# Preface

Sub-Riemannian (also known as Carnot–Carathéodory) spaces are spaces whose metric structure may be viewed as a constrained geometry, where motion is only possible along a given set of directions, changing from point to point. They play a central role in the general program of analysis on metric spaces, while simultaneously continuing to figure prominently in applications from other scientific disciplines ranging from robotic control and planning problems to MRI function to new models of neurobiological visual processing and digital image reconstruction. The quintessential example of such a space is the so-called (first) Heisenberg group. For a precise description we refer the reader to Chapter 2; here we merely remark that this is the simplest instance of a sub-Riemannian space which retains many features of the general case.

The Euclidean isoperimetric problem is the premier exemplar of a problem in the geometric theory of the calculus of variations. In Chapter 1 we review the origins of this celebrated problem and present a spectrum of well-known approaches to its solution. This discussion serves as motivation and foundation for the remainder of this survey, which is devoted to the isoperimetric problem in the Heisenberg group. First formulated by Pierre Pansu in 1982 (see (8.2) in Chapter 8 for the precise statement), the isoperimetric problem in the first Heisenberg group is one of the central questions of sub-Riemannian geometric analysis which has resisted sustained efforts by numerous research groups over the past twenty-five years.

Our goals, in writing this survey, are twofold. First, we want to describe the isoperimetric problem in the Heisenberg group, outline recent progress in this field, and introduce a number of techniques which we think may lead to further understanding of the problem. In accomplishing this program we simultaneously provide a concise and detailed introduction to the basics of analysis and geometry in the setting of the Heisenberg group. Rather than present a general, exhaustive introduction to the field of subelliptic equations, Carnot–Carathéodory metrics and sub-Riemannian geometry, as is done (to different extents) in the standard references [32], [100], [103], [130], [243], [203], [255], and in the forthcoming monograph [114], here we focus on the simplest example of the first Heisenberg group. This seems to us a good starting point for a novice who wants to learn some basic techniques and issues in the field without having to face the most general picture first. At present there are no elementary or introductory texts in this area; we are convinced that there is great need for such a text, to motivate young researchers to work in this area or to clarify to mathematicians working in other fields its principal features. While most of the material in this survey has appeared elsewhere, the approach to the horizontal differential geometry of submanifolds via Riemannian approximation is original; we hope it may be helpful for those who wish to further investigate this interesting line of research.

The structure of this survey is as follows:

**Chapter 1.** We give an abbreviated review of the isoperimetric problem and its solution in Euclidean space, indicating a few proofs for the sharp isoperimetric inequality in the plane arising from diverse areas such as complex analysis, differential geometry, geometric measure theory, nonlinear evolution PDE's (curvature flow), and integral geometry.

**Chapters 2, 3.** We introduce the first Heisenberg group $\mathbb{H}$ and describe in detail its principal metric, analytic and differential geometric features. Our presentation of the sub-Riemannian structure of $\mathbb{H}$ is somewhat nonstandard, as we first introduce an explicit coordinate system and later define the sub-Riemannian metric by referencing this particular set of coordinates. This "hands-on" approach, while not in the coordinate-free approach of modern geometry, fits well with our basic aim as described above.

In Chapter 3 we present a selection of pure and applied mathematical models which feature aspects of Heisenberg geometry: CR geometry, Gromov hyperbolic spaces, jet spaces, path planning for nonholonomic motion, and the functional structure of the mammalian visual cortex.

**Chapter 4.** We turn from the global metric structure of the Heisenberg group $\mathbb{H}$ to a study of the geometry of submanifolds. We introduce the concept of horizontal mean curvature, which gives a sub-Riemannian analog for the classical notion of mean curvature. Computations of the sub-Riemannian differential geometric machinery are facilitated by considering $\mathbb{H}$ as a Gromov–Hausdorff limit of Riemannian manifolds. We illustrate this by computing some of the standard machinery of differential geometry in the Riemannian approximants, and identifying the appropriate sub-Riemannian limits. Typical submanifolds in $\mathbb{H}$ contain an exceptional set, the so-called *characteristic set,* where this sub-Riemannian differential geometric machinery breaks down. In Section 4.4 we work through an extended analysis of the limiting behavior of fundamental ingredients of sub-Riemannian submanifold geometry at the characteristic locus. Such an analysis plays a key role in our later discussion of Pansu's isoperimetric conjecture (see Chapter 8).

**Chapters 5, 6.** Weakening the smoothness requirements of differential geometry leads to the study of geometric measure theory. We give a broad summary of some basic tools of geometric measure theory in $\mathbb{H}$: horizontal Sobolev and BV spaces and the Sobolev embedding theorems, perimeter measure, Hausdorff and Minkowski content and measure, area and co-area formulas, and the Pansu–Rademacher differentiability theorem for Lipschitz functions. This development culminates in Section 6.4, where we present two derivations of the first variation formula for perturbations of the horizontal perimeter. These formulas are essential ingredients in the most recent developments associated with proofs of Pansu's conjecture in certain special cases; our presentation of the first variation formula for the horizontal perimeter is preparatory to our discussion of these developments in Sections 8.5 and 8.6. We conclude Chapter 6 with a brief overview of Mostow's rigidity theorem for cocompact lattices in complex hyperbolic space, emphasizing the role of quasiconformal functions on the Heisenberg group in the proof and building on this to summarize some of the essential aspects of the field of sub-Riemannian geometric function theory which has grown from this application.

**Chapters 7, 8.** With the above tools in hand, we are prepared to begin our discussion of the sub-Riemannian isoperimetric problem in the Heisenberg group. In Chapter 7 we give two proofs for the isoperimetric inequality in $\mathbb{H}$. Neither proof gives the best constant or identifies the extremal configuration. The first proof relies on the equivalence of the isoperimetric inequality with the geometric Sobolev inequality. The second is Pansu's original proof, which relies on an adaptation of an argument of Croke. Chapter 8 is the heart of the survey. We present Pansu's famous conjecture on the isoperimetry extremals, and discuss the current state of knowledge, including various partial results (requiring *a priori* regularity and/or symmetry), and various Euclidean techniques whose natural analogs have been shown to fail in $\mathbb{H}$.

**Chapter 9.** In this concluding chapter, we discuss three other analytic "best constant" problems in the Heisenberg group, whose solutions are known.

We envision this survey as being of use to a variety of audiences and in a variety of ways. Readers who are interested only in obtaining an overview of the general subject area are invited to read Chapters 2–6. These chapters provide a concise introduction to the basic analytic and geometric machinery relevant for the sub-Riemannian metric structure of $\mathbb{H}$. We presuppose a background in Riemannian geometry, PDE and Sobolev spaces (in the Euclidean context), and the basic theory of Lie groups. For those already fluent in sub-Riemannian geometric analysis, Chapters 7 and 8 provide an essentially complete description of the current state of knowledge regarding Pansu's conjecture, and present a wide array of potential avenues for attacks on it and related conjectures. Chapter 9 is essentially independent of the preceding two chapters and can be read immediately following Chapter 6.

We have deliberately aimed at a treatment which is neither comprehensive nor put forth in the most general setting possible, but instead have chosen to work (almost entirely) in the first Heisenberg group, and present those topics and results most closely connected with the isoperimetric problem.

Notable topics which we omit or mention only briefly include:

- The theory of (sub-)Laplacians and the connections between sub-Riemannian geometry, subelliptic PDE and Hörmander's "sums of squares" operators. Similarly, we have very little to say on the subject of potential theory (both linear and nonlinear), apart from some brief results in Chapter 6 connected with the Sobolev embedding theorems.
- Carnot groups as tangent cones of general sub-Riemannian manifolds.
- Further extensions of geometric analysis beyond the sub-Riemannian context, e.g., the emerging theory of "analysis on metric measure spaces".
- *Singular geodesics* in the Martinet (and other sub-Riemannian) distributions.
- Further applications of sub-Riemannian geometry in control theory and nonholonomic mechanics (apart from the discussion in Chapter 3).

These topics are all covered in prior textbooks, which mitigates their omission here. Singular geodesics in sub-Riemannian geometry play a starring role in Montgomery's text [203], and the intricacies of the construction of tangent cones on sub-Riemannian spaces are presented in both [203] and the survey article of Bellaïche [32]. For analysis on metric spaces, the best reference is Heinonen [136]; see also [137]. For nonlinear potential theory (in the Euclidean setting) the principal reference is Heinonen–Kilpeläinen–Martio [139]. In addition to the preceding list, we are also omitting a full discussion of several important recent developments, most notably:

- Rigidity theorems à la Bernstein for minimal surfaces in the Heisenberg group.
- The extraordinary developments in rectifiability and geometric measure theory connected with the extension by Franchi, Serapioni and Serra-Cassano of the structure theorem of de Giorgi to sets of finite perimeter in Carnot groups.

These topics are still very much the subject of active investigation and it is too soon to write their definitive story.

In conclusion, we would be remiss in failing to pay homage to the comprehensive treatise by Gromov [130] on the metric geometry of sub-Riemannian spaces, which provides a wealth of information regarding the structure of these remarkable spaces. Much of the current development in the area represents the working out and elaboration of ideas and notions presented in that work.

## Remarks on notation and conventions

With only a few exceptions, we have attempted to keep our discussion of references, citations, etc. limited to the "Further results" and "Notes" sections of each chapter. In certain cases, particularly when we have used without proof some well-known result which can be found in another textbook, we have deviated from this policy. Despite its size, our reference list still represents only a fraction of the work in this area, and should be viewed merely as a guide to the existing literature.

Our notation and terminology is for the most part standard. The Euclidean space of dimension $n$ and its unit sphere are denoted by $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$, respectively. By $H^n_{\mathbb{A}}$ we denote the hyperbolic space over the division algebra $\mathbb{A}$ (either the real field $\mathbb{R}$, the complex field $\mathbb{C}$, the quaternionic division algebra $\mathbb{K}$ or Cayley's octonions $\mathbb{O}$.) We denote by $B(x,r)$ the (open) metric ball with center $x$ and radius $r$ in any metric space $(X,d)$. We write $\operatorname{diam} A$ for the diameter of any bounded set $A \subset X$, and $\operatorname{dist}(A,B)$ for the distance between any two nonempty sets $A, B \subset X$. If the metric needs to be emphasized we may use a notation of the form $B_d(x,r)$, $\operatorname{diam}_d A$, etc. In the case of the Euclidean metric in $\mathbb{R}^n$, we write $B_E(x,r)$, $\operatorname{diam}_E d$, etc. We always reserve the notation $\langle \cdot, \cdot \rangle$ for the standard Euclidean inner product. An alternate family of inner products, associated to a family of degenerating Riemannian metrics $g_L$ on $\mathbb{R}^3$, will be written $\langle \cdot, \cdot \rangle_L$. The latter family of inner products will play an essential role throughout the survey.

We will use both vector notation and complex notation for points in $\mathbb{R}^2$, switching between the two without further discussion. The unit imaginary element in $\mathbb{C}$ will always be written $\mathbf{i}$. For $v = (v_1, v_2) \in \mathbb{R}^2$ we write $v^\perp = (v_2, -v_1)$.

In any dimension $n$, we write $|A|$ for the Lebesgue measure of a measurable set $A$. For any domain $\Omega \subset \mathbb{R}^n$, we denote by $W^{k,p}(\Omega)$ the Sobolev space of functions on $\Omega$ admitting $p$-integrable distributional derivatives of order at most $k$. The surface area measure on a smooth hypersurface $S$ in a Euclidean space $\mathbb{R}^n$ of any dimension will be denoted $d\sigma$. Finally, we write

$$\omega_{n-1} := \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

for the surface area $\sigma(\mathbb{S}^{n-1})$ of the standard unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

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L.C. (Fayetteville, Arkansas)
D.D. (West Lafayette, Indiana)
S.D.P. (Hanover, New Hampshire)
J.T.T. (Urbana, Illinois)

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