Preface

A *functional identity* (FI) can be informally described as an identical relation involving (arbitrary) elements in a ring together with ("unknown") functions; more precisely, elements are multiplied by values of functions. The goal of the general FI theory is to determine the form of these functions, or, when this is not possible, to determine the structure of the ring admitting the FI in question. This theory has turned out to be a powerful tool for solving a variety of problems in different areas. It is not always easy to recognize that the problem in question can be interpreted through some FI; often this is the most intriguing part of the process. But once one succeeds in discovering an FI that fits into the general theory, this abstract theory then as a rule yields the desired conclusions at a high level of generality.

Among classical algebraic concepts, the one of a polynomial identity (PI) seems to be, at least on the surface, the closest one to the concept of an FI. In fact, a PI is formally just a very special example of an FI (where functions are polynomials). However, the theory of PI's has quite different goals than the theory of FI's. One could say, especially from the point of view of applications, that the two theories are complementary to each other. Under some natural restrictions, PI theory deals with rings that are close to algebras of low dimensions, while FI theory gives definitive answers in algebras of sufficiently large or infinite dimensions.

FI theory is a relatively new one. Its roots lie in the Ph.D. Thesis of the first author from 1990, which was followed in the ensuing years by a series of papers in which he studied some basic FI's, in particular those concerning the so-called commuting maps, and also found first applications. The central figure in establishing the foundations of the general theory was Konstantin I. Beidar. This theory culminated around 2000 in the works by Beidar and the second author on d-free sets. Over the last few years, the emphasis has been mainly on applications of the general theory. A notable example are complete solutions of Herstein's conjectures on Lie homomorphisms and Lie derivations in associative rings (posed in Herstein's 1961 "AMS hour talk"). They were obtained in a series of papers by Beidar and the authors of this book which ended in 2002. Practically all of the advanced FI theory was used in these solutions. In fact, the main motivation for developing this theory was searching for tools that would settle these conjectures.

So far no book has been written exclusively on FI's. Commuting maps and some of their applications were treated in the book *Rings with generalized identities*

(Marcel Dekker, 1996) by Beidar, Martindale and Mikhalev, as well as in the book *Local multipliers of C*-algebras* (Springer, 2003) by Ara and Mathieu, in both cases in some rather special settings so that the overlap with the present book is very small. Up until now the general FI theory has existed only in research journals. We believe that the theory has reached maturity and that a monograph presenting FI's and their applications in a comprehensive and comprehensible manner shall be useful for the interested mathematical community.

Through their applications FI's have connections to different areas, say to Lie algebras, Jordan and other nonassociative algebras, linear algebra, operator theory, functional analysis, and mathematical physics. However, the basic setting of the book is (noncommutative) ring theory. Including non-algebraic topics in the book would make the exposition too diverse. Still, different parts of the book should be of interest to mathematicians having different basic backgrounds. While writing we kept in mind that the reader might not be a ring theorist. The prerequisites needed to follow the exposition are therefore carefully explained. The basic ones are surveyed at the very beginning of the book, while the somewhat more demanding ones appear in four appendices at the end. They consider topics which are more or less widely known and treated in other books. We concentrate on those aspects that are important for this book. Some results in appendices are proved, and some not; in any case we try to explain to the reader, not necessarily in a rigorous manner, the background of the results and concepts that are treated. The book is therefore fairly self-contained and suitable for self-study.

The book consists of three parts. Each of them has its own purpose. Part I is an introductory one, Part II gives a full account of the general FI theory, and Part III is devoted to applications of the general theory. Parts are divided into chapters which are further divided into sections. Chapters end with comments about the literature and the history.

Part I has two chapters. The purpose of Chapter 1 is to introduce the newcomer into the subject and to explain in an informal manner, mainly through examples, what this theory is all about. Chapter 2 contains results that are already of some interest in their own right, and they make it possible for one to use FI theory at a basic level. The concept of *d-freeness* is introduced in its simplest form, and the connections to the concept of the *strong degree* are discussed. The results of Chapter 2 are only partially superseded in Part II; namely, the setting in which they are obtained is somewhat different, and in a certain direction even more general than the setting considered later. In principle, a graduate student should be able to understand Part I without difficulties. One could use some sections as a part of a course in ring theory.

Part II is the core of the book. The general theory has so far been exposed only in numerous papers, which are often very technical and long. They depend upon each other, so it is not always easy to read them. The goal of Part II is to extract from these papers what is, in our opinion, most important and applicable, that is, to show what this theory has to offer. The basic concept upon which everything is based is that of a *d-free set*. Chapter 3 discusses basic properties and constructions of these sets. Chapter 4 is devoted to the study of FI's on d-free sets; the notions of *quasi-polynomials* and *core functions* are of central importance in this context. The purpose of Chapter 5 is to prove the existence of some important classes of d-free sets in prime and semiprime rings, and also to consider some special FI's in these rings. The exposition in Part II is necessarily technical, and in spite of our efforts to be as intelligible as possible, the reader studying the text in detail shall need some patience. It is impossible to avoid long formulas and somewhat complicated notation, so that "love at first sight" with this theory seems a bit unlikely; some effort and time is needed in order to appreciate it and find it enjoyable. On the other hand, the main results have clear statements and so hopefully Part II will serve as a useful source of references.

The meaning of the results of Part II becomes evident in Part III. Applications are the main reason for existence of the general theory. We believe that while this theory is already mature (at least in the direction in which it has been mainly developed), the possibilities for its applications are still far from being exploited. Our intention in Part III is therefore to show through several relevant examples the way this theory can be used. We do not try to present the results in utmost generality, but rather point out various areas where FI's are applicable and demonstrate the methods needed in applications. Chapter 6 deals with *Lie homomorphisms* and other "nonassociative" maps. In particular, solutions of Herstein's Lie map conjectures are discussed. Chapter 7 considers some *linear preserver problems*, particularly commutativity and normality preserving maps. Chapter 8 discusses miscellaneous topics which all, however, belong to the *Lie algebra* framework.

We are grateful to a number of colleagues for fruitful discussions on the subject of the book. It would be impossible to list all whose influence is felt in our writing. But we can not omit mentioning a man we can not thank enough and without whom this book would not exist. Unfortunately he is no longer with us. Konstantin (Kostia) I. Beidar passed away in March 2004. His impact on FI theory was decisive. He wrote more than 20 papers on the subject. Some of them were path-breaking. Kostia himself considered his fundamental results on FI's as some of his best mathematical achievements. It is just incredible that while being so productive in FI theory, at the same time he was also heavily involved in the work in many other fields. But his mathematical strength and knowledge was a kind of a legend among his numerous coauthors and colleagues. Solving problems and inventing new areas seemed so easy when working with Kostia. All three of us had the privilege of cooperating with him on several projects. Our main mathematical tie, however, was FI theory. The plan to write this book was made together with him. In several of our frequent visits to Tainan in Taiwan, where he was working after leaving Moscow State University in 1994, we discussed with him possible concepts of the book. A small portion of the book, a draft of Part I and some other preliminary notes, were written while he was still alive. Arriving at Part II, technically the most entangled part of the book, it was somehow expected that he would become involved in the actual writing. But his early and unexpected

death has forced us to continue without him. We miss him; as a friend and as a mathematician. We hope that this book appropriately represents an important part of Kostia's mathematical legacy.

Maribor, Kent, Philadelphia
February 2007

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http://www.springer.com/978-3-7643-7795-3

Functional Identities
Brešar, M.; Chebotar, M.A.; Martindale, W.S.
2007, XII, 272 p., Softcover
ISBN: 978-3-7643-7795-3
A product of Birkhäuser Basel