5

The Gurov–Reshetnyak Class of Functions

5.1 Embedding in the Gehring Class

The BMO-class is closely related to the class of functions, which was studied first by Gurov and Reshetnyak in [21, 22]. This class could be also defined in terms of mean oscillations of functions in the following way.

Let $Q_0 \subset \mathbb{R}^d$ be a fixed cube. We will say that the non-negative function $f \in L(Q_0)$ satisfies the *Gurov–Reshetnyak condition*, if

$$\Omega(f;Q) \le \varepsilon f_Q, \quad Q \subset Q_0, \tag{5.1}$$

where the constant ε does not depend on the cube Q. The class of all such functions f is called *the Gurov–Reshetnyak class*. We will denote it by $GR \equiv GR(\varepsilon) \equiv GR(\varepsilon, Q_0)$. Often inequality (5.1) is called *the Gurov–Reshetnyak inequality.*

Remark 5.1. Obviously, for $\varepsilon = 2$ inequality (5.1) holds, but if $\varepsilon < 2$ in general it is no more true. Indeed, for the function $f_N(x) = N\chi_{[0,\frac{1}{N}]}(x),\ x \in [0,1] \equiv Q_0$, we have $(f_N)_{Q_0} = 1,\ \Omega(f_N;Q_0) = 2\left(1 - \frac{1}{N}\right)$, so that

$$\frac{\Omega(f_N;Q_0)}{(f_N)_{Q_0}} \to 2, \quad N \to \infty.$$

Hence condition (5.1) is substantial only for $0 < \varepsilon < 2$.

Remark 5.2. For any $\varepsilon < 2$ if the function f satisfies condition (5.1), then it is either positive almost everywhere or equivalent to zero. This fact is a consequence of the following statement.

Proposition 5.3. *If $f \not\sim 0$ is a non-negative locally summable function such that $|\{x:\ f(x) = 0\}| > 0$, then*

$$\sup_Q \frac{\Omega(f;Q)}{f_Q} = 2. \tag{5.2}$$

Proof. Denote $A = \{x : \ f(x) = 0\}$. Since by virtue of Lebesgue theorem 1.1 almost every point of the set A is its density point for any $\delta > 0$ there exists a cube Q such that $f_Q > 0$ and

$$|E| < \frac{\delta}{2}|Q|,$$

for $E \equiv Q \setminus A$. Then the equality

$$\int_{\{x \in E :\, f(x) \le f_Q\}} f(x)dx \le f_Q |\{x \in E :\, f(x) \le f_Q\}| =$$
$$= \frac{|\{x \in E :\, f(x) \le f_Q\}|}{|Q|} \int_E f(x)dx$$

implies

$$\frac{|\{x \in E :\, f(x) > f_Q\}|}{|Q|} \int_E f(x)dx + \int_{\{x \in E :\, f(x) \le f_Q\}} f(x)dx \le$$
$$\le \frac{1}{|Q|} \left(|\{x \in E :\, f(x) > f_Q\}| + |\{x \in E :\, f(x) \le f_Q\}|\right) \int_E f(x)dx =$$
$$= \frac{|E|}{|Q|} \int_E f(x)dx < \frac{\delta}{2} \int_E f(x)dx.$$

Thus

$$\frac{|Q|}{2} (\Omega(f;Q) - (2-\delta) f_Q) =$$
$$= \int_{\{x \in E :\, f(x) > f_Q\}} (f(x) - f_Q)dx - \frac{2-\delta}{2} \int_E f(x)dx =$$
$$= \int_{\{x \in E :\, f(x) > f_Q\}} f(x)dx - f_Q |\{x \in E :\, f(x) > f_Q\}| -$$
$$- \int_E f(x)dx + \frac{\delta}{2} \int_E f(x)dx \ge$$
$$\ge \int_{\{x \in E :\, f(x) \le f_Q\}} f(x)dx - \frac{|\{x \in E :\, f(x) > f_Q\}|}{|Q|} \int_E f(x)dx +$$
$$+ \frac{|\{x \in E :\, f(x) > f_Q\}|}{|Q|} \int_E f(x)dx - \int_{\{x \in E :\, f(x) \le f_Q\}} f(x)dx = 0,$$

i.e.,

$$\frac{\Omega(f;Q)}{f_Q} \ge 2 - \delta.$$

Since $\delta > 0$ is arbitrary equality (5.2) follows. $\square$

The fundamental property of the GR-class, which stipulates numerous applications, is described by the following theorem.

Theorem 5.4 (Gurov, Reshetnyak, [22]). *There exists a number $\varepsilon_0 \equiv \varepsilon_0(d)$, $0 < \varepsilon_0 \le 2$ such that for any ε, $0 < \varepsilon < \varepsilon_0$, one can find $p_0 \equiv p_0(\varepsilon, d) > 1$ such that if the function f satisfies condition* (5.1)*, then $f \in L^p(Q_0)$ for any $p < p_0$.*

Various proofs, generalizations and refinements of this theorem were found by a number of different authors ([27, 4, 15, 76, 13, 14] and others). In what follows we will derive this theorem as a corollary of a more general result (see Corollary 5.8).

Let us introduce the following quantity

$$\nu(f;\sigma) \equiv \sup_{l(Q)\le\sigma} \frac{\Omega(f;Q)}{f_Q}, \quad 0 < \sigma \le l\,(Q_0)\,.$$

The supremum here is taken over all cubes $Q \subset Q_0$ such that their side-lengths $l(Q)$ are less or equal than σ. In addition, if $f_Q = 0$ for some cube $Q \subset Q_0$, then $\Omega(f;Q) = 0$, provided f is non-negative on Q_0. In this case we assume that $\frac{\Omega(f;Q)}{f_Q} = 0$. According to Remark 5.1, for any $f \in L(Q_0)$ we have $\nu(f;\sigma) \le 2$ for $0 < \sigma \le l\,(Q_0)$. Some properties of the function f in terms of $\nu(f;\sigma)$ were studied in [13, 14].

Theorem 5.5 ([37]). *Let $Q_0 \subset \mathbb{R}^d$ be a cube , and let the function $f \in L(Q_0)$ be non-negative. Then*

$$\frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\,du \le 3\cdot 2^d \nu\left(f; 2t^{1/d}\right) f^{**}(t), \quad 0 < t \le 2^{-d}\,|Q_0|\,. \tag{5.3}$$

For the proof of this theorem we will need the following refinement of Calderón–Zygmund lemma 1.14.

Lemma 5.6 (Calderón, Zygmund, [70]). *Let f be a non-negative function, summable on the cube $Q_0 \subset \mathbb{R}^d$, and let $\alpha \ge \frac{1}{|Q_0|}\int_{Q_0} f(x)\,dx$. Then there exist cubes $Q_j \subset Q'_j \subset Q_0$, $j = 1, 2, \ldots$, with pairwise disjoint interiors such that $\left|Q'_j\right| = 2^d\,|Q_j|$,*

$$f_{Q'_j} \le \alpha < f_{Q_j} \le 2^d \alpha, \tag{5.4}$$

and

$$f(x) \le \alpha \ \text{ for almost all } \ x \in Q_0 \setminus \left(\bigcup_{j\ge1} Q_j\right). \tag{5.5}$$

Proof. Essentially the proof of this lemma repeats the proof of Lemma 1.14. We only have to notice that as the cube Q'_j it is enough to take the cube, whose partition results the dyadic cube Q_j, $j = 1, 2, \ldots$. Clearly, in this case the left inequality of (5.4) holds true. The other statements of the lemma follow from Lemma 1.14. $\square$

Proof of Theorem 5.5. Let us fix some t, $0 < t \le 2^{-d}|Q_0|$. Applying Lemma 5.6 with $\alpha = f^{**}(t)$, we obtain the cubes Q_j and Q'_j, which satisfy the properties stated by this lemma.

Denote $E = \cup_{j\ge 1} Q_j$. Using Property 2.1, together with the definition of the rearrangement f^* and (5.5), we obtain

$$\int_0^t |f^*(u) - f^{**}(t)|\, du = 2\int_{\{u:\ f^*(u)>\alpha\}} (f^*(u)-\alpha)\, du =$$

$$= 2\int_{\{x\in Q_0:\ f(x)>\alpha\}} (f(x)-\alpha)\, dx = 2\int_{\{x\in Q_0:\ f(x)>\alpha\}\cap E} (f(x)-\alpha)\, dx.$$

Since the interiors of the cubes Q_j are pairwise disjoint

$$\int_0^t |f^*(u) - f^{**}(t)|\, du = 2\sum_{j\ge 1}\int_{\{x\in Q_0:\ f(x)>\alpha\}\cap Q_j} (f(x)-\alpha)\, dx =$$

$$= 2\sum_{j\ge 1}\int_{\{x\in Q_j:\ f(x)>\alpha\}} (f(x)-\alpha)\, dx =$$

$$= 2\sum_{j\ge 1}\int_{\{x\in Q_j:\ f(x)>\alpha\}} \left(f(x)-f_{Q_j}\right) dx +$$

$$+2\sum_{j\ge 1}\left(f_{Q_j}-\alpha\right)|\{x\in Q_j:\ f(x)>\alpha\}| =$$

$$= 2\sum_{j\ge 1}\int_{\left\{x\in Q_j:\ f(x)>f_{Q_j}\right\}} \left(f(x)-f_{Q_j}\right) dx +$$

$$+2\sum_{j\ge 1}\int_{\left\{x\in Q_j:\ \alpha<f(x)\le f_{Q_j}\right\}} \left(f(x)-f_{Q_j}\right) dx +$$

$$+2\sum_{j\ge 1}\left(f_{Q_j}-\alpha\right)|\{x\in Q_j:\ f(x)>\alpha\}| \equiv S_1+S_2+S_3. \tag{5.6}$$

In order the estimate $S_i,\ i=1,2,3$, let us notice that, by (5.4),

$$\frac{1}{|E|}\int_E f(x)\,dx = \frac{1}{|E|}\sum_{j\ge 1}\int_{Q_j} f(x)\,dx =$$

$$= \frac{1}{|E|}\sum_{j\ge 1}|Q_j|\, f_{Q_j} \ge \frac{\alpha}{|E|}\sum_{j\ge 1}|Q_j| = \alpha.$$

Therefore, by the definition of the rearrangement f^*,

$$\frac{1}{t}\int_0^t f^*(u)\,du = f^{**}(t) = \alpha \le \frac{1}{|E|}\int_E f(x)\,dx \le \frac{1}{|E|}\int_0^{|E|} f^*(u)\,du.$$

From here it follows that

$$|E| \le t, \tag{5.7}$$

provided f^* is monotone, so that $|Q_j| \le t$ and $\left|Q_j'\right| \le 2^d t,\ j = 1,2,\ldots$. Therefore, by Property 2.1,

$$S_1 = 2\sum_{j\ge1}\int_{\left\{x\in Q_j:\ f(x)>f_{Q_j}\right\}}\left(f(x) - f_{Q_j}\right)\,dx = \sum_{j\ge1}\int_{Q_j}\left|f(x) - f_{Q_j}\right|\,dx =$$

$$= \sum_{j\ge1}\frac{\Omega\left(f;Q_j\right)}{f_{Q_j}}\int_{Q_j} f(x)\,dx \le \nu\left(f;t^{1/d}\right)\sum_{j\ge1}\int_{Q_j} f(x)\,dx \le$$

$$\le \nu\left(f;t^{1/d}\right)\sum_{j\ge1}\int_{Q_j'} f(x)\,dx, \tag{5.8}$$

provided $\nu(f;\sigma)$ is monotone. Further, by (5.4),

$$S_3 = 2\sum_{j\ge1}\left(f_{Q_j} - \alpha\right)|\{x\in Q_j:\ f(x)>\alpha\}| \le 2\sum_{j\ge1}\left(f_{Q_j} - f_{Q_j'}\right)|Q_j| \le$$

$$\le 2\sum_{j\ge1}\int_{Q_j}\left|f(x) - f_{Q_j'}\right|\,dx \le 2\sum_{j\ge1}\int_{Q_j'}\left|f(x) - f_{Q_j'}\right|\,dx =$$

$$= 2\sum_{j\ge1}\frac{\Omega\left(f;Q_j'\right)}{f_{Q_j'}}\int_{Q_j'} f(x)\,dx \le 2\nu\left(f;2t^{1/d}\right)\sum_{j\ge1}\int_{Q_j'} f(x)\,dx. \tag{5.9}$$

Taking into account that $S_2 \le 0$, from (5.6), (5.8), (5.9) and from the monotonicity of $\nu(f;\sigma)$ we obtain

$$\int_0^t |f^*(u) - f^{**}(t)|\,du \le 3\nu\left(f;2t^{1/d}\right)\sum_{j\ge1}\int_{Q_j'} f(x)\,dx.$$

Now the application of (5.4) and (5.7) leads to the inequality

$$\int_0^t |f^*(u) - f^{**}(t)|\,du \le 3\nu\left(f;2t^{1/d}\right)\cdot\alpha\sum_{j\ge1}\left|Q_j'\right| =$$

$$= 3\alpha\cdot2^d\nu\left(f;2t^{1/d}\right)\sum_{j\ge1}|Q_j| = 3\alpha\cdot2^d\nu\left(f;2t^{1/d}\right)|E| \le 3\alpha\cdot2^d\nu\left(f;2t^{1/d}\right)\cdot t.$$

Since $\alpha = f^{**}(t)$ the last inequality is equivalent to (5.3). $\square$

Inequality (5.3) leads to the following estimate of the rearrangement of the function f in terms of $\nu(f;\sigma)$.

Theorem 5.7 ([37]). *There exist constants* $c_1 \equiv c_1(d)$ *and* $c_2 \equiv c_2(d)$ *such that for any cube* $Q_0 \subset \mathbb{R}^d$ *and for any non-negative function* $f \in L(Q_0)$

$$f^{**}(t) \le c_1 f_{Q_0} \cdot \exp\left(c_2 \int_{t^{1/d}}^{l(Q_0)} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}\right), \quad 0 < t \le |Q_0|\,. \tag{5.10}$$

Proof. Applying Lemma 2.2 to the function $\varphi = f^*$ with $a = 2$ and using Theorem 5.5, for $0 < t \le 2^{-d}|Q_0|$ we get

$$f^{**}\left(\frac{t}{2}\right) - f^{**}(t) \le \frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\,du \le 3\cdot 2^d \nu\left(f; 2t^{1/d}\right) f^{**}(t),$$

or, equivalently,

$$f^{**}\left(\frac{t}{2}\right) \le \left(1 + 3\cdot 2^d \nu\left(f; 2t^{1/d}\right)\right) f^{**}(t), \quad 0 < t \le 2^{-d}|Q_0|\,.$$

The recurrent application of the last inequality yields

$$f^{**}\left(2^{-d-s}|Q_0|\right) \le f^{**}\left(2^{-d}|Q_0|\right)\cdot \prod_{i=1}^{s}\left(1 + 3\cdot 2^d \nu\left(f; \left(2^{-s+i}|Q_0|\right)^{1/d}\right)\right) =$$

$$= f^{**}\left(2^{-d}|Q_0|\right)\cdot \exp\left(\sum_{i=0}^{s-1} \ln\left(1 + 3\cdot 2^d \nu\left(f; 2^{-i/d} l\,(Q_0)\right)\right)\right) \le$$

$$\le f^{**}\left(2^{-d}|Q_0|\right)\cdot \exp\left(3\cdot 2^d \sum_{i=0}^{s-1} \nu\left(f; 2^{-i/d} l\,(Q_0)\right)\right), \quad s = 1, 2, \dots$$

On the other hand,

$$\nu\left(f; 2^{-i/d} l\,(Q_0)\right) \le \frac{2^{1/d}}{2^{1/d}-1}\int_{2^{-i/d}l(Q_0)}^{2^{-(i-1)/d}l(Q_0)} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}, \quad i = 1, \dots, s-1,$$

provided $\nu(f;\sigma)$ is monotone. Therefore, taking into account that $\nu(f; l\,(Q_0)) \le 2$, we get

$$f^{**}\left(2^{-d-s}|Q_0|\right) \le \exp\left(3\cdot 2^{d+1}\right) f^{**}\left(2^{-d}|Q_0|\right) \times$$

$$\times \exp\left(3\cdot 2^d \cdot \frac{2^{1/d}}{2^{1/d}-1}\sum_{i=1}^{s-1}\int_{2^{-i/d}l(Q_0)}^{2^{-(i-1)/d}l(Q_0)} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}\right) \le$$

$$\leq 2^d \exp\left(3 \cdot 2^{d+1}\right) f_{Q_0} \exp\left(3 \cdot 2^d \cdot \frac{2^{1/d}}{2^{1/d}-1} \int_{2^{-(s-1)/d} l(Q_0)}^{l(Q_0)} \nu(f;\sigma) \frac{d\sigma}{\sigma}\right).$$

Fix some $t \in (0, |Q_0|]$ and choose $s \in \mathbb{N}$ such that $2^{-s} |Q_0| < t \leq 2^{-s+1} |Q_0|$. Then, since f^{**} is monotone the last inequality implies

$$f^{**}(t) \leq f^{**}\left(2^{-s} |Q_0|\right) \leq f^{**}\left(2^{-d-s} |Q_0|\right) \leq$$

$$\leq 2^d \exp\left(3 \cdot 2^{d+1}\right) f_{Q_0} \exp\left(3 \cdot 2^d \cdot \frac{2^{1/d}}{2^{1/d}-1} \int_{2^{(-s+1)/d} l(Q_0)}^{l(Q_0)} \nu(f;\sigma) \frac{d\sigma}{\sigma}\right) \leq$$

$$\leq c_1 \cdot f_{Q_0} \exp\left(c_2 \int_{t^{1/d}}^{l(Q_0)} \nu(f;\sigma) \frac{d\sigma}{\sigma}\right),$$

where $c_1 = 2^d \exp\left(3 \cdot 2^{d+1}\right)$, $c_2 = 3 \cdot 2^d \cdot \frac{2^{1/d}}{2^{1/d}-1}$. □

Corollary 5.8 (Gurov-Reshetnyak theorem 5.4**).** *For any ε such that*

$$0 < \varepsilon < \varepsilon_0(d) \equiv \frac{d}{c_2} = \frac{d\left(2^{1/d}-1\right)}{3 \cdot 2^d \cdot 2^{1/d}},$$

there exists $p_0 \equiv p_0(\varepsilon, d) > 1$ such that if the function f satisfies condition (5.1)*, then $f \in L^p(Q_0)$ for any $p < p_0$.*

Proof. Condition (5.1) is equivalent to the inequality $\nu(f;\sigma) \leq \varepsilon$ for $0 < \sigma \leq l(Q_0)$. Thus (5.10) implies

$$f^{**}(t) \leq c_1 \cdot f_{Q_0} \cdot \exp\left(\frac{c_2}{d} \cdot \varepsilon \cdot \ln \frac{|Q_0|}{t}\right) = c_1 \cdot f_{Q_0} \left(\frac{|Q_0|}{t}\right)^{\frac{\varepsilon}{\varepsilon_0(d)}}, \quad 0 < t \leq |Q_0|. \tag{5.11}$$

Since $\varepsilon < \varepsilon_0(d)$ we have

$$p_0(\varepsilon, d) \equiv \frac{\varepsilon_0(d)}{\varepsilon} = \frac{1}{\varepsilon} \cdot \frac{d\left(2^{1/d}-1\right)}{3 \cdot 2^d \cdot 2^{1/d}} > 1. \tag{5.12}$$

If $p < p_0(\varepsilon, d)$, then, by (5.11),

$$\left(f^{**}(t)\right)^p \leq c_1^p \left(f_{Q_0}\right)^p \left(\frac{|Q_0|}{t}\right)^{\frac{\varepsilon}{\varepsilon_0(d)} \cdot p}, \quad 0 < t \leq |Q_0|.$$

As $p < p_0(\varepsilon, d) = \frac{\varepsilon_0(d)}{\varepsilon}$, we have

$$\|f\|_p^p = \int_0^{|Q_0|} \left(f^*(t)\right)^p dt \leq \int_0^{|Q_0|} \left(f^{**}(t)\right)^p dt \leq c_1^p \frac{\varepsilon_0(d)}{\varepsilon_0(d) - p\varepsilon} |Q_0| \left(f_{Q_0}\right)^p. \; \square \tag{5.13}$$

Remark 5.9. Fix the cube $Q_0 \subset \mathbb{R}^d$ and choose some $Q \subset Q_0$. Then (5.1) implies that $f \in GR(\varepsilon, Q)$. Hence, applying Corollary 5.8 to the cube Q, one can rewrite inequality (5.13) in the following way:

$$\left\{ \frac{1}{|Q|} \int_Q f^p(x)\, dx \right\}^{1/p} \le c_3 \frac{1}{|Q|} \int_Q f(x)\, dx, \quad Q \subset Q_0, \tag{5.14}$$

where $1 < p < p_0(\varepsilon, d)$, $c_3 = c_3(\varepsilon, p, d)$, and c_3 does not depend on $Q \subset Q_0$. Inequality (5.14) is called *the reverse Hölder inequality*, or *the Gehring inequality* [18].

Corollary 5.10. *If the function f is non-negative on the cube $Q_0 \subset \mathbb{R}^d$ and satisfies Gurov–Reshetnyak condition* (5.1) *for some $\varepsilon < \varepsilon_0(d)$, then it also satisfies Gehring inequality* (5.14) *for all $p < p_0(\varepsilon, d)$.*

Remark 5.11. In Gurov–Reshetnyak theorem 5.4 we have found that $p_0(\varepsilon, d) = \frac{\varepsilon_0(d)}{\varepsilon} \to \infty$ as $\varepsilon \to 0+$. The estimate $p_0(\varepsilon, d) = \underline{O}\left(\frac{1}{\varepsilon}\right)$, $\varepsilon \to 0+$, was first obtained in [4, 76]. As it was noticed in [4], it turns out that this limiting behavior cannot be improved. In what follows we will find the maximal possible value of $p_0(\varepsilon, 1)$ (see Corollary 5.35).

Remark 5.12. The proof of inequality (5.10) is based on the application of Lemma 2.2 with $a = 2$. Generally speaking, the parameter $a > 1$ in Lemma 2.2 could be chosen in the "better way" in order to minimize the value of the constant c_2 in the exponent in right-hand side of (5.10). This could allow to slightly increase the values of $\varepsilon_0(d)$ and $p_0(\varepsilon, d)$, obtained in Corollary 5.8. However, this method does not lead to the desired result (i.e. $\varepsilon_0(d) = 2$ and maximal possible $p_0(\varepsilon, d)$), because in order to prove (5.9) we have used estimate (5.3), which is overstated in the sense of constants.

From Theorem 5.7 one can derive the following

Corollary 5.13 (Franciosi, [13]). *If the function $f \in L(Q_0)$ is non-negative on the cube $Q_0 \subset \mathbb{R}^d$ and $\nu(f;\sigma) \to 0$ as $\sigma \to 0+$, then $f \in L^p(Q_0)$ for any $p < \infty$.*

Proof. Take some $p < \infty$ and choose t_0, $0 < t_0 \le |Q_0|$, such that $\nu(f;\sigma) \le \frac{d}{2c_2 p}$ for $0 < \sigma \le t_0^{1/d}$. Then, by (5.10), for $0 < t \le t_0$

$$f^{**}(t) \le c_1 \cdot f_{Q_0} \cdot \exp\left(c_2 \int_{t_0^{1/d}}^{|Q_0|} \nu(f;\sigma)\, \frac{d\sigma}{\sigma} \right) \cdot \left(\frac{t_0}{t} \right)^{\frac{1}{2p}}.$$

Obviously this implies that the function $(f^{**})^p$ is summable on $[0, |Q_0|]$, and hence $f \in L^p(Q_0)$. □

It is clear that inequality (5.10) provides the sufficient conditions for f to belong to the various Orlicz spaces in terms of the order of $\nu(f;\sigma)$ as $\sigma \to 0+$. If $\nu(f;\sigma)$ is such that

$$\int_0^{l(Q_0)} \nu(f;\sigma)\,\frac{d\sigma}{\sigma} < \infty, \tag{5.15}$$

then (5.10) immediately implies the following result.

Corollary 5.14. *If the function* $f \in L(Q_0)$ *is non-negative on the cube* $Q_0 \subset \mathbb{R}^d$ *and satisfies* (5.15)*, then* $f \in L^\infty(Q_0)$*.*

Corollary 5.14 can be sharpen. In order to show this, we will need one result due to Spanne [69] (see also [7, 57]). Denote

$$\nu_1(f;\delta) = \sup_{Q\subset Q_0,\ l(Q)\le\delta} \Omega(f;Q), \quad 0<\delta\le l(Q_0).$$

Theorem 5.15 (Spanne, [69]). *If* $f \in L(Q_0)$ *is such that*

$$\int_0^{l(Q_0)} \nu_1(f;\delta)\,\frac{d\delta}{\delta} < \infty, \tag{5.16}$$

then to any cube $Q \subset Q_0$ *with* $l(Q) \le \frac{1}{2}l(Q_0)$

$$\operatorname*{ess\,sup}_{x\in Q} |(f-f_Q)(x)| \le c_d \int_0^{2l(Q)} \nu_1(f;\delta)\,\frac{d\delta}{\delta}, \tag{5.17}$$

where the constant c_d *depends only on the dimension* d *of the space.*

Proof. Fix the cube $Q \subset Q_0$ such that $l(Q) \le l(Q_0)/2$. In order to prove (5.17) without loss of generality we can assume $f_Q = 0$. According to Lebesgue theorem 1.1, we have $f(x) = \lim_{j\to\infty} f_{Q_j}$ for almost every $x \in Q$, where Q_j is a sequence of dyadic (with respect to Q) cubes of order $j = 1, 2, \ldots,$ contractible to x. Thus, for the proof of (5.17) it is enough to show that

$$\left|f_{Q_j}\right| \le c_d \int_0^{2l(Q)} \nu_1(f;\delta)\,\frac{d\delta}{\delta}, \quad j = 1, 2, \ldots \tag{5.18}$$

Let us prove (5.18). For $i \ge 1$

$$\left|f_{Q_{i+1}} - f_{Q_i}\right| \le \frac{1}{|Q_{i+1}|}\int_{Q_{i+1}} |f(x) - f_{Q_i}|\,dx \le$$

$$\le 2^d \frac{1}{|Q_i|}\int_{Q_i} |f(x) - f_{Q_i}|\,dx \le 2^d \nu_1\left(f; l(Q_i)\right),$$

while the condition $f_Q = 0$ implies

$$|f_{Q_1}| \le 2^d \nu_1(f; l(Q)).$$

Therefore

$$\left|f_{Q_j}\right| \le \left|f_{Q_1}\right| + \sum_{i=1}^{j-1} \left|f_{Q_{i+1}} - f_{Q_i}\right| \le 2^d \left(\nu_1(f; l(Q)) + \sum_{i=1}^{\infty} \nu_1\left(f; l(Q_i)\right)\right) \le$$

$$\le 2^d \left(\frac{1}{l(Q)} \int_{l(Q)}^{2l(Q)} \nu_1(f;\delta)\, d\delta + \frac{1}{l(Q) - l(Q_1)} \int_{l(Q_1)}^{l(Q)} \nu_1(f;\delta)\, d\delta + \right.$$

$$\left. + \sum_{i=2}^{\infty} \frac{1}{l(Q_{i-1}) - l(Q_i)} \int_{l(Q_i)}^{l(Q_{i-1})} \nu_1(f;\delta)\, d\delta \right) =$$

$$= 2^d \left(\frac{1}{l(Q)} \int_{l(Q)}^{2l(Q)} \nu_1(f;\delta)\, d\delta + \frac{1}{l(Q_1)} \int_{l(Q_1)}^{l(Q)} \nu_1(f;\delta)\, d\delta + \right.$$

$$\left. + \sum_{i=2}^{\infty} \frac{1}{l(Q_i)} \int_{l(Q_i)}^{l(Q_{i-1})} \nu_1(f;\delta)\, d\delta \right) \le$$

$$\le 2^{d+1} \left(\int_{l(Q)}^{2l(Q)} \nu_1(f;\delta)\, \frac{d\delta}{\delta} + \int_{l(Q_1)}^{l(Q)} \nu_1(f;\delta)\, \frac{d\delta}{\delta} + \sum_{i=2}^{\infty} \int_{l(Q_i)}^{l(Q_{i-1})} \nu_1(f;\delta)\, \frac{d\delta}{\delta} \right) =$$

$$= 2^{d+1} \int_0^{2l(Q)} \nu_1(f;\delta)\, \frac{d\delta}{\delta}. \quad \square$$

From Spanne's theorem 5.15 and condition (5.16) it follows immediately, that the function f is equivalent to some function g, which is continuous on Q_0 and such that its *modulus of continuity* $\omega(g;\sigma)$ satisfies the condition

$$\omega(g;\sigma) \equiv \sup_{x,y \in Q_0,\ |x-y| \le \sigma} |g(x) - g(y)| \le c_d \int_0^{2\sigma} \nu_1(f;\delta)\, \frac{d\delta}{\delta}, \quad 0 < \sigma \le \frac{1}{2} l(Q_0). \tag{5.19}$$

Indeed, by Lebesgue theorem 1.1, the function $g(x) \equiv \lim_{l(Q)\to 0} f_Q$ for $x \in Q_0$ is equivalent to f. If $x, y \in Q_0$ and $|x - y| \le \sigma$, then we choose the cube $Q \subset Q_0$, which contains x, y, and such that $l(Q) \le \sigma$. Then

$$|g(x) - g(y)| \le |g(x) - f_Q| + |g(y) - f_Q| \le 2 \operatorname*{ess\,sup}_{x \in Q} \left|\left(f - f_Q\right)(x)\right|$$

and Theorem 5.15 imply (5.19).

Now let us suppose that f is non-negative on the cube Q_0 and satisfies (5.15). Then (5.10) implies

$$\|f\|_\infty \le K \equiv c_1 f_{Q_0} \exp\left(c_2 \int_0^{l(Q_0)} \nu(f;\sigma)\, \frac{d\sigma}{\sigma}\right).$$

Hence for any cube $Q \subset Q_0$

$$f_Q \le \|f\|_\infty \le K.$$

Therefore

$$\nu(f;\delta) = \sup_{l(Q)\le\delta} \frac{\Omega(f;Q)}{f_Q} \ge \frac{1}{K} \sup_{l(Q)\le\delta} \Omega(f;Q) = \frac{1}{K}\nu_1(f;\delta), \quad 0<\delta\le l(Q_0). \tag{5.20}$$

So condition (5.15) implies

$$\int_0^{l(Q_0)} \nu_1(f;\delta)\,\frac{d\delta}{\delta} \le K \int_0^{|Q_0|} \nu(f;\delta)\,\frac{d\delta}{\delta} < \infty,$$

i.e., condition (5.16) is also satisfied. Now, applying Spanne theorem 5.15, from (5.20) we obtain the following statement.

Theorem 5.16. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying condition* (5.15)*. Then for any cube $Q \subset Q_0$ with $l(Q) \le \frac{1}{2} l(Q_0)$*

$$\operatorname*{ess\,sup}_{x\in Q} |(f - f_Q)(x)| \le c_d K \int_0^{2l(Q)} \nu(f;\delta)\,\frac{d\delta}{\delta}.$$

In the same way as the estimate of the modulus of continuity follows from Spanne theorem 5.15, the last Theorem implies

Corollary 5.17 ([37]). *If the function f is non-negative on the cube $Q_0 \subset \mathbb{R}^d$ and satisfies condition* (5.15)*, then f is equivalent to the function g, which is continuous on Q_0 and*

$$\omega(g;\delta) = \underline{O}\left(\int_0^{\delta} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}\right), \quad \delta\to 0.$$

Remark 5.18. Inequality (5.20) was obtained under assumption (5.15). This assumption is necessary to guarantee that $\nu(f;\delta)$ dominates $\nu_1(f;\delta)$. Indeed, let us consider the function $f_0(x) = \ln\frac{1}{x}$, $0 < x \le \beta_0$, where $\beta_0 > 0$ is small enough. Then, as it was shown in Example 2.24, $\nu_1(f_0;\delta) = \Omega(f;[0,\delta]) = \frac{2}{\mathrm{e}}$, and

$$\nu(f_0;\delta) \le \frac{1}{\ln\frac{1}{\beta_0}}\Omega(f_0;[0,\delta]) = \frac{2/\mathrm{e}}{\ln\frac{1}{\beta_0}}, \quad 0<\delta\le\beta_0.$$

So, for the function f_0 inequality (5.20) fails for any K, which does not depend on β_0.

Let us come back to Gurov–Reshetnyak theorem 5.4. As we remarked in Corollary 5.8, Gurov–Reshetnyak condition (5.1) for $0 < \varepsilon < \varepsilon_0(d)$ implies that f is summable for some $p > 1$. On the other hand, condition (5.1) is non-trivial for any $\varepsilon < 2$. In this context the following question is natural. *For which $\varepsilon < 2$ one can increase the exponent of summability of the function f, using condition* (5.1)*?* The next theorem provides the answer to this question.

Theorem 5.19 (Coifman, Fefferman, [8]). *Let $f \in L(Q_0)$ be a non-negative summable function on the cube $Q_0 \subset \mathbb{R}^d$. If*

$$|\{x \in Q : \ f(x) > \sigma \cdot f_Q\}| > \theta \cdot |Q|, \quad Q \subset Q_0, \tag{5.21}$$

where the constants $0 < \sigma, \theta < 1$ do not depends on Q, then there exists $r \equiv r(\sigma, \theta, d) > 0$ such that $f \in L^{1+r}(Q_0)$ and

$$\left\{ \frac{1}{|Q_0|} \int_{Q_0} f^{1+r}(x)\, dx \right\}^{\frac{1}{1+r}} \leq c \frac{1}{|Q_0|} \int_{Q_0} f(x)\, dx \tag{5.22}$$

with $c = c(\sigma, \theta, d, r)$.

Proof. First we prove the inequality

$$\int_{\{x \in Q_0 : \ f(x) > \alpha\}} f(x)\, dx \leq c' \cdot \alpha \, |\{x \in Q_0 : \ f(x) > \sigma \cdot \alpha\}| \,, \tag{5.23}$$

where $\alpha \geq f_{Q_0}$ and the constant c' depends only on θ and d. Let us fix some $\alpha \geq f_{Q_0}$ and apply Caldrón–Zygmund lemma 1.14. Then we obtain a collection of cubes $Q_j \subset Q_0, \ j = 1, 2, \ldots$, with pairwise disjoint interiors such that

$$\alpha < \frac{1}{|Q_j|} \int_{Q_j} f(x)\, dx \leq 2^d \alpha,$$

and $f(x) \leq \alpha$ for almost all $x \in Q_0 \setminus \left(\bigcup_{j \geq 1} Q_j \right)$. From here, using (5.21), we have

$$\int_{\{x \in Q_0 : \ f(x) > \alpha\}} f(x)\, dx \leq \sum_{j \geq 1} \int_{Q_j} f(x)\, dx \leq$$

$$\leq 2^d \alpha \sum_{j \geq 1} |Q_j| \leq \frac{2^d \alpha}{\theta} \sum_{j \geq 1} \left| \{x \in Q_j : \ f(x) > \sigma \cdot f_{Q_j}\} \right| \leq$$

$$\leq \frac{2^d}{\theta} \cdot \alpha \sum_{j \geq 1} |\{x \in Q_j : \ f(x) > \sigma \cdot \alpha\}| \leq c' \cdot \alpha \cdot |\{x \in Q_0 : \ f(x) > \sigma \cdot \alpha\}| \,,$$

where $c' = 2^d/\theta$, and this proves (5.23).

Now, multiplying (5.23) by α^{r-1} and integrating, we find that

$$\int_{f_{Q_0}}^{\infty} \alpha^{r-1} \left(\int_{\{x \in Q_0 : \ f(x) > \alpha\}} f(x)\, dx \right) d\alpha \leq$$

$$\leq c' \int_0^\infty \alpha^r \left|\{x \in Q_0 : \ f(x) > \sigma \cdot \alpha\}\right| d\alpha =$$

$$= \frac{c'}{\sigma^{1+r}} \int_0^\infty \tau^r \left|\{x \in Q_0 : \ f(x) > \tau\}\right| d\tau = c'' \int_{Q_0} f^{1+r}(x)\, dx, \tag{5.24}$$

where $c'' = c' \cdot \sigma^{-1-r}/(1+r)$. On the other hand, using the Fubini theorem one can get the following estimate for the left hand-side of the last inequality

$$\int_{f_{Q_0}}^\infty \alpha^{r-1} \left(\int_{\{x \in Q_0: \ f(x) > \alpha\}} f(x)\, dx \right) d\alpha =$$

$$= \int_{\left\{x \in Q_0: \ f(x) > f_{Q_0}\right\}} f(x) \left(\int_{f_{Q_0}}^{f(x)} \alpha^{r-1}\, d\alpha \right) dx =$$

$$= \frac{1}{r} \int_{\left\{x \in Q_0: \ f(x) > f_{Q_0}\right\}} f(x) \left(f^r(x) - (f_{Q_0})^r\right) dx \geq$$

$$\geq \frac{1}{r} \int_{Q_0} f(x) \left(f^r(x) - (f_{Q_0})^r\right) dx = \frac{1}{r} \int_{Q_0} f^{1+r}(x)\, dx - \frac{|Q_0|}{r} (f_{Q_0})^{1+r} .$$

From here and (5.24) it follows that

$$\left(\frac{1}{r} - c'' \right) \frac{1}{|Q_0|} \int_{Q_0} f^{1+r}(x)\, dx \leq \frac{1}{r} (f_{Q_0})^{1+r} .$$

Choosing $r > 0$ so small that $\frac{1}{r} - c'' > 0$, we obtain (5.22). $\square$

Remark 5.20. The presented proof of Theorem 5.19 needs a small refinement. Indeed, in (5.24) we a priori assumed that $f \in L^{1+r}(Q_0)$. This vicious circle can be avoided in the following way.

For $N > f_{Q_0}$ let us consider *the cut-off function* $[f]_N(x) = \min(N, f(x)) \leq f(x)$, $x \in Q_0$. If $f_{Q_0} \leq \alpha < N$, then

$$\{x \in Q_0 : \ f(x) > \alpha\} = \{x \in Q_0 : \ [f]_N(x) > \alpha\}$$

and

$$\{x \in Q_0 : \ f(x) > \sigma \cdot \alpha\} = \{x \in Q_0 : \ [f]_N(x) > \sigma \cdot \alpha\} ,$$

so that, by (5.23),

$$\int_{\{x \in Q_0: \ [f]_N(x) > \alpha\}} [f]_N(x)\, dx \leq c' \cdot \alpha \left|\{x \in Q_0 : \ [f]_N(x) > \sigma \cdot \alpha\}\right| . \tag{5.25}$$

Otherwise, if $\alpha \geq N$, then (5.25) is trivial because the domain of the definition of the integral in the left inequality is empty. So, (5.23) implies (5.25) for all $\alpha \geq f_{Q_0}$ and $N > f_{Q_0}$. It is also clear that $[f]_N \in L^{1+r}(Q_0)$ for any $r > 0$.

Now we repeat the proof of Theorem 5.19. In the proof of (5.24), substituting f by $[f]_N$, we have

$$\left\{\frac{1}{|Q_0|}\int_{Q_0}[f]_N^{1+r}(x)\,dx\right\}^{\frac{1}{1+r}} \le c\frac{1}{|Q_0|}\int_{Q_0}[f]_N(x)\,dx, \quad N > f_{Q_0}.$$

Finally, since $\frac{1}{|Q_0|}\int_{Q_0}[f]_N(x)\,dx \le \frac{1}{|Q_0|}\int_{Q_0} f(x)\,dx$ in order to complete the proof of (5.22) it is enough to send $N \to \infty$ and use the Levi theorem.

Remark 5.21. Condition (5.21) is one of the equivalent forms of the so-called A_∞*–Muckenhoupt condition* (see [8, 72]).

We see that condition (5.21) allows to increase the exponent of summability of the function f. Therefore, in order to proof Gurov–Reshetnyak theorem 5.4 it is enough to show that Gurov–Reshetnyak condition (5.1) implies (5.21). Actually it turn out that conditions (5.1) and (5.21)are equivalent. This fact is the content of the next theorem.

Theorem 5.22 ([42]). *Let $f \in L(Q_0)$ be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$. Then*

(i) *if for some ε, $0 < \varepsilon < 2$, the function f satisfies Gurov-Reshetnyak condition* (5.1)*, then there exist σ and θ, $0 < \sigma, \theta < 1$, which depend only on ε and such that* (5.21) *holds true;*

(ii) *if for some σ and θ, $0 < \sigma, \theta < 1$, the function f satisfies* (5.21)*, then*

$$\Omega(f;Q) \le 2(1-\sigma\theta)f_Q, \quad Q \subset Q_0.$$

Proof. To prove (i) let us choose a number λ such that $\varepsilon < \lambda < 2$. Fix some cube $Q \subset Q_0$ and consider the set $E = \left\{x \in Q : \ f(x) > \frac{\lambda-\varepsilon}{\lambda} \cdot f_Q\right\}$. Since $\frac{\varepsilon}{\lambda} \cdot f_Q \le f_Q - f(x)$ for all $x \in Q \setminus E$ we have

$$\frac{\varepsilon}{\lambda} \cdot f_Q \le \inf_{x \in Q\setminus E}(f_Q - f(x)) \le \frac{1}{|Q \setminus E|}\int_{Q\setminus E}(f_Q - f(x))\,dx.$$

On the other hand, $\frac{\lambda-\varepsilon}{\lambda} < 1$, so that

$$Q \setminus E = \left\{x \in Q : \ f(x) \le \frac{\lambda-\varepsilon}{\lambda} \cdot f_Q\right\} \subset \{x \in Q : \ f(x) < f_Q\}.$$

Now applying Property 2.1 and condition (5.1) to the last inequality, we get

$$\frac{\varepsilon}{\lambda} \cdot f_Q \le \frac{1}{|Q \setminus E|}\int_{\{x\in Q:\ f(x)<f_Q\}}(f_Q - f(x))\,dx =$$

$$= \frac{1}{2} \cdot \frac{|Q|}{|Q \setminus E|} \cdot \Omega(f;Q) \le \frac{\varepsilon}{2} \cdot \frac{|Q|}{|Q \setminus E|} \cdot f_Q.$$

Hence $|Q \setminus E| \le \frac{\lambda}{2} \cdot |Q|$, i.e. $|E| \ge \left(1 - \frac{\lambda}{2}\right) \cdot |Q|$. This inequality coincides with (5.21) for $\sigma = \frac{\lambda - \varepsilon}{\lambda}$ and $\theta = 1 - \frac{\lambda}{2}$.

Let us prove (ii). Fix some cube $Q \subset Q_0$. By Property 2.1 and condition (5.21),

$$\Omega(f;Q) = \frac{2}{|Q|} \int_{\{x \in Q:\ f(x) \le f_Q\}} (f_Q - f(x))\ dx =$$

$$= \frac{2}{|Q|} \int_{\{x \in Q:\ \sigma f_Q < f(x) \le f_Q\}} (f_Q - f(x))\ dx +$$

$$+ \frac{2}{|Q|} \int_{\{x \in Q:\ f(x) \le \sigma f_Q\}} (f_Q - f(x))\ dx \le$$

$$\le \frac{2}{|Q|}(1-\sigma)\,|\{x \in Q:\ f(x) > \sigma f_Q\}| + \frac{2}{|Q|} f_Q\,|\{x \in Q:\ f(x) \le \sigma f_Q\}| =$$

$$= \frac{2}{|Q|} f_Q \Big[(1-\sigma)\big(|Q| - |\{x \in Q:\ f(x) \le \sigma f_Q\}|\big) + |\{x \in Q:\ f(x) \le \sigma f_Q\}|\Big] =$$

$$= \frac{2}{|Q|} f_Q \Big[|Q| - \sigma|Q| - |\{x \in Q:\ f(x) \le \sigma f_Q\}| + \sigma\,|\{x \in Q:\ f(x) \le \sigma f_Q\}| +$$

$$+ |\{x \in Q:\ f(x) \le \sigma f_Q\}|\Big] \le$$

$$\le \frac{2}{|Q|} f_Q \Big[|Q|(1-\sigma) + \sigma(1-\theta)|Q|\Big] = 2(1 - \sigma\theta) f_Q. \quad \square$$

As we have already mentioned above, the next result is the immediate consequence of Theorems 5.19 and 5.22.

Corollary 5.23 ([42]). *Let* $0 < \varepsilon < 2$ *and let* f *be a non-negative function on the cube* $Q_0 \subset \mathbb{R}^d$*, satisfying Gurov–Reshetnyak condition* (5.1)*. Then there exists* $r > 0$*, which depends only on* ε *and* d*, such that* (5.22) *holds true with* c*, depending only on* ε*,* d *and* r*.*

Condition (5.1) of Corollary 5.23 can be replaced by the weaker condition

$$\Omega(f;Q) \le \varepsilon \cdot f_Q, \quad Q \subset Q_0,\ l(Q) \le \delta \cdot l(Q_0), \tag{5.26}$$

where $\delta \in (0,1]$ is fixed. In other words, in the Gurov–Reshetnyak theorem it is sufficient to require that the condition $\Omega(f;Q) \le \varepsilon \cdot f_Q$ is verified on small enough cubes. Indeed, we have

Corollary 5.24. *Let* $0 < \varepsilon < 2$ *and let* f *be a non-negative function on the cube* $Q_0 \subset \mathbb{R}^d$*, satisfying* (5.26) *for some* $\delta \in (0,1]$*. Then there exists* $r > 0$*, which depends only on* ε *and* d*, such that* $f \in L^{1+r}(Q_0)$ *and*

$$\left\{\frac{1}{|Q_0|}\int_{Q_0} f^{1+r}(x)\,dx\right\}^{\frac{1}{1+r}} \le c_1\frac{1}{|Q_0|}\int_{Q_0} f(x)\,dx. \tag{5.27}$$

Here the constant c_1 depends only on ε, d, r and δ.

Proof. Let us partition the cube Q_0 into $N = \left(\left[\frac{1}{\delta}\right]+1\right)^d$ cubes Q_j with pairwise disjoint interiors, dividing each side of the cube Q_0 into $\left[\frac{1}{\delta}\right]+1$ equal parts. Then $l(Q_j) \le \delta \cdot l(Q_0),\ j = 1,\dots,N$, and by virtue of (5.26) to each cube Q_j we can apply Corollary 5.23. Then

$$\left\{\frac{1}{|Q_j|}\int_{Q_j} f^{1+r}(x)\,dx\right\}^{\frac{1}{1+r}} \le c\frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx, \quad j = 1,\dots,N, \tag{5.28}$$

where $r = r(\varepsilon, d) > 0$ and $c = c(\varepsilon, d, r)$. Since $Q_0 = \bigcup_{j=1}^N Q_j$ inequality (5.28) implies $f \in L^{1+r}(Q_0)$. Further, as $|Q_j| = \frac{|Q_0|}{N},\ j = 1,\dots,N$, (5.28) yields

$$\frac{1}{|Q_0|}\int_{Q_0} f^{1+r}(x)\,dx = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{|Q_j|}\int_{Q_j} f^{1+r}(x)\,dx \le$$

$$\le c^{1+r}\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx\right)^{1+r} \le c^{1+r}\left(\max_{1\le j\le N}\frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx\right)^{1+r} \le$$

$$\le c^{1+r}N^{1+r}\,|Q_0|^{-1-r}\left(\sum_{j=1}^{N}\int_{Q_j} f(x)\,dx\right)^{1+r} =$$

$$= c^{1+r}N^{1+r}\left(\frac{1}{|Q_0|}\int_{Q_0} f(x)\,dx\right)^{1+r}.$$

This implies (5.27) with $c_1 = cN \le c\left(\frac{1}{\delta}+1\right)^d$. $\square$

Theorem 5.22 establish the equivalence of Gurov–Reshetnyak condition (5.1) and A_∞–Muckenhoupt condition (5.21). On the other hand, in [8] it was shown that Muckenhoupt condition (5.21) is equivalent to Gehring inequality (5.22) for some $r > 0$. Thus Gurov–Reshetnyak condition (5.1) and Gehring condition (5.22) are also equivalent. Now we will give another proof of the equivalence of conditions (5.1), (5.21) and (5.22). Namely, besides Theorems 5.19 and 5.22, one has to apply the following statement.

Theorem 5.25 ([44]). *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying the Gehring condition*

$$\left\{\frac{1}{|Q|}\int_{Q} f^p(x)\,dx\right\}^{1/p} \le B\cdot\frac{1}{|Q|}\int_{Q} f(x)\,dx, \quad Q \subset Q_0, \tag{5.29}$$

for some $p,\ B > 1$. Then there exists $\varepsilon,\ 0 < \varepsilon < 2$, which depends only on p and B, such that Gurov–Reshetnyak inequality (5.1) *holds true.*

Proof. Let us fix an arbitrary cube $Q \subset Q_0$ and denote $E = \{x \in Q : f(x) \geq f_Q\}$. We can assume that $f_Q > 0$. By the Hölder inequality,

$$\frac{\Omega(f;Q)}{f_Q} = \frac{1}{f_Q}\frac{2}{|Q|}\int_E (f(x) - f_Q)\, dx = 2\frac{|E|}{|Q|}\frac{1}{f_Q}\frac{1}{|E|}\int_E f(x)\, dx - 2\frac{|E|}{|Q|} \leq$$

$$\leq 2\frac{|E|}{|Q|}\frac{1}{f_Q}\left\{\frac{1}{|E|}\int_E f^p(x)\, dx\right\}^{1/p} - 2\frac{|E|}{|Q|} \leq$$

$$\leq 2\left(\frac{|E|}{|Q|}\right)^{1-1/p}\frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_Q f^p(x)\, dx\right\}^{1/p} - 2\frac{|E|}{|Q|}.$$

On the other hand, by condition (5.29),

$$(f_Q)^{-1}\left\{\frac{1}{|Q|}\int_Q f^p(x)\, dx\right\}^{1/p} \leq B,$$

so that

$$\frac{\Omega(f;Q)}{f_Q} \leq 2\left(B\left(\frac{|E|}{|Q|}\right)^{1-1/p} - \frac{|E|}{|Q|}\right). \tag{5.30}$$

Let us consider the function $\varphi(\lambda) = B \cdot \lambda^{1-1/p} - \lambda,\ \lambda > 0$. The analysis of the derivative shows, that φ is increasing on $(0, \lambda_0)$ and decreasing on $(\lambda_0, +\infty)$, where $\lambda_0 = (B(p-1)/p)^p$. We also notice that from the trivial inequality

$$\frac{\Omega(f;Q)}{f_Q} = 2\frac{|Q \setminus E|}{|Q|}\frac{1}{|Q \setminus E|}\int_{Q \setminus E}\left(1 - \frac{f(x)}{f_Q}\right) dx \leq 2\frac{|Q \setminus E|}{|Q|}$$

it follows that

$$\frac{|E|}{|Q|} = 1 - \frac{|Q \setminus E|}{|Q|} \leq 1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}. \tag{5.31}$$

First let us consider the case

$$B < \left(\frac{p}{p-1}\right)^{(p-1)/p}. \tag{5.32}$$

Then

$$\lambda_0 < \left(\left(\frac{p}{p-1}\right)^{(p-1)/p}\frac{p-1}{p}\right)^p = \frac{p-1}{p} < 1.$$

Assume that

$$\frac{\Omega(f;Q)}{f_Q} \geq 2\,(1 - \lambda_0)\,. \tag{5.33}$$

Then $\lambda_0 \geq 1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}$, and (5.30) and (5.31) imply

$$\frac{\Omega(f;Q)}{f_Q} \le 2\varphi\left(\frac{|E|}{|Q|}\right) \le 2\varphi\left(1-\frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right) =$$

$$= 2\left(B\left(1-\frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} - \left(1-\frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)\right) =$$

$$= 2B\left(1-\frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} - 2 + \frac{\Omega(f;Q)}{f_Q},$$

provided φ is monotone on $(0,\lambda_0)$. Thus

$$B\left(1-\frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} \ge 1,$$

or, equivalently,

$$\frac{\Omega(f;Q)}{f_Q} \le 2\left(1-B^{-p/(p-1)}\right). \tag{5.34}$$

Comparing the last inequality with (5.33), we find

$$1-\lambda_0 \le 1-B^{-p/(p-1)}. \tag{5.35}$$

But $\lambda_0 = (B(p-1)/p)^p$, so that (5.35) is equivalent to

$$B \ge \left(\frac{p}{p-1}\right)^{(p-1)/p},$$

which contradicts (5.32). Therefore, condition (5.32) excludes (5.33) and implies

$$\frac{\Omega(f;Q)}{f_Q} < 2\left(1-\lambda_0\right), \tag{5.36}$$

and so $\lambda_0 < 1-\frac{1}{2}\frac{\Omega(f;Q)}{f_Q}$. Taking into account that λ_0 is a point of maximum of φ, from (5.30) we obtain

$$\frac{\Omega(f;Q)}{f_Q} \le 2\varphi\left(\frac{|E|}{|Q|}\right) \le 2\varphi\left(\lambda_0\right) = 2\varphi\left(\left(B\frac{p-1}{p}\right)^p\right) =$$

$$= 2\left[B\left(\left(B\frac{p-1}{p}\right)^p\right)^{1-1/p} - \left(B\frac{p-1}{p}\right)^p\right] = 2B^p\frac{(p-1)^{p-1}}{p^p} =$$

$$= \frac{2}{p-1}\lambda_0 < \frac{2}{p-1}\left(1-\frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right) = \frac{2}{p-1} - \frac{1}{p-1}\frac{\Omega(f;Q)}{f_Q}.$$

This implies

$$\left(1+\frac{1}{p-1}\right)\frac{\Omega(f;Q)}{f_Q}\le\frac{2}{p-1},$$

i.e.,

$$\frac{\Omega(f;Q)}{f_Q}\le\frac{2}{p}. \tag{5.37}$$

Notice, that condition (5.32) is equivalent to the following one

$$\frac{2}{p}<2\left(1-\lambda_0\right).$$

This means that bound (5.37) is stronger than (5.36).

It remains to consider the case

$$B\ge\left(\frac{p}{p-1}\right)^{(p-1)/p}. \tag{5.38}$$

If we suppose that condition (5.33) is satisfied, then, as before, we come to inequality (5.34). Otherwise we obtain the opposite to (5.33) inequality (5.36). Notice also that (5.38) implies

$$1-\lambda_0\le\frac{1}{p}\le 1-B^{-p/(p-1)}.$$

So, we conclude, that among estimates (5.34), (5.36) and (5.37), the estimate provided by (5.34) is the best one that can be achieved in the case (5.38).

Finally, setting

$$\varepsilon\equiv\varepsilon(B,p)=\begin{cases}\frac{2}{p}, & \text{if } B<(p/(p-1))^{(p-1)/p},\\ 2\left(1-B^{-p/(p-1)}\right), & \text{if } B\ge(p/(p-1))^{(p-1)/p},\end{cases}$$

we get (5.1). $\square$

For $1<p<\infty$ and $B>1$ we denote by $G_p(B)$ the class of all functions f, which are non-negative on the cube $Q_0\subset\mathbb{R}^d$ and satisfy the Gehring inequality

$$\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p}\le B\frac{1}{|Q|}\int_Q f(x)\,dx,\quad Q\subset Q_0.$$

The set $G_p\equiv\bigcup_{B>1}G_p(B)$ is called *the Gehring class* .

Remark 5.26. In the proof of Theorem 5.25 we have found that $\varepsilon(B,p)\to 2-0$ as $B\to\infty$ for any fixed $p>1$. This fact is essential. Indeed, let us consider the function $f_b(x)=b\chi_{(-\infty,0)}(x)+\chi_{[0,+\infty)}(x),\ x\in\mathbb{R}$ for $b>1$. An easy computation shows that $f_b\in GR\left(\varepsilon_0\right)$ with the minimal possible value $\varepsilon_0\equiv\varepsilon_0(b)=2\frac{\sqrt{b}-1}{\sqrt{b}+1}\to 2-0$ as $b\to\infty$. In addition, $f_b\in G_p$ for any $p>1$.

This means that the Gurov–Reshetnyak class $GR(\varepsilon)$ does not contain the Gehring class G_p for all $\varepsilon < 2$. Moreover, this example shows that for $\varepsilon < 2$ the set $\bigcap_{p>1} G_p$ does not belong to any $GR(\varepsilon)$. On the other hand, if $p > 1$ and $B \to 1$, then the value $\varepsilon(B,p) = \frac{2}{p}$, obtained in the proof of Theorem 5.25, does not tend to zero. In this sense the constant $\varepsilon(B,p)$ from Theorem 5.25 is overestimated. Indeed, as it was remarked in [4], for $p \geq 2$ and $1 < B < \sqrt{5}$ the condition $f \in G_p(B)$ implies $f \in G_2(B)$, so that by Hölder inequality

$$\Omega^2(f;Q) \leq \frac{1}{|Q|}\int_Q |f(x) - f_Q|^2\, dx = \frac{1}{|Q|}\int_Q f^2(x)\, dx - \left(\frac{1}{|Q|}\int_Q f(x)\, dx\right)^2 \leq$$

$$\leq \left(B^2 - 1\right)\left(\frac{1}{|Q|}\int_Q f(x)\, dx\right)^2 .$$

So, if $G_p(B) \subset GR(\varepsilon)$ for $p \geq 2$, then one can assure that $\varepsilon \to 0$ as $B \to 1$.

Remark 5.27. Let us fix $B > 1$. Then $\varepsilon(B,p) \to 2\frac{B-1}{B}$ as $p \to \infty$. In other words, we have

$$\bigcap_{1<p<\infty} G_p(B) \subset \bigcap_{\varepsilon_1<\varepsilon<2} GR(\varepsilon), \tag{5.39}$$

where $\varepsilon_1 = 2\frac{B-1}{B} > 0$. We do not know the minimal value of $\varepsilon_1(B)$ (possibly depending on d), which guarantees (5.39). Notice, that (5.39) fails for $\varepsilon_1 = 0$. Moreover, for the function f_b, defined in Remark 5.26, if $b = B$ then we have $f_B \in G_p(B)$ for any $p > 1$. On the other hand, from Remark 5.26 we know that $f_B \notin GR(\varepsilon)$ for any $\varepsilon < \varepsilon_0(B) = 2\frac{\sqrt{B}-1}{\sqrt{B}+1}$. Hence (5.39) fails if $\varepsilon_1 < 2\frac{\sqrt{B}-1}{\sqrt{B}+1}$, and moreover, it is easy to see that this fact is valid in the space of any dimension $d \geq 1$. Therefore, if $\varepsilon_1(B)$ is the minimal value such that (5.39) holds, then

$$2\frac{\sqrt{B}-1}{\sqrt{B}+1} \leq \varepsilon_1(B) \leq 2\frac{B-1}{B}.$$

Let us consider the other limit case $p \to 1+0$. For some fixed $B > 1$ we have $\varepsilon(B,p) \to 2-0$, i.e.,

$$\bigcup_{1<p<\infty} G_p(B) \subset \bigcup_{0<\varepsilon<2} GR(\varepsilon). \tag{5.40}$$

The same example of the function f_b, defined in Remark 5.26, shows that the constant 2 in the right-hand side of (5.40) is sharp. Indeed, it is easy to see that $f_b \in G_p\left(B_{p,b}\right)$ for the fixed value $b > 1$, where

$$B_{p,b} = \frac{(p-1)^{(p-1)/p}}{p}\,\frac{b^p - 1}{(b-1)^{1/p}}\,\frac{1}{(b^p - b)^{(p-1)/p}} \to 1+0 \text{ as } p \to 1 \tag{5.41}$$

is the minimal possible value. Fix some $B > 1$, $\varepsilon_1 < 2$ and choose $b > 1$ so big, that $\varepsilon_0(b) = 2\frac{\sqrt{b}-1}{\sqrt{b}+1} > \varepsilon_1$. Then, due to (5.41), for this value of b there exists $p > 1$ such that $B_{p,b} < B$ and $f_b \in \bigcup_{1<p<\infty} G_p(B)$. Obviously, $f_b \notin \bigcup_{0<\varepsilon<\varepsilon_1} GR(\varepsilon)$.

Another proof of Theorem 5.25. Without loss of generality we can assume

$$B \geq \frac{p}{p-1}. \tag{5.42}$$

Fix an arbitrary cube $Q \subset Q_0$ and denote $E = \{x \in Q : \ f(x) \geq f_Q\}$. Then the properties of the oscillations together with the Hölder inequality imply

$$\frac{\Omega(f;Q)}{f_Q} = \frac{1}{f_Q}\frac{2}{|Q|}\int_E (f(x) - f_Q)\, dx \leq$$

$$\leq 2\frac{|E|}{|Q|}\frac{1}{f_Q}\left\{\frac{1}{|E|}\int_E f^p(x)\, dx\right\}^{1/p} - 2\frac{|E|}{|Q|} \leq$$

$$\leq 2\left(\frac{|E|}{|Q|}\right)^{1-1/p}\frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_Q f^p(x)\, dx\right\}^{1/p} - 2\frac{|E|}{|Q|}.$$

Applying (5.29) to the integral in the right-hand side we have

$$\frac{\Omega(f;Q)}{f_Q} \leq 2\left[B\left(\frac{|E|}{|Q|}\right)^{1-1/p} - \frac{|E|}{|Q|}\right]. \tag{5.43}$$

Further, from the inequality

$$\frac{\Omega(f;Q)}{f_Q} = 2\frac{|Q\setminus E|}{|Q|}\frac{1}{|Q\setminus E|}\int_{Q\setminus E}\left(1 - \frac{f(x)}{f_Q}\right)\, dx \leq 2\frac{|Q\setminus E|}{|Q|}$$

it follows that

$$\frac{|E|}{|Q|} = 1 - \frac{|Q\setminus E|}{|Q|} \leq 1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}. \tag{5.44}$$

It is easy to see, that the function $\varphi(\lambda) = B\lambda^{1-1/p} - \lambda,\ \lambda > 0$ increases on $(0, \lambda_0)$, $\lambda_0 = \left(B\frac{p-1}{p}\right)^p$. Notice that due to (5.42), $\lambda_0 > 1$. Since the right-hand side of (5.44) is less or equal than 1 inequalities (5.43) and (5.44) yield

$$\frac{\Omega(f;Q)}{f_Q} \leq 2\varphi\left(\frac{|E|}{|Q|}\right) \leq 2\varphi\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right) =$$

$$= 2\left(B\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} - \left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)\right) =$$

$$= 2B\left(1-\frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} - 2 + \frac{\Omega(f;Q)}{f_Q}.$$

This inequality implies

$$B\left(1-\frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} \geq 1,$$

or, equivalently,

$$\frac{\Omega(f;Q)}{f_Q} \leq 2\left(1 - B^{-p/(p-1)}\right).$$

Therefore, the function f satisfies inequality (5.1) for

$$\varepsilon = 2\left(1 - \left[\max\left(B, \frac{p}{p-1}\right)\right]^{-p/(p-1)}\right) < 2. \quad \square$$

We conclude this section by one interesting property of functions that satisfy the Gehring condition.

Theorem 5.28 (Iwaniec, [28]). *Let $f \in L^p\left(\mathbb{R}^d\right)$, $p > 1$ be a non-negative function, satisfying the Gehring condition*

$$\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{\frac{1}{p}} \leq B\frac{1}{|Q|}\int_Q f(x)\,dx, \quad Q \subset \mathbb{R}^d, \tag{5.45}$$

where the constant $B > 1$ does not depend on the cube Q. Then f is equivalent to zero on $\mathbb{R}^d$.

Proof. Let us assume the contrary. Without loss of generality assume that

$$c \equiv \int_{Q_0} f^p(x)\,dx > 0$$

on the cube $Q_0 \equiv [-1,1]^d$. Then, by condition (5.45), for any cube $Q \supset Q_0$ centered in the origin we have

$$B\frac{1}{|Q|}\int_Q f(x)\,dx \geq \left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{\frac{1}{p}} \geq$$

$$\geq \left\{\frac{1}{|Q|}\int_{Q_0} f^p(x)\,dx\right\}^{\frac{1}{p}} = |Q|^{-\frac{1}{p}}c^{\frac{1}{p}}.$$

Thus

$$\int_Q f(x)\,dx \geq c_1|Q|^{1-\frac{1}{p}}, \tag{5.46}$$

where $c_1 = B^{-1} c^{\frac{1}{p}} > 0$. Let us show that (5.46) contradicts the condition $f \in L^p\left(\mathbb{R}^d\right)$. Since the cube Q_0 is fixed we can construct by induction a sequence of cubes $Q_k \supset Q_{k-1},\ k = 1, 2, \ldots$ such that

$$|Q_k| \geq \left(\frac{2}{c_1} \int_{Q_{k-1}} f(x)\, dx \right)^{\frac{p}{p-1}} \tag{5.47}$$

and

$$\frac{|Q_k|}{|Q_k \setminus Q_{k-1}|} \leq \frac{3}{2}. \tag{5.48}$$

By (5.46),

$$c_1 \, |Q_k|^{-\frac{1}{p}} \leq \frac{1}{|Q_k|} \int_{Q_k} f(x)\, dx =$$

$$= \frac{1}{|Q_k|} \left(\int_{Q_k \setminus Q_{k-1}} f(x)\, dx + \int_{Q_{k-1}} f(x)\, dx \right) =$$

$$= \frac{|Q_k \setminus Q_{k-1}|}{|Q_k|} \frac{1}{|Q_k \setminus Q_{k-1}|} \int_{Q_k \setminus Q_{k-1}} f(x)\, dx + \frac{1}{|Q_k|} \int_{Q_{k-1}} f(x)\, dx.$$

Now, using (5.47) and (5.48), we obtain

$$\frac{1}{|Q_k \setminus Q_{k-1}|} \int_{Q_k \setminus Q_{k-1}} f(x)\, dx \geq$$

$$\geq \frac{|Q_k|}{|Q_k \setminus Q_{k-1}|} \left(c_1 \, |Q_k|^{-\frac{1}{p}} - \frac{1}{|Q_k|} \int_{Q_{k-1}} f(x)\, dx \right) \geq$$

$$\geq \frac{|Q_k|}{|Q_k \setminus Q_{k-1}|} \left(c_1 \, |Q_k|^{-\frac{1}{p}} - \frac{c_1}{2} \, |Q_k|^{-\frac{1}{p}} \right) \geq \frac{c_1}{2} \, |Q_k|^{-\frac{1}{p}} \geq$$

$$\geq \frac{c_1}{2} \left(\frac{3}{2} \right)^{-\frac{1}{p}} |Q_k \setminus Q_{k-1}|^{-\frac{1}{p}} \equiv c_2 \, |Q_k \setminus Q_{k-1}|^{-\frac{1}{p}} .$$

Therefore, by the Hölder inequality,

$$\int_{\mathbb{R}^d} f^p(x)\, dx = \int_{Q_0} f^p(x)\, dx + \sum_{k=1}^{\infty} \int_{Q_k \setminus Q_{k-1}} f^p(x)\, dx =$$

$$= \int_{Q_0} f^p(x)\, dx + \sum_{k=1}^{\infty} |Q_k \setminus Q_{k-1}| \frac{1}{|Q_k \setminus Q_{k-1}|} \int_{Q_k \setminus Q_{k-1}} f^p(x)\, dx \geq$$

$$\geq \int_{Q_0} f^p(x)\,dx + \sum_{k=1}^{\infty} |Q_k \setminus Q_{k-1}| \left(\frac{1}{|Q_k \setminus Q_{k-1}|} \int_{Q_k \setminus Q_{k-1}} f(x)\,dx \right)^p \geq$$

$$\geq \int_{Q_0} f^p(x)\,dx + \sum_{k=1}^{\infty} |Q_k \setminus Q_{k-1}|\, c_2^p\, |Q_k \setminus Q_{k-1}|^{-1} = \infty,$$

and this completes the proof. □

5.1.1 One-Dimensional Case

Let us consider more in detail the case $d = 1$. If in the proof of Theorem 5.5 instead of Calderón–Zygmund lemma 5.6 we use "rising sun lemma" 1.16, which is sharper in the one-dimensional case, then we obtain the following statement.

Theorem 5.29 ([38]). *Let f be a non-negative function, summable on $I_0 \subset \mathbb{R}$. Then*

$$\frac{1}{t} \int_0^t |f^*(u) - f^{**}(t)|\; du \leq \nu(f;t) f^{**}(t), \quad 0 < t \leq |I_0|. \tag{5.49}$$

Proof. Essentially we will follow the proof of Theorem 5.5. Moreover, in the present case the calculations are even simpler.

Let us fix some t, $0 < t \leq |I_0|$, and apply Lemma 1.16 with $\alpha = f^{**}(t)$. As the result we obtain a family of pairwise disjoint intervals $I_j \subset I_0,\ j = 1, 2, \ldots,$ such that

$$\frac{1}{|I_j|} \int_{I_j} f(x)\,dx = \alpha, \tag{5.50}$$

$$f(x) \leq \alpha \quad \text{for almost all} \quad x \in I_0 \setminus E, \tag{5.51}$$

where $E = \cup_{j \geq 1} I_j$. Using Property 2.1, the definition of the rearrangement f^*, formulas (5.50), (5.51) and the monotonicity of $\nu(f;\sigma)$, we obtain

$$\int_0^t |f^*(u) - f^{**}(t)|\; du = 2 \int_{\{u:\ f^*(u) > \alpha\}} (f^*(u) - \alpha)\; du =$$

$$= 2 \int_{\{x \in I_0:\ f(x) > \alpha\}} (f(x) - \alpha)\; dx = 2 \int_{\{x \in I_0:\ f(x) > \alpha\} \cap E} (f(x) - \alpha)\; dx =$$

$$= 2 \sum_{j \geq 1} \int_{\{x \in I_j:\ f(x) > \alpha\}} (f(x) - \alpha)\; dx =$$

$$= 2 \sum_{j \geq 1} \int_{\{x \in I_j:\ f(x) > \alpha\}} \big(f(x) - f_{I_j}\big)\; dx =$$

$$= \sum_{j\geq 1} |I_j|\, \Omega\,(f; I_j) \leq \sum_{j\geq 1} \nu\,(f; |I_j|)\, |I_j|\, f_{I_j} \leq$$

$$\leq \nu(f; |E|) \sum_{j\geq 1} |I_j|\, f_{I_j} = \alpha \cdot |E| \cdot \nu(f; |E|). \tag{5.52}$$

On the other hand, (5.50) and the properties of the rearrangement imply

$$\frac{1}{t}\int_0^t f^*(u)\, du = f^{**}(t) = \alpha = \frac{1}{|E|}\int_E f(x)\, dx \leq \frac{1}{|E|}\int_0^{|E|} f^*(u)\, du,$$

so that $|E| \leq t$. Therefore the monotonicity of $\nu(f;\sigma)$ and (5.52) yield

$$\int_0^t |f^*(u) - f^{**}(t)|\; du \leq \alpha \cdot t \cdot \nu(f; t),$$

i.e. (5.49). $\square$

We will proceed with the further detalization of the case $d = 1$ in the following two directions.

1). Refinement of Theorem 5.7 (i.e. sharpening of the constants in inequality (5.10)).

2). Refinement of Gurov–Reshetnyak Theorem 5.4.

The next theorem is the refined analog of Theorem 5.7 for the case $d = 1$.

Theorem 5.30 ([38]). *Let f be a non-negative function, summable on $I_0 \subset \mathbb{R}$. Then*

$$f^{**}(t) \leq c \cdot f_{I_0} \cdot \exp\left(\frac{\mathrm{e}}{2}\int_t^{|I_0|} \nu(f;\sigma)\, \frac{d\sigma}{\sigma}\right), \quad 0 < t \leq |I_0|\,, \tag{5.53}$$

where $c = \exp(1 + \mathrm{e})$, and in general the coefficient $\mathrm{e}/2$ is sharp.

Proof. Let $a > 1$ (we will find the optimal value of this constant later). Applying Lemma 2.2 to the function $\varphi = f^*$ and using Theorem 5.29, for $0 < t \leq |I_0|$ we have

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \leq \frac{a}{2}\frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\; du \leq \frac{a}{2}\nu(f;t) f^{**}(t),$$

or, equivalently,

$$f^{**}\left(\frac{t}{a}\right) \leq \left(1 + \frac{a}{2}\nu(f;t)\right) f^{**}(t), \quad 0 < t \leq |I_0|\,. \tag{5.54}$$

Let us fix some $t \in \left(0, \frac{|I_0|}{a}\right]$ and denote $s = \left[\ln^{-1} a \cdot \ln \frac{|I_0|}{t}\right]$ (here the square brackets denote the integer part function). By (5.54),

$$f^{**}\left(\frac{t}{a}\right) \le f^{**}\left(\frac{|I_0|}{a}\right) \prod_{k=0}^{s} \left(1 + \frac{a}{2}\nu\left(f; a^k t\right)\right) \le$$

$$\le a \cdot f_{I_0} \prod_{k=0}^{s} \exp\left(\frac{a}{2}\nu\left(f; a^k\right)\right) = a \cdot f_{I_0} \exp\left(\frac{a}{2} \sum_{k=0}^{s} \nu\left(f; a^k t\right)\right). \tag{5.55}$$

On the other hand,

$$\nu\left(f; a^k t\right) \cdot \ln a \le \int_{a^k t}^{a^{k+1} t} \nu(f;\sigma)\, \frac{d\sigma}{\sigma}, \quad k = 0, 1, \dots, s-1,$$

provided $\nu(f;\sigma)$ is monotone. Hence, taking into account the inequality $\nu(f;|I_0|) \le 2$, from (5.55) we obtain

$$f^{**}\left(\frac{t}{a}\right) \le a \cdot f_{I_0} \exp\left(\frac{1}{2}\frac{a}{\ln a}\left(\sum_{k=0}^{s-1} \int_{a^k t}^{a^{k+1} t} \nu(f;\sigma)\, \frac{d\sigma}{\sigma} + \nu\left(f; |I_0|\right)\right)\right) \le$$

$$\le a \cdot \exp\left(\frac{a}{\ln a}\right) \cdot f_{I_0} \exp\left(\frac{1}{2}\frac{a}{\ln a} \int_t^{|I_0|} \nu(f;\sigma)\, \frac{d\sigma}{\sigma}\right).$$

Since the function $\psi(a) \equiv a/\ln a$ for $a > 1$ achieves its minimal value at $a = \mathrm{e}$ we have

$$f^{**}\left(\frac{t}{\mathrm{e}}\right) \le c \cdot f_{I_0} \cdot \exp\left(\frac{\mathrm{e}}{2} \int_t^{|I_0|} \nu(f;\sigma)\, \frac{d\sigma}{\sigma}\right), \tag{5.56}$$

where $c = \exp(1 + \mathrm{e})$. This, together with the monotonicity of f^{**}, implies (5.53) for $0 < t \le \frac{|I_0|}{\mathrm{e}}$. If $t \in \left(\frac{|I_0|}{\mathrm{e}}, |I_0|\right]$, then (5.53) follows from (5.56) because $f^{**}(t) \le f^{**}\left(\frac{|I_0|}{\mathrm{e}}\right)$.

It remains to show that the coefficient $2/\mathrm{e}$ in the right-hand side of (5.53) cannot be decreased. For this let us consider the function $f_0(x) = \ln \frac{1}{x}$, $0 < x \le \beta_0$, where the constant $\beta_0 > 0$ is sufficiently small, we will define it later in Proposition 5.31. In addition, we will show there that

$$\nu(f_0;\sigma) = \frac{2/\mathrm{e}}{1 + \ln \frac{1}{\sigma}}, \quad 0 < \sigma \le \beta_0.$$

Thus if we put some constant $a < \frac{\mathrm{e}}{2}$ in the exponent in (5.53), then the right-hand side becomes

$$c \cdot (f_0)_{[0,\beta_0]} \cdot \exp\left(a \int_t^{\beta_0} \nu\left(f_0;\sigma\right)\, \frac{d\sigma}{\sigma}\right) =$$

$$= c\left(1 + \ln\frac{1}{\beta_0}\right) \exp\left(\frac{2a}{\mathrm{e}} \int_t^{\beta_0} \frac{d\sigma}{\sigma\left(1 + \ln\frac{1}{\sigma}\right)}\right) =$$

$$= c\left(1+\ln\frac{1}{\beta_0}\right)^{1-2a/\mathrm{e}} \cdot \left(1+\ln\frac{1}{t}\right)^{2a/\mathrm{e}} = \overline{c}\left(\ln\frac{1}{t}\right)^{2a/\mathrm{e}}, \quad t \to 0.$$

On the other hand, $f_0^{**}(t) = 1 + \ln\frac{1}{t},\ 0 < t \le \beta_0$. Comparing this equality with the previous one, we see that for $a < \frac{\mathrm{e}}{2}$ inequality (5.53) fails. □

Proposition 5.31. *For the function $f(x) = \ln\frac{1}{x},\ 0 < x \le \beta_0$, where β_0 is a positive constant,*

$$\nu(f;\sigma) = \frac{2/\mathrm{e}}{1+\ln\frac{1}{\sigma}}, \quad 0 < \sigma \le \beta_0.$$

Proof. Let us suppose that $\beta_0 = \mathrm{e}^{-M}$, where the number $M > 1$ is to be defined later. Further, let $0 < \sigma \le \beta_0$ and $a \ge 0$ be such that $a + \sigma \le \beta_0$. Denote $I = [a, a+\sigma]$. Then

$$f_I = \frac{1}{\sigma}\int_a^{a+\sigma} \ln\frac{1}{x}\,dx = 1 + \ln\frac{1}{a+\sigma} - \frac{a}{\sigma}\ln\left(1+\frac{\sigma}{a}\right).$$

Let $x_0 = \mathrm{e}^{-f_I}$. Then $\ln\frac{1}{x_0} = f_I$ and

$$\Omega(f;I) = \frac{1}{|I|}\int_I |f(x) - f_I|\,dx = \frac{2}{\sigma}\int_a^{x_0}\left(\ln\frac{1}{x} - f_I\right)dx =$$

$$= 2\left(1+\frac{\sigma}{a}\right)\left[\exp\left(\frac{\ln(1+\sigma/a)}{\sigma/a} - 1\right) - \frac{\ln(1+\sigma/a)}{\sigma/a}\right].$$

In order to proof the proposition, it is enough to show that

$$\frac{\Omega(f;I)}{f_I} \le \frac{2/\mathrm{e}}{1+\ln 1/\sigma} \equiv \frac{\Omega(f;[0,\sigma])}{f_{[0,\sigma]}}. \tag{5.57}$$

Let us denote

$$\alpha = \frac{\sigma}{a}, \quad \beta = \frac{1}{1+\ln 1/\sigma}.$$

Then the conditions $0 < \sigma \le \beta_0 \equiv \mathrm{e}^{-M}$ and $a + \sigma \le \beta_0 = \mathrm{e}^{-M}$ become

$$0 < \alpha < +\infty,$$

$$\beta \le \frac{1}{1+M+\ln\left(1+\frac{1}{\alpha}\right)} < \frac{1}{1+M}, \tag{5.58}$$

while inequality (5.57) can be rewritten in the following way

$$\frac{\mathrm{e}\left(1+\frac{1}{\alpha}\right)\left[\frac{\ln(1+\alpha)}{\alpha}-\exp\left(\frac{\ln(1+\alpha)}{\alpha}-1\right)\right]+1}{\frac{\ln(1+\alpha)}{\alpha}+\ln\left(1+\frac{1}{\alpha}\right)} \geq \beta. \tag{5.59}$$

Assume $0 < \alpha < 1$. Then, by virtue of (5.58), inequality (5.59) can be derived from the inequality

$$\frac{\mathrm{e}\left(1+\frac{1}{\alpha}\right)\left[\frac{\ln(1+\alpha)}{\alpha}-\exp\left(\frac{\ln(1+\alpha)}{\alpha}-1\right)\right]+1}{\frac{\ln(1+\alpha)}{\alpha}+\ln\left(1+\frac{1}{\alpha}\right)} \geq \frac{1}{1+M+\ln\left(1+\frac{1}{\alpha}\right)},$$

which is equivalent to the following one

$$\mathrm{e}\left(1+\frac{1}{\alpha}\right)\left[\frac{1}{\exp\left(1-\frac{\ln(1+\alpha)}{\alpha}\right)}-\frac{\ln(1+\alpha)}{\alpha}\right] \leq \frac{1+M-\frac{\ln(1+\alpha)}{\alpha}}{1+M+\ln\left(1+\frac{1}{\alpha}\right)}. \tag{5.60}$$

Let us prove (5.60). In order to find the upper bound for the left-hand side of (5.60), set $t \equiv 1-\frac{\ln(1+\alpha)}{\alpha}$. Notice that $0 < t < 1-\ln 2$. Moreover,

$$\frac{1}{\exp\left(1-\frac{\ln(1+\alpha)}{\alpha}\right)}-\frac{\ln(1+\alpha)}{\alpha} = \mathrm{e}^{-t}+t-1 \leq \frac{1}{2}t^2.$$

Hence we have the following bound for the left-hand side of (5.60) (we denote it by L)

$$L \leq \frac{\mathrm{e}}{2}\frac{\alpha+1}{\alpha}\left[1-\frac{\ln(1+\alpha)}{\alpha}\right]^2 = \frac{\mathrm{e}}{2}\frac{\alpha+1}{\alpha^3}\left[\alpha-\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\alpha^k}{k}\right]^2 =$$

$$= \frac{\mathrm{e}}{2}\frac{\alpha+1}{\alpha^3}\left[\sum_{k=2}^{\infty}(-1)^k\frac{\alpha^k}{k}\right]^2 = \frac{\mathrm{e}}{2}(\alpha+1)\alpha\left[\sum_{k=2}^{\infty}(-1)^k\frac{\alpha^{k-2}}{k}\right]^2.$$

Since

$$\left|\sum_{k=2}^{\infty}(-1)^k\frac{\alpha^{k-2}}{k}\right| \leq \frac{1}{2}, \quad 0 < \alpha < 1$$

we see that

$$L \leq \frac{\mathrm{e}}{8}\alpha(\alpha+1) \leq \frac{\mathrm{e}\alpha}{4}.$$

The right-hand side of (5.60) can be estimate as follows

$$\frac{1+M-\frac{\ln(1+\alpha)}{\alpha}}{1+M+\ln\left(1+\frac{1}{\alpha}\right)} \geq \frac{M}{1+M+\ln\left(1+\frac{1}{\alpha}\right)}.$$

Therefore, (5.60) follows from the inequality

$$\frac{\mathrm{e}\alpha}{4} \leq \frac{M}{1+M+\ln\left(1+\frac{1}{\alpha}\right)},$$

or, which is the same, from

$$\ln\left(1+\frac{1}{\alpha}\right) \leq \frac{1}{\alpha}\frac{4M}{\mathrm{e}} - M - 1. \tag{5.61}$$

Inequality (5.61) is valid for $\alpha = 1$ whenever $M \geq M_1$, where $M_1 \equiv (\ln 2 + 1) / \left(\frac{4}{\mathrm{e}} - 1\right)$. Otherwise, if $0 < \alpha < 1$, then (5.61) follows from the fact that the function

$$\varphi(\alpha) = \frac{1}{\alpha}\frac{4M}{\mathrm{e}} - M - 1 - \ln\left(1+\frac{1}{\alpha}\right)$$

is decreasing on $(0,1)$. This can be easily checked by calculation of the derivative of φ.

In order to prove (5.59) in the case $\alpha \geq 1$ we rewrite it in the following form

$$\alpha\left(\exp\left(\frac{\ln(1+\alpha)}{\alpha}\right) - 1\right) \leq$$

$$\leq \ln(1+\alpha)\left[\mathrm{e}\left(1+\frac{1}{\alpha}\right) - \beta\right] - \alpha\beta\ln\left(1+\frac{1}{\alpha}\right) - \exp\left(\frac{\ln(1+\alpha)}{\alpha}\right). \tag{5.62}$$

Let us estimate the last two terms of the right-hand side. Denote $t = \frac{\ln(1+\alpha)}{\alpha}$. Then $0 < t \leq \ln 2$. Using the inequality $\mathrm{e}^t - 1 \leq \frac{t}{\ln 2}$, $0 < t \leq \ln 2$, we have

$$\mathrm{e}^t \leq 1 + \frac{t}{\ln 2} \iff -\exp\left(\frac{\ln(1+\alpha)}{\alpha}\right) \geq -1 - \frac{1}{\ln 2}\frac{\ln(1+\alpha)}{\alpha},$$

$$\alpha\ln\left(1+\frac{1}{\alpha}\right) \leq 1 \iff -\beta\alpha\ln\left(1+\frac{1}{\alpha}\right) \geq -\beta.$$

Therefore the right-hand side of (5.62) admits the following lower bound

$$\ln(1+\alpha)\left[\mathrm{e}\left(1+\frac{1}{\alpha}\right) - \beta\right] - \alpha\beta\ln\left(1+\frac{1}{\alpha}\right) - \exp\left(\frac{\ln(1+\alpha)}{\alpha}\right) \geq$$

$$\geq \ln(1+\alpha)\left[\mathrm{e}\left(1+\frac{1}{\alpha}\right) - \beta\right] - \beta - 1 - \frac{1}{\ln 2}\frac{\ln(1+\alpha)}{\alpha}.$$

To estimate the left-hand side of (5.62) observe that

$$\mathrm{e}^t \leq 1 + \frac{t}{\ln 2},\ 0 < t \leq \ln 2.$$

Hence

$$\alpha\left(\exp\left(\frac{\ln(1+\alpha)}{\alpha}\right)-1\right)\le\alpha\frac{1}{\ln 2}\frac{\ln(1+\alpha)}{\alpha}=\frac{\ln(1+\alpha)}{\alpha},\quad \alpha\ge 1.$$

So, in order to prove (5.62) it is enough to show that

$$\frac{\ln(1+\alpha)}{\ln 2}\le\ln(1+\alpha)\left[\mathrm{e}\left(1+\frac{1}{\alpha}\right)-\beta\right]-\beta-1-\frac{1}{\ln 2}\frac{\ln(1+\alpha)}{\alpha},$$

or, equivalently,

$$\ln(1+\alpha)\ge\frac{1+\beta}{\left(\mathrm{e}-\frac{1}{\ln 2}\right)\left(1+\frac{1}{\alpha}\right)-\beta}. \tag{5.63}$$

The proof of (5.63) splits into the following two cases.

1. $1\le\alpha\le 2$; in this case $\frac{3}{2}\le 1+\frac{1}{\alpha}\le 2,\ \ln(1+\alpha)\ge\ln 2$. The inequality

$$\ln 2>\frac{1}{\frac{3}{2}\left(\mathrm{e}-\frac{1}{\ln 2}\right)}$$

implies that for the sufficiently small β $\left(\beta\le\frac{1}{M_2+1}\right)$

$$\ln(1+\alpha)\ge\ln 2>\frac{1+\beta}{\frac{3}{2}\left(\mathrm{e}-\frac{1}{\ln 2}\right)-\beta}\ge\frac{1+\beta}{\left(\mathrm{e}-\frac{1}{\ln 2}\right)\left(1+\frac{1}{\alpha}\right)-\beta},$$

and so (5.63) follows.

2. $\alpha\ge 2$; in this case $1+\frac{1}{\alpha}\ge 1,\ \ln(1+\alpha)\ge\ln 3$. Since

$$\ln 3>\frac{1}{\mathrm{e}-\frac{1}{\ln 2}}$$

we have that for any sufficiently small β $\left(\beta\le\frac{1}{M_3+1}\right)$

$$\ln(1+\alpha)\ge\ln 3>\frac{1+\beta}{\mathrm{e}-\frac{1}{\ln 2}-\beta}\ge\frac{1+\beta}{\left(\mathrm{e}-\frac{1}{\ln 2}\right)\left(1+\frac{1}{\alpha}\right)-\beta},$$

and (5.63) follows in this case as well.

Setting $M=\max(M_1,M_2,M_3)$, we obtain (5.57). □

Let us come back to Gurov–Reshetnyak theorem 5.4. Theorem 5.30 has the following immediate corollary.

Corollary 5.32 ([38]). *Let f be a non-negative function on $I_0\subset\mathbb{R}$, satisfying condition (5.1) for some $\varepsilon<2/\mathrm{e}$. Then $f\in L^p(I_0)$ for any $p<p_0'$, where $p_0'\equiv p_0'(\varepsilon)=\frac{2}{\mathrm{e}}\cdot\frac{1}{\varepsilon}$.*

Remark 5.33. We have already mentioned (see Remark 5.11) that in Gurov-Reshetnyak theorem 5.4 the limiting exponent of summability of the function, satisfying Gurov–Reshetnyak condition (5.1), is equal to $p_0(\varepsilon,d)=\frac{\varepsilon_0(1)}{\varepsilon}$ (see (5.12)). Moreover, $p_0(\varepsilon,d)=\underline{O}\left(\frac{1}{\varepsilon}\right)$ as $\varepsilon\to 0+$, and this limiting behavior cannot be improved. In the case $d = 1$ Corollary 5.32 provides the bigger limiting exponent of summability: $p_0'(\varepsilon) = \frac{2}{\mathrm{e}} \cdot \frac{1}{\varepsilon} > p_0(\varepsilon, 1)$. The value $p_0'(\varepsilon)$ is the maximal possible also in the sense of equivalence. In what follows we will derive this fact as a corollary of Theorem 5.34. Moreover, Corollary 5.32 states that Gurov–Reshetnyak condition (5.1) assures the possibility to increase the exponent of summability of f only for $\varepsilon < \frac{2}{\mathrm{e}}$, and it leaves open the problem in the case $\frac{2}{\mathrm{e}} \leq \varepsilon < 2$, though the possibility of a certain increment of the exponent of summability for any $\varepsilon < 2$ is provided by Corollary 5.23.

Theorem 5.34 ([34]). *Let $\varepsilon,\ 0 < \varepsilon < 2$ be fixed, and let $p_0'' \equiv p_0''(\varepsilon) > 1$ be a root of the equation*

$$\frac{p^p}{(p-1)^{p-1}} = \frac{2}{\varepsilon}. \tag{5.64}$$

Then

(i) if f is a non-negative function on $I_0 \subset \mathbb{R}$, satisfying Gurov–Reshetnyak condition (5.1) *with the given ε, then*

$$f^{**}(t) \leq c \cdot f^{**}\left(|I_0|\right) \cdot \left(\frac{t}{|I_0|}\right)^{-1/p_0''}, \quad 0 < t \leq |I_0|, \tag{5.65}$$

where the constant c depends only on ε;

(ii) there exists a function $f_0 \in L([0,1])$, satisfying (5.1) *such that*

$$t^{1/p_0''} \cdot f_0^{**}(t) \geq c > 0, \quad 0 < t \leq 1. \tag{5.66}$$

Proof. Let us denote

$$\varphi(p) = \frac{p^p}{(p-1)^{p-1}}, \quad p > 1.$$

It is easy to see, that $\varphi'(p) > 0$, $\lim_{p\to 1+0} \varphi(p) = 1$, $\lim_{p\to+\infty} \varphi(p) = +\infty$, i.e. φ is continuous and increasing between 1 and $+\infty$. Hence for any $\varepsilon,\ 0 < \varepsilon < 2$, the equation (5.64) has a unique root $p_0'' = p_0''(\varepsilon) > 1$.

Let us prove (i). As condition (5.1) means that $\nu(f;t) \leq \varepsilon,\ 0 < t \leq |I_0|$, hence, by Theorem 5.29,

$$\frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\, du \leq \varepsilon \cdot f^{**}(t), \quad 0 < t \leq |I_0|.$$

Assume $a > 1$. According to Lemma 2.2,

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \leq \frac{a}{2} \cdot \frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\, du \leq \frac{a}{2} \cdot \varepsilon \cdot f^{**}(t),$$

so that

$$f^{**}\left(\frac{t}{a}\right) \le \left(\frac{a}{2}\cdot\varepsilon+1\right) f^{**}(t), \quad 0 < t \le |I_0|. \tag{5.67}$$

Set

$$a = \left(\frac{p_0''}{p_0''-1}\right)^{p_0''} > 1.$$

Then, by (5.64), we have $\left(\frac{a}{2}\cdot\varepsilon+1\right)^{p_0''} = a$ and hence (5.67) takes the form

$$f^{**}\left(\frac{t}{a}\right) \le a^{1/p_0''}\cdot f^{**}(t), \quad 0 < t \le |I_0|.$$

The successive application of this inequality leads to the following one

$$f^{**}\left(a^{-j}\,|I_0|\right) \le a^{j/p_0''}\cdot f^{**}\left(|I_0|\right), \quad j = 1, 2, \ldots \tag{5.68}$$

If $t,\ 0 < t \le |I_0|$ is given, then we can choose j such that $a^{-j}|I_0| < t \le a^{-j+1}|I_0|$. Then (5.68) yields

$$f^{**}(t) \le f^{**}\left(a^{-j}\,|I_0|\right) \le \left(a^j\right)^{1/p_0''}\cdot f^{**}\left(|I_0|\right) \le$$

$$\le \left(a\frac{|I_0|}{t}\right)^{1/p_0''}\cdot f^{**}\left(|I_0|\right) = c\cdot f^{**}\left(|I_0|\right)\cdot\left(\frac{t}{|I_0|}\right)^{-1/p_0''}$$

with $c = a^{1/p_0''} = \frac{p_0''}{p_0''-1}$. Clearly, c depends only on ε.

In order to prove (ii), denote $q = p_0''$ and set $f_0(x) = x^{-1/q} + B$ with $B = \frac{q}{q-1}$. Then for $0 < t \le 1$

$$t^{1/q}\cdot f_0^{**}(t) = t^{1/q}\cdot\frac{1}{t}\int_0^t f_0(x)\,dx = t^{1/q}\cdot\left(\frac{t^{-1/q}}{1-\frac{1}{q}}+B\right) \ge \frac{q}{q-1} \equiv c,$$

so that (5.66) holds true. It remains to show that the function f_0 satisfies condition (5.1).

Denote $g(x) = x^{-1/q},\ x > 0$. Let $I \subset [0,1]$. If $(f_0)_I \le (f_0)_{[0,1]}$, then we choose $h,\ 0 < h \le 1$, such that $(f_0)_I = (f_0)_{[0,h]}$. According to Property 2.15, the monotonicity of f_0 on $[0,h]$ implies $\Omega(f_0; I) \le \Omega(f_0;[0,h])$. Further, $\Omega(f_0;[0,h]) = \Omega(g;[0,h])$. Since, as it was shown in Example 2.28,

$$\Omega(g;[0,h]) = 2h^{-1/q}\frac{(q-1)^{q-2}}{q^{q-1}}$$

we have

$$\frac{\Omega\left(f_0; I\right)}{(f_0)_I} \le \frac{\Omega(g;[0,h])}{(f_0)_{[0,h]}} = \frac{2h^{-1/q}\cdot\frac{(q-1)^{q-2}}{q^{q-1}}}{h^{-1/q}\cdot\frac{q}{q-1}+B} \le 2\cdot\frac{(q-1)^{q-1}}{q^q} = \varepsilon,$$

where the last equality follows from (5.64). In the case $(f_0)_I > (f_0)_{[0,1]}$ one can find $h > 1$ such that $(f_0)_I = (f_0)_{[0,h]}$. Then the same arguments lead to the inequality

$$\Omega\,(f_0; I) = \Omega(g; I) \le \Omega(g; [0,h]) = 2h^{-1/q} \cdot \frac{(q-1)^{q-2}}{q^{q-1}} \le$$

$$\le 2 \cdot \frac{(q-1)^{q-1}}{q^q} \cdot \frac{q}{q-1} = \varepsilon \cdot B \le \varepsilon \cdot (f_0)_I\,.$$

Therefore, (5.1) is true for all intervals $I \subset [0,1]$. □

Using the same arguments as in the proof of Proposition 5.9, from part (i) of Theorem 5.34 we immediately obtain the following *one-dimensional Gurov–Reshetnyak theorem* 5.4 *with exact limiting exponent of summability.*

Corollary 5.35 ([34]). *Let f be a non-negative function on $I_0 \subset \mathbb{R}$, satisfying the condition*

$$\Omega(f; I) \le \varepsilon \cdot f_I, \quad I \subset I_0,$$

for some $\varepsilon < 2$. Then f satisfies the Gehring inequality

$$\left\{\frac{1}{|I|}\int_I f^p(x)\,dx\right\}^{1/p} \le c \cdot \frac{1}{|I|}\int_I f(x)\,dx, \quad I \subset I_0,$$

for any $p < p_0''$, where $p_0'' = p_0''(\varepsilon) > 1$ is a root of equation (5.64)*, and the constant $c \equiv c\,(\varepsilon, p)$ depends only on ε and p (for example, one can take $c = \frac{(p_0'')^{1+1/p}}{(p_0''-1)(p_0''-p)^{1/p}}$).*

On the other hand, part (ii) of Theorem 5.34 shows, that in general the limiting exponent p_0'' in Corollary 5.35 cannot be increased.

Remark 5.36. It is easy to see that the root $p_0''(\varepsilon)$ of equation (5.64) satisfies the following relations:

$$p_0''(\varepsilon) > \frac{2}{\mathrm{e}} \cdot \frac{1}{\varepsilon} \quad \text{and} \quad p_0''(\varepsilon) \sim \frac{2}{\mathrm{e}} \cdot \frac{1}{\varepsilon}, \quad \varepsilon \to 0+\,.$$

Therefore Corollary 5.35 revises Corollary 5.32. On the other hand, since the value $p_0''(\varepsilon)$ is the maximal possible in Corollary 5.35 it follows that, as we mentioned in Remark 5.33, the exponent of summability $p_0'(\varepsilon) = \frac{2}{\mathrm{e}} \cdot \frac{1}{\varepsilon}$, obtained in Corollary 5.32, is equivalent to the maximal exponent $p_0''(\varepsilon)$ as $\varepsilon \to 0+$.

5.1.2 Anisotropic Case

Let us come back to the multidimensional case. By analogy with BMO^R-class, let us define *the anisotropic Gurov–Reshetnyak class* $GR^R \equiv GR^R(\varepsilon) \equiv GR^R(\varepsilon, R_0)$ as a class of all functions f that are non-negative on the segment $R_0 \subset \mathbb{R}^d$ and satisfy the Gurov–Reshetnyak condition

$$\Omega(f;R) \le \varepsilon \cdot f_R, \quad R \subset R_0, \tag{5.69}$$

where the constant ε, $0 < \varepsilon < 2$, does not depend on R. Clearly, $GR^R(\varepsilon) \subset GR(\varepsilon)$ for any $\varepsilon \in (0,2)$. The following example shows that in general the opposite inclusion is not true.

Example 5.37. For a given $\varepsilon \in (0,2)$ let us construct a function from the Gurov–Reshetnyak class $GR(\varepsilon)$, which does not belong $GR^R(\varepsilon_1)$ for any $\varepsilon_1 < 2$.

As in Example 2.32, we set

$$f(x) = \sum_{k=1}^{\infty} \chi_{[0,2^{-k+1}]\times\left[0,\frac{1}{k}\right]}(x), \quad x \equiv (x_1,x_2) \in [0,1]^2 \equiv Q_0.$$

In Example 2.32 it was already shown that $f \in BMO\,(Q_0)$. Therefore there exists B such that $\Omega(f;Q) \le B$ for any cube $Q \subset Q_0$. Choose some $\varepsilon \in (0,2)$ and set $g(x) = f(x) + \frac{B}{\varepsilon}$, $x \in Q_0$. Then

$$\frac{\Omega(g;Q)}{g_Q} = \frac{\Omega(f;Q)}{f_Q + \frac{B}{\varepsilon}} \le \frac{B}{f_Q + \frac{B}{\varepsilon}} \le \varepsilon, \quad Q \subset Q_0,$$

so that the function g satisfies Gurov–Reshetnyak condition (5.1), i.e. $g \in GR\,(\varepsilon, Q_0)$.

On the other hand, let us show that $g \notin GR^R(\varepsilon_1)$ for any $\varepsilon_1 < 2$. For this we will use the following inequalities obtained in Example 2.32 ($k \ge 100$):

$$L_k \equiv [\ln(k+1)] \le \ln(k+1) \le f_{R_k} = \sum_{s=1}^{k} \frac{1}{s} + \sum_{s=1}^{\infty} \frac{1}{s+k} \cdot 2^{-s} \le 2 + \ln k,$$

$$\Omega\,(f;R_k) \ge 2L_k - 2 - 2\ln\left(L_k+1\right).$$

Here $R_k = \left[0, 2^{-k+1}\right] \times [0,1]$. Thus

$$\frac{\Omega\,(g;R_k)}{g_{R_k}} = \frac{\Omega\,(f;R_k)}{f_{R_k} + \frac{B}{\varepsilon}} \ge \frac{2L_k - 2 - 2\ln\left(L_k+1\right)}{2 + \ln k + \frac{B}{\varepsilon}} \sim \frac{2\ln k - 2\ln\ln k}{\ln k} \ \rightarrow\ 2$$

as $k \to \infty$. Hence the function g does not belong to the Gurov–Reshetnyak class $GR^R(\varepsilon_1)$ for any $\varepsilon_1 < 2$. $\square$

Let $R_0 \subset \mathbb{R}^d$ be a segment, and let f be a non-negative function on R_0. Define

$$\nu^R(f;\sigma) = \sup_{|R|\le\sigma} \frac{\Omega(f;R)}{f_R}, \quad 0 < \sigma \le |R_0|,$$

where the supremum is taken over all segments $R \subset R_0$ of measure smaller than σ. As before, if $f_R = 0$, then we assume $\frac{\Omega(f;R)}{f_R} = 0$.

Theorem 5.38 ([43]). *Let f be a non-negative function, summable on the segment $R_0 \subset \mathbb{R}^d$. Then*

$$\frac{1}{t}\int_0^t |f^*(u)-f^{**}(t)|\,du \le \nu^R(f;t)\cdot f^{**}(t), \quad 0<t\le |R_0|. \tag{5.70}$$

Proof. We will follow the same way as in the proof of Theorem 5.29. Fix $t,\ 0<t\le |R_0|$, and apply Lemma 1.30 with $\alpha = f^{**}(t)$. Then we obtain at most countable family of segments $R_j \subset R_0,\ j=1,2,\dots$ such that

$$\frac{1}{|R_j|}\int_{R_j} f(x)\,dx = \alpha, \tag{5.71}$$

$$f(x)\le\alpha \quad \text{for almost all} \quad x\in R_0\setminus E, \tag{5.72}$$

where $E=\cup_{j\ge1}R_j$. Hence

$$\int_0^t |f^*(u)-f^{**}(t)|\,du = 2\int_{\{u:\ f^*(u)>\alpha\}} (f^*(u)-\alpha)\ du =$$

$$= 2\int_{\{x\in R_0:\ f(x)>\alpha\}} (f(x)-\alpha)\ dx = 2\int_{\{x\in R_0:\ f(x)>\alpha\}\cap E} (f(x)-\alpha)\ dx =$$

$$= 2\sum_{j\ge1}\int_{\{x\in R_j:\ f(x)>\alpha\}} (f(x)-\alpha)\ dx =$$

$$= 2\sum_{j\ge1}\int_{\{x\in R_j:\ f(x)>\alpha\}} \left(f(x)-f_{R_j}\right)\ dx =$$

$$= \sum_{j\ge1}|R_j|\,\Omega\left(f;R_j\right) \le \sum_{j\ge1}\nu^R\left(f;|R_j|\right)|R_j|\cdot f_{R_j} \le$$

$$\le \nu^R\left(f;|E|\right)\sum_{j\ge1}|R_j|\cdot f_{R_j} = \alpha\cdot|E|\cdot\nu^R(f;|E|).$$

But (5.71) implies

$$\frac{1}{t}\int_0^t f^*(u)\,du = f^{**}(t) = \alpha = \frac{1}{|E|}\int_E f(x)\,dx \le \frac{1}{|E|}\int_0^{|E|} f^*(u)\,du, \tag{5.73}$$

so that $|E|\le t$. Therefore, using the monotonicity of $\nu^R(f;\sigma)$, from (5.73) we get

$$\int_0^t |f^*(u)-f^{**}(t)|\,du \le \alpha\cdot t\cdot\nu^R(f;t). \quad \square$$

Theorem 5.39 ([43]). *Let $\varepsilon,\ 0<\varepsilon<2$ be given, and assume that $p_0''\equiv p_0''(\varepsilon)>1$ is a root of equation (5.64). Then* *(i) if f is non-negative on the segment $R_0 \subset \mathbb{R}^d$ and satisfies Gurov–Reshetnyak condition* (5.69) *with the given ε, then*

$$f^{**}(t) \leq c \cdot f^{**}(|R_0|) \cdot \left(\frac{t}{|R_0|}\right)^{-1/p_0''}, \quad 0 < t \leq |R_0|, \tag{5.74}$$

where the constant c depends only on ε;

(ii) there exists a function $f_0 \in L\left([0,1]^d\right)$, satisfying (5.69) *such that*

$$t^{1/p_0''} \cdot f_0^{**}(t) \geq c > 0, \quad 0 < t \leq 1.$$

Proof. For the prof of (i) we will use Theorem 5.38. Condition (5.69) implies that $\nu^R(f;t) \leq \varepsilon,\ 0 < t \leq |R_0|$, and so (5.70) becomes

$$\frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\, du \leq \varepsilon \cdot f^{**}(t), \quad 0 < t \leq |R_0|.$$

Further, as in the proof of Theorem 5.34, for $a = \left(p_0''/\left(p_0''-1\right)\right)^{p_0''} > 1$

$$f^{**}\left(\frac{t}{a}\right) \leq \left(\frac{a}{2}\cdot\varepsilon + 1\right) f^{**}(t), \quad 0 < t \leq |R_0|.$$

Now the same arguments as in the proof of Theorem 5.34 lead to (5.74).

Part (ii) can be proved in the same way as part (ii) of Theorem 5.34. Indeed, it is enough to consider the function

$$f_0(x_1, \ldots, x_d) = x_1^{-1/p_0''} + \frac{p_0''}{p_0''-1}, \quad (x_1, \ldots, x_d) \in [0,1]^d. \quad \square$$

Let $d \geq 1$. Fix some segment $R_0 \subset \mathbb{R}^d$. The application of Theorem 5.39 to an arbitrary segment $R \subset R_0$ immediately leads to the following *multidimensional analog of the Gurov–Reshetnyak theorem with exact limiting exponent of summability.*

Corollary 5.40. *Let f be a non-negative function on the segment $R_0 \subset \mathbb{R}^d$ such that*

$$\Omega(f;R) \leq \varepsilon \cdot f_R, \quad R \subset R_0,$$

for some $\varepsilon < 2$. Then

(i) f satisfies the Gehring inequality

$$\left\{\frac{1}{|R|}\int_R f^p(x)\, dx\right\}^{1/p} \leq c \cdot \frac{1}{|R|}\int_R f(x)\, dx, \quad R \subset R_0,$$

for any $p < p_0''$, where $p_0''(\varepsilon) > 1$ is a root of equation (5.64) *and the constant $c = c(\varepsilon, p)$ depends only on ε and p;*

(ii) the value of p_0'' in (i) cannot be increased.

5.2 Embedding in the Muckenhoupt Class

By analogy with the Gehring class, let us consider the class $A_q \equiv A_q(C)$ of the non-negative functions f, satisfying *the reverse Hölder inequality* with the negative exponent

$$\frac{1}{|Q|}\int_Q f(x)\,dx \left\{\frac{1}{|Q|}\int_Q f^{-1/(q-1)}(x)\,dx\right\}^{q-1} \le C, \quad Q \subset Q_0. \tag{5.75}$$

Here the cube $Q_0 \subset \mathbb{R}^d$ is fixed, and the constants $q,\ C > 1$ do not depend on the cube $Q \subset Q_0$. Condition (5.75) is called *the A_q–Muckenhoupt condition* ([59]). The classes of Muckenhoupt functions are closely related to Gehring classes. Namely, every Gehring class is contained in some Muckenhoupt class and vice versa (see [8, 72]). In Section 5.1 we saw that Gehring condition (5.14) is equivalent to Gurov–Reshetnyak condition (5.1). Therefore Muckenhoupt condition (5.75) is also equivalent to Gurov–Reshetnyak condition (5.1). Here we will give the direct proof of this fact. First we prove that every Muckenhoupt class is contained in some Gurov–Reshetnyak class.

Theorem 5.41 ([43]). *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Muckenhoupt condition* (5.75) *for some $q, C > 1$. Then f belongs to the Gurov–Reshetnyak class $GR(\varepsilon)$ with $\varepsilon = 2\left(1-(qC)^{-1}\right),\ 0<\varepsilon<2$.*

Proof. Fix some cube $Q \subset Q_0$. Due to condition (5.75), for $0 < u \le 1$ we have

$$f_Q \le C\left\{\frac{1}{|Q|}\int_Q f^{-1/(q-1)}(x)\,dx\right\}^{-(q-1)} =$$

$$= C\left\{\frac{1}{|Q|}\int_0^{|Q|} (f\chi_Q)_*^{-1/(q-1)}(t)\,dt\right\}^{-(q-1)} \le$$

$$\le C\left\{\frac{1}{|Q|}\int_0^{u|Q|} (f\chi_Q)_*^{-1/(q-1)}(t)\,dt\right\}^{-(q-1)} \le C\,(f\chi_Q)_*(u|Q|)\cdot u^{-(q-1)}.$$

Thus,

$$(f\chi_Q)_*(t) \ge \frac{1}{C} f_Q \left(\frac{t}{|Q|}\right)^{q-1}, \quad 0 < t \le |Q|.$$

Therefore, according to Property 2.1,

$$\Omega(f;Q) = \frac{2}{|Q|}\int_{\{x\in Q:\ f(x)<f_Q\}} (f_Q - f(x))\,dx =$$

$$= \frac{2}{|Q|}\int_{\left\{t\in(0,|Q|):\ (f\chi_Q)_*(t)<f_Q\right\}} \left(f_Q - (f\chi_Q)_*(t)\right)\,dt \le$$

$$\leq \frac{2}{|Q|} f_Q \int_0^{|Q|} \left(1 - \frac{1}{C}\left(\frac{t}{|Q|}\right)^{q-1}\right) dt = 2\left(1 - \frac{1}{qC}\right) f_Q.$$

This means that $f \in GR(\varepsilon)$, where $\varepsilon = 2\left(1-(qC)^{-1}\right) < 2$. $\square$

Now let us prove that every Gehring class is contained in some Muckenhoupt class. This fact, joined to Theorem 5.41 and the results of Section 5.1, will complete the proof of the equivalence of the Gurov–Reshetnyak and Muckenhoupt classes.

Theorem 5.42 (Coifman, Fefferman, [8]). *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying the Gehring condition*

$$\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p} \leq B \cdot \frac{1}{|Q|}\int_Q f(x)\,dx, \quad Q \subset Q_0 \tag{5.76}$$

for some $p,\ B > 1$. Then there exist $q,\ C > 1$, which depend only on $p,\ B$ and d such that Muckenhoupt inequality (5.75) holds true.

In [8] this theorem was obtain as a consequence of a series of propositions. Here we reconstruct the original proof from [8]. First we need some auxiliary lemmas (see [8]).

Lemma 5.43. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Gehring condition (5.76). Then for every $\theta,\ 0<\theta<1$, there exists $\sigma,\ 0<\sigma<1$ such that for any cube $Q \subset Q_0$ and for any measurable subset $E \subset Q$ the condition $\frac{|E|}{|Q|} \geq 1-\sigma$ implies*

$$\frac{\int_E f(x)\,dx}{\int_Q f(x)\,dx} \geq 1-\theta. \tag{5.77}$$

Proof. Let $\theta,\ 0<\theta<1$ and set $\sigma = \left(\frac{\theta}{B}\right)^{p/(p-1)}$, $0<\sigma<1$. Let $E_1 \subset Q$ be a measurable set such that $\frac{|E_1|}{|Q|} < \sigma$. Then, by the Hölder inequality, from (5.76) we obtain

$$\frac{1}{|Q|}\int_{E_1} f(x)\,dx = \frac{1}{|Q|}\int_Q f(x)\chi_{E_1}(x)\,dx \leq$$

$$\leq \left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p}\left\{\frac{1}{|Q|}\int_Q \chi_{E_1}^{p/(p-1)}(x)\,dx\right\}^{(p-1)/p} \leq$$

$$\leq B\frac{1}{|Q|}\int_Q f(x)\,dx\left(\frac{|E_1|}{|Q|}\right)^{(p-1)/p} \leq \theta\frac{1}{|Q|}\int_Q f(x)\,dx.$$

So,

$$\frac{|E_1|}{|Q|} < \left(\frac{\theta}{B}\right)^{p/(p-1)} \quad \Longrightarrow \quad \frac{\int_{E_1} f(x)\,dx}{\int_Q f(x)\,dx} \leq \theta. \tag{5.78}$$

Now consider the measurable set $E \subset Q,\ |E| \geq \left(1 - \left(\frac{\theta}{B}\right)^{p/(p-1)}\right) |Q|$. Denote $E_1 = Q \setminus E$. Then

$$\frac{|E_1|}{|Q|} = 1 - \frac{|E|}{|Q|} \leq \left(\frac{\theta}{B}\right)^{p/(p-1)},$$

so that, by virtue of (5.78),

$$\frac{\int_E f(x)\,dx}{\int_Q f(x)\,dx} = 1 - \frac{\int_{E_1} f(x)\,dx}{\int_Q f(x)\,dx} \geq 1 - \theta. \quad \square$$

Lemma 5.44. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Gehring condition (5.76). Then for any $\theta,\ 0 < \theta < 1$, and for any cube $Q \subset Q_0$*

$$\int_Q f(x)\,dx \leq \frac{1}{1-\theta} \int_{\left\{x \in Q:\ f(x) \leq \frac{B^{p/(p-1)}}{\theta^{1/(p-1)}} f_Q\right\}} f(x)\,dx. \tag{5.79}$$

Proof. For σ, defined in the previous lemma, set

$$\beta = \frac{\sigma^{1/p}}{B} = \frac{\left(\frac{\theta}{B}\right)^{1/(p-1)}}{B} = \frac{\theta^{1/(p-1)}}{B^{p/(p-1)}}$$

and denote

$$E' = \left\{x \in Q:\ f(x) > \frac{f_Q}{\beta}\right\}.$$

Then condition (5.76) implies

$$\frac{1}{\beta}\left(\frac{|E'|}{|Q|}\right)^{1/p} = \frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_{E'}\left(\beta\frac{1}{f_Q}\right)^{-p} dx\right\}^{1/p} \leq$$

$$\leq \frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_{E'}\left(\frac{1}{f(x)}\right)^{-p} dx\right\}^{1/p} \leq \frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_{Q}\left(\frac{1}{f(x)}\right)^{-p} dx\right\}^{1/p} =$$

$$= \frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p} \leq B,$$

i.e.,

$$\frac{|E'|}{|Q|} \leq (\beta B)^p = \sigma.$$

Denote $E = \left\{x \in Q:\ f(x) \leq \frac{f_Q}{\beta}\right\}$. Then $\frac{|E|}{|Q|} = 1 - \frac{|E'|}{|Q|} \geq 1 - \sigma$ and from (5.77) we obtain

$$\frac{\int_E f(x)\,dx}{\int_Q f(x)\,dx} \geq 1-\theta,$$

which is equivalent to (5.79). $\square$

Lemma 5.45. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Gehring condition* (5.76). *Then for any θ, $0<\theta<1$, and for any cube $Q_1 \subset Q \subset Q_0$ such that $|Q_1| = t|Q|$, $0<t<1$,*

$$\int_{Q_1} f(x)\,dx \geq (1-\theta)^{\left(\ln \frac{1}{t}\right)/\left(\ln \frac{1}{1-\sigma}\right)+1} \int_Q f(x)\,dx, \tag{5.80}$$

where σ is defined in Lemma 5.43.

Proof. Consider the cubes $Q_1 \subset Q \subset Q_0$, $|Q_1| = t|Q|$, $0<t<1$. Let us construct the cubes $Q_1 \subset Q_2 \subset \cdots \subset Q_k \subset Q$ such that $|Q_i| \geq (1-\sigma)\,|Q_{i+1}|$, $i=1,\dots,k-1$, $|Q_k| \geq (1-\sigma)|Q|$. Then

$$|Q| \leq \frac{1}{1-\sigma}\,|Q_k| \leq \left(\frac{1}{1-\sigma}\right)^2 |Q_{k-1}| \leq \cdots \leq \left(\frac{1}{1-\sigma}\right)^k |Q_1|,$$

where k is chosen in such a way that $(1-\sigma)^{k+1} < t \leq (1-\sigma)^k$, i.e.

$$k \leq \frac{\ln t}{\ln(1-\sigma)} < k+1,$$

$$k = \left[\frac{\ln \frac{1}{t}}{\ln \frac{1}{1-\sigma}}\right] + 1.$$

Then, by Lemma 5.43,

$$\int_{Q_1} f(x)\,dx \geq (1-\theta) \int_{Q_2} f(x)\,dx \geq \cdots \geq (1-\theta)^k \int_Q f(x)\,dx,$$

i.e.,

$$\int_{Q_1} f(x)\,dx \geq (1-\theta)^{\left(\ln \frac{1}{t}\right)/\left(\ln \frac{1}{1-\sigma}\right)+1} \int_Q f(x)\,dx. \quad \square$$

The next lemma is similar to Calderón–Zygmund lemma 1.14. But unlikely the Calderón–Zygmund lemma, the Gehring condition in this case is essential.

Lemma 5.46. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Gehring condition* (5.76). *Then for any cube $Q \subset Q_0$ and any $\lambda > \frac{1}{f_Q}$ there exists a collection of cubes $Q_j \subset Q$, $j=1,2,\dots,$ with pairwise disjoint interiors such that*

$$\delta \frac{1}{\lambda} \leq \frac{1}{|Q_j|} \int_{Q_j} f(x)\,dx \leq \frac{1}{\lambda}, \tag{5.81}$$

$$f(x) \geq \frac{1}{\lambda} \quad \textit{for almost all} \quad x \in Q_0 \setminus \left(\bigcup_{j\geq 1} Q_j\right), \tag{5.82}$$

where $\delta = \delta(B,\theta,d) > 0$.

Proof. Fix some cube $Q \subset Q_0$. Let us partition Q into 2^d congruent cubes, dividing in halves each side of the cube Q. Assume that Q' is one of the obtained cubes. If $f_{Q'} > \frac{1}{\lambda}$, then we partition the cube Q' again in the next step. Otherwise, if $f_{Q'} \leq \frac{1}{\lambda}$, then we assign to Q' the next number j. Then, taking into account the equality $\frac{|Q'|}{|Q|} = 2^{-d}$ and (5.80), we have

$$\frac{1}{|Q'|}\int_{Q'} f(x)\,dx = 2^d \frac{1}{|Q|}\int_{Q'} f(x)\,dx \geq$$

$$\geq 2^d\,(1-\theta)^{\left(\ln 2^d\right)/\left(\ln \frac{1}{1-\sigma}\right)+1}\,\frac{1}{|Q|}\int_Q f(x)\,dx > \delta\frac{1}{\lambda},$$

where $\delta = 2^d\,(1-\theta)^{\left(\ln 2^d\right)/\left(\ln \frac{1}{1-\sigma}\right)+1}$ and σ is defined in Lemma 5.43. This means that the left inequality of (5.81) holds true. Sorting out all cubes Q' in this way, we pass to the next step.

As the result of the described process we obtain a collection of cubes Q_j with pairwise disjoint interiors, which satisfy (5.81). Let $x \in Q \setminus \left(\bigcup_{j\geq 1} Q_j\right)$. Then one can choose a sequence of cubes $\overline{Q}_i$, contractible to x such that $f_{\overline{Q}_i} > \frac{1}{\lambda}$. Then (5.82) follows from Lebesgue theorem 1.1. □

Proof of Theorem 5.42. Fix an arbitrary θ, $0 < \theta < 1$. Let $Q \subset Q_0$, and assume that δ is as defined in Lemma 5.46 and $\lambda > \frac{1}{f_Q}$. Now we apply successively condition (5.82), the left inequality of (5.81), condition (5.79) and the right inequality of (5.81). Then

$$\left|\left\{x \in Q:\; f(x) < \frac{1}{\lambda}\right\}\right| \leq \sum_{j\geq 1}|Q_j| \leq \frac{1}{\delta}\lambda\sum_{j\geq 1}\int_{Q_j} f(x)\,dx \leq$$

$$\leq \frac{1}{\delta}\frac{1}{1-\theta}\lambda\sum_{j\geq 1}\int_{\left\{x\in Q_j:\; f(x)<\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}f_{Q_j}\right\}} f(x)\,dx \leq$$

$$\leq \frac{1}{\delta}\frac{1}{1-\theta}\lambda\sum_{j\geq 1}\int_{\left\{x\in Q_j:\; f(x)<\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\frac{1}{\lambda}\right\}} f(x)\,dx \leq$$

$$\leq \frac{1}{\delta}\frac{1}{1-\theta}\lambda\int_{\left\{x\in Q:\; f(x)<\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\frac{1}{\lambda}\right\}} f(x)\,dx, \tag{5.83}$$

provided the interiors of the cubes Q_j, obtained in Lemma 5.46, are pairwise disjoint. Now let $0 < \varepsilon < 1$ (we will choose it later). Then (5.83) yields

$$\int_{1/f_Q}^{\infty} \lambda^{\varepsilon-1}\left|\left\{x \in Q:\; \frac{1}{f(x)} > \lambda\right\}\right|\,d\lambda \leq$$

$$\leq \frac{1}{\delta}\frac{1}{1-\theta}\int_{1/f_Q}^{\infty}\lambda^{\varepsilon}\int_{\left\{x\in Q:\; f(x)<\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\frac{1}{\lambda}\right\}} f(x)\,dx\,d\lambda \leq$$

$$\le \frac{1}{\delta}\frac{1}{1-\theta}\int_0^\infty \lambda^\varepsilon \int_{\left\{x\in Q:\ \frac{1}{f(x)}>\frac{\theta^{1/(p-1)}}{B^{p/(p-1)}}\lambda\right\}} f(x)\,dx\,d\lambda =$$

$$= \frac{1}{\delta}\frac{1}{1-\theta}\left(\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\right)^{1+\varepsilon}\int_0^\infty u^\varepsilon \int_{\left\{x\in Q:\ \frac{1}{f(x)}>u\right\}} f(x)\,dx\,du. \tag{5.84}$$

Applying the Fubini theorem to the integral in the right-hand side, we get

$$\int_0^\infty u^\varepsilon \int_{\left\{x\in Q:\ \frac{1}{f(x)}>u\right\}} f(x)\,dx\,du = \int_Q f(x)\int_0^{1/f(x)} u^\varepsilon\,du\,dx =$$

$$= \frac{1}{1+\varepsilon}\int_Q f(x)\left(\frac{1}{f(x)}\right)^{1+\varepsilon}dx = \frac{1}{1+\varepsilon}\int_Q f^{-\varepsilon}(x)\,dx.$$

Therefore, (5.84) implies

$$\int_{1/f_Q}^\infty \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)}>\lambda\right\}\right| d\lambda \le$$

$$\le \frac{1}{\delta}\frac{1}{1-\theta}\left(\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\right)^{1+\varepsilon}\frac{1}{1+\varepsilon}\int_Q f^{-\varepsilon}(x)\,dx. \tag{5.85}$$

Transforming the left-hand side of (5.85) we get

$$\int_{1/f_Q}^\infty \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)}>\lambda\right\}\right| d\lambda =$$

$$= \int_0^\infty \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)}>\lambda\right\}\right| d\lambda -$$

$$- \int_0^{1/f_Q} \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)}>\lambda\right\}\right| d\lambda =$$

$$= \frac{1}{\varepsilon}\int_Q f^{-\varepsilon}(x)\,dx - \int_0^{1/f_Q} \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)}>\lambda\right\}\right| d\lambda. \tag{5.86}$$

The last integral can be estimated as follows:

$$\int_0^{1/f_Q} \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)}>\lambda\right\}\right| d\lambda \le$$

$$\le |Q|\int_0^{1/f_Q}\lambda^{\varepsilon-1}\,d\lambda = \frac{|Q|}{\varepsilon}\left(\frac{1}{f_Q}\right)^\varepsilon = \frac{|Q|}{\varepsilon}(f_Q)^{-\varepsilon},$$

so that (5.86) becomes

$$\int_{1/f_Q}^{\infty} \lambda^{\varepsilon-1} \left| \left\{ x \in Q : \frac{1}{f(x)} > \lambda \right\} \right| d\lambda \geq \frac{1}{\varepsilon} \int_Q f^{-\varepsilon}(x)\, dx - \frac{|Q|}{\varepsilon} (f_Q)^{-\varepsilon}.$$

Substitution of this estimate into (5.85) gives

$$\left(\frac{1}{\varepsilon} - \frac{1}{\delta} \frac{1}{1-\theta} \left(\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}} \right)^{1+\varepsilon} \frac{1}{1+\varepsilon} \right) \int_Q f^{-\varepsilon}(x)\, dx \leq \frac{|Q|}{\varepsilon} (f_Q)^{-\varepsilon}, \quad (5.87)$$

where $\varepsilon > 0$ is so small, that

$$c_1 \equiv \frac{1}{\varepsilon} - \frac{1}{\delta} \frac{1}{1-\theta} \left(\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}} \right)^{1+\varepsilon} \frac{1}{1+\varepsilon} > 0.$$

From (5.87) it follows that

$$\frac{1}{|Q|} \int_Q f^{-\varepsilon}(x)\, dx \leq \frac{1}{c_1 \varepsilon} \left\{ \frac{1}{|Q|} \int_Q f(x)\, dx \right\}^{-\varepsilon}. \quad (5.88)$$

Setting $\varepsilon = \frac{1}{q-1}$, i.e. $q = 1 + \frac{1}{\varepsilon}$, we rewrite (5.88) in the form

$$\frac{1}{|Q|} \int_Q f^{-1/(q-1)}(x)\, dx \leq \frac{1}{c_1 \varepsilon} \left\{ \frac{1}{|Q|} \int_Q f(x)\, dx \right\}^{-1/(q-1)},$$

or, equivalently,

$$\frac{1}{|Q|} \int_Q f(x)\, dx \left\{ \frac{1}{|Q|} \int_Q f^{-1/(q-1)}(x)\, dx \right\}^{q-1} \leq \left(\frac{1}{c_1 \varepsilon} \right)^{q-1} \equiv C,$$

and this completes the proof of Muckenhoupt inequality (5.75). □

5.2.1 One-Dimensional Case

Theorem 5.47 ([43]). *Let f be a non-negative function on $I_0 \subset \mathbb{R}$, satisfying Gurov–Reshetnyak condition* (5.1) *for some ε, $0 < \varepsilon < 2$. Then*

$$\frac{1}{t} \int_0^t |f_*(u) - f_{**}(t)|\, du \leq \varepsilon \cdot f_{**}(t), \quad 0 < t \leq |I_0|. \quad (5.89)$$

Proof. Fix some $t \in (0, |I_0|]$ and let $\alpha = f_{**}(t) \leq f_{I_0}$. Using Lemma 1.18, let us construct a collection of pairwise disjoint intervals I_j, $j = 1, 2, \ldots$, such that

$$\frac{1}{|I_j|} \int_{I_j} f(x)\, dx = \alpha, \quad (5.90)$$

$$f(x) \geq \alpha \quad \text{for almost all} \quad x \in I_0 \setminus E, \tag{5.91}$$

where $E = \cup_{j \geq 1} I_j$. By (5.90),

$$\frac{1}{t} \int_0^t f_*(u)\, du = f_{**}(t) = \frac{1}{|E|} \sum_{j \geq 1} |I_j|\, f_{I_j} =$$

$$= \frac{1}{|E|} \sum_{j \geq 1} \int_{I_j} f(x)\, dx = \frac{1}{|E|} \int_E f(x)\, dx \geq \frac{1}{|E|} \int_0^{|E|} f_*(u)\, du.$$

This implies $|E| \leq t$, provided f_* is monotone. Hence, by (5.91) and (5.1),

$$\int_0^t |f_*(u) - f_{**}(t)|\, du = 2 \int_{\{u:\ f_*(u) < f_{**}(t)\}} (f_{**}(t) - f_*(u))\, du =$$

$$= 2 \int_{\{x \in I_0:\ f(x) < f_{**}(t)\}} (f_{**}(t) - f(x))\, dx = \int_E |f(x) - f_{**}(t)|\, dx =$$

$$= \sum_{j \geq 1} \int_{I_j} \left| f(x) - f_{I_j} \right| dx \leq \varepsilon \sum_{j \geq 1} |I_j|\, f_{I_j} = \varepsilon \cdot f_{**}(t) |E| \leq \varepsilon \cdot t \cdot f_{**}(t),$$

and inequality (5.89) follows. $\square$

The next theorem is the analog of Theorem 5.34 for the exact embedding of the Gurov–Reshetnyak class in the Muckenhoupt class.

Theorem 5.48 ([43]). *Let $\varepsilon,\ 0 < \varepsilon < 2$, and let $q_0'' = q_0''(\varepsilon) > 1$ be a root of the equation*

$$(q-1) q^{-q/(q-1)} = \frac{\varepsilon}{2}. \tag{5.92}$$

Then

(i) if f is a non-negative function on $I_0 \subset \mathbb{R}$, satisfying Gurov–Reshetnyak condition (5.1) with the given ε, then

$$f_{**}(t) \geq c \cdot f_{**}(|I_0|) \left(\frac{|I_0|}{t} \right)^{-\left(q_0''-1\right)}, \quad 0 < t \leq |I_0|, \tag{5.93}$$

where the constant $c > 0$ depends only on ε;

(ii) there exists $f_0 \in L([0,1])$, which satisfies (5.1), and such that

$$(f_0)_{**}(t) \leq c_1 t^{q_0''-1}, \quad 0 < t \leq 1, \tag{5.94}$$

where the constant c_1 does not depend on t.

Proof. The function

$$\varphi(q) = (q-1)q^{-q/(q-1)}, \quad q > 1,$$

is continuous on $(1, +\infty)$, $\lim_{q\to 1+0} \varphi(q) = 0$ and $\lim_{q\to+\infty} \varphi(q) = 1$. Moreover, the analysis of the derivative shows that φ is strictly increasing on $(1, +\infty)$. These properties of φ imply that for any ε, $0 < \varepsilon < 2$, equation (5.92) has a unique root $q_0'' = q_0''(\varepsilon) > 1$.

Let us prove (i). Applying Theorem 5.47 and Lemma 2.3 to the function f_*, for $a > 1$ we have

$$f_{**}(t) - f_{**}\left(\frac{t}{a}\right) \le \frac{a}{2}\frac{1}{t}\int_0^t |f_*(u) - f_{**}(t)|\, du \le \frac{a\varepsilon}{2} f_{**}(t), \quad 0 < t \le |I_0|,$$

or, equivalently,

$$f_{**}\left(\frac{t}{a}\right) \ge \left(1 - \frac{a\varepsilon}{2}\right) f_{**}(t), \quad 0 < t \le |I_0|. \tag{5.95}$$

Later we will choose the constant $a > 1$ in such a way, that

$$1 - \frac{a\varepsilon}{2} > 0. \tag{5.96}$$

By (5.95),

$$f_{**}\left(a^{-j}|I_0|\right) \ge \left(1 - \frac{a\varepsilon}{2}\right)^j f_{**}(|I_0|), \quad j = 1, 2, \ldots \tag{5.97}$$

Let $q_0'' > 1$ be the root of equation (5.92). Set $a = (q_0'')^{1/\left(q_0''-1\right)} > 1$. Then

$$1 - \frac{a\varepsilon}{2} = 1 - (q_0'')^{1/\left(q_0''-1\right)}\left(q_0''-1\right)(q_0'')^{-q_0''/\left(q_0''-1\right)} = \frac{1}{q_0''} > 0,$$

so that (5.96) follows. In addition,

$$\left(1 - \frac{a\varepsilon}{2}\right)^{1/\left(q_0''-1\right)} = (q_0'')^{-1/\left(q_0''-1\right)} = \frac{1}{a},$$

and hence (5.97) can be rewritten in the following form

$$f_{**}\left(a^{-j}|I_0|\right) \ge (a^j)^{-\left(q_0''-1\right)} f_{I_0}, \quad j = 1, 2, \ldots \tag{5.98}$$

Now for the given $t \in (0, |I_0|]$ we choose $j \ge 1$ such that $a^{-j}|I_0| < t \le a^{-j+1}|I_0|$. Then from (5.98), by virtue of the monotonicity of f_{**}, we get

$$f_{**}(t) \ge f_{**}\left(a^{-j}|I_0|\right) \ge \left(a^j\right)^{-\left(q_0''-1\right)} f_{I_0} \ge a^{-\left(q_0''-1\right)}\left(\frac{|I_0|}{t}\right)^{-\left(q_0''-1\right)} f_{I_0},$$

which is exactly (5.93) with $c = a^{-\left(q_0''-1\right)}$, i.e., c depends only on ε.

Let us prove (ii). Assume $0 < \varepsilon < 2$ and let $q_0'' > 1$ be defined by (5.92). Set $\alpha = q_0'' - 1 > 0$. Clearly, the function $f(x) = x^\alpha$, $0 \le x \le 1$ satisfies (5.94). Therefore it remains to show that $f \in GR(\varepsilon)$, i.e., it remains to check the inequality (5.1).

Consider an arbitrary $I \subset [0,1]$. Let us choose $t > 0$ such that $J \equiv [0,t] \supset I$ and $f_J = f_I$. Then, by Property 2.15,

$$\left(f_I\right)^{-1} \Omega(f;I) \le \left(f_J\right)^{-1} \Omega(f;J) = \left(f_{[0,1]}\right)^{-1} \Omega(f;[0,1]) =$$

$$= 2\alpha(\alpha+1)^{-(\alpha+1)/\alpha} = 2\left(q_0''-1\right)\left(q_0''\right)^{-q_0''/\left(q_0''-1\right)} = \varepsilon,$$

and this completes the proof of the theorem. □

Part (i) of Theorem 5.48 has the following corollary.

Corollary 5.49 ([43]). *Let f be a non-negative function on $I_0 \subset \mathbb{R}$ such that*

$$\Omega(f;I) \le \varepsilon \cdot f_I, \quad I \subset I_0,$$

for some $\varepsilon < 2$. Then f satisfies the Muckenhoupt inequality

$$\frac{1}{|I|}\int_I f(x)\,dx \left\{\frac{1}{|I|}\int_I f^{-1/(q-1)}(x)\,dx\right\}^{q-1} \le c, \quad I \subset I_0, \tag{5.99}$$

for any $q > q_0''$, where $q_0'' = q_0''(\varepsilon) > 1$ is a root of equation (5.92), *and the constant c depends only on ε and q.*

Proof. Clearly, it is enough to give the proof for the case $I = I_0$. Let $q > q_0''$. Then for $0 < t \le |I_0|$ from (5.93) we obtain

$$f_{**}^{-1/(q-1)}(t) \le a^{\left(q_0''-1\right)/(q-1)} \left(\frac{|I_0|}{t}\right)^{\left(q_0''-1\right)/(q-1)} \left(f_{I_0}\right)^{-1/(q-1)},$$

where $a = a(\varepsilon)$ is defined in the proof of Theorem 5.47. Integrating from 0 to $|I_0|$ we find

$$\int_0^{|I_0|} f_{**}^{-1/(q-1)}(t)\,dt \le c_1\,|I_0|\left(f_{I_0}\right)^{-1/(q-1)},$$

where $c_1 = a^{\left(q_0''-1\right)/(q-1)}(q-1)/\left(q-q_0''\right)$ depends only on q and ε. Therefore

$$\frac{1}{|I_0|}\int_{I_0} f^{-1/(q-1)}(x)\,dx = \frac{1}{|I_0|}\int_0^{|I_0|} f_*^{-1/(q-1)}(t)\,dt \le$$

$$\le \frac{1}{|I_0|}\int_0^{|I_0|} f_{**}^{-1/(q-1)}(t)\,dt \le c_1\left(f_{I_0}\right)^{-1/(q-1)}.$$

The last inequality implies (5.99) with $c = c_1^{q-1}$, which depends only on ε and q. □

Remark 5.50. Part (ii) of Theorem 5.48 implies that for $q = q_0''$ Corollary 5.49 fails.

Remark 5.51. From equation (5.92) it is easy to see, that for $q_0'' \equiv q_0''(\varepsilon)$ and $\varepsilon \to 0$

$$q_0'' - 1 \sim \frac{2\varepsilon}{\mathrm{e}}.$$

Remark 5.52. Set $q = \frac{p}{p-1}$. Then equation (5.92) becomes

$$\frac{p^p}{(p-1)^{p-1}} = \frac{2}{\varepsilon}.$$

This is exactly equation (5.64). It defines the limiting exponent of Gehring class, containing a function which satisfies the Gurov–Reshetnyak condition.

5.2.2 Anisotropic Case

For $d \geq 2$ one can prove the following analog of Theorem 5.47.

Theorem 5.53 ([46]). *Let f be a non-negative function on the segment $R_0 \subset \mathbb{R}^d$, satisfying Gurov–Reshetnyak condition* (5.69) *for some ε, $0 < \varepsilon < 2$. Then*

$$\frac{1}{t}\int_0^t |f_*(u) - f_{**}(t)|\, du \leq \varepsilon \cdot f_{**}(t), \quad 0 < t \leq |R_0|. \tag{5.100}$$

Proof. Essentially it is enough to repeat the proof of Theorem 5.47 with the only difference that now, instead of one-dimensional Lemma 1.18, one has to apply Lemma 1.31, obtaining the following analogs of (5.90) and (5.91) respectively

$$\frac{1}{|R_j|}\int_{R_j} f(x)\, dx = \alpha, \quad j = 1, 2, \ldots, \tag{5.101}$$

$$f(x) \geq \alpha \quad \text{for almost all} \quad x \in R_0 \setminus E. \tag{5.102}$$

Here $E = \cup_{j\geq 1} R_j$ and the interiors of the segments $R_j \subset R_0$ are pairwise disjoint. The rest of the proof just repeats the proof of Theorem 5.47. □

As in the case $d = 1$, Theorem 5.53 implies the following results.

Theorem 5.54 ([46]). *Let ε, $0 < \varepsilon < 2$ be given, and let $q_0'' = q_0''(\varepsilon) > 1$ be a root of the equation*

$$(q-1)q^{-q/(q-1)} = \frac{\varepsilon}{2}. \tag{5.103}$$

Then

(i) *if f is a non-negative function on the segment $R_0 \subset \mathbb{R}^d$, satisfying Gurov–Reshetnyak condition* (5.69) *with some given ε, then*

$$f_{**}(t) \geq c \cdot f_{**}\left(|I_0|\right) \left(\frac{|R_0|}{t}\right)^{-\left(q_0''-1\right)}, \quad 0 < t \leq |R_0|\,,$$

where the constant $c > 0$ *depends only on* ε*;*
(*ii*) *there exists* $f_0 \in L([0,1]^d)$*, satisfying* (5.69) *such that*

$$\left(f_0\right)_{**}(t) \leq c_1 t^{q_0''-1}, \quad 0 < t \leq 1,$$

where c_1 *does not depend on* t*.*

Corollary 5.55 ([46]). *Let* f *be a non-negative function on the segment* $R_0 \subset \mathbb{R}^d$ *such that*

$$\Omega(f;R) \leq \varepsilon \cdot f_R, \quad R \subset R_0,$$

for some $\varepsilon < 2$*. Then* f *verifies the Muckenhoupt inequality*

$$\frac{1}{|R|}\int_R f(x)\,dx \left\{\frac{1}{|R|}\int_R f^{-1/(q-1)}(x)\,dx\right\}^{q-1} \leq c, \quad R \subset R_0,$$

for any $q > q_0''$*, where* $q_0'' = q_0''(\varepsilon) > 1$ *is a root of equation* (5.103)*, while the constant* c *depends only on* ε *and* q*.*

Remark 5.56. Part (*ii*) of Theorem 5.54 implies that for $q = q_0''$ Corollary 5.55 fails.

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