Introduction

Let X be a compact complex manifold and L be a holomorphic line bundle on X. We denote by $H^q(X, L)$ the qth cohomology group of the sheaf of holomorphic sections of L on X.

Many important results in algebraic and complex geometry are derived by combining a vanishing property with an index theorem, or from the asymptotic results on the tensor powers L^p when $p \to \infty$. One of the most famous examples is the Kodaira–Serre vanishing theorem which asserts that if L is positive, then $H^q(X, L^p)$ vanish for $q \geqslant 1$ and large p. The key remark is that the spectrum of the Kodaira–Laplace operator $\square_p$ acting on $(0, q)$-forms, $q \geqslant 1$, with values in the tensor powers L^p, shifts to the right linearly in the tensor power p. As a consequence the kernel of $\square_p$ is trivial on forms of higher degree and the vanishing theorem follows by the Hodge theory and the Dolbeault isomorphism. Moreover, the Riemann–Roch–Hirzebruch theorem implies that L^p has a lot of holomorphic sections on X for large p, which indeed embed the manifold X in a projective space.

An important generalization which we will emphasize is the asymptotic holomorphic Morse inequalities of Demailly. They give asymptotic bounds on the Morse sums of the $\overline{\partial}$-Betti numbers $\dim H^q(X, L^p)$ in terms of certain integrals of the curvature form of L. The holomorphic Morse inequalities provide a useful tool in complex geometry. They are again based on the asymptotic spectral behavior of the Kodaira–Laplace operator $\square_p$ for large p.

The applications of these vanishing theorems and holomorphic Morse inequalities are numerous. Let us mention here only the Kodaira embedding theorem, the classical Lefschetz hyperplane theorem for projective manifolds, the computation of the asymptotics of the Ray-Singer analytic torsion by Bismut and Vasserot, as well as the solution of the Grauert–Riemenschneider conjecture by Siu and Demailly or the compactification of complete Kähler manifolds of negative Ricci curvature by Nadel and Tsuji. Donaldson's work on the existence of symplectic submanifolds was inspired by the same circle of ideas.

The holomorphic Morse inequalities are global statements which can be deduced from local information such as the behavior of the heat or Bergman kernels. In this refined form we can establish the asymptotic expansion of the Bergman kernel associated to L^p as $p \to \infty$, which have had a tremendous impact on research in the last years. Especially, let's single out its applications in Donaldson's approach to the existence of Kähler metrics with constant scalar curvature in relation to the Mumford–Chow stability which was mainly motivated by a conjecture of Yau. Other applications include the convergence of the induced Fubini–Study metrics, the distribution of zeroes of random sections, the Berezin–Toeplitz quantization and sampling problems.

Another important operator which we will study, also in view of the generalization to symplectic manifolds, is the Dirac operator acting on high tensor powers of L on symplectic manifolds. For a Kähler manifold the square of the Dirac operator is twice the Kodaira Laplacian.

In the present book we will give for the first time a self-contained and unified treatment to the holomorphic Morse inequalities and the asymptotic expansion of the Bergman kernel by using heat kernels, and we present also various applications. Our point of view comes from the local index theory, especially from the analytic localization techniques developed by Bismut–Lebeau. Basically, the holomorphic Morse inequalities are a consequence of the small time asymptotic expansion of the heat kernel. The Bergman kernel corresponds to the limit of the heat kernel when the time parameter goes to infinity, and the asymptotic is more sophisticated. A simple principle in this book is that the existence of the spectral gap of the operators implies the existence of the asymptotic expansion of the corresponding Bergman kernel, no matter if the manifold X is compact or not, or singular, or with boundary. Moreover, we will present a general and algorithmic way to compute the coefficients of the expansion.

Let us now give a rapid account of the main results discussed in this book.

In the first chapter we introduce the basic material. After giving a self-contained presentation of the connections on the tangent bundle, Dirac operator and Lichnerowicz formula, we specify them for the Kodaira Laplacian, especially we study in detail the Bochner–Kodaira–Nakano formula without and with boundary term. These various formulas are fundamental and have a lot of applications. We will use them repeatedly throughout the text. As a direct application, we establish immediately classical vanishing results and the spectral gap property for Kodaira Laplacians and modified Dirac operators. The latter will play an essential role in our approach to the asymptotic expansion of Bergman kernel.

The last two sections of this chapter are dedicated to Demailly's holomorphic Morse inequalities. They originally arose in connection with the generalization of the Kodaira vanishing theorem for Moishezon manifolds proposed by Grauert and Riemenschneider, who conjectured that a compact connected complex manifold X possessing a semi-positive line bundle L, which is positive at at least one point, is Moishezon. The conjecture was solved by Siu and Demailly. The solution of Demailly involves the following strong Morse inequalities:

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(X,L^p)\leqslant\frac{p^n}{n!}\int_{X(\leqslant q)}(-1)^q\left(\tfrac{\sqrt{-1}}{2\pi}R^L\right)^n+o(p^n)\qquad(1)$$

as $p \longrightarrow \infty$, where R^L is the curvature of L (cf. (1.5.15)), and $X(\leqslant q)$ is the set of points where $\dot{R}^L \in \mathrm{End}(T^{(1,0)}X)$, defined by $R^L(u,\overline{v}) = g^{TX}(\dot{R}^L u, \overline{v})$ for $u, v \in T^{(1,0)}X$ and a Riemannian metric g^{TX} on TX, is non-degenerate and has at most q negative eigenvalues. For $q = n$ we have equality, so we obtain an asymptotic Riemann–Roch–Hirzebruch formula.

Demailly's discovery was triggered by Witten's influential analytic proof of the standard Morse inequalities. Witten analyzes the spectrum of the Schrödinger operator $\Delta_t = \Delta + t^2|df|^2 + tV$, where $t > 0$ is a real parameter, Δ is the Bochner Laplacian acting on forms on X, f is a Morse function on X and V is a 0-order operator. For $t \longrightarrow \infty$, the spectrum of Δ_t approaches the spectrum of a sum of harmonic oscillators attached to the critical points of f. In Demailly's holomorphic Morse inequalities, the role of the Morse function is played by the Hermitian metric on the line bundle and the Hessian of the Morse function becomes the curvature of the bundle. The original proof was based on the study of the semi-classical behavior as $p \to \infty$ of the spectral counting functions of the Kodaira Laplacians $\square_p$ on L^p. Subsequently, Bismut gave a heat kernel proof which involves probability theory, and then Demailly and Bouche were able to replace the probability technique by a classical heat kernel argument.

We present here a new approach based on the asymptotic of the heat kernel of the Kodaira Laplacian, $\exp(-\frac{u}{p}\square_p)$. The analytic core follows in Section 1.6 where, inspired by the work of Bismut–Lebeau, we present a new proof for the asymptotic of the heat kernel. In Section 1.7 we apply these results to obtain a heat equation proof of the holomorphic Morse inequalities following Bismut.

In Chapter 2 we study the properties of the field of meromorphic functions. We establish further two fundamental results about Moishezon manifolds. Then we give the proof of the Siu–Demailly criterion which answers the Grauert–Riemenschneider conjecture. For $q = 1$, the Morse inequalities (1) give

$$\dim H^0(X, L^p) \geqslant \frac{p^n}{n!} \int_{X(\leqslant 1)} \Big(\tfrac{\sqrt{-1}}{2\pi} R^L\Big)^n + o(p^n)\,, \quad p \longrightarrow \infty\,. \tag{2}$$

Therefore if L satisfies

$$\int_{X(\leqslant 1)} \Big(\tfrac{\sqrt{-1}}{2\pi} R^L\Big)^n > 0\,, \tag{3}$$

(in particular, if L is semi-positive and positive at at least one point), there are a lot of sections in $H^0(X, L^p)$, which by taking quotients deliver n independent meromorphic functions, i.e., X is Moishezon.

In Section 2.4 we present an algebraic reformulation of the holomorphic Morse inequalities.

In Chapter 3 we prove the Morse inequalities for the Dolbeault L^2-cohomology spaces for a non-compact manifold satisfying the fundamental estimate (Poincaré inequality) at infinity. Using this more abstract formulation of the Morse inequalities, we can find a lower bound for the growth of the holomorphic section space for uniformly positive line bundles (Theorem 3.3.5) and an extension of the Siu–Demailly criterion for compact complex spaces with isolated singularities.

We end the chapter with a study of a class of manifolds satisfying pseudoconvexity conditions in the sense of Andreotti–Grauert, namely q-convex and weakly 1-complete manifolds and also covering manifolds. Pseudoconvex manifolds are very important in complex geometry and analysis.

In Chapter 4, we study the asymptotic expansion of the Bergman kernel. We assume now that L is positive, equivalently, there exists a Hermitian metric h^L on L, such that $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ defines a Kähler form on X, where R^L is the curvature of the holomorphic Hermitian connection ∇^L on (L, h^L). In the rest of the Introduction we denote by g^{TX} the associated Kähler metric to ω on TX. We also let E be a holomorphic vector bundle on X with a Hermitian metric h^E.

Since L is positive, the Kodaira–Serre vanishing theorem shows that

$$H^q(X, L^p \otimes E) = 0 \tag{4}$$

for p large enough and $q \geqslant 1$. Thus the whole cohomology of $L^p \otimes E$ concentrates in degree zero.

The Bergman kernel $P_p(x, x')$ associated to $L^p \otimes E$ for p large enough, is the smooth kernel of the orthogonal projection P_p from $\mathscr{C}^\infty(X, L^p \otimes E)$, the space of smooth sections of tensor powers $L^p \otimes E$, on the space of holomorphic sections of $L^p \otimes E$, or, equivalently, on the kernel of the Kodaira Laplacian $\square_p$ on $L^p \otimes E$. More precisely, let $\{S_i^p\}_{i=1}^{d_p}$ be any orthonormal basis of $H^0(X, L^p \otimes E)$ with respect to the global inner product induced by g^{TX}, h^L and h^E (cf. (1.3.14)). Then for p large enough,

$$P_p(x, x') = \sum_{i=1}^{d_p} S_i^p(x) \otimes (S_i^p(x'))^* \in (L^p \otimes E)_x \otimes (L^p \otimes E)^*_{x'}. \tag{5}$$

Especially,

$$P_p(x, x) = \sum_{i=1}^{d_p} |S_i^p(x)|^2, \quad \text{if } E = \mathbb{C}. \tag{6}$$

The Bergman kernel has been studied by Tian, Yau, Bouche, Ruan, Catlin, Zelditch, Lu, Wang, and many others, in various generalities, establishing the asymptotic expansion for high powers of L. Moreover, it was discovered that the coefficients in the asymptotic expansion encode geometric information about the underlying complex projective manifolds.

Our approach to the study of the asymptotic expansion continues the method applied in Chapter 1. We treat both the Dirac operator and the Kodaira Laplacian in the same time by means of the modified Dirac operator. The key point of our method is that the spectrum $\mathrm{Spec}(\square_p)$ of $\square_p$ (or of the half of the square of the

Dirac operator) has a spectral gap, cf. Section 1.5. This means that there exists $C > 0$ such that for $p \geqslant 1$,

$$\mathrm{Spec}(\Box_p) \subset \{0\} \cup]2\pi p - C, +\infty[. \tag{7}$$

We can divide our approach in three steps. The first step is to establish the spectral gap property (7). The second is the localization: the spectral gap property (7) and the finite propagation speed of solutions of hyperbolic equations allow us first to localize the asymptotic of $P_p(x_0, x')$ in the neighborhood of x_0. We pull-back and extend the operator to $T_{x_0}X \cong \mathbb{R}^{2n}$, and verify that it inherits also the spectral gap property. The third step is to work on $\mathbb{R}^{2n}$. Here we combine the spectral gap property, the rescaling of the coordinates and functional analysis techniques, to conclude the proof of our final result. Moreover, by using a formal power series trick, we get a general and algorithmic way to compute the coefficients in the expansion. Certainly, for the last two steps it makes no difference whether the manifold X is compact or not. Thus in various new situations, we only need to verify the spectral gap property (cf. Chapters 5, 6, 8).

We obtain finally the following asymptotic expansion (cf. Theorem 4.1.2):

$$P_p(x, x) \sim \sum_{r=0}^{\infty} \boldsymbol{b}_r(x) p^{n-r}, \tag{8}$$

where $\boldsymbol{b}_r(x) \in \mathrm{End}(E)_x$ are smooth coefficients, which are polynomials in R^{TX}, R^E and their derivatives with order $\leqslant 2r - 2$. Moreover

$$\boldsymbol{b}_0 = \mathrm{Id}_E, \quad \boldsymbol{b}_1 = \frac{1}{4\pi}\Big[2R^E(w_j, \overline{w}_j) + \frac{1}{2} r^X \,\mathrm{Id}_E\Big], \tag{9}$$

where r^X is the scalar curvature of (TX, g^{TX}) and $\{w_j\}_{j=1}^n$ is an orthonormal basis of $T^{(1,0)}X$. In the case of trivial bundle E the term $\boldsymbol{b}_1$ was calculated by Lu and used by Donaldson in his work on the existence of Kähler metrics with constant scalar curvature.

We also find the full off-diagonal expansion of the Bergman kernel $P_p(x, x')$ with the help of the heat kernel.

In Chapter 5, we study in detail the metric aspect of the Kodaira map as an application of the asymptotic expansion of the Bergman kernel. First, we present an analytic proof of the Kodaira embedding theorem following an original idea of Bouche, and we study the convergence of the induced Fubini–Study metric. Then the Kodaira map $\Phi_p : X \longrightarrow \mathbb{P}(H^0(X, L^p)^*)$, defined by $\Phi_p(x) = \{s \in H^0(X, L^p) : s(x) = 0\}$ for $x \in X$, is an embedding for p large enough and for any $l \in \mathbb{N}$, there exists $C_l > 0$ such that

$$\left|\frac{1}{p}\, \Phi_p^*(\omega_{FS}) - \omega\right|_{\mathscr{C}^l(X)} \leqslant \frac{C_l}{p^2}\,, \tag{10}$$

where ω_{FS} is the Fubini–Study form on $\mathbb{P}(H^0(X, L^p)^*)$.

By using the Kodaira embedding, we also discuss briefly the relation of the Bergman kernel and the existence of Kähler metrics with constant scalar curvature. Then, as an easy consequence of our approach, we describe the asymptotic expansion of the Bergman kernel on complex orbifolds, and the metric aspect of the Kodaira map.

Finally, we give an introduction to the Ray-Singer analytic torsion and study its asymptotic behavior. The analytic torsions have a lot of applications, especially in Arakelov geometry. This seems to be quite independent of our subject, but in fact, Donaldson has used the analytic torsion in his study of the existence of Kähler metrics with constant scalar curvature.

In Chapter 6 we establish the existence of the expansion on compact sets of a non-compact manifold, as long as the spectral gap exists. One interesting situation is the case of Zariski open sets in compact complex spaces endowed with the generalized Poincaré metric. The expansion of the Bergman kernel implies a new proof of the Shiffman–Ji–Bonavero–Takayama criterion for a Moishezon manifold. Then we obtain again Morse inequalities which are suitable for the study of the compactification of complete Kähler manifolds with pinched negative curvature.

In Chapter 7, using the full off-diagonal expansion of the Bergman kernel, we study the properties of Toeplitz operators and the Berezin–Toeplitz quantization. For $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, we define the Toeplitz operator $\{T_{f,p}\}$ as the family of linear operators

$$T_{f,p} : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E)\,, \quad T_{f,p} = P_p\, f\, P_p\,. \tag{11}$$

One of our main goals is to show that the set of Toeplitz operators is closed under the composition of operators, so they form an algebra. More precisely, let $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, then there exist $C_r(f,g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ with

$$T_{f,p}\, T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathcal{O}(p^{-\infty}), \tag{12}$$

where C_r are differential operators. In particular $C_0(f,g) = fg$.

If $f, g \in \mathscr{C}^\infty(X)$, then

$$[T_{f,p}\,, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \mathcal{O}(p^{-2}), \tag{13}$$

here $\{f,g\}$ is the Poisson bracket of f, g on $(X, 2\pi\omega)$.

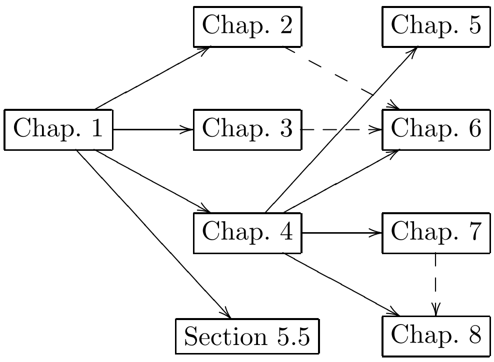
In Chapter 8, we find the asymptotic expansion of the Bergman kernel associated to the modified Dirac operator and the renormalized Bochner Laplacian, as well as their applications.

We hope the material of this book can also be used by graduate students. To help the readers, we add five appendices. In Appendix A, we recall the Sobolev embedding theorems and basic elliptic estimates. In Appendix B, we present useful material from Hermitian geometry. We also introduce the basics of Chern–Weil and Chern–Simons theories. In Appendix C, we collect some facts about self-adjoint operators. In Appendix D, we explain in detail the relation of the heat kernel and the finite propagation speed of solutions of hyperbolic equations. Finally, in Appendix E, we explain the basic facts about the harmonic oscillator.

The book should also serve as an analytic introduction to the applications to algebraic geometry of the holomorphic Morse inequalities as developed by Demailly and his school, as well as to Donaldson's approach to the existence of Kähler metrics of constant scalar curvature.

To keep the book within reasonable size, we list several classical results without proofs, and we indicate the corresponding references in the bibliographic notes of each chapter. The literature concerning the various themes we treat is quite vast and contains many important contributions. We could not include them all in the Bibliography, and restrained to the references which directly influenced our work.

Prerequisites for this book are a course on differentiable manifolds and vector bundles. This book is not necessarily meant to be read sequentially. The reader is encouraged to go directly to the chapter of interest. Basically, Chapters 1 and 4 introduce the main technical ideas, and other chapters are various generalizations and applications. Here is a roadmap for our book.



Notation

We denote by $\mathbb{C}, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}$ the complex, natural, rational, real, integer numbers, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\mathbb{R}_+ = [0, \infty[$, $\mathbb{R}_+^* =]0, \infty[$, $\mathbb{Q}_+ = \mathbb{Q} \cap \mathbb{R}_+$. For $u \in \mathbb{R}$, we denote by $\lfloor u \rfloor$ the integer part of u.

For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $B = (B_1, \ldots, B_m) \in \mathbb{C}^m$, we write by

$$|\alpha| = \sum_{j=1}^m \alpha_j, \quad \alpha! = \prod_j (\alpha_j!), \quad B^\alpha = \prod_j B_j^{\alpha_j}.$$

$SL(n, \mathbb{C})$ is the space of $\mathbb{C}$-valued $n \times n$ matrices with determinant 1. $O(n)$ is the orthogonal group of degree n over $\mathbb{R}$. $U(n)$ is the unitary group of degree n over $\mathbb{C}$.

We denote by dim or $\dim_{\mathbb{C}}$ the complex dimension of a complex (vector) space. We denote also by $\dim_{\mathbb{R}}$ the real dimension of a space.

For a complex vector bundle E on a manifold X, $\mathrm{rk}(E)$ denotes its rank, and Id_E the identity morphism. Also, $\det(E) := \Lambda^{\mathrm{rk}(E)}(E)$ is its determinant line bundle, E^* its dual bundle and $\mathrm{End}(E) := E \otimes E^*$. The space of smooth sections of E over X is denoted by $\mathscr{C}^\infty(X, E)$.

If Q is an operator, we denote by $\mathrm{Ker}(Q)$ its kernel, $\mathrm{Im}(Q)$ its image set.

If U is a subset of V, we write $U \subset V$. If U is a relatively compact subset of V, we write $U \Subset V$. The characteristic function 1_U of U is defined as 1 on U and 0 on the complement of U.

In the whole book, if there is no other specific notification, when in a formula a subscript index appears two times, then we sum up with this index.

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In particular, our collaboration started at the Humboldt Universität zu Berlin, where, in his Thesis from 1922, Stefan Bergman discovered the kernel function in response to a question of Erhard Schmidt. We reproduce here the account of Menahem Max Schiffer from 1981[1]: "Bergman participated in Schmidt's seminar and was charged to give a lecture on the development of arbitrary functions with finite square integrals in terms of an orthogonal set. As he told me, he misunderstood the task and instead of dealing with real functions over a real interval, he attacked the problem for analytic functions over a complex domain. He found the task hard but attacked it courageously and carried it through. This was the genesis of his famous theory of the kernel function."

[1]M.M. Schiffer, *Stefan Bergman* (1885–1977) *in memoriam*, Ann. Pol. Math. 39 (1981), 5–9.

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