

# Preface

The new field of noncommutative geometry (see [29, 63]) applies ideas from geometry to mathematical structures determined by noncommuting variables, and vice versa. Typically, a crucial part of the information is encoded in a noncommutative algebra whose elements represent these noncommuting variables. Such algebras are naturally associated — for instance as algebras of differential or pseudo-differential operators, algebras of intertwining operators for representations, Hecke algebras, algebras of observables in quantum mechanics — with many different geometric structures arising from subjects ranging from mathematical physics and differential geometry to number theory. The fundamental tools for the study of topological invariants attached to noncommutative structures are given by K-theory and cyclic homology. These generalised homology theories are naturally given as bivariant theories, that is, as functors of two variables. For instance, bivariant K-theory specialises both to ordinary topological K-theory and to its dual, K-homology.

This book grew out of an Oberwolfach Seminar organised by the three authors in May 2005. Our aim in this seminar was to introduce young mathematicians to the various forms of topological K-theory for (noncommutative) algebras without assuming too much background on the part of our audience. A second aim was to sketch some typical applications of these techniques, including bivariant versions of the Atiyah–Singer Index Theorem, twisted K-theory, some applications to mathematical physics, and the Baum–Connes conjecture.

An important part of our book is devoted to a complete and unified description of a formalism that has been developed over the past 10 years in [36, 37, 39], and which allows us to construct topological K-theory and associated bivariant theories with good properties for many different categories of algebras over  $\mathbb{R}$  or  $\mathbb{C}$  such as  $C^*$ -algebras, Banach algebras, locally convex algebras, Ind- or Pro-Banach algebras. Since the construction has to be adapted to the different possible categories, one first problem that we have to address is to fix the setting in which to present the construction. Here we have settled for the category of bornological algebras. This setting has been advocated in various contexts in [82, 84, 85]. It is particularly flexible and elegant and covers many interesting examples (for instance it is especially well suited for smooth group algebras). Another argument for this choice is the fact that the construction of bivariant K-theory for locally convex algebras is already available in published form in [36, 37, 39]. So we can

use this opportunity to spell out the (minor) changes that have to be made in the bornological setting.

We start from scratch with a discussion of elementary topological K-theory. We do this in a bornological setting, which is more general than the one of Banach algebras. A good choice for this turns out to be the class of local Banach algebras. These algebras are essentially inductive limits of Banach algebras and provide a class of bornological algebras which allow functional calculus. For these algebras basic topological K-theory can be developed in complete analogy with the case of Banach algebras. We present proofs of Bott periodicity and of the Pimsner–Voiculescu exact sequence, and we show how K-theory can be computed in examples. We also briefly discuss the computation of K-theory for group  $C^*$ -algebras and the Baum–Connes conjecture. This topic is taken up again in Chapters 10 and 13.

The next chapters treat bivariant K-theory for bornological algebras following [36, 37, 39]. The original arguments have been improved and streamlined in various places. We have made an effort to present everything with complete technical detail. As a consequence, this book contains the most comprehensive and technically complete account to date of this approach to bivariant K-theory. We also introduce the framework of triangulated categories. It fits perfectly to describe the kind of bivariant theories we are discussing and helps to understand their nature. In fact, different bivariant K-theories can be described as different localisations of a version of stable homotopy.

An account of topological K-theory and its bivariant forms cannot be complete without a discussion of the situation for  $C^*$ -algebras, and notably of Kasparov’s KK-theory — the origin of many of the ideas and concepts in the field. We survey some of the different theories and techniques that have been developed for  $C^*$ -algebras and some other theories — algebraic dual K-theory, homotopy-theoretic KK — that can be defined whenever one-variable topological K-theory is available.

We also discuss twisted K-theory in the setting of  $C^*$ -algebras as the K-theory of a bundle with fibres that are elementary  $C^*$ -algebras. This involves continuous-trace algebras, the Dixmier–Douady class, and related topics such as the Brauer group. In the setting of  $C^*$ -algebras we further discuss the K-theory of crossed products by  $\mathbb{R}$  (Connes’ Thom Isomorphism Theorem) and its relation to the Pimsner–Voiculescu sequence.

The last five chapters of the book are devoted to applications. These chapters are largely independent of one another, except that Chapters 9 and 10 are needed in Chapter 11. Readers interested in index theory may want to concentrate on Chapter 12, while Chapter 11 deals with mathematical physics (in particular with T-duality) and Chapter 13 treats the Universal Coefficient Theorem for KK and the Baum–Connes conjecture via localisation of triangulated categories. Some easier cases of Baum–Connes conjecture are already treated from a more down-to-earth point of view in Chapters 5 and 10.

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