Preface

Let G be a group and suppose that G has an abelian normal subgroup A. If $H = G/A$, then H acts on A by $ah = a^g$, where $h = gA \in H$ and $a \in A$, and this action transforms A into a $\mathbb{Z}H$-module (see all details below). If A is periodic, then very often we may replace A by one of its primary $p-$components. This allows us to assume that A is a p-subgroup, where p is a prime. This way we arrive at a p-module over the ring $\mathbb{Z}H$. In this case, the structure of the lower layer

$$P_1 = \Omega_1(A) = \{a \in A \mid pa = 0\}$$

of A has a significant influence on the structure of A. Since P_1 is an elementary abelian p-subgroup, we may think of P_1 as a module over the ring $\mathbb{F}_pH$, where $\mathbb{F}_p$ is a prime field of order p.

The above approach allows one to employ module and ring-theoretical methods for the characterization of the groups considered. This relatively old idea has shown itself to be very effective in the theory of finite groups. Progress in the study of finite groups naturally led to the implementation of this approach in infinite groups that are closely related to finite groups, specifically, in infinite groups with some finiteness conditions. It is well known that in the theory of rings many significant results are related to finiteness conditions, especially the classical conditions of minimality and maximality. Thus, both artinian and noetherian rings are the main subjects of the largest and the richest branches of the theories of commutative and non-commutative rings. The minimality and the maximality conditions were introduced in groups side by side and played a crucial role in the development of the theory of infinite groups. The study of groups with the maximal condition on all subgroups (the Max condition) led to the fundamental theory of polycyclic groups and applications to other areas (see, [255]). On the other hand, the exploration of groups with the minimal condition on all subgroups (the Min condition) was extremely fruitful and generated among many other the research associated with the well-known problems stated by O.Yu. Schmidt and S.N. Chernikov (see [43, 311]).

After very detailed investigations, many group theorists conjectured that groups with the Max condition should be polycyclic-by-finite and the groups with the Min condition should be Chernikov. Refuting this, A.Yu. Ol'shanskii surprisingly developed a series of his famous *monsters* ([208, Chapter 28]), emphasizing the inexhaustible wealth of Infinite Group Theory.

These examples also solved other famous problems in Group Theory. Other infinite, finitely generated, residually finite, periodic groups having many unusual properties have been constructed by R. I. Grigorchuk (see, for example the survey [93]). Observe that for his first example [92], R.I. Grigorchuk produced a clear and relatively simple construction which compares favorably with examples that existed before. These brilliant creations of A.Yu. Olshanskii and R.I. Grigorchuk show that methods and approaches that are traditional for the theory of generalized soluble groups do not work effectively beyond this theory. They developed a clear understanding that the theory of generalized soluble groups is just a proper specific part of the general group theory. Along with other branches such as finite groups, abelian groups, and linear groups, the theory of generalized soluble groups has its own specific subject of research, rich history and sophisticated methodology.

Together with the ordinary maximal and minimal conditions on all subgroups, the maximal and minimal conditions on normal subgroups (Max-n and Min-n) began to be studied. It turns out that in locally nilpotent groups these latter conditions coincided with the respective ordinary conditions. However, even soluble groups with Max-n and Min-n required new approaches. The classical papers of P. Hall [95, 96, 98] played a major role in the implementation of both module and ring-theoretical methods in the study of soluble groups. The investigation of abelian-by-nilpotent groups that satisfy the maximal condition on normal subgroups led P. Hall to the consideration of noetherian modules over a ring of the form $\mathbb{Z}H$, where H is a finitely generated nilpotent group. The basis connection here between groups, rings and modules is established by the remarkable theorem due to P. Hall: If R is a noetherian ring and G is a polycyclic-by-finite group, then the group ring RG is likewise noetherian. This result stimulated further development of the theory of group rings of polycyclic- by-finite groups as well as the theory of modules over polycyclic-by-finite groups (see, for example, C.J. Brookes [28], K.A. Brown [30], S. Donkin [57, 58], D.R. Farkas [68], K.W. Gruenberg [94], A.V. Jategaonkar [118, 120] , I.N. Musson [195, 196], D. Passman [216, 217, 218], J.E. Roseblade [248, 249, 250, 251, 252], R.L. Snider [267, 268], and many others).

Exploration of the dual condition — that is, investigation of groups with the minimal condition on normal subgroups — began significantly later. The first investigations showed that there exist metabelian groups satisfying the minimal condition on normal subgroups that are not Chernikov. Investigation of such groups determined the necessity of the study of artinian modules over rings of type $\mathbb{Z}H$, where H is an abelian Chernikov group. The excellent paper by B. Hartley and D. McDougall [110] contains the description of such modules and groups. This paper was also a starting point for the investigation of artinian modules over group rings. It is worth noting here that the situation with artinian modules is rather different. Actually, the group ring of a group that is not polycyclic-by-finite loses the valuable property of being noetherian. Therefore, we have no well-developed deep theory of the respective group rings, and we need to build one. At first glance, the criteria of complementability of modules, which arise from the classical theorems of Maschke and Fitting, seem as likely candidates for this. A result of L.G. Kovacs and M.F. Newman [137] is one of the first important generalizations of Maschke's Theorem on infinite groups. Since then, the following steps of the theory were correlated with the criteria of semisimplicity of artinian modules, which are related to conditions of injectivity of simple modules. Investigation of the questions mentioned above developed approaches to the description of some artinian modules over a ring of the form FG, where F is a field. Note that the transition from a scalar field to a ring significantly complicates the problem because there are not too many situations in which one can get a good description of artinian modules over group rings. Sometimes it is only possible to obtain the description of their injective envelopes, and just in the cases in which they are also artinian modules. This way we arrive at the following important problem: to find conditions under which an injective envelope of an artinian module is likewise artinian.

At present the theory of modules over group rings is a very well developed algebraic theory that is rich in many important results and has its own goals and themes of a different nature. Many famous algebraists have made their contributions to this theory. The main aim of this book is to highlight some important results within the framework of the described circle of issues outlined above. Because of the voluntarily limited scope of this book, we were unable to include all valuable accomplishments. We focused our study on artinian modules because noetherian modules are presented well enough elsewhere. The last chapter is dedicated to some group theoretical results about the splitting of a group over its locally nilpotent residual. Such theorems about splitting of a group over its abelian generalized nilpotent radical are very useful in many different investigations. In particular, we found them to be very effective in the study of just non-$\mathfrak{X}$-groups (see L. A. Kurdachenko, J. Otal and I.Ya. Subbotin [157]). Among these results we chose a general theorem proven by D.J.S. Robinson [246]. In [246] the author's proof is based on homological methods. In the current book we develop a new proof having applied only results pertaining groups and artinian modules. This aptly illustrates the effectiveness of the artinian condition.

We want to note that originally many important results have been obtained for integral group rings. However, quite often in the applications, one needs to deal with other group rings RG. For example, the cases in which $R = F\langle x\rangle$ is the group ring of an infinite cyclic group over a (finite) field F, or $R = \mathbb{Z}_{p^\infty}$ is the ring of integer p-adics, or $R = F[[X]]$ is the ring of the power series over a (finite) field F frequently occur. Therefore, in the majority of situations, we consider artinian modules over a group ring DG, where the ring D of scalars is a Dedekind domain. This requires the insertion of some valuable results of the theory of modules over Dedekind domains.

Of course, the choice of content has also been determined by the interests and tastes of the authors. Our selections have been influenced by the work of many people, and the authors especially owe their gratitude to B. Hartley and D.I. Zaitsev. The contribution of B. Hartley to the development of the theory of artinian modules over group rings is difficult to overestimate. His well-known papers in this and other areas, as well as the work of his numerous students and collaborators around the world, have been incredibly influential. Many of his important contributions are mentioned in this book, but many others are not. For example, we have omitted the detailed construction of the fundamental important counterexamples of uncountable artinian modules over certain nilpotent and soluble groups. D.I. Zaitsev's interest in the implementation of ring and module-theoretical results to groups, and his outstanding achievements in this area, inspired a series of works dedicated to the development of his productive methods (see the survey L.S. Kazarin and L.A. Kurdachenko [132]). In many cases, his influence determined the content of this book. For the first author, D.I. Zaitsev was a mentor and a good friend. Unfortunately, the Chernobyl Nuclear Catastrophe undermined his health, and D.I. Zaitsev passed away in 1990 at the age of forty eight. This catastrophe was also an original cause of death of our friends and colleagues V.E. Goretsky, S.S. Levishenko and V.V. Pylaev, who were at the top of their careers when passing away due to different medical complications brought on by this disaster. The stress experienced during the Chernobyl Catastrophe was the main reason for the heart attack that took away the life of one of the main founders of Infinite Group Theory and the head of the Kiev Group Theory School, S.N. Chernikov. This great mathematician was a teacher for many Ukranian algebraists, including the authors of this book. His influence on the development of Infinite Group Theory could not be overestimated.

We, with genuine gratitude, remember another brilliant algebraist, whose extremely various scientific interests and personality made a great impact on the forming of the first and the third authors, namely Z.I. Borevich. He was a professor of St Petersburg University, but he was born in Ukraine and was interested in the development of Ukrainian algebra and greatly supported many Ukrainian mathematicians, who have became leading researches nowadays.

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