Hardy–Littlewood–Sobolev for sub-Laplacians). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$, and let d be an $\mathcal{L}$-gauge. Suppose $1 < \alpha < Q$ and $1 < p < \frac{Q}{\alpha}$. Let $q > p$ be defined by*

$$1/q = 1/p - \alpha/Q.$$

Then there exists a positive constant $C = C(\alpha, p, \mathbb{G}, \mathcal{L}, d)$ such that

$$\|I_\alpha(f)\|_q \le C\,\|f\|_p \quad \text{for every } f \in L^p(\mathbb{G}).$$

Here, we use the notation $\|\cdot\|_r$ to denote the L^r norm in $\mathbb{G} \equiv \mathbb{R}^N$ with respect to the Lebesgue measure.

In the classical Euclidean setting, several proofs of this theorem are known. The simplest proof seems to be due to L.I. Hedberg [Hed72] and it makes use of the maximal function theorem by Hardy–Littlewood–Wiener. This last theorem holds in general metric spaces equipped with a *doubling measure*.[8] In particular, it holds in our context.

Fixed a sub-Laplacian $\mathcal{L}$ and an $\mathcal{L}$-gauge d, we define the $\mathcal{L}$-*maximal function* $M_{\mathcal{L}}(f)$ of a function $f \in L^p(\mathbb{G}, \mathbb{C})$, $1 < p < \infty$, as follows

$$M_{\mathcal{L}}(f)(x) := \sup_{r>0} \fint_{B_d(x,r)} |f(y)|\,\mathrm{d}y, \qquad x \in \mathbb{G},$$

where we used the notation

$$\fint_D = \frac{1}{|D|}\int_D .$$

The function $x \mapsto M_{\mathcal{L}}(f)(x)$ is lower semicontinuous. Indeed, if $M_{\mathcal{L}}(f)(x) > \alpha$, there exists $r > 0$ such that $\fint_{B_d(x,r)} |f(y)|dy > \alpha$. Then, since

$$x \mapsto \fint_{B_d(x,r)} |f(y)|\,\mathrm{d}y$$

is a continuous function (see Ex. 5 at the end of this chapter), there exists $\delta > 0$ such that

[8] A Radon measure μ on a quasi-metric space (X, d) is *doubling* if there exists a positive constant C_d such that

$$0 < \mu(B(x, 2r)) \le C_d\,\mu(B(x, r))$$

for every d-ball with center at x and radius r.

$$M_{\mathcal{L}}(f)(z) \geq \fint_{B_d(z,r)} |f(y)|\,\mathrm{d}y > \alpha \qquad \text{if } d(x^{-1} \circ z) < \delta.$$

The *$\mathcal{L}$-maximal function theorem* is the following one.

$\mathcal{L}$-Maximal Function Theorem. *Let $1 < p < \infty$. With the above notation, there exists a positive constant $C = C(p, \mathbb{G}, \mathcal{L}, d)$ such that*

$$\|M_{\mathcal{L}}(f)\|_p \leq C\,\|f\|_p \qquad \textit{for every } f \in L^p(\mathbb{G}, \mathbb{C}).$$

A proof of this theorem in general doubling metric spaces (which is out of our scopes here) can be found in the monograph [Ste81, Chapter 2], by E.M. Stein.

Starting from this result, we now prove Theorem 5.9.1 by using the idea in L.I. Hedberg's paper [Hed72].

Proof (of Theorem 5.9.1). For every fixed $t > 0$ and $x \in \mathbb{G}$, we have

$$\begin{aligned} I_\alpha(f)(x) &= \left(\int_{d(x,y)\leq t} + \int_{d(x,y)\geq t}\right) |f(y)|\,(d(x,y))^{\alpha-Q}\,\mathrm{d}y \\ &=: I_\alpha^{(t)}(f)(x) + E_\alpha^{(t)}(f)(x). \end{aligned}$$

The Hölder inequality gives

$$\begin{aligned} E_\alpha^{(t)}(f)(x) &\leq \|f\|_p \left(\int_{d(x,y)\leq t} (d(x,y))^{(\alpha-Q)p/(p-1)}\,\mathrm{d}y\right)^{1-1/p} \\ &= (\text{see } (5.36))\;\; C\,\|f\|_p \left(\int_t^\infty \tau^{Q-1+(\alpha-Q)p/(p-1)}\,\mathrm{d}\tau\right)^{1-1/p} \\ &= C'\,\|f\|_p\, t^{\alpha-Q/p}. \end{aligned}$$

On the other hand, one has

$$\begin{aligned} I_\alpha^{(t)}(f)(x) &= \sum_{k=0}^\infty \int_{\{2^{-k-1} < d(x,y)/t \leq 2^{-k}\}} |f(y)|\,(d(x,y))^{\alpha-Q}\,\mathrm{d}y \\ &\leq C \sum_{k=0}^\infty \left(t/2^k\right)^{\alpha-Q} \int_{\{d(x,y)\leq t2^{-k}\}} |f(y)|\,\mathrm{d}y \\ &\leq C' \sum_{k=0}^\infty \left(t/2^k\right)^{\alpha-Q} \left(2^{-k}\,t\right)^Q M_{\mathcal{L}}(f)(x) \\ &= C'\,t^\alpha\,M_{\mathcal{L}}(f)(x) \sum_{k=0}^\infty 2^{-k\alpha}. \end{aligned}$$

Then, summing up the above estimates, we derive

$$I_\alpha(f)(x) \leq C\left(t^{\alpha-Q/p}\,\|f\|_p + t^\alpha\,M_{\mathcal{L}}(f)(x)\right), \qquad x \in \mathbb{G},$$

for every $t > 0$. We now choose

$$t = \left(\frac{\|f\|_p}{M_{\mathcal{L}}(f)(x)} \right)^{p/Q},$$

so that

$$I_\alpha(f)(x) \le C \, \|f\|_p^{(p\,\alpha)/Q} (M_{\mathcal{L}}(f)(x))^{1-(p\,\alpha)/Q}.$$

Hence, being $q\,(1 - p\alpha/Q) = p$, we have

$$\begin{aligned} \|I_\alpha(f)\|_q^q &\le C \|f\|_p^{(pq\alpha)/Q} \int_{\mathbb{G}} (M_{\mathcal{L}}(f)(x))^p \,\mathrm{d}x \\ &\qquad \text{(by the } \mathcal{L}\text{-maximal function theorem)} \\ &\le C \|f\|_p^{(p\,q\,\alpha)/Q} \|f\|_p^p = C \|f\|_p^q, \end{aligned}$$

since $(p\,q\,\alpha)/Q + p = q$. The theorem is proved. $\square$

From this theorem and the representation formula (5.16), one easily obtains an inequality extending the classical Sobolev embedding theorem to the homogeneous Carnot groups.

Theorem 5.9.2 (Sobolev–Stein embedding). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$ of homogeneous dimension Q.*

Suppose $1 < p < Q$. Then there exists a positive constant $C = C(p, \mathbb{G}, \mathcal{L})$ such that

$$\|u\|_q \le C \, \|\nabla_{\mathcal{L}} u\|_p \qquad \textit{for every } u \in C_0^\infty(\mathbb{R}^N, \mathbb{R}),$$

where

$$1/q = 1/p - 1/Q \quad \left(\textit{i.e. } q = \frac{Qp}{Q-p} \right).$$

Proof. Let $u \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$. Using the representation formula (5.16), we have

$$u(x) = -\int_{\mathbb{G}} \Gamma(x^{-1} \circ y) \, \mathcal{L}u(y) \,\mathrm{d}y.$$

Keeping in mind that $\mathcal{L} = \sum_{j=1}^m X_j^2$ and $X_j^* = -X_j$, by integrating by parts at the right-hand side, we obtain

$$u(x) = \int_{\mathbb{R}^N} (\nabla_{\mathcal{L}} \Gamma)(x^{-1} \circ y) \, \nabla_{\mathcal{L}} u(y) \,\mathrm{d}y. \tag{5.79}$$

On the other hand, out of the origin, we have

$$\nabla_{\mathcal{L}} \Gamma = \beta_d \, \nabla_{\mathcal{L}} (d^{2-Q}) = (2-Q)\, \beta_d \, d^{1-Q} \, \nabla_{\mathcal{L}} d,$$

so that, since $\nabla_{\mathcal{L}} d$ is smooth in $\mathbb{G} \setminus \{0\}$ and δ_λ-homogeneous of degree zero,

$$|\nabla_{\mathcal{L}} \Gamma| \le C \, d^{1-Q},$$

for a suitable constant $C > 0$ depending only on $\mathcal{L}$. Using this inequality in (5.79), we get

$$|u(x)| \leq C \int_{\mathbb{G}} |\nabla_{\mathcal{L}} u(y)| \, d(x,y)^{1-Q} \, \mathrm{d}y = C \, I_1(|\nabla_{\mathcal{L}} u|)(x).$$

Then, by the Hardy–Littlewood–Sobolev Theorem 5.9.1,

$$\|u\|_q \leq C \, \|I_1(|\nabla_{\mathcal{L}} u|)\|_q \leq C \, \|\nabla_{\mathcal{L}} u\|_p,$$

where

$$1/q = 1/p - 1/Q \quad \left(\Leftrightarrow q = \frac{Qp}{Q-p} \right).$$

This ends the proof. □

5.10 Some Remarks on the Analytic Hypoellipticity of Sub-Laplacians

In this section, we collect some results on the hypoellipticity (especially in the analytic sense) of sub-Laplacians. It is far from our scopes here to give proofs of the results of this section, the interested reader will be properly referred to the existing literature.

First of all, we recall the relevant definition.

Definition 5.10.1 ((Analytic) hypoellipticity). *We say that a differential operator L defined on an open set $\Omega \subseteq \mathbb{R}^N$ is hypoelliptic* (*respectively, analytic hypoelliptic*) *in Ω if, for every open set $\Omega' \subseteq \Omega$ and every $f \in C^\infty(\Omega', \mathbb{R})$* (*respectively, f real analytic in Ω'*)*, any solution u to the equation*

$$Lu = f \text{ on } \Omega' \text{ (in the weak sense of distributions)}$$

is of class $C^\infty(\Omega', \mathbb{R})$ (*respectively, is real analytic on Ω'*).

In the sequel, we shall write $u \in C^\omega(\Omega)$ to mean that u is real analytic on Ω. Moreover, we may also use the notation C^∞*-hypoelliptic* and C^ω*-hypoelliptic* to mean, respectively, hypoelliptic and analytic hypoelliptic.

In the very special case of a homogeneous differential operator L with *constant coefficients* in $\mathbb{R}^N$, the problem of hypoellipticity is completely solved by the following result (see, e.g. [Hor69]).

Let L be a homogeneous differential operator with constant coefficients in $\mathbb{R}^N$. Then the following statements are equivalent:

1) *L is C^∞-hypoelliptic in $\mathbb{R}^N$,*
2) *L is C^ω-hypoelliptic in $\mathbb{R}^N$,*
3) *L is elliptic in $\mathbb{R}^N$.*

Moreover, if L has constant coefficients (but it is not necessarily homogeneous), then the equivalence of (2) and (3) still holds true.

As a consequence of the above result, all sub-Laplacians on the Euclidean group $\mathbb{E} = (\mathbb{R}^N, +)$ (see Section 4.1.1) are C^∞ and C^ω-hypoelliptic.

We next focus our attention to the C^∞-hypoellipticity. Let $\{X_1, \dots, X_m\}$ be C^∞-vector fields on $\mathbb{R}^N$. We recall the well-known *rank* (or *bracket*) *condition* (also referred to as *Hörmander's hypoellipticity condition*):

$$\dim\bigl(\mathrm{Lie}\{X_1, \dots, X_m\} I(x)\bigr) = N \qquad \forall\, x \in \mathbb{R}^N,$$

i.e. for every $x \in \mathbb{R}^N$, there exists a set of N iterated brackets of the X_i's which are linearly independent at x. The following celebrated result holds (see Hörmander [Hor67]).

If $\{X_1, \dots, X_m\}$ are C^∞-vector fields on $\mathbb{R}^N$, then the rank condition is sufficient for the C^∞-hypoellipticity of the operator $L = \sum_{j=1}^n X_j^2$. Moreover, if the coefficients of the X_j's are analytic, then the rank condition is also necessary for the C^∞-hypoellipticity of L.

(See also Derridj [Der71], Helffer–Nourrigat [HN79], Kohn [Koh73], Oleĭnik–Radkevič [OR73], Rothschild–Stein [RS76].) See also Bony [Bon69,Bon70] for a partial converse of the rank condition, namely

If the sum of squares L of smooth vector fields is C^∞-hypoelliptic, then the rank condition holds on an open set, dense in $\mathbb{R}^N$.

(See also Fediĭ [Fed70], Kusuoka–Stroock [KuSt85], Bell–Mohammed [BM95] for examples of sums of squares which are C^∞-hypoelliptic but do not satisfy the rank condition everywhere.)

The problem of C^∞-hypoellipticity is thus completely solved for any sub-Laplacian $\mathcal{L}$ on any homogeneous Carnot group, since $\mathcal{L}$ is a sum of squares of polynomial vector fields satisfying the rank condition. It is also interesting to remark that, for any homogeneous left-invariant differential operator $\mathcal{L}$ on a stratified Lie group (hence in particular for our sub-Laplacians), the C^∞-hypoellipticity of $\mathcal{L}$ is *equivalent* to a Liouville-type property for $\mathcal{L}$, namely the property stating that *the only bounded functions u on $\mathbb{G}$ such that $\mathcal{L}u = 0$ are the constant functions.* (See [Rot83]; see also [HN79,Gel83].)

We finally turn our attention to the C^ω-hypoellipticity. The problem of analytic hypoellipticity is more involved and only partial results are known. To begin with, we consider the rank condition, which played a central rôle for C^∞-hypoellipticity. Unfortunately, if L is a sum of squares of analytic vector fields, then the rank-condition is *not sufficient* for analytic hypoellipticity. (See, for instance, Trèves [Tre78], Tartakoff [Tar80], Grigis–Sjöstrand [GS85]; see also explicit counterexamples in [BG72,Hel82,PR80,HH91,Chr91].) In the sequel of the section, we collect some of these results (in particular, several explicit negative ones) for our sub-Laplacians on Carnot groups. The first one is encouraging:

The canonical sub-Laplacian on the Heisenberg–Weyl group $\mathbb{H}^n$ is analytic hypoelliptic.

It is not difficult to prove this result making use of the real analyticity (out of the origin) of the fundamental solution Γ for the canonical sub-Laplacian on $\mathbb{H}^n$, for instance ($Q = 2n + 2$ denotes the homogeneous dimension of $\mathbb{H}^n$)

$$\Gamma(x, t) = c_Q \, \frac{1}{(|x|^4 + |t|^2)^{(Q-2)/4}}. \tag{5.80}$$

Since (see Kaplan [Kap80]; see also Example 5.4.7, page 250, and Chapter 18) the fundamental solution Γ for the canonical sub-Laplacian on every H-type group has exactly the same form as in (5.80), then *the canonical sub-Laplacian on any H-type group is analytic-hypoelliptic.*

This is only a partial result of what is true on (a class of operators containing the) Heisenberg-type groups, as we shall see below; but, unfortunately (as we shall also see in a moment), it must be soon realized that C^ω-hypoellipticity rarely occurs within the non-Euclidean setting of Carnot groups.

To this end, we cite a first "negative" result (see Helffer [Hel82]):

If $\mathbb{G}$ is a Carnot group of step two, and $\mathcal{L}$ is a sub-Laplacian on $\mathbb{G}$, then a necessary condition for $\mathcal{L}$ to be C^ω-hypoelliptic is that $\mathbb{G}$ is a HM-group. (See Definition 3.7.3, page 174, for the definition of HM-group.)

Hence, at least within the setting of step-two Carnot groups, in order to find a C^ω-hypoelliptic sub-Laplacian, we must restrict our attention to the sub-class of HM-groups. Fortunately, a complete answer on the C^ω-hypoellipticity for sub-Laplacians is available on HM-groups. This is given by the following result (see Métivier [Met81]).

If $\mathbb{G}$ is a HM-group and $L \in \mathfrak{U}_m(\mathfrak{g})$ (i.e. L is a homogeneous operator of degree m in the relevant enveloping algebra), then L is C^ω-hypoelliptic if and only if L is C^∞-hypoelliptic.

Consequently, since any sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ belongs to $\mathfrak{U}_2(\mathfrak{g})$ and $\mathcal{L}$ is C^∞-hypoelliptic by the rank condition, then $\mathcal{L}$ is also C^ω-hypoelliptic.

This result covers quite large classes of remarkable cases: for example, since the Heisenberg–Weyl groups, the Iwasawa-groups, and (more generally) the H-type groups all belong to the HM-group class, we now know that *all* their sub-Laplacians are C^ω-hypoelliptic. On the converse, it is easy to exhibit a step-two group where *no* sub-Laplacian is real analytic: take any Carnot group where the first layer has odd dimension (for, in that case, it cannot be a HM-group; see Example 5.10.2 below).

The problem finally rest on the investigation of Carnot groups of step *strictly greater than two*. Unfortunately, a complete answer to the analytic-hypoellipticity in that case is still an *open problem*. Quoting Rothschild [Rot84], it is reasonable to conjecture that if $\mathbb{G}$ is not a HM-group, then there is no $L \in \mathfrak{U}_m(\mathfrak{g})$ (hence, no sub-Laplacian) which is analytic-hypoelliptic. For example, from a result by M. Christ [Chr93, Theorem 1.5] we infer that if $\mathbb{G}$ is a filiform Carnot group of dimension ≥ 4, then no sub-Laplacian on $\mathbb{G}$ is analytic-hypoelliptic. In Examples 5.10.4, 5.10.5, 5.10.6 below, we exhibit some other explicit negative results.

To begin, we give two examples: the first (respectively, the second) is an example of a sub-Laplacian on a homogeneous Carnot group of step two which is not (respectively, which is) analytic-hypoelliptic; in the latter case, we explicitly write the (analytic) fundamental solution.

Example 5.10.2. Let $\mathbb{R}^4$ (the points are denoted by (x, y, z, t) with $x, y, z, t \in \mathbb{R}$) be equipped with the dilation $\delta_\lambda(x, y, z, t) = (\lambda x, \lambda y, \lambda z, \lambda^2 t)$ and the composition law

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \circ \begin{pmatrix} \xi \\ \eta \\ \zeta \\ \tau \end{pmatrix} = \begin{pmatrix} x + \xi \\ y + \eta \\ z + \zeta \\ t + \tau + x\,\eta \end{pmatrix}.$$

Then $\mathbb{G} = (\mathbb{R}^4, \circ, \delta_\lambda)$ is a homogeneous Carnot group of step two, and

$$\Delta_{\mathbb{G}} = (\partial_x)^2 + (\partial_y + x\,\partial_t)^2 + (\partial_z)^2$$

is its canonical sub-Laplacian. It is easily seen that $\mathbb{G}$ is isomorphic to the sum (in the sense of Section 4.1.5) of the Heisenberg-Weyl group $\mathbb{H}^1$ on $\mathbb{R}^3$ and the usual Euclidean group $(\mathbb{R}, +)$ (note that the canonical sub-Laplacians on both these groups are analytic hypoelliptic!). *It can be proved that $\Delta_{\mathbb{G}}$ is not analytic hypoelliptic!* Indeed, by the cited result of Helffer [Hel82], this follows from the fact that $\mathbb{G}$ is not a HM-group, since the first layer of the stratification has odd dimension.

This example is a particular case of the family of *not analytic hypoelliptic* sub-Laplacians (see Rothschild [Rot84])

$$\mathcal{L} = \sum_{j=1}^{n} \left((\partial_{x_j})^2 + (\partial_{y_j} + x_j\,\partial_t)^2 \right) + (\partial_z)^2, \tag{5.81}$$

which, in turn, are inspired by a famous counterexample by Baouendi–Goulaouic [BG72]. Indeed, in [BG72] it is proved that the operator

$$L := \sum_{j=1}^{n} \left((\partial_{x_j})^2 + (x_j\,\partial_t)^2 \right) + (\partial_z)^2$$

on $\mathbb{R}^{n+2}$ (the points are denoted by (x, z, t), $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $z \in \mathbb{R}$, $t \in \mathbb{R}$) is not C^ω-hypoelliptic on $\mathbb{R}^{n+2}$. Now, we notice that the operator $\mathcal{L}$ in (5.81) is a "lifted" version of L, so that it is easy to prove that $\mathcal{L}$ is not C^ω-hypoelliptic on $\mathbb{R}^{2n+2}$ if L is not C^ω-hypoelliptic on $\mathbb{R}^{n+2}$.

Indeed, suppose to the contrary that $\mathcal{L}$ is C^ω-hypoelliptic on $\mathbb{R}^{2n+2}$. Then take a function $f = f(x, z, t)$ real analytic on an open set $\Omega \subseteq \mathbb{R}^{n+2}$ and a solution $u = u(x, z, t)$ to $Lu = f$ on Ω. If we set $\widetilde{f}(x, y, z, t) := f(x, z, t)$ and $\widetilde{u}(x, y, z, t) := u(x, z, t)$, then we notice that

$$\mathcal{L}\widetilde{u}(x, y, z, t) = Lu(x, z, t) = f(x, z, t) = \widetilde{f}(x, y, z, t) \tag{5.82}$$

on the open set $\widetilde{\Omega} := \{(x, y, z, t) \,:\, (x, z, t) \in \Omega,\ y \in \mathbb{R}\}$. Since $\widetilde{f}$ is clearly real analytic on $\widetilde{\Omega}$, then (5.82) and the supposed C^ω-hypoellipticity of $\mathcal{L}$ imply $\widetilde{u} \in C^\omega(\widetilde{\Omega})$. This obviously means that u is real analytic on Ω. Thus we have shown that L is C^ω-hypoelliptic on $\mathbb{R}^{n+2}$, contrarily to what is proved in [BG72]. $\square$

Example 5.10.3. This example is taken from Balogh–Tyson [BT02]. Let us consider the group $\mathbb{G}$ on $\mathbb{R}^5$ (the points are denoted by $(x_1, x_2, x_3, x_4, t) \in \mathbb{G}$, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ corresponds to the first layer of the stratification, $t \in \mathbb{R}$ to the second one) with dilation $\delta_\lambda(x_1, x_2, x_3, x_4, t) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda^2 t)$ and the composition law

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ t \end{pmatrix} \circ \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \tau \end{pmatrix} = \begin{pmatrix} x_1 + \xi_1 \\ x_2 + \xi_2 \\ x_3 + \xi_3 \\ x_4 + \xi_4 \\ t + \tau + \frac{1}{2}(x_2 \xi_1 - x_1 \xi_2 + 2x_4 \xi_3 - 2x_3 \xi_4) \end{pmatrix}.$$

Following our conventional notation, $\mathbb{G}$ is the homogeneous Carnot group of step two (with $m = 4$ generators and $n = 1$) with relevant matrix

$$B^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

Then $\mathbb{G}$ is obviously a HM-group, for B is a non-singular skew-symmetric matrix. In particular, by the cited result of Métivier [Met81], the canonical sub-Laplacian $\mathcal{L}$ is C^ω-hypoelliptic. We can see this directly, for the relevant fundamental solution Γ has been explicitly written by Balogh–Tyson in [BT02] (making use of a remarkable formula by Beals–Gaveau–Greiner, see [BGG96]; we describe this formula closely in Section 5.12, page 291): it is apparent that Γ is analytic out of the origin! Indeed, it holds

$$\Gamma(x_1, x_2, x_3, x_4, t) = c\, d^{2-Q}(x_1, x_2, x_3, x_4, t),$$

where c is a suitable positive constant, $Q = 6$ is the homogeneous dimension of $\mathbb{G}$, and d is the homogeneous norm defined by

$$\begin{aligned} &d(x_1, x_2, x_3, x_4, t) \\ &\quad = \left(\left(\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_3^2 + x_4^2 \right)^2 + t^2 \right)^{1/8} \\ &\quad \cdot \left(\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \sqrt{\left(\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_3^2 + x_4^2 \right)^2 + t^2} \right)^{3/8} \\ &\quad \cdot \left(\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_3^2 + x_4^2 + \sqrt{\left(\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_3^2 + x_4^2 \right)^2 + t^2} \right)^{-1/8}. \end{aligned} \tag{5.83}$$

This formula gives the remarkable example of an *explicit* fundamental solution of a group which is *not* a H-type group!

We then turn our attention to groups of step greater than two. Following the idea in the argument at the end of Example 5.10.2, we can give infinite examples of groups of arbitrarily high step with a sub-Laplacian which is not C^ω-hypoelliptic: it suffices to take the sum (in the sense of Section 4.1.5) of the group $\mathbb{G}$ of Example 5.10.2 with another Carnot group. Let us now quote some other examples taken or inspired by the existing literature.

Example 5.10.4. This example is taken from a paper by Christ [Chr95] (see also [Hel82,PR80]). Consider the Carnot group on $\mathbb{R}^4$ with the composition [9] low

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \circ \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} x_1 + \xi_1 \\ x_2 + \xi_2 \\ x_3 + \xi_3 - \xi_2 x_1 \\ x_4 + \xi_4 + 2\xi_3 x_1 - \xi_2 x_1^2 \end{pmatrix}.$$

It is easily seen that $\mathbb{G}$ is a *filiform* homogeneous Carnot group of step three and two generators, with dilations

$$\delta_\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4),$$

and such that the first two vector fields of the relevant Jacobian basis are

$$X_1 = \partial_{x_1}, \qquad X_2 = \partial_{x_2} - x_1 \partial_{x_3} - x_1^2 \partial_{x_4}.$$

Then, it can be proved that *the canonical sub-Laplacian on* $\mathbb{G}$

$$\Delta_{\mathbb{G}} = X_1^2 + X_2^2 = (\partial_{x_1})^2 + \left(\partial_{x_2} - x_1 \partial_{x_3} - x_1^2 \partial_{x_4}\right)^2$$

is not C^ω*-hypoelliptic.* (Compare the group in this example to the Bony-type sub-Laplacian with $N = 2$ in Section 4.3.3, page 202.) □

Example 5.10.5. It is known (see [Chr91,Chr93,HH91,PR80]) that in $\mathbb{R}^3$ (with coordinates (t, s, x)) the operator

$$L = (\partial_t)^2 + (\partial_s - t^m \partial_x)^2 \tag{5.84}$$

is not C^ω-hypoelliptic for any $m \in \mathbb{N}$, $m \geq 2$. In this example, fixed m as above, we give a suitable sub-Laplacian $\mathcal{L}$ "lifting" L: as a consequence (arguing as at the end of Example 5.10.2), $\mathcal{L}$ cannot be C^ω-hypoelliptic, since L does not possess this property.

Take $N \in \mathbb{N}$, $N \geq m \geq 2$, and consider the following Bony-type sub-Laplacian (see Section 4.3.3, page 202): we equip $\mathbb{R}^{2+N}$ (whose points are denoted by (t, s, x), $t,\ s \in \mathbb{R}$, $x \in \mathbb{R}^N$) by the composition law

$$(t, s, x_1, x_2, x_3, \dots, x_N) \circ (\tau, \sigma, \xi_1, \xi_2, \xi_3, \dots, \xi_N)$$
$$= \begin{pmatrix} t + \tau \\ s + \sigma \\ x_1 + \xi_1 + \sigma\, t \\ x_2 + \xi_2 + \xi_1\, t + \sigma\, \frac{t^2}{2!} \\ x_3 + \xi_3 + \xi_2\, t + \xi_1\, \frac{t^2}{2!} + \sigma\, \frac{t^3}{3!} \\ \vdots \\ x_N + \xi_N + \xi_{N-1}\, t + \xi_{N-2}\, \frac{t^2}{2!} + \cdots + \xi_1\, \frac{t^{N-1}}{(N-1)!} + \sigma\, \frac{t^N}{N!} \end{pmatrix}$$

[9] Compare to Ex. 3, Chapter 4, page 216.

and the group of dilations defined by

$$\delta_\lambda(t, s, x_1, x_2, x_3, \dots, x_N) = (\lambda t, \lambda s, \lambda^2 x_1, \lambda^3 x_2, \lambda^4 x_3, \dots, \lambda^{N+1} x_N).$$

Then, $\mathbb{G} = (\mathbb{R}^{2+N}, \circ, \delta_\lambda)$ is a filiform homogeneous Carnot group of step $N+1$ (note that, since $N \geq m \geq 2$, then the step $N+1$ is ≥ 3) and two generators, and

$$\Delta_{\mathbb{G}} = (\partial_t)^2 + \left(\partial_s + t\, \partial_{x_1} + \frac{t^2}{2!}\, \partial_{x_2} + \cdots + \frac{t^N}{N!}\, \partial_{x_N} \right)^2$$

is its canonical sub-Laplacian.

We now prove that $\Delta_{\mathbb{G}}$ is not C^ω-hypoelliptic (starting from the fact that L in (5.84) is not). Indeed, since L is not C^ω-hypoelliptic, there exists an open set $\Omega \subseteq \mathbb{R}^3$ and a function $u = u(t, s, x)$ on Ω such that $Lu \in C^\omega(\Omega)$ but $u \notin C^\omega(\Omega)$. Let us now consider the function (recall that $m \leq N$)

$$\widetilde{u} = \widetilde{u}(t, s, x_1, \dots, x_N) := u(t, s, -m! x_m)$$

defined on the open subset of $\mathbb{R}^{2+N}$

$$\widetilde{\Omega} := \left\{ (t, s, x_1, \dots, x_N) \mid (t, s, -m!\, x_m) \in \Omega \right\}.$$

As a consequence, it holds

$$\Delta_{\mathbb{G}} \big(\widetilde{u}(t, s, x_1, \dots, x_N) \big) = (Lu)(t, s, -m!\, x_m) \text{ on } \widetilde{\Omega},$$

whence $\Delta_{\mathbb{G}} \widetilde{u} \in C^\omega(\widetilde{\Omega})$ (for $Lu \in C^\omega(\Omega)$) and $\widetilde{u} \notin C^\omega(\widetilde{\Omega})$ (for $u \notin C^\omega(\Omega)$). This proves that $\Delta_{\mathbb{G}}$ is not analytic-hypoelliptic. □

Example 5.10.6. Let $m, k \in \mathbb{N}$ be such that $0 \leq m \leq k$. Consider the operator on $\mathbb{R}^3$ (whose points are denoted by (x_1, x_2, x_3))

$$L = (\partial_{x_1})^2 + (x_1^m\, \partial_{x_2})^2 + (x_1^k\, \partial_{x_3})^2. \tag{5.85}$$

O.A. Oleĭnik and E.V. Radkevič [OR72] (see also [Him98]) proved that L is C^ω-hypoelliptic if and only if $m = k$. Our aim in this example is to "lift" (in a suitable sense) the vector fields on $\mathbb{R}^3$ appearing in (5.85)

$$\partial_{x_1}, \qquad x_1^m\, \partial_{x_2}, \qquad x_1^k\, \partial_{x_3} \tag{5.86}$$

to three vector fields (in a larger space, namely $\mathbb{R}^{3+m+k}$) generating a homogeneous Carnot group. It will easily follow (arguing as in the last paragraph of Example 5.10.5) that the relevant sub-Laplacian is not C^ω-hypoelliptic if L is not.

When $m = k = 0$, then $L = \Delta_{\mathbb{R}^3}$ is the ordinary Laplace operator on $\mathbb{R}^3$; when $m = 0$, $k = 1$, we obtain a non-analytic-hypoelliptic operator already considered in Example 5.10.2 (see also Baouendi–Goulaouic [BG72]). Let us now suppose that $k > m \geq 1$. We equip $\mathbb{R}^{3+m+k}$ with the following coordinates (the semicolon will denote different layers in a suitable homogeneous Carnot group structure)

$$P = (x_1, y_1, z_1; y_2, z_2; y_3, z_3; \dots, y_m, z_m; \; x_2, z_{m+1}; z_{m+2}; z_{m+3}; \dots; z_k; x_3).$$

We define a group of dilations by setting (recall that $k \geq m + 1$)

$$\begin{aligned}\delta_\lambda(P) = \; \big(&\lambda x_1, \lambda y_1, \lambda z_1;\\ &\lambda^2 y_2, \lambda^2 z_2; \lambda^3 y_3, \lambda^3 z_3; \dots; \lambda^m y_m, \lambda^m z_m;\\ &\lambda^{m+1} x_2, \lambda^{m+1} z_{m+1};\\ &\lambda^{m+2} z_{m+2}; \lambda^{m+3} z_{m+3}; \dots; \lambda^k z_k;\\ &\lambda^{k+1} x_3\big).\end{aligned}$$

Let us also consider on $\mathbb{R}^{3+m+k}$ the following vector fields

$$\begin{aligned}X &= \partial_{x_1},\\ Y &= \partial_{y_1} + x_1 \partial_{y_2} + x_1^2 \partial_{y_3} + \cdots + x_1^{m-1} \partial_{y_m} + x_1^m \partial_{x_2},\\ Z &= \partial_{z_1} + x_1 \partial_{z_2} + x_1^2 \partial_{z_3} + \cdots + x_1^{k-1} \partial_{z_k} + x_1^k \partial_{x_3}.\end{aligned}$$

It is then not difficult to see that X, Y, Z are δ_λ-homogeneous of degree one and they fulfill hypotheses **(H0)–(H1)–(H2)** of page 191. Hence, by the results of Section 4.2 (page 191), we can define a suitable homogeneous Carnot group structure on $\mathbb{R}^{3+m+k}$ such that

$$\Delta_{\mathbb{G}} = X^2 + Y^2 + Z^2$$

is the relevant canonical sub-Laplacian. Now, since

$$\Delta_{\mathbb{G}}(u(x_1, x_2, x_3)) = (Lu)(x_1, x_2, x_3)$$

for any smooth function u on $\mathbb{R}^{3+m+k}$ depending only on x_1, x_2, x_3, we can argue as in the last paragraph of Example 5.10.5 to infer that $\Delta_{\mathbb{G}}$ is not C^ω-hypoelliptic (since L is not). $\square$

5.11 Harmonic Approximation

Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. In this section, we give some conditions ensuring that a $\mathcal{L}$-harmonic function defined in a neighborhood of a compact set K contained in an open set Ω can be uniformly approximated on K by a sequence of $\mathcal{L}$-harmonic functions in Ω.

In some of the results of this section, we assume that $\mathcal{L}$ is *analytic-hypoelliptic* (see Section 5.10). $\Gamma = d^{2-Q}$, will denote its fundamental solution. To begin with, we prove the following lemma.

Lemma 5.11.1. *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$, and let $\Gamma = d^{2-Q}$ be its fundamental solution. Suppose also that $\mathcal{L}$ is analytic-hypoelliptic.*

Let $K \subseteq \mathbb{G}$ be compact. Then there exists $R = R(K, \mathbb{G}, \mathcal{L}) > 0$ such that, for every $z \in \mathbb{G}$ with $d(z) > R$, it holds

$$\Gamma(z^{-1} \circ y) = \sum_{\alpha \in (\mathbb{N} \cup \{0\})^N} C_\alpha(z) \cdot y^\alpha, \qquad C_\alpha(z) = \frac{1}{\alpha!} D^\alpha \Big|_{y=0} \big(\Gamma(z^{-1} \circ y)\big),$$

the series converging uniformly on the d-disc $\{y \in \mathbb{G} \, : \, d(y) < d(K)\}$, where $d(K) := \sup_{z \in K} d(z)$.

Proof. Let $\zeta \in \mathbb{G}$ be such that $d(\zeta) = 1$. Since $\eta \mapsto \Gamma(\zeta^{-1} \circ \eta)$ is analytic close to $\eta = 0$ and $\partial B_d(0, 1) = \{\zeta \in \mathbb{G} \, : \, d(\zeta) = 1\}$ is compact, there exists $\rho > 0$, independent of ζ, such that

$$\Gamma(\zeta^{-1} \circ \eta) = \sum_{\alpha \in (\mathbb{N} \cup \{0\})^N} a_\alpha(\zeta) \cdot \eta^\alpha, \qquad \text{uniformly on } B_d(0, \rho),$$

where

$$a_\alpha(\zeta) = \frac{1}{\alpha!} D^\alpha \Big|_{\eta=0} \big(\Gamma(\zeta^{-1} \circ \eta)\big).$$

Let us now choose $R > 0$ such that $R > d(K)/\rho$. If $z \in \mathbb{G}$, $d(z) > R$, then

$$\begin{aligned}
\Gamma(z^{-1} \circ y) &= d^{2-Q}(z) \, \Gamma\big((\delta_{d^{-1}(z)} z^{-1}) \circ \delta_{d^{-1}(z)}(y)\big) \\
&= d^{2-Q}(z) \sum_{\alpha \in (\mathbb{N} \cup \{0\})^N} a_\alpha\big(\delta_{d^{-1}(z)} z^{-1}\big) \cdot \big(\delta_{d^{-1}(z)}(y)\big)^\alpha \\
&=: \sum_{\alpha \in (\mathbb{N} \cup \{0\})^N} C_\alpha(z) \cdot y^\alpha.
\end{aligned}$$

The series converges uniformly for $y \in \mathbb{G}$ such that

$$d\big(\delta_{d^{-1}(z)}(y)\big) = \frac{d(y)}{d(z)} < \rho.$$

In particular, this holds for

$$y \in B_d(0, R\,\rho) \supseteq B_d(0, d(K)). \quad \square$$

The next lemma does not require the analytic-hypoellipticity of $\mathcal{L}$.

Lemma 5.11.2. *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$, and let $\Gamma = d^{2-Q}$ be its fundamental solution. Let u be an $\mathcal{L}$-harmonic function on the ball $B_d(0, r)$. Suppose*

$$u(x) = \sum_{k=1}^{\infty} u_k(x), \qquad x \in B_d(0, r),$$

where u_k is a continuous δ_λ-homogeneous function in the whole $\mathbb{G}$ of δ_λ-degree m_k, $k \in \mathbb{N}$. If the series is uniformly convergent on the compact subsets of $B_d(0, r)$ and $m_k \neq m_h$ for every $k \neq h$, then every u_k is $\mathcal{L}$-harmonic in $\mathbb{G}$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ with $\operatorname{supp}\varphi \subseteq B_d(0,1)$. Since u is $\mathcal{L}$-harmonic in $B_d(0,r)$, we have

$$0 = \int_{B_d(0,r)} u(x)\, \mathcal{L}\big(\varphi(\delta_{\lambda^{-1}}(x))\big)\, \mathrm{d}x \qquad \forall\, \lambda < r.$$

The homogeneity of $\mathcal{L}$ and the change of variable $y = \delta_{\lambda^{-1}}(x)$ give

$$\begin{aligned}
0 &= \lambda^{Q-2} \int_{B_d(0,r/\lambda)} u(\delta_\lambda(y))\, \mathcal{L}\varphi(y)\, \mathrm{d}y \\
&= \lambda^{Q-2} \sum_{k=1}^{\infty} \int_{B_d(0,r/\lambda)} u_k(\delta_\lambda(y))\, \mathcal{L}\varphi(y)\, \mathrm{d}y \\
&= \lambda^{Q-2} \sum_{k=1}^{\infty} \lambda^{m_k} \int_{B_d(0,r/\lambda)} u_k(y)\, \mathcal{L}\varphi(y)\, \mathrm{d}y \\
&= \lambda^{Q-2} \sum_{k=1}^{\infty} \lambda^{m_k} \int_{B_d(0,1)} u_k(y)\, \mathcal{L}\varphi(y)\, \mathrm{d}y.
\end{aligned}$$

Note that, in the last equality, we were able to replace $B_d(0, r/\lambda)$ by $B_d(0,1)$, since $\lambda < r$ and φ is supported in a compact set in $B_d(0,1)$. Then

$$\int_{B_d(0,1)} u_k(y)\, \mathcal{L}\varphi(y)\, \mathrm{d}y = 0 \qquad \forall\, \varphi \in C_0^\infty(B_d(0,1))\ \forall\, k \in \mathbb{N}.$$

This means that $\mathcal{L}u_k = 0$ in $B_d(0,1)$ in the weak sense of distributions. Since $\mathcal{L}$ is hypoelliptic, this implies the $\mathcal{L}$-harmonicity of u_k in $B_d(0,1)$, and so in $\mathbb{G}$, due to the δ_λ-homogeneity of u_k, $k \in \mathbb{N}$. □

We are now ready to prove the announced approximation theorem.

Theorem 5.11.3 (An approximation theorem). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Suppose that $\mathcal{L}$ is analytic-hypoelliptic.*

Let $\Omega \subseteq \mathbb{G}$ be an open set such that $\partial\Omega = \partial\overline{\Omega}$. Let $K \subseteq \Omega$ be compact and satisfy the following condition:

$$\text{if } \omega \text{ is a bounded connected component of } \Omega \setminus K, \text{ then } \partial\omega \subseteq \partial\Omega. \tag{5.87}$$

Then, for every function h which is $\mathcal{L}$-harmonic in a neighborhood of K, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{L}$-harmonic functions in Ω such that

$$\lim_{n\to\infty} h_n = h, \quad \text{uniformly on } K.$$

Proof. By general results from functional analysis, it is well known that it suffices to prove the following statement.[10]

[10] This is also known as "Caccioppoli's completeness method".

Let μ be a signed Radon measure supported on K and satisfying

$$\int_K h \,\mathrm{d}\mu = 0 \tag{5.88}$$

for every $\mathcal{L}$-harmonic function h in Ω. Then (5.88) holds for every function h which is $\mathcal{L}$-harmonic just in a neighborhood of K.

The crucial part of the proof is the following assertion.

Claim. If μ satisfies the previous hypotheses, then

$$u(x) := \int_K \Gamma(y^{-1} \circ x) \,\mathrm{d}\mu(y) = \int_K \Gamma(x^{-1} \circ y) \,\mathrm{d}\mu(y) \tag{5.89}$$

is identically zero in $\Omega \setminus K$.

We first show how to complete the proof of the theorem by using this claim. Let h be an $\mathcal{L}$-harmonic function in an open set $\Omega_0 \supseteq K$, $\Omega_0 \subseteq \Omega$. Choose a function $\phi \in C_0^\infty(\Omega_0)$ such that $\phi = 1$ in an open set $\Omega_1 \supseteq K$, $\overline{\Omega}_1 \subseteq \Omega_0$. Then $\phi\, h \in C_0^\infty(\Omega_0)$ and $\phi\, h = h$ in Ω_1. By the representation formula (5.16), we have

$$\begin{aligned}(\phi\, h)(y) &= -\int_{\Omega_0} \Gamma(x^{-1} \circ y)\mathcal{L}(\phi\, h)(x) \,\mathrm{d}x \\ &= -\int_{\Omega_0 \setminus \overline{\Omega}_1} \Gamma(x^{-1} \circ y)\, \mathcal{L}(\phi\, h)(x) \,\mathrm{d}x, \qquad y \in \Omega_0.\end{aligned}$$

It follows that

$$\begin{aligned}\int_K h(y) \,\mathrm{d}\mu(y) &= \int_K (\phi\, h)(y) \,\mathrm{d}\mu(y) \\ &= -\int_K \left(\int_{\Omega_0 \setminus \overline{\Omega}_1} \Gamma(x^{-1} \circ y)\, \mathcal{L}(\phi\, h)(x) \,\mathrm{d}x \right) \mathrm{d}\mu(y) =: (\star).\end{aligned}$$

Thus, by interchanging the integrals and keeping in mind (5.89), we infer

$$(\star) = -\int_{\Omega_0 \setminus \overline{\Omega}_1} \mathcal{L}(\phi\, h)(x) u(x) \,\mathrm{d}x = 0,$$

since $u = 0$ in $\Omega \setminus K \supseteq \Omega_0 \setminus \overline{\Omega}_1$.

Thus, we are left with the proof of the *Claim*. Let ω be a connected component of $\Omega \setminus K$. We have to prove that $u \equiv 0$ in ω.

We first suppose that ω is bounded. Then $\emptyset \neq \partial\omega \subseteq \partial\Omega$. In particular, this implies that $\partial\Omega \neq \emptyset$, hence $\overline{\Omega} \neq \mathbb{G}$ for $\partial\overline{\Omega} = \partial\Omega$. For every $x_0 \in \mathbb{G} \setminus \overline{\Omega}$, the function $y \mapsto \Gamma(x_0^{-1} \circ y)$ is $\mathcal{L}$-harmonic in Ω. Therefore, by the assumption (5.88),

$$u(x_0) = \int_K \Gamma(y^{-1} \circ x_0) \,\mathrm{d}\mu(y) = 0.$$

This proves that $u \equiv 0$ in $\mathbb{G} \setminus \overline{\Omega}$. As a consequence,

$$D^\alpha u(x) = 0 \qquad \text{for every multi-index } \alpha \tag{5.90}$$

and for every $x \in \mathbb{G} \setminus \overline{\Omega}$. Since $u \in C^\infty(\mathbb{G} \setminus K)$, it follows that (5.90) also holds at any point $x \in \partial\overline{\Omega} = \partial\Omega$. In particular, since $\partial\omega \subseteq \partial\Omega$, (5.90) holds at some point $x \in \partial\omega$. Then $\omega \equiv 0$ in a neighborhood of x, since u is $\mathcal{L}$-harmonic, hence real analytic, close to x. The connectedness of ω and again the analyticity of u imply that $u \equiv 0$ in ω.

Let us now assume that ω is unbounded. By Lemma 5.11.1, for every $x \in \omega$ with $d(x)$ sufficiently large, we have

$$\Gamma(x^{-1} \circ y) = \sum_{\alpha \in (\mathbb{N} \cup \{0\})^N} C_\alpha(x) \cdot y^\alpha, \qquad \text{uniformly in } B_d(0, d(K)).$$

Then

$$u(x) = \int_K \Gamma(x^{-1} \circ y)\, \mathrm{d}\mu(y) = \sum_{m=0}^{\infty} \int_K u_m(x, y)\, \mathrm{d}\mu(y),$$

where

$$u_m(x, y) = \sum_{|\alpha|_G = k} C_\alpha(x) \cdot y^\alpha.$$

The function u_m is δ_λ-homogeneous of degree m and the series

$$\sum_{m=0}^{\infty} u_m(x, \cdot)$$

is uniformly convergent on $B_d(0, d(K))$ to the $\mathcal{L}$-harmonic function $y \mapsto \Gamma(x^{-1} \circ y)$. Then, by Lemma 5.11.2, $u_m(x, \cdot)$ is $\mathcal{L}$-harmonic in $\mathbb{G}$, so that, by the assumption (5.88),

$$\int_K u_m(x, y)\, \mathrm{d}\mu(y) = 0 \qquad \forall\, m \geq 0.$$

Thus we have proved that $u(x) = 0$ for every $x \in \omega$, with $d(x)$ sufficiently large. Since ω is connected and u is analytic in ω, this implies $u \equiv 0$ in ω and completes the proof of the theorem. $\square$

5.12 An Integral Representation Formula for the Fundamental Solution on Step-two Carnot Groups

The aim of this section is to state a remarkable result in the paper [BGG96] by R. Beals, B. Gaveau and P. Greiner. This result provides a somewhat explicit integral representation formula of the fundamental solution of the canonical sub-Laplacian on a *general Carnot group of step two*.

We shall see in Section 16.3 (page 637 in Part III) that, given a Carnot group $\mathbb{G}_1$ and an arbitrary sub-Laplacian $\mathcal{L}$ on $\mathbb{G}_1$, there exists a Carnot group $\mathbb{G}_2$ isomorphic to $\mathbb{G}_1$ such that $\mathcal{L}$ corresponds (via the related isomorphism in the relevant Lie algebras) to the canonical sub-Laplacian $\Delta_{\mathbb{G}_2}$ on $\mathbb{G}_2$. Moreover, if $\mathbb{G}_2$ (whence $\mathbb{G}_1$) has step two, we saw in Proposition 3.5.1 (page 168) that we can perform another Lie-group isomorphism sending $\mathbb{G}_2$ into the homogeneous Carnot group $\mathbb{G}_3$ such that the composition law on $\mathbb{G}_3$ is given by (we follow our usual notation)

$$(x,t)\circ(\xi,\tau)=\left(x+\xi, t_1+\tau_1+\frac{1}{2}\langle B^{(1)}x,\xi\rangle,\dots,t_n+\tau_n+\frac{1}{2}\langle B^{(n)}x,\xi\rangle\right) \quad (5.91)$$

and the matrices $B^{(i)}$'s are skew-symmetric. This last isomorphism also sends the canonical sub-Laplacian of $\mathbb{G}_2$ into the canonical sub-Laplacian of $\mathbb{G}_3$. Now, the cited result in [BGG96] furnishes an integral formula for the *canonical* sub-Laplacian on a homogeneous Carnot group of step two whose composition law $\circ$ has precisely the above form and the matrices $B^{(i)}$'s are *skew-symmetric*. Our argument above shows that (and how) we can obtain a representation formula for the fundamental solution of any (not necessarily canonical) sub-Laplacian on any homogeneous Carnot group of step two.

We now state the remarkable result in [BGG96]. We explicitly remark that in [BGG96] the formalism of complex Hamiltonian mechanics is followed: we slightly change the therein notation.

Theorem 5.12.1 (Beals–Gaveau–Greiner, [BGG96]). *Let $\mathbb{G}=\mathbb{R}^{m+n}$ (whose points are denoted by (x,t), $x\in\mathbb{R}^m$, $t\in\mathbb{R}^n$) be equipped with a homogeneous Carnot group structure by the dilation $\delta_\lambda(x,t)=(\lambda x,\lambda^2 t)$ and the composition law in* (5.91), *where the $B^{(k)}$'s are n skew-symmetric linearly independent matrices of order $m\times m$. Consider the canonical sub-Laplacian $\Delta_{\mathbb{G}}=\sum_{i=1}^n X_i^2$, where*

$$X_i=\partial/\partial x_i+\frac{1}{2}\sum_{k=1}^n\left(\sum_{l=1}^m b_{i,l}^{(k)}x_l\right)\partial/\partial t_k,\qquad i=1,\dots,m$$

(here $b_{i,l}^{(k)}$ denotes the entry of position (i,l) of $B^{(k)}$). Then, for every (x,t) with $x\neq 0$, the fundamental solution Γ of $\Delta_{\mathbb{G}}$ is given by

$$\Gamma(x,t)=c_Q\int_{\mathbb{R}^n}\frac{\sqrt{\det(\mathbf{V}(B(\tau)))}}{\left(\frac{1}{2}\langle\mathbf{W}(B(\tau))\cdot x,x\rangle-\iota\,\langle t,\tau\rangle\right)^{Q/2-1}}\,d\tau, \qquad (5.92)$$

where ι is the imaginary unit of $\mathbb{C}$ and c_Q is the dimensional constant

$$c_Q=\frac{\mathbf{\Gamma}(\frac{Q}{2}-1)}{2\,(2\pi)^{Q/2}}. \qquad (5.93)$$

Here we used the following notation: $Q=m+2n$ is the homogeneous dimension of $\mathbb{G}$, $\mathbf{\Gamma}$ in (5.93) *is Euler's Gamma function, $\tau=(\tau_1,\dots,\tau_n)\in\mathbb{R}^n$,*

$$B(\tau)=\frac{1}{2}\,(\tau_1\,B^{(1)}+\dots+\tau_n\,B^{(n)}),$$

V *and* **W** *are the real-analytic functions prolonging* $z/\sin(z)$ *and* $z/\tan(z)$, *respectively, at* $z = 0$, *i.e.*

$$\mathbf{V}(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j}, \qquad \mathbf{W}(z) = \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^{2j}\ B_{2j}}{(2j)!} z^{2j}$$

(*here the* B_{2j}*'s are the Bernoulli numbers*). *Moreover, for every* $t \in \mathbb{R}^n \setminus \{0\}$, *we have*

$$\Gamma(0,t) = \lim_{0 \neq x \to 0} \Gamma(x,t).$$

We remark that a general integral formula for Γ is provided in [BGG96] comprising the case $x = 0$ too, by shifting the contour $\mathbb{R}^n$ into the complex domain $\mathbb{C}^n$ (see [BGG96, Theorem 3, page 315]).

The δ_λ-homogeneity of Γ in (5.92) (of degree $2 - Q$) should be noted.

As we saw in Example 5.10.3 (page 283), formula (5.92) can, in some cases, give explicit fundamental solutions. This can be done by using the fact that, if $\lambda_1(\tau), \dots, \lambda_m(\tau)$ and $v_1(\tau), \dots, v_m(\tau)$ denote the eigenvalues and corresponding eigenvectors of the matrix $B(\tau)$ (over the complex field), normalized in such a way that $|v_j(\tau)| = 1$ for $j = 1, \dots, m$, we have

$$\det\big(\mathbf{V}(B(\tau))\big) = \prod_{j=1}^{m} \frac{\lambda_j(\tau)}{\sin(\lambda_j(\tau))},$$

$$\big\langle \mathbf{W}(B(\tau)) \cdot x, x \big\rangle = \sum_{j=1}^{m} \frac{\lambda_j(\tau)}{\tan(\lambda_j(\tau))}\ \big|\langle x, v_j(\tau)\rangle\big|^2.$$

(In the last formula, the inner product is, of course, that of $\mathbb{C}^m$.)

5.13 Appendix A. Maximum Principles

In this section, we shall prove some *weak* and *strong* maximum principles for $\mathcal{L}$, an arbitrary sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$.

To begin with, we prove some elementary lemmas.

Lemma 5.13.1. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set and let* $u : \Omega \to \mathbb{R}$ *be an arbitrary function. Then there exists a point* $x_0 \in \overline{\Omega}$ *such that*

$$\limsup_{x \to x_0} u(x) = \sup_{\Omega} u. \tag{5.94}$$

Proof. We argue by contradiction and assume that (5.94) is false. Then, for every $x \in \overline{\Omega}$, there exists an open neighborhood V_x of x such that

$$\sup_{\Omega \cap V_x} u < \sup_{\Omega} u. \tag{5.95}$$

The family $\{V_x \,:\, x \in \overline{\Omega}\}$ is an open covering of $\overline{\Omega}$, so that, since $\overline{\Omega}$ is compact, we have

$$\overline{\Omega} \subseteq \bigcup_{j=1}^{p} V_{x_j}, \quad p \in \mathbb{N},$$

for suitable $x_1, \dots, x_p \in \overline{\Omega}$. Then

$$\sup_{\Omega} u = \max\left\{ \sup_{\Omega \cap V_{x_j}} u \,:\, j = 1, \dots, p \right\}. \tag{5.96}$$

On the other hand, by (5.95), the right-hand side of (5.96) is strictly less than $\sup_{\Omega} u$. This contradiction proves the lemma. □

Lemma 5.13.2. *Let A and B be $N \times N$ symmetric matrices with constant real entries. Assume $A \geq 0$ and $B \leq 0$. Then* $\mathrm{trace}(A \cdot B) \leq 0$.

Proof. Let $R := A^{1/2}$ be a symmetric square root of A. Then $\mathrm{trace}(A \cdot B) = \mathrm{trace}(R \cdot R \cdot B) = \mathrm{trace}(R \cdot B \cdot R) = \mathrm{trace}(R^T \cdot B \cdot R) \leq 0$, since $B \leq 0$. □

Lemma 5.13.3. *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let $\Omega \subseteq \mathbb{G}$ be an arbitrary open set, and let $u : \Omega \to \mathbb{R}$ be a C^2 real function. Assume that u has a local maximum at $x_0 \in \Omega$. Then*

$$\mathcal{L}u(x_0) \leq 0. \tag{5.97}$$

Proof. We know that $\mathcal{L} = \mathrm{div}(A \cdot \nabla^T)$, where A is a $N \times N$ symmetric matrix with polynomial entries and $A(x) \geq 0$ at any point $x \in \mathbb{R}^N$. Then

$$\mathcal{L} = \mathrm{trace}(A \cdot D^2 u) + \langle b, \nabla u \rangle, \tag{5.98}$$

where $D^2 u = (\partial_{x_i x_j})_{i,j \leq N}$ is the Hessian matrix of u and b is the vector-valued function whose j-th component is given by

$$b_j = \sum_{i=1}^{N} \partial_{x_i} a_{i,j}. \tag{5.99}$$

Since u has a local maximum at x_0, we have $\nabla u(x_0) = 0$ and $D^2 u(x_0) \leq 0$. Then, by Lemma 5.13.2,

$$\mathcal{L}u(x_0) = \mathrm{trace}(A(x_0) \cdot D^2 u(x_0)) \leq 0.$$

This ends the proof. □

We are now able to give a simple proof of the following weak maximum principle.

Theorem 5.13.4 (Weak maximum principle). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let Ω be a bounded open subset of $\mathbb{G}$. Let $u : \Omega \to \mathbb{R}$ be a C^2 function such that*

$$\begin{cases} \mathcal{L}u \geq 0 & \text{in } \Omega, \\ \limsup_{x \to y} u(x) \leq 0 & \text{for every } y \in \partial\Omega. \end{cases} \tag{5.100}$$

Then $u \leq 0$ in Ω.

Proof. We know that the matrix A in (5.98) has the following block form (see also (1.91), page 64)

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix},$$

where $A_{1,1} = (a_{i,j})_{i,j \leq m}$ is a constant $m \times m$ symmetric matrix strictly positive definite. Then $a_{1,1} > 0$. Let b_1 be given by (5.99) with $j = 1$. Define

$$\lambda := 2 \sup_{x \in \Omega} \frac{b_1(x)}{a_{1,1}}, \qquad M := \sup_{(x_1,\dots,x_N) \in \Omega} \exp(\lambda\, x_1),$$

and

$$h(x) = h(x_1, \dots, x_N) := M - \exp(\lambda\, x_1).$$

A trivial computation shows that

$$h(x) \geq 0 \quad \text{and} \quad \mathcal{L}h(x) < 0 \text{ for every } x \in \Omega. \tag{5.101}$$

For an arbitrary $\varepsilon > 0$, let us now consider the function $u_\varepsilon := u - \varepsilon\, h$. Due to inequalities (5.101) and condition (5.100), we have

$$\mathcal{L}u_\varepsilon > 0 \text{ in } \Omega \quad \text{and} \quad \limsup_{x \to y} u_\varepsilon(x) \leq 0 \text{ for every } y \in \partial\Omega. \tag{5.102}$$

By Lemma 5.13.1, there exists a point $x_0 \in \overline{\Omega}$ such that

$$\limsup_{x \to x_0} u_\varepsilon(x) = \sup_\Omega u_\varepsilon. \tag{5.103}$$

We want to show that $x_0 \in \partial\Omega$. Arguing by contradiction, we assume $x_0 \in \Omega$. Then, by the continuity of u in Ω,

$$u_\varepsilon(x_0) = \limsup_{x \to x_0} u_\varepsilon(x),$$

so that, by (5.103), $u_\varepsilon(x_0) = \max_\Omega u_\varepsilon$. As a consequence, by Lemma 5.13.3, $\mathcal{L}u_\varepsilon(x_0) \leq 0$. This contradicts the first inequality in (5.102). Thus $x_0 \in \partial\Omega$. Then, by (5.103) and the second condition in (5.102),

$$\sup_\Omega u_\varepsilon = \limsup_{x \to x_0} u_\varepsilon(x) \leq 0.$$

Hence, $u - \varepsilon\, h = u_\varepsilon \leq 0$ in Ω for every $\varepsilon > 0$. Letting ε tend to zero, we obtain $u \leq 0$ in Ω. The theorem is thus completely proved. $\square$

Note 5.13.5. The previous proof can be applied also to *continuous* functions satisfying the inequality $\mathcal{L}u \geq 0$ in the asymptotic sense of Exercises 8 and 9 at the end of the chapter (see also Ex. 10).

Corollary 5.13.6. *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let Ω be an unbounded open subset of $\mathbb{G}$. Let $u : \Omega \to \mathbb{R}$ be a C^2 function such that*

$$\begin{cases} \mathcal{L}u \geq 0 & \textit{in } \Omega, \\ \limsup_{x \to y} u(x) \leq 0 & \textit{for every } y \in \partial\Omega, \\ \limsup_{|x| \to \infty} u(x) \leq 0. \end{cases} \tag{5.104}$$

Then $u \leq 0$ in Ω.

Proof. Let $\varepsilon > 0$ be arbitrary but fixed. The third condition in (5.104) implies the existence of a real positive constant R such that

$$u(x) - \varepsilon < 0 \quad \text{in } \Omega \setminus \Omega_R, \tag{5.105}$$

where $\Omega_R := \{x \in \Omega \,:\, |x| < R\}$. It follows that

$$\begin{cases} \mathcal{L}(u - \varepsilon) = \mathcal{L}u \geq 0 & \text{in } \Omega_R, \\ \limsup_{x \to y} u(x) \leq 0 & \text{for every } y \in \partial\Omega_R. \end{cases}$$

Then, by Theorem 5.13.4, $u - \varepsilon \leq 0$ in Ω_R. This inequality, together with (5.105), gives $u \leq \varepsilon$ in Ω for every $\varepsilon > 0$. Hence $u \leq 0$ in Ω. □

A particular case of Corollary 5.13.6 is the following one.

Corollary 5.13.7. *If $\mathcal{L}$ is as in Corollary* 5.13.6, *the only entire $\mathcal{L}$-harmonic function vanishing at infinity is the null function.*

Proof. Let $u : \mathbb{G} \to \mathbb{R}$ be an entire $\mathcal{L}$-harmonic function vanishing at infinity, i.e. $u \in C^\infty(\mathbb{G}, \mathbb{R})$ satisfies

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \mathbb{G}, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

Then, by applying Corollary 5.13.6 both to u and $-u$, we get $u \equiv 0$. □

The rest of this section is devoted to the proof of the following strong maximum principle.

Theorem 5.13.8 (Strong maximum principle). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let Ω be a connected open subset of $\mathbb{G}$. Let $u : \Omega \to \mathbb{R}$ be a C^2 function such that*

$$u \leq 0 \quad \textit{and} \quad \mathcal{L}u \geq 0 \quad \textit{in } \Omega. \tag{5.106}$$

Suppose there exists a point $x_0 \in \Omega$ such that $u(x_0) = 0$. Then $u(x) = 0$ for every $x \in \Omega$.

The proof of this theorem requires several preliminary results. In what follows, we shall denote by $|\cdot|$ the standard Euclidean norm and by $D(z, r)$ the ball

$$D(z, r) := \{x \in \mathbb{R}^N \, : \, |x - z| < r\}.$$

Definition 5.13.9. *Let F be a relatively closed subset of Ω. We say that a vector $\nu \in \mathbb{R}^N \setminus \{0\}$ is* orthogonal to F at a point $y \in \Omega \cap \partial F$ *if*

$$\overline{D(y + \nu, |\nu|)} \subseteq (\Omega \setminus F) \cup \{y\}. \tag{5.107}$$

If this inclusion holds, we shall write $\nu \perp F$ at y. We also put

$$F^* := \big\{y \in \Omega \cap \partial F \,|\, \text{there exists } \nu\text{: } \nu \perp F \text{ at } y\big\}.$$

With the above notation, we explicitly remark that $F^* \neq \emptyset$ if $F \subset \Omega$, $F \neq \Omega$. Indeed, since Ω is connected, $\Omega \cap \partial F$ is not empty. Take a point $z \in \Omega \cap \partial F$, a ball $D(z, R) \subseteq \Omega$ and a point $x_0 \in D(z, R/2)$. Let $y \in \Omega \cap \partial F$ be such that

$$r := |x_0 - y| = \operatorname{dist}(x_0, \partial F).$$

Then $y \in F^*$ and $\nu := \frac{r}{2}\,(x_0 - y) \perp F$ at y.

The following Hopf-type lemma will be crucial for the proof of the strong maximum principle in Theorem 5.13.8.

Lemma 5.13.10 (A Hopf-type lemma for sub-Laplacians). *Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let $\Omega \subseteq \mathbb{G}$ be open, and let $u : \Omega \to \mathbb{R}$ be a C^2 function satisfying the inequalities in* (5.106). *Let*

$$F := \{x \in \Omega \, : \, u(x) = 0\}. \tag{5.108}$$

Assume $\emptyset \neq F \neq \Omega$. Then, for every $y \in F^*$ and $\nu \perp F$ at y, we have

$$q_{\mathcal{L}}(y, \nu) = 0, \tag{5.109}$$

where $q_{\mathcal{L}}(x, \xi) := \langle A(x) \cdot \xi, \xi \rangle$ is the characteristic form of $\mathcal{L}$ defined in (5.1a).

Proof. Let $y \in F^*$ and $\nu \perp F$ at y. Then

$$\overline{D(y + \nu, |\nu|)} \subseteq (\Omega \setminus F) \cup \{y\}.$$

Since $F^* \subseteq F \cap \Omega$, y is a maximum point for u (see (5.108)). Then $\nabla u(y) = 0$. We now argue by contradiction assuming that (5.109) is false. Hence

$$q_{\mathcal{L}}(y, \nu) > 0.$$

Let us now consider the function

$$h(x) := \exp(-\lambda\, |x - z|^2) - \exp(-\lambda\, r^2),$$

where $z = y + \nu$ and $r = |\nu|$. The positive constant λ will be fixed later on. A direct and easy computation shows that

$$\mathcal{L}h(y) = 4\lambda^2 \exp(-\lambda\, r^2) \cdot \big(q_{\mathcal{L}}(y, \nu) + \mathcal{O}(1/\lambda)\big),$$

as $\lambda \to \infty$. Then, we can choose and fix $\lambda > 0$ in such a way that $\mathcal{L}h > 0$ in a suitable neighborhood V of y. Obviously, we may assume $\overline{V} \subset \Omega$. Let us now consider the bounded open set

$$U := V \cap D(z, r).$$

Note that $\partial U = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = V \cap \partial D(z, r) \text{ and } \Gamma_2 = \overline{D(z, r)} \cap \partial V.$$

Since Γ_2 is a compact subset of $\Omega \setminus F$ and $u < 0$ in $\Omega \setminus F$, there exists $\varepsilon > 0$ such that $u + \varepsilon\, h < 0$ in Γ_2. On the other hand, being $h = 0$ on $\partial D(z, r)$ and $u \le 0$ in Ω, we have $u + \varepsilon\, h \le 0$ on Γ_1. Then, since $\mathcal{L}(u + \varepsilon\, h) \ge \varepsilon\, \mathcal{L}h \ge 0$ in U, from the maximum principle of Theorem 5.13.4, we obtain $u + \varepsilon\, h \le 0$ in U. Since $u(y) = h(y) = 0$, this implies

$$\frac{u(y + t\,\nu) - u(y)}{t} \le -\varepsilon\, \frac{h(y + t\,\nu) - h(y)}{t} \quad \text{for } 0 < t < 1. \tag{5.110}$$

Letting t tend to zero in this inequality, we get

$$\langle \nabla u(y), \nu \rangle \le -\varepsilon\, \langle \nabla h(y), \nu \rangle = -2\varepsilon\, \exp(-\lambda\, r^2)\, r^2.$$

This contradicts the condition $\nabla u(y) = 0$ and completes the proof. □

Corollary 5.13.11. *Let the hypotheses and notation of the previous lemma hold. Let also* $\mathcal{L} = \sum_{j=1}^m X_j^2$. *Then we have*

$$\langle X_j I(y), \nu \rangle = 0 \quad \forall\, y \in F^* \;\forall\, \nu \perp F \text{ at } y$$

and for every $j = 1, \ldots, m$.

Proof. It follows from the previous lemma, by just noticing that

$$q_{\mathcal{L}}(x, \xi) = \sum_{j=1}^m \langle X_j I(y), \xi \rangle^2. \quad \square$$

Another crucial definition is the following one.

Definition 5.13.12 ((Positively) invariant set w.r.t. a vector field). *Let* $X \in T(\mathbb{R}^N)$ *be a smooth vector field in* $\mathbb{R}^N$, *and let* F *be a relatively closed subset of* Ω. *We say that* F *is* positively X-invariant *if, for any integral curve* γ *of* X, $\gamma : [0, T] \to \Omega$ *such that* $\gamma(0) \in F$, *we have* $\gamma(t) \in F$ *for every* $t \in [0, T]$. *We say that* F *is* X-invariant *if it is positively* X*-invariant with respect to both* X *and* $-X$.

It is easy to verify that the condition

$$\langle XI(y), \nu\rangle \leq 0 \quad \forall\, y \in F^* \,\forall\, \nu \perp F \text{ at } y \tag{5.111}$$

is *necessary* for the positive X-invariance of F. Indeed, let $y \in F^*$, $\nu \perp F$ at y and $\gamma : [0, T] \to \Omega$ be an integral curve of X such that $\gamma(0) = y$. Let F be positively X-invariant. Since

$$\overline{D(y+\nu, |\nu|)} \subseteq (\Omega \setminus F) \cup \{y\},$$

we have

$$|\gamma(t) - (y+\nu)|^2 \geq |\nu|^2 \text{ and } |\gamma(0) - (y+\nu)|^2 = |\nu|^2$$

for every $t \in [0, T]$. This means that the real function $t \mapsto |\gamma(t) - (y+\nu)|^2$ has a minimum at $t = 0$. As a consequence,

$$0 \leq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} |\gamma(t) - (y+\nu)|^2 = \langle \dot\gamma(0), \gamma(0) - (y+\nu)\rangle = \langle XI(y), -\nu\rangle.$$

Hence (5.111) holds. We will show that this condition is also *sufficient* for F to be positively X-invariant. To prove this statement, we need the following elementary lemma.

Lemma 5.13.13. *Let $g : [0, T] \to \mathbb{R}$ be a continuous function such that*

$$\limsup_{h\to 0^-} \frac{g(t+h) - g(t)}{h} \leq M \quad \forall\, t \in (0, T], \tag{5.112}$$

for a suitable $M \in \mathbb{R}$. Then

$$g(t) \leq g(0) + M\,t \quad \forall\, t \in [0, T].$$

Proof. Let $\varepsilon > 0$ be fixed. Condition (5.112) implies that the real function

$$t \mapsto g(t) - g(0) - (M+\varepsilon)t$$

has a maximum at $t = 0$. Indeed, suppose to the contrary that there exist $\varepsilon_0 > 0$ and $t_0 \in (0, T]$ such that

$$g(t) - g(0) - (M+\varepsilon)\,t \leq g(t_0) - g(0) - (M+\varepsilon_0)\,t_0 \quad \forall\, t \in [0, T].$$

In particular, for $t = t_0 + h$ and $h < 0$ small enough, this gives

$$\frac{g(t_0+h) - g(t_0)}{h} \geq (M+\varepsilon_0),$$

which contradicts the hypothesis. Then, $g(t) - g(0) - (M+\varepsilon)\,t \leq 0$ for every $t \in [0, T]$. Letting ε tend to zero, we obtain the assertion. □

Proposition 5.13.14 (Nagumo–Bony). *Let $X \in T(\mathbb{R}^N)$ be a smooth vector field in $\mathbb{R}^N$, and let F be a relatively closed subset of Ω. Then F is positively X-invariant if and only if*

$$\langle XI(y), \nu\rangle \leq 0 \quad \forall\, y \in F^* \,\forall\, \nu \perp F \textit{ at } y. \tag{5.113}$$

Proof. We only need to show the "if" part. Let $\gamma : [0, T] \to \Omega$ be an integral curve of X such that $x_0 := \gamma(0) \in F$. Define

$$\delta(t) := \mathrm{dist}(\gamma(t), F), \quad 0 \le t \le T.$$

We have to prove that $\delta(t) = 0$ for every $t \in [0, T]$.

Let V be a bounded neighborhood of x_0 containing $\gamma([0, T])$, and let

$$L := \sup_{x,z\in V,\ x\neq z} \frac{|XI(x) - XI(z)|}{|x - z|} \tag{5.114}$$

be the Lipschitz constant of X on V. We may suppose $L\,T < 1/2$ and $V = D(x_0, r)$ with $D(x_0, 2r) \subseteq \Omega$. We claim that

$$L(t) := \limsup_{h\to 0^-} \frac{\delta(t+h) - \delta(t)}{h} \le L\,\delta(t) \quad \forall\, t \in (0, T]. \tag{5.115}$$

If $\delta(t) = 0$, inequality (5.115) is trivial, since $h < 0$ and $\delta(t + h) \ge 0$. Suppose $\delta(t) > 0$ and choose a sequence $h_n \uparrow 0$ such that

$$L(t) = \lim_{n\to\infty} \frac{\delta(t+h_n) - \delta(t)}{h_n}.$$

Let us now denote $x := \gamma(t)$ and $x_n := \gamma(t + h_n)$. Since $\gamma([0, T]) \subset D(x_0, r)$ and $D(x_0, 2r) \subseteq \Omega$, for every $n \in \mathbb{N}$ there exists a point $z_n \in F \cap \Omega$ such that

$$|x_n - z_n| = \mathrm{dist}(x_n, F) = \delta(t + h_n).$$

Obviously, we may suppose that $z_n \to z \in F \cap \overline{D(x_0, r)}$, so that, since $x_n \to x$,

$$\begin{aligned} |x - z| &= \lim_{n\to\infty} |x_n - z| = \lim_{n\to\infty} \mathrm{dist}(x_n, F) \\ &= \mathrm{dist}(x, F) = \delta(t). \end{aligned}$$

Moreover,

$$\nu := \frac{1}{2}(x - z)\perp F \text{ at } z. \tag{5.116}$$

Then

$$\begin{aligned} \delta(t+h_n) - \delta(t) &= |x_n - z_n| - |x - z| \ge |x_n - z_n| - |x - z_n| \\ &\ge |x_n - x| \ge -\frac{\langle x_n - x, z_n - x\rangle}{|x - z_n|}. \end{aligned}$$

Hence

$$\begin{aligned} L(t) &\le \lim_{n\to\infty} \left\langle \frac{x - z_n}{|x - z_n|}, \frac{x_n - x}{h_n} \right\rangle = \left\langle \dot\gamma(t), \frac{x - z}{|x - z|} \right\rangle \\ &= \frac{2}{|x - z|}\, \langle XI(x), \nu\rangle \\ &= \frac{2}{|x - z|}\, \big(\langle XI(z) - XI(x), \nu\rangle + \langle XI(z), \nu\rangle\big). \end{aligned}$$

From (5.116) and (5.113), together with (5.114), we finally get

$$L(t) \le L\,|x - z| = L\,\delta(t).$$

This completes the proof of (5.115). This inequality, combined with Lemma 5.13.13, gives

$$\delta(t) \le \delta(t) - \delta(0) \le L\,T \sup_{[0,T]} \delta,$$

so that $\sup_{[0,T]} \delta \le 1/2 \cdot \sup_{[0,T]} \delta$. Hence $\delta \equiv 0$, and the proof is complete. $\square$

Corollary 5.13.15. *The closed set F is X-invariant if and only if*

$$\langle XI(y), \nu\rangle = 0, \quad \forall\, y \in F^* \quad \forall\, \nu \perp F \textit{ at } y.$$

Proof. It straightforwardly follows from Proposition 5.13.14 and Definition 5.13.12. $\square$

This corollary, together with Corollary 5.13.11, immediately gives the following result.

Corollary 5.13.16. *Let $\mathcal{L} = \sum_{j=1}^m X_j^2$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let $u : \Omega \to \mathbb{R}$ be a C^2 function satisfying the inequalities* (5.106). *Let $F := \{x \in \Omega : u(x) = 0\}$. Assume $\emptyset \ne F \ne \Omega$. Then F is invariant with respect to $X_1, \ldots, X_m$.*

Proposition 5.13.17. *Let F be a relatively closed subset of the open set $\Omega \subseteq \mathbb{R}^N$. Assume $\emptyset \ne F \ne \Omega$. Then*

$$\mathfrak{a} := \{X \in T(\mathbb{R}^N) : F \textit{ is } X\textit{-invariant}\}$$

is a Lie algebra of vector fields.

Proof. Let $X, Y \in \mathfrak{a}$, and let λ, μ be real constants. By Corollary 5.13.15, we have $\langle XI(y), \nu\rangle = \langle YI(y), \nu\rangle = 0$ for every $y \in F^*$ and for every $\nu \perp F$ at y. Then

$$\langle \lambda\, XI(y) + \mu\, YI(y), \nu\rangle = 0 \quad \forall\, y \in F^* \;\forall\, \nu \perp F \text{ at } y.$$

Hence $\mathfrak{a}$ is a linear space. The next lemma will complete the proof of the proposition. $\square$

Lemma 5.13.18. *In the notation of Proposition* 5.13.17, *if $X, Y \in \mathfrak{a}$, then* $[X, Y] \in \mathfrak{a}$.

Proof. Let $y \in F^*$ and $\nu \perp F$ at y. For every $t > 0$ define

$$\Gamma(t) := \big(\exp(-\sqrt{t}Y) \circ \exp(-\sqrt{t}X) \circ \exp(\sqrt{t}Y) \circ \exp(-\sqrt{t}X)\big)(y).$$

Here $\circ$ denotes the composition of maps, whereas "exp" denotes the exponential of a vector field as introduced in Definition 1.1.2, page 8.

Let $T > 0$ be such that $\Gamma(t) \in \Omega$ for $0 \le t \le T$. By using the Taylor expansion (1.7) of exp (on page 7) with $n = 2$, a direct computation gives

$$\begin{aligned}\Gamma(t) &= y + t\big(\mathcal{J}_{YI}(y) \cdot XI(y) - \mathcal{J}_{XI}(y) \cdot YI(y)\big) + o(t)\\ &= y + t[X, Y]I(y) + o(t), \quad \text{as } t \downarrow 0.\end{aligned}$$

Then

$$\lim_{t\downarrow 0} \frac{\Gamma(t) - y}{t} = [X, Y]I(y). \tag{5.117}$$

On the other hand, since F is X and Y invariant, $\Gamma(t) \in F$ for every $t \in [0, T]$. As a consequence, since $\overline{D(y + \nu, |\nu|)} \subseteq (\Omega \setminus F) \cup \{y\}$ and $\Gamma(0) = y$,

$$|\Gamma(t) - (y + \nu)|^2 \ge |\nu|^2 = |\Gamma(0) - (y + \nu)|^2.$$

Then, by using also (5.117), we have

$$0 \ge \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} |\Gamma(t) - (y + \nu)|^2 = 2\langle [X, Y]I(y), \nu\rangle.$$

Hence

$$\langle [X, Y]I(y), \nu\rangle \le 0 \quad \forall\, y \in F^* \;\forall\, \nu \perp F \text{ at } y.$$

By swapping X with Y, we also get $\langle [Y, X]I(y), \nu\rangle \le 0$, so that

$$\langle [X, Y]I(y), \nu\rangle = 0 \quad \forall\, y \in F^* \;\forall\, \nu \perp F \text{ at } y.$$

Then, by Corollary 5.13.15, $[X, Y] \in \mathfrak{a}$. $\square$

Finally, we are in the position to give the proof of Theorem 5.13.8.

Proof (of Theorem 5.13.8). Let us define

$$F := \{x \in \Omega \,:\, u(x) = 0\}.$$

By hypothesis, $x_0 \in F$. Then F is a non-empty relatively closed subset of Ω. We have to prove that $F = \Omega$. By contradiction, assume $F \ne \Omega$. Then, since Ω is connected, $F^* \ne \emptyset$. Let $y \in F^*$, and let $\nu \perp F$ at y. By Proposition 5.13.17,

$$\langle ZI(y), \nu\rangle = 0 \quad \forall\, Z \in \mathfrak{g}.$$

Since $\dim(\mathfrak{g}) = N$, this obviously implies $\nu = 0$. On the other hand, by the very definition of a vector orthogonal to F, we have $\nu \in \mathbb{R}^N \setminus \{0\}$. This contradiction completes the proof. $\square$

5.13.1 A Decomposition Theorem for $\mathcal{L}$-harmonic Functions

In this section, we give a decomposition theorem for $\mathcal{L}$-harmonic functions, resemblant to the decomposition of a holomorphic function on an annulus of $\mathbb{C}$ into the sum of the regular and singular parts from its Laurent expansion (for the classical case of the Laplace operator, see also S. Axler, P. Bourdon, W. Ramey [ABR92]). Our main tool is the maximum principle from the previous section (precisely, we use Corollary 5.13.7, page 296).

In the sequel, we assume $Q \geq 3$. Moreover, in the proof of Theorem 5.13.20, we adopt the following notation: $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is a homogeneous Carnot group, $\mathcal{L}$ is a sub-Laplacian on $\mathbb{G}$, $\Gamma = d^{2-Q}$ is the fundamental solution for $\mathcal{L}$ (see Proposition 5.4.2). If d is the above $\mathcal{L}$-gauge, A is any subset of $\mathbb{G}$ and $\lambda > 0$, we agree to set

$$A_\lambda := \{x \in \mathbb{G} \mid d\text{-dist}(x, A) < \lambda\},$$

where

$$d\text{-dist}(x, A) := \inf\{d(x^{-1} \circ a) \mid a \in A\}.$$

Moreover, we use the following simple lemma.

Lemma 5.13.19. *Let K be a compact subset of $\mathbb{G}$, and let f be bounded on K. Then the function*

$$F : \mathbb{G} \to \mathbb{R}, \quad F(x) := \int_K \Gamma(y^{-1} \circ x)\, f(y)\, \mathrm{d}y$$

is $\mathcal{L}$-harmonic on $\mathbb{G} \setminus K$ and vanishes at infinity. Moreover, if μ is a Radon measure on $\mathbb{R}^N$ with compact support K, the same is true for $G(x) := \int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x)\, \mathrm{d}\mu(y)$.

Proof. It easily follows by differentiation under the integral sign (recall also that Γ is locally integrable and vanishes at infinity). □

We are now ready to state and prove the following assertion.

Theorem 5.13.20 (The decomposition theorem). *Let the hypotheses in the incipit of this section hold. Let $K \subset \Omega \subseteq \mathbb{G}$, with K compact and Ω open. If u is $\mathcal{L}$-harmonic in $\Omega \setminus K$, then u has a decomposition of the form*

$$u = r + s,$$

where r is $\mathcal{L}$-harmonic in Ω and s is $\mathcal{L}$-harmonic in $\mathbb{G} \setminus K$. Furthermore, it can be assumed that s vanishes at infinity, and in this case the above decomposition is unique.

Proof. Suppose the theorem holds true whenever Ω is bounded. We show that it holds true even for an unbounded Ω. Indeed, let $u \in \mathcal{H}(\Omega \setminus K)$, where K is compact and Ω is an (unbounded) open set. Let $R > 0$ be such that $K \subset B_d(0, R)$. Set $\widetilde{\Omega} := \Omega \cap B_d(0, R)$. Then $K \subset \widetilde{\Omega}$ and, by our assumption, u can be decomposed as $u = \widetilde{r} + s$, where $\widetilde{r} \in \mathcal{H}(\widetilde{\Omega})$, $s \in \mathcal{H}(\mathbb{G} \setminus K)$ and $s \to 0$ at infinity. We consider the function $r := u - s$ on Ω. Then r is $\mathcal{L}$-harmonic in $\Omega \setminus K$ and extends $\mathcal{L}$-harmonically across K, since r coincides with $\widetilde{r}$ on $\widetilde{\Omega}$ (which is an open neighborhood of K). This ends the proof, since $u = r + s$, with $r \in \mathcal{H}(\Omega)$, $s \in \mathcal{H}(\mathbb{G} \setminus K)$ and $s \to 0$ at infinity.

We can then suppose that Ω is bounded. We fix the following notation (see also Fig. 5.1):

$$B(\lambda) := \text{closure of } (\partial\Omega)_\lambda, \quad C(\lambda) := \text{closure of } K_\lambda, \quad A(\lambda) := \Omega \setminus (B(\lambda) \cup C(\lambda)).$$

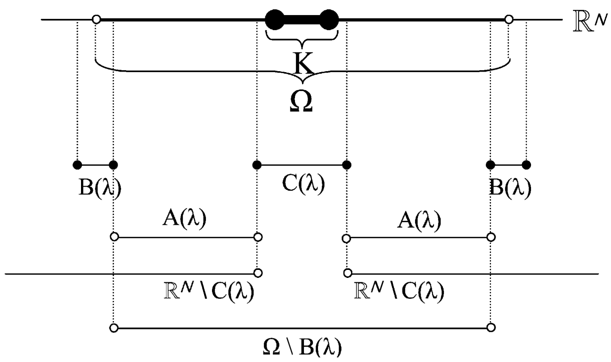

Fig. 5.1. Figure for the proof of Theorem 5.13.20

Since K is compact and $\partial\Omega$ is closed, we can choose $\lambda > 0$ small enough so that

$$B(\lambda) \cap C(\lambda) = \emptyset.$$

Since Ω is bounded, we can choose a cut-off function $\psi_\lambda \in C_0^\infty(\mathbb{R}^N)$ such that

$$\operatorname{supp}(\psi_\lambda) \subset \Omega \setminus K, \qquad \psi_\lambda \equiv 1 \text{ on } A(\lambda).$$

We consider the function $u\,\psi_\lambda$, and we agree to consider this function trivially prolonged on $\mathbb{R}^N$ to be zero. Hence this trivial prolongation belongs to $C_0^\infty(\mathbb{R}^N)$. By (5.16) in Theorem 5.3.3 (page 237) (being $\psi_\lambda \equiv 1$ on $A(\lambda)$),

$$\begin{aligned} u(x) = (u\,\psi_\lambda)(x) &= -\int_{\mathbb{R}^N} \Gamma(x^{-1} \circ y)\, \mathcal{L}(u\,\psi_\lambda)(y)\,\mathrm{d}y \\ &= -\left(\int_{A(\lambda)} + \int_{B(\lambda)} + \int_{C(\lambda)} + \int_{\mathbb{R}^N \setminus (A(\lambda) \cup B(\lambda) \cup C(\lambda))} \right) \\ &= -\left(\int_{B(\lambda)} + \int_{C(\lambda)} \right) \Gamma(y^{-1} \circ x)\, \mathcal{L}(u\,\psi_\lambda)(y)\,\mathrm{d}y \quad \forall\, x \in A(\lambda). \end{aligned} \tag{5.118}$$

In the last equality we used the fact that $\psi_\lambda \equiv 1$ on $A(\lambda)$ jointly with $\mathcal{L}u = 0$ on Ω, and the fact that ψ_λ is supported in Ω. We now set

$$r_\lambda(x) := -\int_{B(\lambda)} \Gamma(y^{-1} \circ x)\, \mathcal{L}(u\,\psi_\lambda)(y)\,\mathrm{d}y \qquad \text{for } x \in \Omega \setminus B(\lambda),$$

$$s_\lambda(x) := -\int_{C(\lambda)} \Gamma(y^{-1} \circ x)\, \mathcal{L}(u\,\psi_\lambda)(y)\,\mathrm{d}y \qquad \text{for } x \in \mathbb{G} \setminus C(\lambda).$$

Hence, (5.118) gives the decomposition

$$u(x) = r_\lambda(x) + s_\lambda(x) \qquad \forall\, x \in A(\lambda). \tag{5.119}$$

From Lemma 5.13.19, we infer that r_λ and s_λ are $\mathcal{L}$-harmonic on the respective sets of definition and that s_λ vanishes at infinity.

Let now $0 < \mu < \lambda$. Then obviously $A(\lambda) \subset A(\mu)$. From the decomposition in (5.119), we get an analogous decomposition

$$u(x) = r_\mu(x) + s_\mu(x) \qquad \forall\, x \in A(\mu). \tag{5.120}$$

We claim that the decompositions (5.119) and (5.120) are compatible, i.e.

$$\begin{aligned} r_\lambda(x) &= r_\mu(x) \qquad \forall\, x \in \Omega \setminus B(\lambda), \\ s_\lambda(x) &= s_\mu(x) \qquad \forall\, x \in \mathbb{G} \setminus C(\lambda). \end{aligned} \tag{5.121}$$

To prove the claim, we first remark that from (5.119) and (5.120) we obtain

$$r_\lambda(x) - r_\mu(x) = s_\mu(x) - s_\lambda(x) \qquad \forall\, x \in A(\lambda). \tag{5.122}$$

Now, let us consider the following function

$$S : \mathbb{G} \to \mathbb{R}, \qquad S(x) := \begin{cases} s_\mu(x) - s_\lambda(x) & \text{for every } x \in \mathbb{G} \setminus C(\lambda), \\ r_\lambda(x) - r_\mu(x) & \text{for every } x \in \Omega \setminus B(\lambda). \end{cases}$$

We claim that S has the following properties:

i) S is well-posed: indeed, thanks to (5.122), $s_\mu - s_\lambda$ coincides with $r_\lambda - r_\mu$ on the set
$$\{\mathbb{G} \setminus C(\lambda)\} \cap \{\Omega \setminus B(\lambda)\} = A(\lambda);$$
ii) S vanishes at infinity: indeed, this is true for s_μ and s_λ;
iii) S is $\mathcal{L}$-harmonic on $\mathbb{G}$: indeed, this is true for $s_\mu - s_\lambda$ on the open set $\mathbb{G} \setminus C(\lambda)$, and this is true for $r_\lambda - r_\mu$ on the open set $\Omega \setminus B(\lambda)$ (recall that $C(\mu) \subset C(\lambda)$ and $B(\mu) \subset B(\lambda)$).

Now, by the maximum principle (precisely, see Corollary 5.13.7, page 296) we infer that $S \equiv 0$, which is equivalent to the claimed (5.121). Now, let us fix a decreasing sequence of positive λ_n's such that $\lambda_n \to 0$ as $n \to \infty$. We set

$$\begin{aligned} r(x) &:= r_{\lambda_n}(x) \quad \forall\, x \in \Omega \quad (\text{where } n \in \mathbb{N} \text{ is such that } x \in \Omega \setminus B(\lambda_n)), \\ s(x) &:= s_{\lambda_n}(x) \quad \forall\, x \in \mathbb{G} \setminus K \quad (\text{where } n \in \mathbb{N} \text{ is such that } x \in \mathbb{G} \setminus C(\lambda_n)). \end{aligned}$$

Thanks to (5.121), the definition of $r(x)$ and $s(x)$ are unambiguous.[11] Now, the decomposition

$$u(x) = r(x) + s(x) \qquad \forall\, x \in \Omega \setminus K$$

[11] We are also using the trivial fact that $\Omega \setminus B(\lambda_n) \uparrow \Omega$ and $\mathbb{R}^N \setminus C(\lambda_n) \downarrow \mathbb{R}^N \setminus K$ as $n \to \infty$.

follows from the analogous decomposition (5.119) (and the fact that $A(\lambda) \uparrow \Omega \setminus K$ as $n \to \infty$). Moreover, it is easily seen that $r \in \mathcal{H}(\Omega)$, $s \in \mathcal{H}(\mathbb{G} \setminus K)$ and s vanishes at infinity. This gives the desired decomposition of u as in the assertion.

Finally, the uniqueness of the decomposition in the assertion (under the assumption that s vanishes at infinity) is another easy consequence of the same maximum principle quoted above. Indeed, suppose we are given two decompositions

$$r_2 + s_2 = u = r_1 + s_1 \qquad \text{on } \Omega \setminus K,$$

where

$$r_i \in \mathcal{H}(\Omega), \quad s_i \in \mathcal{H}(\mathbb{G} \setminus K), \quad \lim_{x \to \infty} s_i(x) = 0, \qquad i = 1, 2.$$

Then setting

$$S : \mathbb{G} \to \mathbb{R}, \qquad S(x) := \begin{cases} s_1(x) - s_2(x) & \text{for every } x \in \mathbb{G} \setminus K, \\ r_2(x) - r_1(x) & \text{for every } x \in \Omega, \end{cases}$$

we see that $S \in \mathcal{H}(\mathbb{G})$, S vanishes at infinity, and we end the proof following the same arguments as in the previous paragraph. □

5.14 Appendix B. The Improved Pseudo-triangle Inequality

Let $\mathbb{G} = (\mathbb{R}^N, \delta_\lambda, \circ)$ be a homogeneous Carnot group, and let d be a symmetric homogeneous norm on $\mathbb{G}$, *smooth* out of the origin.[12] For example, d could be an $\mathcal{L}$-gauge on $\mathbb{G}$ for some sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$.

We know from Proposition 5.1.7 (page 231) that (even without assumptions on smoothness or symmetry of d) d satisfies the pseudo-triangle inequality

$$d(a \circ b) \leq \mathbf{c}\,(d(a) + d(b)) \quad \text{for every } a,\ b \in \mathbb{G}.$$

Here $\mathbf{c} \geq 1$ is a constant depending on d and $\mathbb{G}$.

The aim of this brief appendix is to prove the following improvement of the pseudo-triangle inequality. For the following result, see also [DFGL05, (2.6)].

Proposition 5.14.1 (The improved pseudo-triangle inequality). *Let d be a symmetric homogeneous norm on the homogeneous Carnot group $\mathbb{G}$. Furthermore, suppose d is smooth out of the origin. Then there exists a constant $\beta \geq 1$ depending only on d and $\mathbb{G}$ such that*

$$d(y \circ x) \leq \beta\, d(y) + d(x) \qquad \textit{for every } x,\ y \in \mathbb{G}. \tag{5.123}$$

[12] In other words (see the definition at the beginning of Section 5.1, page 229), the present d has the following properties:
$d \in C(\mathbb{G}, [0, \infty)) \cap C^\infty(\mathbb{G} \setminus \{0\})$; $d(\delta_\lambda(x)) = \lambda\, d(x)$ for every $\lambda > 0$ and $x \in \mathbb{G}$; $d(x) = 0$ iff $x = 0$; $d(x^{-1}) = d(x)$ for every $x \in \mathbb{G}$.

Proof. Since (5.123) holds when $y = 0$, we can suppose $y \neq 0$. Since d is δ_λ-homogeneous of degree one, (5.123) is equivalent to

$$d(\delta_{1/d(y)}(y) \circ \delta_{1/d(y)}(x)) \le \beta + d(\delta_{1/d(y)}(x)). \tag{5.124a}$$

By using the symmetry of d and setting

$$\xi^{-1} = \delta_{1/d(y)}(y), \qquad \eta^{-1} = \delta_{1/d(y)}(x),$$

(5.124a) is equivalent to (note that $d(\delta_{1/d(y)}(y)) = 1$)

$$d(\eta \circ \xi) - d(\eta) \le \beta \qquad \text{for every } \xi,\ \eta \in \mathbb{G}\text{: } d(\xi) = 1. \tag{5.124b}$$

By the usual[13] pseudo-triangle inequality for d, (5.124b) holds when $\eta \in B_d(0, M)$, i.e. $d(\eta) \le M$ (where $M = M(d, \mathbb{G}) \gg 1$ will be chosen in the sequel). Indeed, if $\eta \in B_d(0, M)$, we have

$$d(\eta \circ \xi) - d(\eta) \le \mathbf{c}\,(d(\eta) + d(\xi)) - d(\eta) = (\mathbf{c} - 1)d(\eta) + \mathbf{c} \le (\mathbf{c} - 1)M + \mathbf{c} =: \beta.$$

We can hence suppose $\eta \notin B_d(0, M)$. Roughly speaking, we will show that we can drop η from (5.124b), by an argument of left-translation along curves which are supported away from zero, when M is large enough. Then, (5.124b) will follow from the classical mean value theorem.

We now make this precise. Set $Z := \mathrm{Log}\,(\xi) \in \mathfrak{g}$, where Log is the logarithmic map related to $\mathbb{G}$ and $\mathfrak{g}$ is the Lie algebra of $\mathbb{G}$. Consider the integral curve γ of Z starting from η, i.e. with our usual notation

$$\gamma(t) = \exp(tZ)(\eta) = \eta \circ \exp(tZ)(0) = \eta \circ \mathrm{Exp}\,(tZ).$$

Here we used Corollary 1.2.24 (page 24) and the definition of exponential map (see page 24). Obviously, we have $\gamma(0) = \eta$ and $\gamma(1) = \eta \circ \mathrm{Exp}\,(Z) = \eta \circ \mathrm{Exp}\,(\mathrm{Log}\,(\xi)) = \eta \circ \xi$. If we show that we can choose $M \gg 1$ such that

$$d(\gamma(t)) \ge 1 \qquad \text{for every } t \in [0, 1] \tag{5.124c}$$

(recall that γ depends on η, besides ξ), then $[0, 1] \ni t \mapsto u(t) := d(\gamma(t))$ is smooth (for d is smooth out of the origin by hypothesis) so that the classical Lagrange mean value theorem applies and gives

$$\begin{aligned} d(\eta \circ \xi) - d(\eta) &= u(1) - u(0) \le \sup_{t\in[0,1]} |\dot u(t)| = \sup_{t\in[0,1]} \left|\left\langle (\nabla d)(\gamma(t)), \dot\gamma(t)\right\rangle\right| \\ &= \sup_{t\in[0,1]} \left|\left\langle (\nabla d)(\gamma(t)), (ZI)(\gamma(t))\right\rangle\right| \\ &= \sup_{t\in[0,1]} \left|(Zd)(\gamma(t))\right|. \end{aligned} \tag{5.124d}$$

Let $X_1, \ldots, X_N$ be the Jacobian basis for the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. Hence,

[13] See Proposition 5.1.7-1, page 231.

$$Z = \mathrm{Log}\,(\xi) = \sum_{j=1}^{N} p_j(\xi)\, Z_j, \tag{5.124e}$$

where the p_j's are polynomials, so that there exists a constant C such that

$$\sup_{d(\xi)=1} |p_j(\xi)| \le C_1 \quad \text{for all } j = 1, \dots, N, \tag{5.124f}$$

since $\{\xi \in \mathbb{G} : d(\xi) = 1\}$ is a compact set (see, e.g. Proposition 5.1.4, page 230). Moreover, Z_j is δ_λ-homogeneous of degree $\sigma_j \ge 1$ (see Corollary 1.3.19, page 42), so that $Z_j d$ is δ_λ-homogeneous of degree $1 - \sigma_j \le 0$. Consequently, it is bounded on $\mathbb{G} \setminus B_d(0, 1)$, say

$$\sup_{d(\zeta)\ge 1} |(Z_j d)(\zeta)| \le C_2 \quad \text{for all } j = 1, \dots, N. \tag{5.124g}$$

We now use again the claimed (5.124c) and, by collecting together (5.124f) to (5.124g), we derive that (5.124d) yields

$$\begin{aligned} d(\eta \circ \xi) - d(\eta) &\le \sup_{d(\zeta)\ge 1} \left|(Zd)(\zeta)\right| \\ &= \sup_{d(\zeta)\ge 1} \left| \sum_{j=1}^{N} p_j(\xi)\, (Z_j d)(\zeta) \right| \le N C_1 C_2. \end{aligned}$$

This proves (5.124b). We are then left to prove (5.124c). From the pseudo-triangle inequality for d (see Proposition 5.1.7-2, page 231) we have

$$\begin{aligned} d(\gamma(t)) = d(\eta \circ \mathrm{Exp}\,(tZ)) &\ge \frac{1}{\mathbf{c}}\, d(\eta) - d(\mathrm{Exp}\,(tZ)) \\ &\ge \frac{1}{\mathbf{c}}\, M - \sup_{t\in[0,1],\, d(\xi)=1} d(\mathrm{Exp}\,(t\,\mathrm{Log}\,(\xi))) =: \frac{1}{\mathbf{c}}\, M - m(d, G), \end{aligned}$$

whence (5.124c) follows by choosing $M = \mathbf{c}(1 + m(d, \mathbb{G}))$. The finiteness of $m(d, \mathbb{G})$ follows from

$$\begin{aligned} d(\mathrm{Exp}\,(t\,\mathrm{Log}\,(\xi))) &= d\left(\mathrm{Exp}\left(\sum_{j=1}^{N} t p_j(\xi)\, Z_j\right)\right) \\ &\le \sup_{|\zeta_j|\le C_1} d\left(\mathrm{Exp}\left(\sum_{j=1}^{N} \zeta_j\, Z_j\right)\right) < \infty, \end{aligned}$$

uniformly for $t \in [0, 1]$, $d(\xi) = 1$. Here we used (5.124f) and the fact that

$$q(\zeta) := \mathrm{Exp}\left(\sum_{j=1}^{N} \zeta_j\, Z_j\right)$$

has polynomial coefficient functions (see (1.75a), page 50). This completes the proof. $\square$

Note that, β being the constant in Proposition 5.14.1, if we replace x, y in (5.123) by, respectively, $y^{-1} \circ z$ and $z^{-1} \circ x$, we get

$$d(y^{-1} \circ x) \le \beta\, d(y^{-1} \circ z) + d(z^{-1} \circ x) \qquad \text{for every } x,\ y,\ z \in \mathbb{G}. \tag{5.125a}$$

Moreover, by interchanging y and z in the above inequality (and using the symmetry of d) one gets

$$|d(y^{-1} \circ x) - d(z^{-1} \circ x)| \le \beta\, d(y^{-1} \circ z) \qquad \text{for every } x,\ y,\ z \in \mathbb{G}. \tag{5.125b}$$

5.15 Appendix C. Existence of Geodesics

Let $\mathbb{G} = (\mathbb{R}^N, \delta_\lambda, \circ)$ be a homogeneous Carnot group. Let

$$\mathfrak{g} = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(r)}$$

be a stratification of the Lie algebra of $\mathbb{G}$, as in Remark 1.4.8 (page 59). Let $X = \{X_1, \ldots, X_m\}$ be any basis of $W^{(1)}$. We consider the related Carnot–Carathéodory distance d_X. The aim of this section is to prove that the "inf" defining d_X in (5.6) on page 232 is actually a minimum.

In other words, fixed any $x, y \in \mathbb{G}$, we show the existence of a X-subunit curve $\gamma : [0, T] \to \mathbb{G}$ connecting x and y (i.e. $\gamma(0) = x$, $\gamma(T) = y$) such that $T = d_X(x, y)$. We shall call any such curve a *X-geodesic* (for x and y).

The existence of geodesics can be proved in many general cases (see [Bus55] and [HK00]). Our argument here will make crucial use of the δ_λ-homogeneity and left-invariant properties of the system X. The resulting proof will be quite simple (for a more general proof, see the note after Theorem 5.15.5).

First, we recall that, by Propositions 5.2.4, 5.2.6 and Theorem 5.2.8,

$$d_X(x, y) = d_0(y^{-1} \circ x) \quad \text{for every } x, y \in \mathbb{G}, \tag{5.126}$$

where

$$d_0(z) := d_X(z, 0), \quad z \in \mathbb{G}, \tag{5.127}$$

and d_0 is a homogeneous (symmetric) norm on $\mathbb{G}$. As usual, we denote the dilation of $\mathbb{G}$ by

$$\delta_\lambda(x) = \delta_\lambda(x_1, \ldots, x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N), \quad \lambda > 0,\ x \in \mathbb{G},$$

where $1 = \sigma_1 \le \cdots \le \sigma_N = r$ are consecutive integers and r is the step of nilpotency of $\mathbb{G}$. We are ready to prove the following result.

Proposition 5.15.1. *Let $\mathbb{G}$ be a homogeneous Carnot group, and let d be any homogeneous norm on $\mathbb{G}$.*

For every compact set $K \subset \mathbb{G}$, there exists a constant $\mathbf{c}_K > 0$ such that

$$(\mathbf{c}_K)^{-1} |x - y| \le d(y^{-1} \circ x) \le \mathbf{c}_K\, |x - y|^{1/r} \quad \forall\, x,\ y \in K, \tag{5.128}$$

where r is the step of $\mathbb{G}$ and $|\cdot|$ is the Euclidean norm on $\mathbb{G} \equiv \mathbb{R}^N$.

In particular, (5.128) *holds true if* $d(y^{-1} \circ x)$ *is replaced by* $d_X(x, y)$, *where* d_X *is the control distance related to any basis of generators* (*of the first layer of the stratification of the Lie algebra*) *of* $\mathbb{G}$.

(*Note*. More generally, the estimates in (5.128) also hold when $d = d_X$ (with a suitable r), where X is a system of smooth vector fields satisfying the Hörmander condition, see [Lan83,NSW85,VSC92,Gro96]. The first equality in (5.128) holds for a general Carnot–Carathéodory distance $d = d_X$, see [HK00] and Ex. 25 at the end of the chapter.)

Proof. Once (5.128) has been proved, the last assertion of the proposition follows from (5.126), by taking $d = d_0$ (d_0 as in (5.127)).

We then turn to prove (5.128). If $|\cdot|$ denotes the absolute value on $\mathbb{R}$, set

$$\varrho : \mathbb{G} \to [0, \infty), \quad \varrho(x) = \sum_{j=1}^{N} |x_j|^{1/\sigma_j}. \tag{5.129}$$

Obviously, ϱ is a homogeneous (symmetric) norm on $\mathbb{G}$. By the equivalence of all homogeneous norms on $\mathbb{G}$ (see Proposition 5.1.4), there exists a constant $\mathbf{c} = \mathbf{c}(\varrho, d, \mathbb{G}) \geq 1$ such that

$$\mathbf{c}^{-1}\, \varrho(x) \leq d(x) \leq \mathbf{c}\, \varrho(x) \quad \forall\, x \in \mathbb{G}. \tag{5.130}$$

Thus (5.128) will follow if we show that, given a compact set $K \subset \mathbb{G}$,

$$(\mathbf{c}_K)^{-1} |x - y| \leq \varrho(y^{-1} \circ x) \leq \mathbf{c}_K\, |x - y|^{1/r} \quad \forall\, x,\ y \in K, \tag{5.131}$$

for a suitable constant $\mathbf{c}_K > 0$.

Now, we recall the result in Corollary 1.3.18 (page 41): for every $j \in \{1, \ldots, N\}$ and every $x, y \in \mathbb{G}$, we have

$$(y^{-1} \circ x)_j = x_j - y_j + \sum_{k:\, \sigma_k < \sigma_j} P_k^{(j)}(x, y)\, (x_k - y_k),$$

where $P_k^{(j)}(x, y)$ is a polynomial function. Thus, the very definition of ϱ gives

$$\varrho(y^{-1} \circ x) = \sum_{j=1}^{N} \left| x_j - y_j + \sum_{k:\, \sigma_k < \sigma_j} P_k^{(j)}(x, y)\, (x_k - y_k) \right|^{1/\sigma_j}. \tag{5.132}$$

Since any $P_k^{(j)}$ is a continuous function, we immediately get from (5.132) the estimate from above: for every $x, y \in K$

$$\begin{aligned}\varrho(y^{-1} \circ x) &\leq \sum_{j=1}^{N} \Big(|x_j - y_j| + \mathbf{c} \textstyle\sum_{k:\, \sigma_k < \sigma_j} |x_k - y_k| \Big)^{1/\sigma_j} \\ &\leq \mathbf{c}' \sum_{j=1}^{N} \Big(\sum_{k:\, \sigma_k \leq \sigma_j} |x_k - y_k| \Big)^{1/\sigma_j} \leq \mathbf{c}' \sum_{j=1}^{N} \sum_{k:\, \sigma_k \leq \sigma_j} |x_k - y_k|^{1/\sigma_j}\end{aligned}$$

$$\leq \mathbf{c}'' \sum_{j=1}^{N} |x_j - y_j|^{1/\sigma_j} \leq \mathbf{c}''' \sum_{j=1}^{N} |x_j - y_j|^{1/\sigma_N}$$
$$\leq \mathbf{c}''' N \left(\sum_{j=1}^{N} |x_j - y_j| \right)^{1/\sigma_N} \leq \mathbf{c}''' N^{1+1/\sigma_N} \, |x - y|^{1/\sigma_N}.$$

This gives the estimate from above in (5.131), since $\sigma_N = r$. (Yet another proof of the estimate from above can be obtained by the Lagrange mean value theorem.[14])

The estimate from below is just a little more delicate. We fix the notation: for every $j = 2, \dots, N$, let n_j be the cardinality of the set $\{k : \sigma_k < \sigma_j\}$. Let α be a vector in $\mathbb{R}^n$ with $n = n_2 + \cdots + n_N$, and let us denote α in the following way

$$\alpha = \left(\alpha^{(2)}, \dots, \alpha^{(N)}\right) \quad \text{with} \quad \alpha^{(j)} = \left(\alpha_1^{(j)}, \dots, \alpha_{n_j}^{(j)}\right), \ j = 2, \dots, N.$$

With the notation in (5.132), if K is a compact subset of $\mathbb{G}$, there exists a constant $M \geq 1$ such that (recall that the $P_k^{(j)}$'s are polynomial functions)

$$\sup_{x,y \in K} |P_k^{(j)}(x, y)| \leq M \quad \forall \, j = 2, \dots, N \ \forall \, k : \ \sigma_k < \sigma_j.$$

As a consequence, (5.132) gives (here $\mathbf{M} = n\,M$)

$$\varrho(y^{-1} \circ x) = \sum_{j=1}^{N} \left| x_j - y_j + \sum_{k:\, \sigma_k < \sigma_j} P_k^{(j)}(x, y) \, (x_k - y_k) \right|^{1/\sigma_j}$$

---

[14] Indeed, fix any $y \in \mathbb{G}$ and set

$$f_y : \mathbb{R}^N \to \mathbb{R}, \quad f_y(x) := (y^{-1} \circ x)_j.$$

Given any $x, x_0 \in \mathbb{G}$, by the Lagrange mean value theorem, we have

$$(\star) \qquad |f_y(x) - f_y(x_0)| \leq \sup_{|\xi - x_0| \leq |x - x_0|} |\nabla f_y(\xi)| \, |x - x_0|.$$

Now, take $x_0 = y$ and observe that $f_y(x_0) = f_y(y) = 0$. Moreover, if x, y belong to a compact set K, and if $R_K \gg 1$ is such that $K \subseteq B(0, R_K)$, then

$$\sup_{|\xi - y| \leq |x - y|} |\nabla f_y(\xi)| \leq \sup_{|\xi| \leq R_K + \mathrm{diam}(K)} |\nabla f_y(\xi)| =: M_j < \infty,$$

for $f_y(\xi)$ is a polynomial function in ξ and y. Thus, $(\star)$ gives (set $M = \max_{j \leq N} M_j$)

$$|(y^{-1} \circ x)_j| \leq M \, |x - y| \qquad \forall \, x, y \in K \ \forall \, j \leq N.$$

Then, for every $x, y \in K$, we have

$$\varrho(y^{-1} \circ x) = \sum_{j=1}^{N} |(y^{-1} \circ x)_j|^{1/\sigma_j} \leq \mathbf{c} \textstyle\sum_{j=1}^{N} |x - y|^{1/\sigma_j} \leq \mathbf{c}' |x - y|^{1/r}.$$

$$\geq \inf_{|\alpha|\leq \mathbf{M}} \left\{ \sum_{j=1}^{N} \left| x_j - y_j + \sum_{k:\,\sigma_k<\sigma_j} \alpha_k^{(j)}\,(x_k - y_k) \right|^{1/\sigma_j} \right\} =: (\bigstar).$$

Being $x, y \in K$, we obviously have (whenever $\alpha \leq \mathbf{M}$)

$$\left| x_j - y_j + \sum_{k:\,\sigma_k<\sigma_j} \alpha_k^{(j)}\,(x_k - y_k) \right| \leq \mathbf{c}(K, \mathbf{M}) < \infty,$$

hence (notice that $1/\sigma_j \leq 1$)

$$\begin{aligned} (\bigstar) &\geq \frac{\mathbf{c}(K,\mathbf{M})^{1/r}}{\mathbf{c}(K,\mathbf{M})} \inf_{|\alpha|\leq \mathbf{M}} \left\{ \sum_{j=1}^{N} \left| x_j - y_j + \sum_{k:\,\sigma_k<\sigma_j} \alpha_k^{(j)}\,(x_k - y_k) \right| \right\} \\ &= \mathbf{c}'(K) \inf_{|\alpha|\leq \mathbf{M}} \left\{ f_\alpha(x-y) \right\} =: (\bigstar'), \end{aligned}$$

where

$$f_\alpha : \mathbb{R}^N \to \mathbb{R}, \quad f_\alpha(z) = \sum_{j=1}^{N} \left| z_j + \sum_{k:\,\sigma_k<\sigma_j} \alpha_k^{(j)}\, z_k \right|.$$

We explicitly remark that f_α is homogeneous of degree 1 with respect to the Euclidean dilations on $\mathbb{R}^N$. As a consequence (if $x \neq y$, otherwise there is nothing to prove),

$$f_\alpha(x-y) = |x-y|\, f_\alpha\!\left(\frac{x-y}{|x-y|} \right) \geq |x-y| \inf_{|\xi|=1} f_\alpha(\xi).$$

This gives

$$(\bigstar') \geq |x-y|\,\mathbf{c}'(K) \inf_{|\alpha|\leq \mathbf{M},\, |\xi|=1} f_\alpha(\xi) \geq |x-y|\,\mathbf{c}''(K).$$

Indeed, since $f_\alpha(\xi)$ is a continuous function in ξ and α, and (for every α) $f_\alpha(\cdot)$ is *positive outside the origin* (as a simple inductive argument shows), there exist α_0 and ξ_0 (with $|\alpha_0| \leq \mathbf{M}$ and $|\xi_0| = 1$) such that

$$\inf_{|\alpha|\leq \mathbf{M},\, |\xi|=1} f_\alpha(\xi) = f_{\alpha_0}(\xi_0) > 0.$$

This gives the lower estimate in (5.131) and completes the proof. □

From Proposition 5.15.1 we obtain useful corollaries, as we show below. In the sequel, we write d_E to denote the usual Euclidean metric on $\mathbb{G} \equiv \mathbb{R}^N$.

Corollary 5.15.2. *Let $\mathbb{G}$ be a homogeneous Carnot group. Let d_X be the control distance related to any basis X of generators* (*of the first layer of the stratification of the Lie algebra*) *of $\mathbb{G}$.*

Then $A \subseteq \mathbb{G}$ is bounded in the metric space $(\mathbb{G}, d_X)$ if and only if it is bounded in the Euclidean metric space $(\mathbb{G}, d_E)$. More precisely, there exists a constant $\mathbf{c} \geq 1$ *such that* (*for every* $R > 0$)

$$B_{d_X}(0, R) \subseteq B_{d_E}(0, \mathbf{R}_1(R)) \quad \textit{and} \quad B_{d_E}(0, R) \subseteq B_{d_X}(0, \mathbf{R}_2(R)), \tag{5.133}$$

where

$$\mathbf{R}_1(R) = \mathbf{c}\max\{R, R^r\}, \quad \mathbf{R}_2(R) = \mathbf{c}\max\{R, R^{1/r}\}.$$

(*Note*. The fact that a bounded set in $(\mathbb{R}^N, d_E)$ is also bounded in $(\mathbb{R}^N, d_X)$ holds for a d_X related to a general system of vector fields satisfying the Hörmander condition, see the note after Proposition 5.15.1. The reverse assertion may be false even in the Hörmander case, see Ex. 27.)

Proof. By (5.130), we know that there exists a constant $\mathbf{c} = \mathbf{c}(\varrho, d_X, \mathbb{G}) \geq 1$ such that

$$\mathbf{c}^{-1}\, \varrho(x) \leq d_X(x, 0) \leq \mathbf{c}\, \varrho(x) \quad \forall\, x \in \mathbb{G}, \tag{5.134}$$

where ϱ is as in (5.129). Now, it is obvious that a set $A \subseteq \mathbb{G}$ is bounded in $(\mathbb{G}, d_E)$ iff there exists $R > 0$ such that $A \subseteq B_\varrho(0, R)$. This remark jointly with (5.134) proves the first assertion of the corollary.

Precisely (see the definition of ϱ), it holds (for every $R > 0$)

$$B_{d_E}(0, R) \subseteq B_\varrho\big(0, N \max\{R, R^{1/r}\}\big), \quad B_\varrho(0, R) \subseteq B_{d_E}\big(0, N \max\{R, R^r\}\big).$$

These inclusions, together with (5.134), prove (5.133). □

Obviously, (5.133) also gives the inclusions

$$B_{d_X}(0, \mathbf{R}_3(R)) \subseteq B_{d_E}(0, R) \quad \text{and} \quad B_{d_E}(0, \mathbf{R}_4(R)) \subseteq B_{d_X}(0, R), \tag{5.135}$$

where

$$\mathbf{R}_3(R) = \mathbf{c}^{-1} \min\{R, R^{1/r}\}, \quad \mathbf{R}_1(R) = \mathbf{c}^{-r} \min\{R, R^r\}.$$

Corollary 5.15.3. *Let the hypothesis of Corollary* 5.15.2 *hold.*

Then the map

$$\mathrm{id} : (\mathbb{G}, d_X) \to (\mathbb{G}, d_E), \quad \mathrm{id}(x) = x$$

is a homeomorphism. As a consequence, the topologies of the metric spaces $(\mathbb{G}, d_X)$, $(\mathbb{G}, d_E)$ *coincide.*

(*Note*. The continuity of $\mathrm{id} : (\mathbb{G}, d_X) \to (\mathbb{G}, d_E)$ holds for a general Carnot–Carathéodory distance d_X, see Ex. 25 at the end of the chapter. For a general d_X, the reverse continuity may be false, see Ex. 26 (see also [BR96]). Instead, this reverse continuity holds for all smooth vector fields satisfying the Hörmander condition, see the note after Proposition 5.15.1 and Ex. 25.)

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{G}$. Suppose $x_n \to x$ in $(\mathbb{G}, d_E)$. Then there exists a compact set $K \subset \mathbb{R}^N$ such that $x, x_n \in K$ for every $n \in \mathbb{N}$. Hence, by the second inequality in (5.128), we derive

$$d_X(x, x_n) \le \mathbf{c}_K\, |x - x_n| \quad \forall n \in \mathbb{N},$$

so that, letting $n \to \infty$, $x_n \to x$ in $(\mathbb{G}, d_X)$ too.

Vice versa, suppose $x_n \to x$ in $(\mathbb{G}, d_X)$. Then the sequence $\{x_n\}_n$ is bounded in $(\mathbb{G}, d_X)$. Hence, by Corollary 5.15.2, it is also bounded in $(\mathbb{G}, d_E)$. Consequently, there exists a compact subset K of $\mathbb{R}^N$ such that $x, x_n \in K$ for every $n \in \mathbb{N}$. Hence, by the first inequality in (5.128), we derive

$$|x - x_n| \le \mathbf{c}_K\, d_X(x, x_n) \quad \forall n \in \mathbb{N},$$

so that, letting $n \to \infty$, $x_n \to x$ in $(\mathbb{G}, d_E)$ too.

We have thus proved that

$$x_n \xrightarrow{d_X} x \;\Rightarrow\; \mathrm{id}(x_n) \xrightarrow{d_E} x, \qquad x_n \xrightarrow{d_E} x \;\Rightarrow\; \mathrm{id}^{-1}(x_n) \xrightarrow{d_X} x.$$

As a consequence, by the well-known characterization of continuity (in a *metric* space) as *sequential*-continuity, we infer that $\mathrm{id} : (\mathbb{G}, d_X) \to (\mathbb{G}, d_E)$ is a homeomorphism. This implies that id and id^{-1} are open maps, i.e. $(\mathbb{G}, d_X)$ and $(\mathbb{G}, d_E)$ have the same open sets. □

By collecting together the above lemmas, we obtain the following result.

Proposition 5.15.4. *Let $\mathbb{G}$ be a homogeneous Carnot group. Let d_X be the control distance related to any basis X of generators* (*of the first layer of the stratification of the Lie algebra*) *of $\mathbb{G}$.*

Then, a subset of $\mathbb{G}$ is, respectively, open, closed, bounded or compact in the metric space $(\mathbb{G}, d_X)$ if and only if the same holds in the Euclidean metric space $\mathbb{G} \equiv \mathbb{R}^N$.

In particular, the compact subsets of $(\mathbb{G}, d_X)$ are precisely the closed and bounded subsets of $\mathbb{R}^N$ or, equivalently, the closed and bounded subsets of $(\mathbb{G}, d_X)$.

Proof. Let $A \subseteq \mathbb{G}$. Then, by Corollary 5.15.3, A is, respectively, open, closed or compact in the metric space $(\mathbb{G}, d_X)$ if and only if the same holds in $(\mathbb{G}, d_E)$.

By Corollary 5.15.2, A is bounded in $(\mathbb{G}, d_X)$ if and only if the same holds in $(\mathbb{G}, d_E)$. This also gives the last assertion of the proposition and completes the proof. □

We are in a position to prove the following result.

Theorem 5.15.5 (Existence of X-geodesics). *Let $\mathbb{G}$ be a homogeneous Carnot group. Let d_X be the control distance related to any basis X of generators* (*of the first layer of the stratification of the Lie algebra*) *of $\mathbb{G}$.*

*Then, for every $x, y \in \mathbb{G}$, there exists a X-*geodesic *connecting x and y, i.e. there exists a X-subunit path $\gamma : [0, T] \to \mathbb{G}$ such that $\gamma(0) = x$, $\gamma(T) = y$ and $T = d_X(x, y)$.*

(*Note*. Since we proved that the compact subsets of $(\mathbb{G}, d_X)$ are precisely the closed and bounded subsets of $(\mathbb{G}, d_X)$ (see Proposition 5.15.4) this result also follows from the general results in [Bus55, page 25].)

Proof. Fix $x, y \in \mathbb{G}$. By definition of d_X, there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of X-subunit paths

$$\gamma_n : [0, T_n] \to \mathbb{G}, \quad \gamma_n(0) = x,\ \gamma_n(T_n) = y, \quad T_n \searrow d_X(x, y).$$

By definition of X-subunit path, γ_n is an absolutely continuous curve such that, for every $n \in \mathbb{N}$, there exists $E_n \subseteq [0, T_n]$ such that $F_n := [0, T_n] \setminus E_n$ has vanishing Lebesgue measure and

$$\langle \dot\gamma_n(t), \xi\rangle^2 \le \sum_{j=1}^{m} \langle X_j I(\gamma_n(t)), \xi\rangle^2 \quad \forall\, \xi \in \mathbb{R}^N\ \forall\, t \in E_n. \tag{5.136}$$

Since $T_n \le T_{n+1}$ for all n, it is not restrictive to suppose that every γ_n is defined on $[0, T_1]$, by prolonging γ_n to be y on $(T_n, T_1]$. We still denote this prolongation by γ_n. Observe that the prolongation is still a X-subunit path (connecting x to y) because, for $t \in (T_n, T_1]$, γ_n is constant, whence the far left-hand side in (5.136) is 0. The resulting prolongation is also trivially absolutely continuous.

We claim that the family of functions $\mathcal{F} = \{\gamma_n\}_n$ on $[0, T_1]$ is uniformly bounded and equicontinuous.

• *$\mathcal{F}$ is equicontinuous*: Suppose we have proved that $\mathcal{F}$ is uniformly bounded, i.e.

$$\exists\, M > 0 : \quad \sup_{t\in[0,T_1]} |\gamma_n(t)| \le M \quad \forall\, n \in \mathbb{N}. \tag{5.137}$$

Then the equicontinuity of $\mathcal{F}$ follows. Indeed, for every $t, t' \in [0, T_1]$, it holds

$$\begin{aligned}
|\gamma_n(t) - \gamma_n(t')| &= \left| \int_{t'}^{t} \dot\gamma_n(s)\,\mathrm{d}s \right| \le \int_{t'}^{t} |\dot\gamma_n(s)|\,\mathrm{d}s \\
(\text{see the note}^{15}) &= \int_{t'}^{t} \sup_{\xi\in\mathbb{R}^N:\,|\xi|=1} \left|\langle \dot\gamma_n(s), \xi\rangle\right| \mathrm{d}s \\
(\text{by (5.136)}) &\le \int_{t'}^{t} \sup_{\xi\in\mathbb{R}^N:\,|\xi|=1} \left(\sum_{j=1}^{m} \langle X_j I(\gamma_n(s)), \xi\rangle^2 \right)^{1/2} \mathrm{d}s \\
(\text{Cauchy–Schwartz}) &\le \int_{t'}^{t} \sup_{\xi\in\mathbb{R}^N:\,|\xi|=1} \left(\sum_{j=1}^{m} |X_j I(\gamma_n(s))|^2\, |\xi|^2 \right)^{1/2} \mathrm{d}s
\end{aligned}$$

[15] Here, we use a well-known fact: if $v \in \mathbb{R}^N$, then

$$|v| = \sup_{\xi\in\mathbb{R}^N:\,|\xi|=1} |\langle v, \xi\rangle|.$$

$$
\begin{aligned}
&= \int_{t'}^{t} \left|(X_1 I, \dots, X_m I)(\gamma_n(s))\right| \mathrm{d}s \\
\text{(by (5.137))} \quad &\le |t-t'| \sup_{|\eta|\le M} \left|(X_1 I, \dots, X_m I)(\eta)\right| \le M' \, |t-t'|.
\end{aligned}
$$

In the last inequality, we used the smoothness of the X_j's. Note that this proves more than the equicontinuity, namely, $\mathcal{F}$ is a uniform Lipschitz-continuous family.

• *$\mathcal{F}$ is uniformly bounded*: We are then left to prove (5.137). By the very definition of d_X, for every $t \in [0, T_1]$, we have

$$d_X(x, \gamma_n(t)) \le t \le T_1.$$

As a consequence, by the triangle-inequality for the distance d_X, we get

$$d_X(0, \gamma_n(t)) \le d_X(0, x) + d_X(x, \gamma_n(t)) \le d_X(0, x) + T_1 < \infty.$$

Hence, the set $\{\gamma_n(t) : n \in \mathbb{N}, t \in [0, T_1]\}$ is bounded in $(\mathbb{G}, d_X)$, so that, due to Corollary 5.15.2, the set is bounded in the Euclidean metric. This is precisely (5.137).

We are then entitled to apply the Arzelà–Ascoli theorem, ensuring that there exists a subsequence of $\{\gamma_n\}_{n\in\mathbb{N}}$ which converges uniformly on $[0, T_1]$, say to $\gamma : [0, T_1] \to \mathbb{G}$. For the sake of brevity, we still denote this uniformly convergent subsequence by $\{\gamma_n\}_{n\in\mathbb{N}}$.

Set $T := d_X(x, y)$. We aim to prove that the curve

$$\gamma^* : [0, T] \to \mathbb{G}, \quad \gamma^* := \gamma|_{[0,T]}$$

is X-subunit and γ connects x to y, which will end the proof since γ is defined on $[0, T]$.

First, $\gamma_n(0) = x$ for every $n \in \mathbb{N}$ yields $\gamma(0) = x$, by pointwise convergence. Second, since γ_n converges *uniformly* to γ on $[0, T_1]$, and $\{T_n\}_n$ is a sequence in $[0, T_1]$ converging to T, we infer

$$\gamma_n(T_n) \to \gamma(T),$$

so that

$$\gamma^*(T) = \gamma(T) = \lim_{n\to\infty} \gamma_n(T_n) = y,$$

because $\gamma_n(T_n) = y$ for every $n \in \mathbb{N}$. This proves that γ^* connects x to y.

Finally, we prove that γ is X-subunit. To begin with, we remark that γ *is absolutely continuous*. Indeed, in proving the equicontinuity of $\mathcal{F}$, we showed that there exists a constant M' such that

$$|\gamma_n(t) - \gamma_n(t')| \le M' \, |t-t'| \quad \text{for every } n \in \mathbb{N} \text{ and every } t, t' \in [0, T_1].$$

Letting $n \to \infty$, this shows that γ is Lipschitz continuous, whence it is absolutely continuous. Obviously, γ is X-subunit iff, for every fixed $\xi \in \mathbb{R}^N$, the functions

$$f^{\pm}(t) := \left(\sum_{j=1}^{m} \langle X_j I(\gamma(t)), \xi\rangle^2\right)^{1/2} \pm \langle \dot{\gamma}(t), \xi\rangle$$

are non-negative a.e. on $[0, T_1]$. To this end, it suffices to prove that (notice that $f_{\pm} \in L^1(0, T_1)$)

$$\int_0^{T_1} f^{\pm}(t)\, \varphi(t)\, \mathrm{d}t \geq 0 \quad \forall\, \varphi \in C_0^{\infty}(0, T_1),\ \varphi \geq 0. \tag{5.138}$$

First, being γ_n X-subunit, it holds

$$\begin{aligned} 0 &\leq \int_0^{T_1} \left\{\left(\sum_{j=1}^{m} \langle X_j I(\gamma_n(t)), \xi\rangle^2\right)^{1/2} \pm \langle \dot{\gamma}_n(t), \xi\rangle\right\}\varphi(t)\, \mathrm{d}t \\ &= \int_0^{T_1} \left(\sum_{j=1}^{m} \langle X_j I(\gamma_n), \xi\rangle^2\right)^{1/2} \varphi \pm \int_0^{T_1} \langle \dot{\gamma}_n, \xi\rangle\, \varphi =: A_n \pm B_n \end{aligned}$$

for every $n \in \mathbb{N}$ and every $\varphi \in C_0^{\infty}(0, T_1)$, $\varphi \geq 0$ (exploit (5.136)), whence

$$0 \leq A_n \pm B_n \quad \text{for every } n \in \mathbb{N}. \tag{5.139}$$

Now, by dominated convergence (indeed, recall that X_j is smooth and $\{\gamma_n\}_{n\in\mathbb{N}}$ is uniformly bounded), the limit of A_n is easily obtained:

$$A_n = \int_0^{T_1} \left(\sum_{j=1}^{m} \langle X_j I(\gamma_n), \xi\rangle^2\right)^{1/2} \varphi \overset{n\to\infty}{\longrightarrow} \int_0^{T_1} \left(\sum_{j=1}^{m} \langle X_j I(\gamma), \xi\rangle^2\right)^{1/2} \varphi. \tag{5.140}$$

Furthermore, the limit of B_n can be obtained by recalling that absolutely continuous functions support integration by parts:

$$\begin{aligned} B_n &= \int_0^{T_1} \langle \dot{\gamma}_n, \xi\rangle\, \varphi = \sum_{i=1}^{N} \xi_i \int_0^{T_1} (\dot{\gamma}_n)_i\, \varphi = -\sum_{i=1}^{N} \xi_i \int_0^{T_1} (\gamma_n)_i\, \dot{\varphi} \\ &\overset{n\to\infty}{\longrightarrow} -\sum_{i=1}^{N} \xi_i \int_0^{T_1} \gamma_i\, \dot{\varphi} = \sum_{i=1}^{N} \xi_i \int_0^{T_1} \dot{\gamma}_i\, \varphi = \int_0^{T_1} \langle \dot{\gamma}, \xi\rangle\, \varphi. \end{aligned} \tag{5.141}$$

(Passing to the limit, we used another dominated convergence argument, recalling that $\gamma_n \to \gamma$ uniformly.) As a consequence, letting $n \to \infty$ in (5.139), from (5.140) and (5.141) we infer

$$0 \leq \int_0^{T_1} \left(\sum_{j=1}^{m} \langle X_j I(\gamma), \xi\rangle^2\right)^{1/2} \varphi \pm \int_0^{T_1} \langle \dot{\gamma}, \xi\rangle\, \varphi,$$

which is exactly (5.138) (by the very definition of $f^{\pm}$). This completes the proof. □

As a consequence of Theorem 5.15.5, we have the following *segment property* for the Carnot–Carathéodory distance on a homogeneous Carnot group. (See also B. Franchi and E. Lanconelli [FL83] for the segment property in a sub-elliptic context.)

Corollary 5.15.6 (The segment property). *Let $\mathbb{G}$ be a homogeneous Carnot group. Let d_X be the control distance related to any basis X of generators (of the first layer of the stratification of the Lie algebra) of $\mathbb{G}$.*

Then the metric space $(\mathbb{G}, d_X)$ has the segment property, *i.e. for every $x, y \in \mathbb{G}$, there exists a continuous curve $\gamma : [0, T] \to \mathbb{G}$ with $\gamma(0) = x$, $\gamma(T) = y$, and such that*

$$d_X(x, y) = d_X(x, \gamma(t)) + d_X(\gamma(t), y) \quad \textit{for every } t \in [0, T].$$

For instance, γ can be any X-geodesic connecting x and y, whose existence is granted by Theorem 5.15.5.

Proof. Let $x, y \in \mathbb{G}$ be fixed. Let γ be a X-geodesic connecting x and y: the existence of such a X-geodesic is granted by Theorem 5.15.5. We claim that

$$d_X(x, y) = d_X(x, \gamma(t)) + d_X(\gamma(t), y) \quad \text{for every } t \in [0, T].$$

Fix $t \in [0, T]$. First, by the very definition of d_X (being γ a X-subunit curve), we have

$$(\star) \qquad d_X(x, \gamma(t)) \le t.$$

(Actually, the equality will hold.) We next prove that

$$(\star\star) \qquad d_X(\gamma(t), y) \le T - t.$$

(Actually, the equality will hold.) Indeed, consider the curve

$$\widetilde{\gamma} : [0, T - t] \to \mathbb{G}, \quad \widetilde{\gamma}(s) := \gamma(s + t).$$

Obviously, $\widetilde{\gamma}$ is X-subunit and $\widetilde{\gamma}(0) = \gamma(t)$, $\widetilde{\gamma}(T - t) = \gamma(t) = y$. Hence, again by the definition of d_X, one has

$$d_X(\gamma(t), y) = d_X(\widetilde{\gamma}(0), \widetilde{\gamma}(T - t)) \le T - t,$$

i.e. $(\star\star)$ holds. Consequently, by $(\star)$ and $(\star\star)$ together with the triangle-inequality, we get

$$T = d_X(x, y) \le d_X(x, \gamma(t)) + d_X(\gamma(t), y) \le t + (T - t) = T.$$

Hence, these inequalities are in fact equalities, and the proof is complete. (Incidentally, this also proves that the equality holds in $(\star)$ and $(\star\star)$.) $\square$

Bibliographical Notes. Some of the topics presented in this chapter also appear in [BL01].

For other equivalent definitions of the Carnot–Carathéodory distance, see, e.g. [JSC87,NSW85]. See [HK00] and the references therein for applications of Carnot–Carathéodory distances in PDE's problems. In particular, see the collection of papers [BR96] for an introduction to the geometry of C-C spaces. For explicit estimates of the Carnot–Carathéodory distance for "diagonal" vector fields, see [FL82]; for the segment property, see [FL83]. See also [GN98].

The gauge functions on the Heisenberg group and on H-type groups were discovered by G.B. Folland [Fol75] and A. Kaplan [Kap80], respectively.

For mean value formulas for the Hörmander sum of squares, see [Lan90,CGL93]; see [Gav77] for mean value formulas for the Heisenberg group; for a survey on mean value formulas in the classical setting of Laplace's operator, see [NV94] (see also the list of references therein for related results in a non-Riemannian setting).

The bibliography on Harnack-type inequalities for sub-elliptic operators is extremely vast: see, e.g. [FL83] for the first result on these topics. See also [BM95, Bon69,FGW94,FL82,GL03,LM97a,LK00,SaC90,SCS91,Varo87].

For Liouville-type theorems for homogeneous operators, see [Gel83,KoSt85, LM97a,Luo97]. For results on the maximum principles and propagation, see [Ama79, Bon69,Hil70,Hop52,PW67,Red71,Spe81,Tai88].

5.16 Exercises of Chapter 5

Ex. 1) Give a detailed proof of Lemma 5.13.19, page 303.

Ex. 2) The following operator

$$\mathcal{L} := \sum_{j=1}^{n} \left((\partial_{x_j})^2 + (\partial_{y_j} + x_j\, \partial_t)^2\right) + (\partial_z)^2$$

in $\mathbb{R}^{2n+2}$ (the points are denoted by (x, y, z, t) with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $z \in \mathbb{R}$, $t \in \mathbb{R}$) is not analytic-hypoelliptic (see [Rot84]). Find an explicit homogeneous Carnot group $\mathbb{G}$ such that $\mathcal{L}$ is the canonical sub-Laplacian of $\mathbb{G}$.

Ex. 3) Construct explicitly the Carnot group referred to in Example 5.10.6, page 286. (*Hint*: Model the needed composition law on the composition law in Example 5.10.5, page 285.)

Ex. 4) Consider the Kolmogorov-type group in Ex. 7 of Chapter 3, page 179. By means of formula (5.92), verify that the fundamental solution of its canonical sub-Laplacian is given by (when $x \neq 0$)

$$\begin{aligned}
&\Gamma(x,t)\\
&\quad = c \int_{\mathbb{R}^2} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \, \frac{|\tau|}{\sinh|\tau|}\\
&\qquad \times \left(\frac{\{(\tau_2 x_2 - \tau_1 x_3)^2 + 2\,\frac{|\tau|}{\tanh|\tau|}\,(|\tau|^2\,x_1^2 + (\tau_1 x_2 + \tau_2 x_3)^2)\}}{\{\ldots \textit{the above braces} \ldots\}^2 + (t_1\tau_1 + t_2\tau_2)^2} \right)^{\frac{5}{2}}
\end{aligned}$$

where c is the dimensional constant

$$c = \frac{3}{2^7\,\pi^3}.$$

Ex. 5) Prove that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, then

$$x \mapsto \fint_{B_d(x,r)} |f(y)|\,\mathrm{d}y$$

is a continuous function.

Ex. 6) Provide a detailed proof of (5.61) at page 264.

Ex. 7) Let $u \in C^2(B_d(0,R)) \cap \mathcal{H}\big(B_d(0,R) \setminus \overline{B_d(0,r)}\big)$, $0 < r < R$. Prove that there exist constants C_0, C_1 such that

$$\mathcal{M}_\rho(u)(0) = C_0 + \frac{C_1}{\rho^{Q-2}} \quad \text{for every } \rho \in (r,R).$$

Ex. 8) **(The surface asymptotic sub-Laplacian).** Let $\Omega \subseteq \mathbb{G}$ be open, and let $u \in C(\Omega,\mathbb{R})$. Given $x \in \Omega$, we say that the *surface asymptotic sub-Laplacian* of u at x is non-negative, written $\mathcal{A}\mathcal{L}u(x) \geq 0$, if

$$\liminf_{r\to 0^+} \frac{\mathcal{M}_r(u)(x) - u(x)}{r^2} \geq 0.$$

If $\mathcal{A}\mathcal{L}(-u)(x) \geq 0$, we write $\mathcal{A}\mathcal{L}u(x) \leq 0$. Prove the following statements:

(i) If $\mathcal{A}\mathcal{L}u \geq 0$ and $\mathcal{A}\mathcal{L}v \geq 0$, then $\mathcal{A}\mathcal{L}(\lambda\,u + \mu\,v) \geq 0$ for every $\lambda, \mu \geq 0$.

(ii) If $\mathcal{A}\mathcal{L}u \geq 0$ and $\lambda \leq 0$, then $\mathcal{A}\mathcal{L}(\lambda\,u) \leq 0$.

(iii) If $u \in C^2(\Omega,\mathbb{R})$ and $\mathcal{L}u \geq 0$, then $\mathcal{A}\mathcal{L}u(x) \geq 0$ for every $x \in \Omega$.

(iv) If $x_0 \in \Omega$ is a local maximum point for u, then $\mathcal{A}\mathcal{L}u(x_0) \leq 0$.

Ex. 9) **(The solid asymptotic sub-Laplacian).** Let $\Omega \subseteq \mathbb{G}$ be open, and let $u \in C(\Omega,\mathbb{R})$. Given $x \in \Omega$, we say that the *solid asymptotic sub-Laplacian* of u at x is non-negative, written $\mathrm{A}\mathcal{L}u(x) \geq 0$, if

$$\liminf_{r\to 0^+} \frac{\mathrm{M}_r(u)(x) - u(x)}{r^2} \geq 0.$$

If $\mathrm{A}\mathcal{L}(-u)(x) \geq 0$, we write $\mathrm{A}\mathcal{L}u(x) \leq 0$. Prove the statements of the previous exercise with $\mathcal{A}\mathcal{L}$ replaced by $\mathrm{A}\mathcal{L}$.

Ex. 10) **(Maximum principle for $\mathcal{AL}$ and $\mathrm{A}\mathcal{L}$).** Let $\Omega \subseteq \mathbb{G}$ be open and bounded. Let $u \in C(\Omega, \mathbb{R})$ be such that

$$\begin{cases} \mathcal{AL}u \geq 0 & \text{in } \Omega, \\ \limsup\limits_{y \to x} u(y) \leq 0 & \forall\, x \in \partial\Omega, \end{cases} \quad \text{or} \quad \begin{cases} \mathrm{A}\mathcal{L}u \geq 0 & \text{in } \Omega, \\ \limsup\limits_{y \to x} u(y) \leq 0 & \forall\, x \in \partial\Omega. \end{cases}$$

Prove that $u \leq 0$ in Ω. (*Hint*: Proceed as in the proof of the weak maximum principle, Theorem 5.13.4.)

Ex. 11) Provide the detailed proof of the result in Remark 5.6.4.

Ex. 12) Let $\mathcal{L}$ be a sub-Laplacian on a homogeneous Carnot group $\mathbb{G}$ on $\mathbb{R}^N$. Suppose that $\mathcal{L}$ has no first order differential terms, i.e. $\mathcal{L}(x_k) \equiv 0$ for every $k = 1, \ldots, N$. Let $\psi \in C^2(\mathbb{G}, \mathbb{G})$ be such that

$$\mathcal{L}(u \circ \psi) = (\mathcal{L}\, u) \circ \psi \qquad \forall\, u \in C^2(\mathbb{G}). \tag{5.142}$$

Following the usual notation for the "stratification" of the variables of $\mathbb{G}$, we set

$$\psi = (\psi^{(1)}, \ldots, \psi^{(r)})$$

(here $\psi^{(i)} \in C^2(\mathbb{G}, \mathbb{R}^{N_i})$, $N_1 + \cdots + N_r = N$ and N_i is the dimension of the i-th layer in the stratification of the algebra of $\mathbb{G}$). Prove that, for every $1 \leq j \leq N_1$, it holds

$$\begin{cases} 0 = \mathcal{L}\, \psi_j^{(1)} \\ 1 = \left|\nabla_{\mathcal{L}}\, \psi_j^{(1)}\right|^2. \end{cases} \tag{5.143}$$

(*Hint*: Use (1.112) in Ex. 12, Chapter 1, page 83.)

Let now $u_j^{(1)} := (\psi_j^{(1)})^2$. Then $u_j^{(1)} \geq 0$, and from (5.143) we have $\mathcal{L}u_j^{(1)} = 2$. Using Liouville Theorem 5.8.4, derive that $u_j^{(1)}$ is a polynomial of $\mathbb{G}$-degree ≤ 2 and therefore $u_j^{(1)} = p_1(x^{(1)}) + p_2(x^{(2)})$, where p_1 and p_2 are polynomials of ordinary degrees 2 and 1, respectively. Deduce that $p_2 \equiv 0$. Since $|\psi_j^{(1)}| = \sqrt{u_j^{(1)}}$, using, for example, Theorem 5.8.8, derive that $\psi_j^{(1)}$ is a polynomial of $\mathbb{G}$-degree ≤ 1, depending at most on $x^{(1)}$.

Now suppose by induction that $\psi^{(1)}, \ldots, \psi^{(n)}$ have polynomial component functions of $\mathbb{G}$-degree respectively $1, \ldots, n$ at most. Write $\mathcal{L} = \sum_{k=1}^p X_k^2$ with $X_k = \sum_{i=1}^N \sigma_i^{(k)}\, \partial_i$. Prove that (using again (1.112)) if $i = n + 1$ and $1 \leq j \leq N_{i+1}$, it holds

$$\begin{cases} 0 = \mathcal{L}\, \psi_j^{(n+1)} \\ \sum_{k=1}^p \left(\sigma_{n+1,\, j}^{(k)}\right)^2(\psi) = \left|\nabla_{\mathcal{L}}\, \psi_j^{(n+1)}\right|^2. \end{cases} \tag{5.144}$$

Let $u_j^{(n+1)} := (\psi_j^{(n+1)})^2$; then $u_j^{(n+1)} \geq 0$ and, from (5.144), derive

$$\mathcal{L}\, u_j^{(n+1)} = 2 \sum_{k=1}^p \left(\sigma_{n+1,\, j}^{(k)}\right)^2(\psi).$$

On the other hand, $\sigma^{(k)}_{n+1,\,j}$ is a polynomial of $\mathbb{G}$-degree n thus depending only on $x^{(1)}, \dots, x^{(n)}$. Consequently, by induction hypothesis,

$$\big(\sigma^{(k)}_{n+1,\,j}\big)^2(\psi) = \big(\sigma^{(k)}_{n+1,\,j}(\psi^{(1)}, \dots, \psi^{(n)})\big)^2$$

has $\mathbb{G}$-degree at most $2\,n$. Therefore (why?), $u_j^{(n+1)}$ is a polynomial of $\mathbb{G}$-degree at most $2 + 2n$, consequently $|\psi_j^{(n+1)}| = \sqrt{u_j^{(n+1)}} \le C(1 + |x|)^{n+1}$. This last estimate, together with $\mathcal{L}\psi_j^{(n+1)} = 0$ and Theorem 5.8.2, proves that $\psi_j^{(n+1)}$ is a polynomial of $\mathbb{G}$-degree at most $n + 1$. We have proved the following assertion.

Proposition 5.16.1. *Let ψ be a $C^2(\mathbb{G}, \mathbb{G})$ map such that*

$$\mathcal{L} \circ \psi = \psi \circ \mathcal{L},$$

where $\mathcal{L}$ is a sub-Laplacian without the first order differential terms. Following the usual notation of the coordinates on $\mathbb{G}$, we let $\psi = (\psi^{(1)}, \dots, \psi^{(r)})$, where $\psi^{(j)} \in C^2(\mathbb{G}, \mathbb{R}^{N_j})$ for every $j = 1, \dots, r$. Then each component of $\psi^{(j)}$ is a polynomial function of $\mathbb{G}$-degree less or equal to j.

Ex. 13) (Maps commuting with Laplace operator $\boldsymbol{\Delta}$). Consider the Laplace operator $\Delta = \sum_{i=1}^N \partial_{i,i}^2$ on $\mathbb{R}^N$. Let $\psi \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ be such that

$$\Delta(u \circ \psi) = (\Delta\, u) \circ \psi \qquad \forall\, u \in C^2(\mathbb{R}^N). \tag{5.145}$$

With the aid of Ex. 12, Chapter 1, page 83, prove that (5.145) holds if and only if

$$\begin{cases} \Delta\psi_i = 0, & i = 1, \dots, N, \\ |\nabla\psi_i|^2 = 1, & i = 1, \dots, N, \\ \langle \nabla\psi_i, \nabla\psi_j \rangle = 0, & 1 \le i, j \le N, \quad i \ne j. \end{cases}$$

Derive that there exists an $N \times N$ *orthogonal* matrix M and a vector $C \in \mathbb{R}^N$ such that

$$\psi(x) = Mx + C.$$

This means that the set of the maps commuting with the Laplace operator coincides with the group of the isometries of $\mathbb{R}^N$.

Ex. 14) (Maps commuting with $\boldsymbol{\Delta_{\mathbb{H}^1}}$). Let us consider the Kohn Laplacian $\Delta_{\mathbb{H}^1}$ on the Heisenberg-Weyl group $\mathbb{H}^1 \equiv \mathbb{R}^3$. We characterize the maps $\psi \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\Delta_{\mathbb{H}^1}(u \circ \psi) = (\Delta_{\mathbb{H}^1}\, u) \circ \psi \qquad \forall\, u \in C^2(\mathbb{H}^1). \tag{5.146}$$

Set $\psi(x, y, t) = (\xi(x, y, t), \eta(x, y, t), \tau(x, y, t))$. With the aid of Ex. 12, Chapter 1, page 83, prove that (5.146) is equivalent to

$$\begin{cases} \text{(i)} & \begin{cases} 1 = |\nabla_{\mathbb{H}^1}\xi|^2, \\ 0 = \Delta_{\mathbb{H}^1}\xi, \end{cases} \\ \text{(ii)} & \begin{cases} 1 = |\nabla_{\mathbb{H}^1}\eta|^2, \\ 0 = \Delta_{\mathbb{H}^1}\eta, \end{cases} \\ \text{(iii)} & \begin{cases} 4\,(\xi^2+\eta^2) = |\nabla_{\mathbb{H}^1}\tau|^2, \\ 0 = \Delta_{\mathbb{H}^1}\tau, \end{cases} \\ \text{(iv)} & 2\,\eta = \langle \nabla_{\mathbb{H}^1}\xi, \nabla_{\mathbb{H}^1}\tau \rangle, \\ \text{(v)} & -\,2\,\xi = \langle \nabla_{\mathbb{H}^1}\eta, \nabla_{\mathbb{H}^1}\tau \rangle, \\ \text{(vi)} & 0 = \langle \nabla_{\mathbb{H}^1}\xi, \nabla_{\mathbb{H}^1}\eta \rangle. \end{cases}$$

Using Liouville Theorem 5.8.4, derive that ξ and η are polynomials of $\mathbb{H}^1$-degree ≤ 1, τ is a polynomial of $\mathbb{H}^1$-degree ≤ 2. Hence, there exist constants such that

$$\begin{aligned} \xi(x,y,t) &= c_0 + c_1\,x + c_2\,y, \\ \eta(x,y,t) &= d_0 + d_1\,x + d_2\,y, \\ \tau(x,y,t) &= e_0 + e_1\,x + e_2\,y + e_3\,t. \end{aligned}$$

From the first equation in (i) and the first equation in (ii) and from (vi) it follows that there exists $\theta \in [0, 2\,\pi[$ such that

$$(c_1, c_2) = (\cos\theta, \sin\theta), \qquad (d_1, d_2) = \pm(-\sin\theta, \cos\theta).$$

From (iv) and (v) one gets

$$\begin{pmatrix} e_1 + 2\,y\,e_3 \\ e_2 - 2\,x\,e_3 \end{pmatrix} = \begin{pmatrix} 2d_0c_1 - 2d_1c_0 \pm 2\,y \\ 2d_0c_2 - 2d_2c_0 \mp 2\,x \end{pmatrix},$$

i.e.

$$e_3 = \pm 1, \quad e_1 = 2d_0c_1 - 2d_1c_0, \quad e_2 = 2d_0c_2 - 2d_2c_0.$$

Recalling that on $\mathbb{H}^1$ it holds

$$(\alpha, \beta, \gamma) \circ (x, y, t) = (x + \alpha, y + \beta, t + \gamma + 2x\beta - 2y\alpha),$$

we have proved that *the only maps commuting with* $\Delta_{\mathbb{H}^1}$ *have the form*

$$\begin{aligned} \psi_+(x,y,t) &= \begin{pmatrix} c_0 \\ d_0 \\ e_0 \end{pmatrix} \circ \begin{pmatrix} M_+ \cdot \binom{x}{y} \\ t \end{pmatrix}, \\ \psi_-(x,y,t) &= \begin{pmatrix} c_0 \\ d_0 \\ e_0 \end{pmatrix} \circ \begin{pmatrix} M_- \cdot \binom{x}{y} \\ -t \end{pmatrix}, \end{aligned}$$

where M_+ *and* M_- *are* (*respectively*) *isometries with the determinant equal to* $+1$ (*respectively*, -1) *and* (c_0, d_0, e_0) *is a fixed point in* $\mathbb{H}^1$.

Ex. 15) **(Translation-formula in surface integrals over a d-sphere).** Let $f : \mathbb{G} \to \mathbb{R}$ be non-negative (or with suitable summability properties). Let d be a smooth, homogeneous and symmetric norm on $\mathbb{G}$, and let $S(x_0, r)$ be the d-sphere with center x_0 and radius $r > 0$. Finally, let H^{N-1} denote the Hausdorff $(N-1)$-dimensional measure on $\mathbb{R}^N$. Prove that

$$\int_{S(x_0,r)} f(x)\,\mathrm{d}H^{N-1}(x) = \int_{S(0,r)} f(x_0 \circ y)\,K_d(x_0, y)\,\mathrm{d}H^{N-1}(y),$$

where

$$K_d(x_0, y) := \frac{|\nabla(d(x_0^{-1} \circ \cdot))|(x_0 \circ y)}{|\nabla d(y)|}.$$

Prove also that

$$K_d(x_0, y) = \frac{|\nabla d(y) \cdot \mathcal{J}_{\tau_{x_0^{-1}}}(x_0 \circ y)|}{|\nabla d(y)|} = \left| \frac{\nabla d(y)}{|\nabla d(y)|} \cdot \mathcal{J}_{\tau_{x_0^{-1}}}(x_0 \circ y) \right|.$$

(*Hint*: Consider the identity (why does it hold?)

$$\int_{S(x_0,r)} f(x)\,\mathrm{d}H^N(x) = \int_{S(0,r)} f(x_0 \circ y)\,\mathrm{d}H^N(y)$$

rewritten (by the coarea formula) as

$$\int_0^r \left(\int_{S(x_0,\lambda)} f(x)\,\frac{\mathrm{d}H^{N-1}(x)}{|\nabla d(x_0^{-1} \circ \cdot)|(x)} \right) \mathrm{d}r$$
$$= \int_0^r \left(\int_{S(0,\lambda)} f(x_0 \circ y)\,\frac{\mathrm{d}H^{N-1}(y)}{|\nabla d(y)|} \right) \mathrm{d}r$$

and differentiate w.r.t. r.)

Ex. 16) In the sequel, d is a symmetric homogeneous norm on $\mathbb{G}$. If $x \in \mathbb{G}$ and $A \subseteq \mathbb{G}$ is any set, we call the *d-distance of x to A*, the following real non-negative number

$$\mathrm{dist}_d(x, A) := \inf_{a \in A} d(x^{-1} \circ a).$$

Prove the following result.

Lemma 5.16.2. *Let $A \subset \mathbb{G}$ be any set. For every $x \in \mathbb{G}$, there exists $\overline{a}_x \in \overline{A}$ (the closure of A) such that*

$$\mathrm{dist}_d(x, A) = d(x^{-1} \circ \overline{a}_x).$$

(*Hint*: By definition, there exists $\{a_j\}_j$ in A such that $d(x^{-1} \circ a_j) \to \mathrm{dist}_d(x, A)$. Obviously, $\{a_j\}_j$ is bounded, otherwise $d(x^{-1} \circ a_j) \geq \frac{1}{\mathbf{c}}\,d(a_j) - d(x) \to \infty$. Then, extract a subsequence $a_{j_n} \to \overline{a}_x \in \overline{A}$, as $n \to \infty$, and use the continuity of d.)

Then prove the following result.

Proposition 5.16.3. *Let $A \subseteq \mathbb{G}$ be any set. The following assertions hold*:
(a) *For every $x \in \mathbb{G}$, we have* $\mathrm{dist}_d(x, A) = \mathrm{dist}_d(x, \overline{A})$;
(b) *The d-distance from A, i.e. the function*

$$\mathrm{dist}_d(\cdot, A) : \mathbb{G} \to [0, \infty), \qquad x \mapsto \mathrm{dist}_d(x, A),$$

is a continuous function.

Hint: (a). Let $\alpha \in \overline{A}$ be such that $\mathrm{dist}_d(x, \overline{A}) = d(x^{-1} \circ \alpha)$. Let $a_j \in A$ be such that $a_j \to \alpha$ as $j \to \infty$. Then consider the following inequalities:

$$\begin{aligned}\mathrm{dist}_d(x, A) &\geq \mathrm{dist}_d(x, \overline{A}) = d(x^{-1} \circ \alpha)\\ &= \lim_{j\to\infty} d(x^{-1} \circ a_j) \geq \mathrm{dist}_d(x, A).\end{aligned}$$

The last inequality follows from $d(x^{-1} \circ a_j) \geq \mathrm{dist}_d(x, A)$ for every $j \in A$, since $a_j \in A$, and by the very definition of $\mathrm{dist}_d(x, A)$.
(b). Provide the details of the following arguments: Let $x_0 \in \mathbb{G}$ be fixed. It suffices to show that from every sequence $\{x_j\}_{j\in\mathbb{N}}$ in $\mathbb{G}$ converging to x_0 we can extract a subsequence $\{x_{j_n}\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \mathrm{dist}_d(x_{j_n}, A) = \mathrm{dist}_d(x_0, A). \tag{5.147}$$

It is not restrictive to suppose that A is closed. For every $j \in \mathbb{N}$, there exists $a_j \in A$ such that

$$\mathrm{dist}_d(x_j, A) = d(x_j^{-1} \circ a_j). \tag{5.148}$$

It is easy to see that $\{a_j\}_{j\in\mathbb{N}}$ is bounded. Hence, we can extract a converging subsequence from $\{a_j\}_{j\in\mathbb{N}}$, say $a_{j_n} \to a \in A$ as $n \to \infty$. From (5.148) we infer

$$\lim_{n\to\infty} \mathrm{dist}_d(x_{j_n}, A) = \lim_{n\to\infty} d(x_{j_n}^{-1} \circ a_{j_n}) = d(x_0^{-1} \circ a). \tag{5.149}$$

Thus, (5.147) will follow if we show that $d(x_0^{-1} \circ a) = \mathrm{dist}_d(x_0, A)$. Suppose to the contrary that we have $d(x_0^{-1} \circ a) \neq \mathrm{dist}_d(x_0, A)$. This may occur iff

$$\text{(i)} : d(x_0^{-1} \circ a) < \mathrm{dist}_d(x_0, A) \qquad \text{or} \qquad \text{(ii)} : d(x_0^{-1} \circ a) > \mathrm{dist}_d(x_0, A).$$

Case (i) is impossible by the very definition of $\mathrm{dist}_d(x_0, A)$ (since $a \in A$). Case (ii) is absurd too, as we show below. Let $a_0 \in A$ be such that $\mathrm{dist}_d(x_0, A) = d(a_0^{-1} \circ x_0)$. Then, by (5.149), we derive

$$\begin{aligned}d(x_0^{-1} \circ a) &= \lim_{n\to\infty} \mathrm{dist}_d(x_{j_n}, A) \leq \lim_{n\to\infty} d(a_0^{-1} \circ x_{j_n})\\ &= d(a_0^{-1} \circ x_0) = \mathrm{dist}_d(x_0, A).\end{aligned}$$

This contradicts (ii).

Ex. 17) Prove the following improvement of Proposition 5.16.3 above, when the extra hypothesis *d is smooth* holds.

Proposition 5.16.4. *Let d be a smooth homogeneous norm as in Proposition* 5.14.1, *and let $A \subseteq \mathbb{G}$. Prove that the function* $\mathrm{dist}_d(x, A)$ *is Lipschitz continuous with respect to d, i.e.*

$$|\mathrm{dist}_d(\xi, A) - \mathrm{dist}_d(\eta, A)| \leq \beta\, d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathbb{G},$$

where β is the same constant as in Proposition 5.14.1.

(*Hint*: Let $a \in A$, and write (5.125a) with $x = a$, $y = \xi$, $z = \eta$,

$$d(\xi^{-1} \circ a) \leq d(\eta^{-1} \circ a) + \beta\, d(\xi^{-1} \circ \eta).$$

Then take the infimum over $a \in A$, and derive $\mathrm{dist}_d(\xi, A) \leq \mathrm{dist}_d(\eta, A) + \beta\, d(\xi, \eta)$. Then interchange ξ and η.)

Ex. 18) **(T: Another mean integral operator).** Let $\mathcal{L}$ be a sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$. Let $\Omega \subseteq \mathbb{G}$ be a bounded open set. For every $x \in \Omega$, let us put

$$r_x := \frac{1}{2}\mathrm{dist}_d(x, \partial\Omega),$$

where d is an $\mathcal{L}$-gauge function. For every $u \in L^1_{\mathrm{loc}}(\Omega)$, define

$$T(u)(x) := \mathrm{M}_{r_x}(u)(x), \tag{5.150}$$

where M_r is the average operator defined in (5.50f).
Prove the following statements:

(i) $T(u)$ is continuous in Ω,
(ii) $u \leq v \Rightarrow T(u) \leq T(v)$,
(iii) $u \in L^\infty(\Omega) \Rightarrow \sup_\Omega |T(u)| \leq \operatorname{ess\,sup}_\Omega |u|$.

Ex. 19) **(The strong maximum principle related to T).** Let T be as in the previous exercise, and let u be a continuous function in Ω such that $T(u) \leq u$. If Ω is connected and u attains its minimum in Ω, then $u =$ constant in Ω.

Ex. 20) **(The weak maximum principle related to T).** Let T be as in the previous exercises, and let u be a continuous function in Ω such that

$$\begin{cases} T(u) \leq u & \text{in } \Omega, \\ \liminf_{x \to y} u(x) \geq 0 & \forall\, y \in \partial\Omega. \end{cases}$$

Prove that $u \geq 0$ in Ω.

Ex. 21) Let u be an $\mathcal{L}$-harmonic function in an open set $\Omega \subseteq \mathbb{G}$, $\Omega \neq \mathbb{G}$. Assume $u \in L^p(\Omega)$, $1 \leq p \leq \infty$. Then

$$|u(x)| \leq \frac{c}{\mathrm{dist}_d(x, \partial\Omega)}\, \|u\|_{L^p(\Omega)},$$

where c is independent of u and Ω, and d is an $\mathcal{L}$-gauge function.
(*Hint*: Formula (5.52) may prove useful.)

Ex. 22) **(Another Koebe-type result).** Let $\Omega \subseteq \mathbb{G}$ be open, and let $u \in C(\Omega, \mathbb{R})$. Suppose that one of the following conditions is satisfied:

(i) $\mathcal{M}_\rho(u)(x) = \mathcal{M}_r(u)(x)$ for every $\rho, r > 0$ such that $0 < \rho \le r$ and $\overline{B_d(x,r)} \subset \Omega$,

(ii) $\mathrm{M}_\rho(u)(x) = \mathrm{M}_r(u)(x)$ for every $\rho, r > 0$ such that $0 < \rho \le r$ and $\overline{B_d(x,r)} \subset \Omega$.

Show that $u \in C^\infty(\Omega, \mathbb{R})$ and $\mathcal{L}u = 0$ in Ω.

Ex. 23) Prove the following *Harnack inequality on d-spheres*.

Theorem 5.16.5. *There exists a constant $C > 1$ such that*

$$\sup_{\partial B_d(0,r)} h \le C \inf_{\partial B_d(0,r)} h \tag{5.151}$$

for every $0 < r \le 1/2$ and every $\mathcal{L}$-harmonic non-negative function h on $B_d(0,1) \setminus \{0\}$.

Ex. 24) Let d be any homogeneous norm on $\mathbb{G}$, smooth on $\mathbb{G} \setminus \{0\}$. Prove that

$$\int_{d(x)=r} \frac{\mathrm{d}H^{N-1}(x)}{|\nabla d(x)|} = c_d\, r^{Q-1} \quad \text{for every } r > 0,$$

where $c_d = Q\, H^N(B_d(0,1))$. (*Hint*: By the coarea formula and the δ_λ-homogeneity of d, we have

$$\int_0^r \left(\int_{\{d(x)=\rho\}} \frac{\mathrm{d}H^{N-1}(x)}{|\nabla d(x)|} \right) \mathrm{d}\rho = \int_{\{d(x)<r\}} \mathrm{d}H^N = r^Q\, H^N(B_d(0,1)).$$

Then, take the derivative w.r.t. r on both sides.)

Ex. 25) In this exercise, we follow the arguments from [HK00] in Lemma 11.1 and Proposition 11.2, jointly with some of our results in Section 5.2.

Lemma 5.16.6. *Let $X = \{X_1, \ldots, X_m\}$ be a system of locally Lipschitz-continuous vector fields on $\mathbb{R}^N$.*
Let $B_E(x, R) = \{z \in \mathbb{R}^N : |z - x| < R\}$ denote the Euclidean ball centered at x with radius R. With reference to (5.8a) (*page 233*) *set*

$$M(x,R) := \sup_{z \in B_E(x,R)} \left\{ \sum_{j=1}^n |X_j I(z)| \right\}.$$

Suppose $\gamma : [0,T] \to \mathbb{R}^N$ is X-subunit, $\gamma(0) = x$ and $T < R/M(x,R)$. Then $\gamma([0,T]) \subseteq B_E(x,R)$.

(*Hint*: Suppose to the contrary that $\gamma([0,T]) \nsubseteq B_E(x,R)$. Hence, there exists a least $t \in (0,T]$, such that $y := \gamma(t) \notin B_E(x,R)$. From the minimality of t and the continuity of γ, we certainly have $y \in \partial B_E(x,R)$, i.e.

$|y-x| = R$. Notice also that $d_X(x,y) \le t \le T$. Hence, (5.8b) on page 233 gives

$$R = |x-y| \le M(x, |x-y|)\, d_X(x,y) \le M(x,R)T < R.$$

This is absurd.)
Deduce the following result.

Corollary 5.16.7. *Let $X = \{X_1, \dots, X_m\}$ be a system of locally Lipschitz-continuous vector fields on $\mathbb{R}^N$.*
Then, for every bounded set $D \subset \mathbb{R}^N$, there exist positive numbers $R_0 \ll 1$, $R_1 \gg 1$ (depending only on D and X) such that $B_{d_X}(x,R) \subseteq B_E(0,R_1)$ for every $x \in D$ and every $R \in [0,R_0]$. In particular, for every $x_0 \in \mathbb{R}^N$, every Carnot–Carathéodory ball $B_{d_X}(x_0,R)$ with small radius (dependently on x_0) is bounded in the Euclidean metric.

(*Hint*: With the notation of Lemma 5.16.6, let

$$\mathbf{M} := \sup\{M(x,1) : x \in \overline{D}\}.$$

Obviously, $\mathbf{M} < \infty$, for the X_j's are continuous. Set $R_0 := (4\,\mathbf{M})^{-1}$. Let $x \in D$ and $R \le R_0$. (Note that $\mathbf{M} \ge M(x,1)$, since $x \in D$.) Let also $y \in B_{d_X}(x,R)$, i.e. $d_X(x,y) < R$. Then there exists a X-subunit curve $\gamma : [0,T] \to \mathbb{R}^N$ such that $\gamma(0) = x$, $\gamma(T) = y$ and

$$T < d_X(x,y) + (4\mathbf{M})^{-1} < 2\,R_0 = (2\mathbf{M})^{-1} < 1/M(x,1).$$

Then, we can apply Lemma 5.16.6 with $R = 1$ and derive that $\gamma([0,T]) \subseteq B_E(x,1)$. In particular, this gives $|y-x| = |\gamma(T)-x| < 1$, i.e. $y \in B_E(x,1)$. Due to the arbitrariness of $y \in B_{d_X}(x,R)$, this proves

$$B_{d_X}(x,R) \subseteq B_E(x,1) \subseteq D_1 := \bigcup_{x\in D} B_E(x,1) \Subset \mathbb{R}^N.$$

This ends the proof, by finding a suitable $R_1 \gg 1$ such that $D_1 \subseteq B_E(0,R_1)$.)

Proposition 5.16.8. *Let $X = \{X_1, \dots, X_m\}$ be a system of locally Lipschitz-continuous vector fields on $\mathbb{R}^N$.*
Let K be a compact subset of $\mathbb{R}^N$. Then there exists a constant $\mathbf{c} = \mathbf{c}(K,X) > 0$ such that

$$d_X(x,y) \ge \mathbf{c}\,|x-y| \quad \textit{for every } x,y \in K.$$

Proof. If $d_X(x,y) = \infty$, there is nothing to prove. Hence, we can suppose $d_X(x,y) < \infty$. Let $\gamma : [0,T] \to \mathbb{R}^N$ be any X-subunit curve such that $\gamma(0) = x$, $\gamma(T) = y$. Let $R := |x-y|$. Set

$$M := \sup \left\{ \sum_{j=1}^{n} |X_j I(z)| \ : \ z \in B_E(x, 1 + \mathrm{diam}(K)), \ x \in K \right\}.$$

Note that $M = M(K, X) < \infty$, since K is compact and the X_j's are continuous. Obviously, we do not have $\gamma([0, T]) \subseteq B_E(x, R)$, because $|\gamma(T) - x| = |y - x| = R$ is not $< R$. Hence, by Lemma 5.16.6,

$$T \geq R/M(x, R) \geq R/M =: \mathbf{c}\, R = \mathbf{c}|x - y|.$$

(Indeed, $M(x, R) \leq M$.) Passing to the infimum over the above γ's, we get $d_X(x, y) \geq \mathbf{c}\, |x - y|$. $\square$

Derive the following assertion from Corollary 5.16.7 and Proposition 5.16.8.

Proposition 5.16.9. *Let $X = \{X_1, \dots, X_m\}$ be locally Lipschitz-continuous vector fields on $\mathbb{R}^N$. Suppose $\mathbb{R}^N$ is X-connected. Let d_X be the relevant Carnot–Carathéodory distance. Then the map*

$$\mathrm{id} : (\mathbb{G}, d_X) \to (\mathbb{G}, d_E)$$

is continuous.

(*Hint*: Dealing with metric spaces, we can prove sequential continuity. Let $x_n \to x_0$ in $(\mathbb{G}, d_X)$. We aim to prove that $x_n \to x_0$ in $(\mathbb{G}, d_E)$. By Corollary 5.16.7, there exists $\varepsilon = \varepsilon(x_0) > 0$ such that $B_{d_X}(x_0, \varepsilon)$ is bounded in $(\mathbb{G}, d_E)$. By definition of limit, there exists $\overline{n} = \overline{n}(\varepsilon) \in \mathbb{N}$ such that $x_n \in B_{d_X}(x_0, \varepsilon)$ whenever $n \geq \overline{n}$. Hence the set $\{x_n \ : \ n \geq \overline{n}\} \cup \{x_0\}$ is contained in a compact subset of $\mathbb{R}^N$. We can thus apply Proposition 5.16.8 to derive that, for a suitable $\mathbf{c} > 0$,

$$\mathbf{c}\, |x_n - x_0| \leq d_X(x_n, x_0) \quad \text{for all } n \geq \overline{n}.$$

Letting $n \to \infty$, we get $x_n \to x_0$ in $(\mathbb{G}, d_E)$.)
Now, consider a system X of Hörmander vector fields in $\mathbb{R}^N$ and the relevant d_X. Recall that, by the Carathéodory–Chow–Rashevsky theorem (see Chapter 19 for suitable references), $\mathbb{R}^N$ is X-connected. Moreover, by known results (see, e.g. [NSW85]) the inequalities in Proposition 5.15.1 hold (for a suitable $r > 0$) when $d(y^{-1} \circ x)$ is replaced by $d_X(x, y)$. In particular, given a compact subset K of $\mathbb{R}^N$, there exist r, $\mathbf{c} > 0$ (depending on K, X) such that

$$d_X(x, y) \leq \mathbf{c}\, |x - y|^{1/r} \quad \forall\, x, \ y \in K.$$

Arguing as in the first paragraph of the proof of Corollary 5.15.3, derive the following assertion:
Given a system X of Hörmander vector fields in $\mathbb{R}^N$, the map $\mathrm{id} : (\mathbb{G}, d_E) \to (\mathbb{G}, d_X)$ *is continuous* (indeed, it is a homeomorphism, see Proposition 5.16.9).

Ex. 26) Consider the vector fields in $\mathbb{R}^2$ defined by

$$X_1 = \partial_{x_1}, \quad X_2 = \max\{0, x_1\}\, \partial_{x_2}.$$

Consider the relevant d_X, where $X = \{X_1, X_2\}$. Is $\mathbb{R}^2$ X-connected? Prove that the identity map $\mathrm{id} : (\mathbb{R}^2, d_E) \to (\mathbb{R}^2, d_X)$ is not continuous.
(*Hint*: It may be useful to notice that the integral curves of X_1 are the lines parallel to the x_1 axis; the integral curve of X_2 through (x_0, y_0) is

$$t \mapsto \begin{cases} (x_0, y_0) & \text{if } x_0 \le 0, \\ (x_0, y_0 + x_0\, t) & \text{if } x_0 > 0, \end{cases}$$

i.e. the single point (x_0, y_0) if $x_0 \le 0$ or the lines parallel to the x_2 axis if $x_0 > 0$.)

Ex. 27) Consider the vector fields in $\mathbb{R}^2$ defined by

$$X_1 = \partial_{x_1}, \quad X_2 = (1 + x_2^2)\, \partial_{x_2}.$$

Verify that the system $X = \{X_1, X_2\}$ satisfies the Hörmander condition. Consider the relevant d_X. Is $\mathbb{R}^2$ X-connected? Prove that there exists a set $\Omega \subset \mathbb{R}^2$ which is bounded w.r.t. d_X but unbounded in the Euclidean metric. *Hint*: Observe that the integral curve of X_2 (hence, a X-subunit path!) starting from (x_0, y_0) is

$$t \mapsto \big(x_0, \tan(t + \arctan(y_0))\big).$$

Hence, for example,

$$d_X\left((0,0), \left(0, \tan\left(\frac{\pi}{2} - \frac{1}{n}\right)\right)\right) \le \pi/2 - 1/n < \pi/2,$$

but

$$d_E\left((0,0), \left(0, \tan\left(\frac{\pi}{2} - \frac{1}{n}\right)\right)\right) \longrightarrow \infty \quad \text{as } n \to \infty.$$

Ex. 28) Prove the following linear algebra result.

Lemma 5.16.10. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *be vectors in* $\mathbb{R}^N$. *Suppose* $\mathbf{w} \in \mathbb{R}^N$ *is such that*

$$(\star) \qquad \langle \mathbf{w}, \mathbf{x} \rangle^2 \le \sum_{j=1}^{m} \langle \mathbf{v}_j, \mathbf{x} \rangle^2 \quad \forall\, \mathbf{x} \in \mathbb{R}^N.$$

Then there exist scalars $\alpha_1, \ldots, \alpha_m$ *such that*

$$\mathbf{w} = \sum_{j=1}^{m} \alpha_j\, \mathbf{v}_j \quad \textit{and} \quad \sum_{j=1}^{m} \alpha_j^2 \le 1.$$

Proof. Let us first prove that $\mathbf{w} \in \mathrm{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\} =: V$. From the decomposition $\mathbb{R}^N = V \oplus V^\perp$ it follows

$$\mathbf{w} = \mathbf{a} + \mathbf{b}, \qquad \mathbf{a} \in V, \quad \mathbf{b} \in V^\perp.$$

If we choose $\mathbf{x} := \mathbf{b}$ in ($\star$), we get

$$\|\mathbf{b}\|^4 = \langle \mathbf{a} + \mathbf{b}, \mathbf{b} \rangle^2 \le \sum_{j=1}^{m} \langle \mathbf{v}_j, \mathbf{b} \rangle^2 = 0,$$

whence $\mathbf{b} = 0$, i.e. $\mathbf{w} = \mathbf{a} \in V$.
Up to a permutation of the $\mathbf{v}_j$'s, it is not restrictive to suppose that $(\mathbf{v}_1, \dots, \mathbf{v}_q)$ is a basis of V. Then we have

$$\mathbf{v}_j = \sum_{k=1}^{q} \beta_{j,k}\, \mathbf{v}_k \quad \forall\, j = 1, \dots, m.$$

The $m \times q$ matrix B whose (j, k)-th entry is $\beta_{j,k}$ has the block form

$$B = \begin{pmatrix} \mathbb{I}_q \\ \widehat{B} \end{pmatrix},$$

where $\mathbb{I}_q$ is the identity matrix of order q. Let $\gamma_1, \dots, \gamma_q \in \mathbb{R}$ be such that $\mathbf{w} = \sum_{k=1}^{q} \gamma_k\, \mathbf{v}_k$. We characterize *all* the scalars $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ such that $\mathbf{w} = \sum_{j=1}^{m} \alpha_j\, \mathbf{v}_j$. From the identity

$$\sum_{k=1}^{q} \gamma_k\, \mathbf{v}_k = \mathbf{w} = \sum_{k=1}^{q} \left(\sum_{j=1}^{m} \alpha_j\, \beta_{j,k} \right) \mathbf{v}_k$$

and the linear independence of $\mathbf{v}_1, \dots, \mathbf{v}_q$ we infer

$$(\bullet) \qquad \gamma_k = \sum_{j=1}^{m} \alpha_j\, \beta_{j,k} \quad \forall\, k = 1, \dots, q.$$

Let us suppose that, besides ($\bullet$), there also exists a solution $\bar{\mathbf{x}} \in \mathbb{R}^N$ to the m-equation system

$$(\mathrm{SL}) \qquad \begin{cases} \langle \mathbf{v}_j, \mathbf{x} \rangle = \alpha_j, \\ j = 1, \dots, m. \end{cases}$$

This will give

$$\left(\sum_{j=1}^{m} \alpha_j^2 \right)^2 = \left(\sum_{j=1}^{m} \alpha_j\, \langle \mathbf{v}_j, \bar{\mathbf{x}} \rangle \right)^2 = \left(\left\langle \sum_{j=1}^{m} \alpha_j\, \mathbf{v}_j, \bar{\mathbf{x}} \right\rangle \right)^2$$
$$= \langle \mathbf{w}, \bar{\mathbf{x}} \rangle^2 \le \sum_{j=1}^{m} \langle \mathbf{v}_j, \bar{\mathbf{x}} \rangle^2 = \sum_{j=1}^{m} \alpha_j^2,$$

whence $\sum_{j=1}^{m} \alpha_j^2 \le 1$, and the proof is complete.

We remark that, in order to (SL) to be solvable, it is necessary and sufficient that the rank of the coefficient-matrix of (SL) (i.e. q) equals the rank of the complete-matrix of (SL): thanks to the dependence of $\mathbf{v}_{q+1}, \dots, \mathbf{v}_m$ w.r.t. $\mathbf{v}_1, \dots, \mathbf{v}_q$, this is equivalent to

$$(\bullet\bullet) \qquad \alpha_j = \sum_{k=1}^{q} \beta_{j,k}\, \alpha_k \quad \forall\, j = q+1, \dots, m.$$

Hence, the proof is complete if we show that there exists $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ satisfying $(\bullet)$ and $(\bullet\bullet)$, i.e. a solution to the m-equation and m-indeterminate system $\alpha_1, \dots, \alpha_m$

$$\text{(SL)'} \qquad \begin{cases} \displaystyle\sum_{j=1}^{m} \alpha_j\, \beta_{j,k} = \gamma_k, & k = 1, \dots, q, \\ \displaystyle\sum_{k=1}^{q} \beta_{j,k}\, \alpha_k - \alpha_j = 0, & j = q+1, \dots, m. \end{cases}$$

The coefficient-matrix of (SL)' is

$$\begin{pmatrix} \mathbb{I}_q & {}^t\widehat{B} \\ \widehat{B} & -\mathbb{I}_{m-q} \end{pmatrix}.$$

We show that this matrix is invertible, whence (SL)' is solvable.
In general, if A is a real $q \times (m-q)$ matrix, the block matrix

$$P := \begin{pmatrix} \mathbb{I}_q & A \\ {}^tA & -\mathbb{I}_{m-q} \end{pmatrix} \quad \text{satisfies} \quad P^2 = \mathbb{I}_m + \begin{pmatrix} A & 0 \\ 0 & {}^tA \end{pmatrix} \cdot \begin{pmatrix} {}^tA & 0 \\ 0 & A \end{pmatrix}.$$

Observe that P^2 is the sum of a positive-definite matrix plus a positive semi-definite one. In particular, P^2 is positive-definite, hence it is not singular. This gives $|\det P| = \sqrt{\det P^2} > 0$, so that P is not singular too. This ends the proof. $\square$

By means of Lemma 5.16.10, derive the following equivalent characterization of X-subunit curve.

Proposition 5.16.11. *Let $X = \{X_1, \dots, X_m\}$ be a system of locally Lipschitz-continuous vector fields on an open set $\Omega \subseteq \mathbb{R}^N$.*
Let $\gamma : [0, T] \to \Omega$ be an absolutely continuous curve. Then γ is X-subunit if and only if there exist measurable functions $c_j : [0, T] \to \mathbb{R}$, $j = 1, \dots, m$, such that, almost everywhere on $[0, T]$,

$$\dot{\gamma}(t) = \sum_{j=1}^{m} c_j(t)\, X_j I(\gamma(t)), \quad \textit{and} \quad \sum_{j=1}^{m} \big(c_j(t)\big)^2 \le 1. \tag{5.152}$$

Proof. Suppose $\gamma : [0, T] \to \Omega$, satisfies (5.152). The Cauchy-Schwartz inequality in $\mathbb{R}^m$ immediately yields

$$\langle \dot\gamma(t), \xi \rangle^2 \le \left(\sum_{j=1}^{m} (c_j(t))^2 \right) \cdot \left(\sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 \right) \le \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2,$$

i.e. γ is a X-subunit curve. Vice versa, let γ be X-subunit. If we apply Lemma 5.16.10 with the choice

$$\mathbf{v}_j := X_j(\gamma(t)), \quad j = 1, \dots, m, \quad \mathbf{w} := \dot\gamma(t),$$

then (5.152) is satisfied for suitable scalar functions c_j's defined a.e. on $[0, T]$. It is easy to prove[16] the measurability of the c_j's. □

Ex. 29) Let u be an $\mathcal{L}$-harmonic function in an open set $\Omega \subseteq \mathbb{G}$. Let $\overline{B_d(x_0, r)} \subset \Omega$. Then, for every multi-index α, there exists a constant $C_\alpha > 0$ (independent of u, x_0 and r) such that

$$\left| X^\alpha u(x_0) \right| \le C_\alpha \, r^{-|\alpha|_{\mathbb{G}}} \sup_{B_d(x_0, r)} |u|.$$

(*Hint*: Since u is $\mathcal{L}$-harmonic, the representation formula (5.50a) (see also (5.50d)) gives

$$u(x_0) = \int_{\mathbb{R}^N} u(z) \, \phi_r(x^{-1} \circ z) \, \mathrm{d}z = \int_{\mathbb{R}^N} u(z) \, \widehat{\phi_r}(x^{-1} \circ z) \, \mathrm{d}z,$$

where $\widehat{\phi_r}(\zeta) = \phi_r(\zeta^{-1})$. Now, derive the integral.)

[16] Indeed, notice that $\dot\gamma$ is measurable (since γ is a absolutely continuous), $t \mapsto X_j(\gamma(t))$ is continuous and the operations which provide the components of a vector w.r.t. a system of vectors are continuous. We leave the details to the reader.

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