Chapter 1

Introduction

1.1 Walks and the metric theory of ordinals

This book is devoted to a particular recursive method of constructing mathematical structures that live on a given ordinal θ, using a single transformation $\xi \mapsto C_\xi$ which assigns to every ordinal $\xi < \theta$ a set C_ξ of smaller ordinals that is closed and unbounded in the set of ordinals $< \xi$. The transfinite sequence

$$C_\xi \ (\xi < \theta)$$

which we call a 'C-sequence' and on which we base our recursive constructions may have a number of 'coherence properties' and we shall give a detailed study of them and the way they influence these constructions. Here, 'coherence' usually means that the C_ξ's are chosen in some canonical way, beyond the already mentioned and natural requirement that C_ξ is closed and unbounded in ξ for all ξ. For example, choosing a canonical 'fundamental sequence' of sets $C_\xi \subseteq \xi$ for $\xi < \varepsilon_0$, relying on the specific properties of the Cantor normal form for ordinals below the first ordinal satisfying the equation $x = \omega^x$, is a basis for a number of important results in proof theory. In set theory, one is interested in longer sequences as well and usually has a different perspective in applications, so one is naturally led to use some other tools besides the Cantor normal form. It turns out that the sets C_ξ can not only be used as 'ladders' for climbing up in recursive constructions but also as tools for 'walking' from an ordinal β to a smaller one α,

$$\beta = \beta_0 > \beta_1 > \cdots > \beta_{n-1} > \beta_n = \alpha,$$

where the 'step' $\beta_i \to \beta_{i+1}$ is defined by letting β_{i+1} be the minimal point of C_{β_i} that is bigger than or equal to α. This notion of a 'walk' and the corresponding 'characteristics' and 'distance functions' constitute the main body of study in this book. We show that the resulting 'metric theory of ordinals' is a theory of considerable intrinsic interest which provides not only a unified approach to a number of classical problems in set theory but is also easily applicable to other areas of mathematics. For example, highly applicable characteristics of the walk are defined on the basis of the corresponding 'traces'. The most natural trace of the walk is its 'upper trace' defined simply to be the set

$$\mathrm{Tr}(\alpha,\beta) = \{\beta_0 > \beta_1 > \cdots > \beta_{n-1} > \beta_n\}$$

of places visited along the way, which is of course most naturally enumerated in decreasing order. Another important trace of the walk is its 'lower trace,' the set

$$\Lambda(\alpha,\beta) = \{\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-2} \leq \lambda_{n-1}\},$$

where $\lambda_i = \max(\bigcup_{j=0}^{i} C_{\beta_j} \cap \alpha)$ for $i < n$. The traces are usually used in defining various binary operations on ordinals $< \theta$, the most prominent of which is the 'square-bracket operation' that gives us a way to transfer the quantifier 'for every unbounded set' to the quantifier 'for every closed and unbounded set'. It is perhaps not surprising that this reduction of quantifiers has proven to be quite useful in constructions of mathematical structures on θ where one needs to have some grip on substructures of cardinality θ.

From the metric theory of ordinals based on analysis of walks, one also learns that the triangle inequality of an ultrametric

$$\varrho(\alpha,\gamma) \leq \max\{\varrho(\alpha,\beta), \varrho(\beta,\gamma)\}$$

has three versions, depending on the natural ordering between the ordinals α, β and γ. The three versions of the inequality are in fact of a quite different character and occur in quite different places and constructions in set theory. For example, the most frequent occurrence is the case $\alpha < \beta < \gamma$, when the triangle inequality becomes something that one can call 'transitivity' of ϱ. Considerably more subtle is the case $\alpha < \gamma < \beta$ of this inequality[1]. It is this case of the inequality that captures most of the coherence properties found in this article. It is also an inequality that has proven to be quite useful in applications.

A large portion of the book is organized as a discussion of four basic characteristics of the walk $\rho, \rho_0, \rho_1, \rho_2$ and ρ_3. The reader may choose to follow the analysis of any of these functions in various contexts. The characteristic $\rho_0(\alpha,\beta)$ codes the entire walk $\beta = \beta_0 > \beta_1 > \cdots > \beta_{n-1} > \beta_n = \alpha$ by simply listing the positions of β_{i+1} in the set C_{β_i} for $i < n$. While this looks simple-minded, the resulting mapping ρ_0 is a rather remarkable object. For example, in the realm of the space ω_1 of countable ordinals, it gives us a canonical example of a special Aronszajn tree of increasing sequences of rationals which has the additional remarkable property that, when ordered lexicographically, its cartesian square can be covered by countably many chains. In other words, the single characteristic ρ_0 of walks on countable ordinals gives two critical structures, one in the class of

[1]It appears that the third case $\beta < \alpha < \gamma$ of this inequality is rarely a reasonable assumption to be made in this context.

so-called Lipschitz trees and the other in the class of linear orderings. For higher cardinals θ, analysis of ρ_0 leads us to some interesting finitary characterizations of hyper inaccessible cardinals. This is given in some detail in Chapter 6 of this book.

The characteristic $\rho_1(\alpha, \beta)$ loses a considerable amount of information about the walk as it records only the maximal order type among the sets

$$\{C_{\beta_0} \cap \alpha, C_{\beta_1} \cap \alpha, \ldots, C_{\beta_{n-1}} \cap \alpha\}.$$

Nevertheless it gives us the first example of what we call a 'coherent mapping'. The class of coherent mappings and trees in the case $\theta = \omega_1$ exhibits an unexpected structure that we study in great detail in Chapter 4 of the book. The fine structure in the class of 'coherent trees' is based on the metric notion of a 'Lipschitz mapping' between trees. The profusion of such mappings between coherent trees eventually leads us to the so-called 'Lipschitz Map Conjecture' that has proven crucial for the final resolution of the basis problem for uncountable linear orderings and that is presented in the same chapter. For higher cardinals θ the characteristic ρ_1 and its local versions offer a rich source of so-called 'unbounded functions' that have some applications.

The characteristic $\rho_2(\alpha, \beta)$ simply counts the number of steps of the walk from β to α. While this also looks rather simple minded, the remarkable properties of the corresponding function ρ_2 become especially apparent on higher cardinals θ. Important properties of this characteristic are its coherence and its unboundedness. The coherence property of ρ_2 requires the corresponding C-sequence C_ξ $(\xi < \theta)$ to be 'coherent' in the sense that $C_\alpha = C_\beta \cap \alpha$ whenever α is a limit point of C_β. On the other hand, the unboundedness of ρ_2 translates into a requirement that the corresponding C-sequence C_ξ $(\xi < \theta)$ be 'nontrivial'[2], a condition that eventually leads us to a simple and natural characterization of weakly compact cardinals that we choose to reproduce in some detail in Chapter 6.

Finally, the characteristic $\rho_3(\alpha, \beta)$ attaches one of the digits 0 or 1 to the walk according to the behavior of the last step $\beta_{n-1} \to \beta_n = \alpha$. The full analysis of this characteristic is currently available only in the reals of the space ω_1, where ρ_3 becomes a rather canonical example of a sequence-coherent mapping with values in $\{0, 1\}$ and with properties reminiscent of those appearing in the well-known notion of a Hausdorff gap in the quotient algebra $\mathcal{P}(\omega)/\mathrm{fin}$ (another critical object that shows up in many problems about this quotient structure).

The true 'metric theory of ordinals' comes only with development of the characteristic $\rho(\alpha, \beta)$ of the walk that takes advantage of the so-called 'full lower trace' of the walk. The depth of this characteristic is apparent even in the space ω_1 of countable ordinals, but its full power comes at higher cardinals θ and especially at θ that are successors of singular cardinals. The full analysis of the characteristic ρ requires C_ξ $(\xi < \theta)$ to be a so-called 'square sequence' or in other words requires

[2]We say that C_ξ $(\xi < \theta)$ is *nontrivial* if there is no closed and unbounded set $C \subseteq \theta$ such that. for all limit points α of C, there is $\beta \geq \alpha$ such that $C \cap \alpha \subseteq C_\beta$.

the most widely known coherence condition on this sequence, which says that if α is a limit point of C_β, then $C_\alpha = C_\beta \cap \alpha$. It is not surprising that this characteristic has the largest number of applications, many of which are reproduced in this book. We have already mentioned that its development in the case $\theta = \omega_1$ was the initial impulse for development of the so-called metric theory of countable ordinals that has already a rich spectrum of applications. At higher cardinals θ the characteristic ρ can be used in facilitating set-theoretic forcing constructions of rather special objects and we shall reproduce some of these constructions in Chapter 7 of this book. While, as said above, the full development of ρ requires C_ξ $(\xi < \theta)$ to be a 'square sequence', the function ρ itself holds considerable information about the notion of 'square sequences.' For example in Chapter 7, we use ρ to turn an arbitrary square sequence C_ξ $(\xi < \theta)$ into a non-special one by expressing the usual order relation among ordinals $< \theta$ as an increasing union of tree-orderings that come from square sequences on θ themselves. This again leads us to some applications that we choose to reproduce in detail at the end of Chapter 7.

We have already mentioned that one of the important outcomes of our study of walks on ordinals is the 'square-bracket operation', a transformation which to every pair $\alpha < \beta$ of ordinals $< \theta$ assigns an ordinal $[\alpha\beta]$ belonging to the upper trace $\mathrm{Tr}(\alpha, \beta)$ of the walk from β to α. We have also mentioned that the choice of $[\alpha\beta]$ has to be rather careful in order to reduce an arbitrary unbounded subset A of θ to the corresponding set of values

$$\{[\alpha\beta] : \alpha, \beta \in A \text{ and } \alpha < \beta\}$$

that contains a closed and unbounded subset of θ relative to some fixed stationary set $\Gamma \subseteq \theta$, which the C-sequence C_ξ $(\xi < \theta)$ avoids[3]. We present several variations on the way $[\alpha\beta]$ is chosen, each of which works best in some particular context. The common feature of these definitions of $[\alpha\beta]$ is that they are all based on the oscillation mapping

$$\mathrm{osc} : \mathcal{P}(\theta)^2 \to \mathrm{Card}$$

defined by

$$\mathrm{osc}(x, y) = |x \setminus (\sup(x \cap y) + 1)/\sim|,$$

where $\sim$ is the equivalence relation on $x \setminus (\sup(x \cap y) + 1)$ defined by letting $\alpha \sim \beta$ if and only if the closed interval determined by the ordinals α and β contains no point from y. In other words, $\mathrm{osc}(x, y)$ is simply the number of convex pieces that the set $x \setminus (\sup(x \cap y) + 1)$ is split into by the set y. The original theory of the oscillation mapping osc has been developed in the realm of partial functions from θ into θ. In other words, there is a well-developed theory of the oscillation mapping

$$\mathrm{osc}(s, t) = |\{\xi : s(\xi) \leq t(\xi) \text{ but } s(\xi^+) > t(\xi^+)\}|, {}^4$$

[3] C_ξ $(\xi < \theta)$ *avoids* Γ if $C_\alpha \cap \Gamma = \emptyset$ for all limit ordinals $\alpha < \theta$.

[4] Here, ξ^+ is the immediate successor of ξ in the common domain of s and t.

(see, for example, [111]), but the general theory works equally well and it will be in part reproduced here in Chapters 8 and 9. The common feature of all results of such a theory is identification of the notion of 'unbounded', in either of the two contexts, in such a way that the typical oscillation result would say that the set of values $\operatorname{osc}(x, y)$ the oscillation mapping takes when x and y run inside two 'unbounded' sets is in some sense rich. In our context of defining the square-bracket operation, the sets x and y in $\operatorname{osc}(x, y)$ are members of our C-sequence C_ξ $(\xi < \theta)$ on which we base the notion of walk, and the notion of 'unbounded' becomes the familiar notion of nontriviality of C_ξ $(\xi < \theta)$. This makes the square-bracket operation $[\alpha\beta]$ well defined in a wide variety of contexts and therefore quite applicable.

Judging from the applications found so far, it appears that in order to make a particular variation of the oscillation mapping or the square-bracket operation useful, one needs to be be able to give a quite precise estimate of its behavior, not only on unbounded subsets of θ but also on families of θ pairwise-disjoint finite subsets of θ. It is for this reason that definite results about any particular variation of osc and $[\alpha\beta]$ presented in this book will typically be about families A of pairwise-disjoint finite subsets of θ. Given such a family A of cardinality θ, by going to a subfamily, we may assume that elements of A have some fixed finite cardinality n. So, given a in A one can view it enumerated increasingly as $a(0), \ldots, a(n-1)$. All variations of the square-bracket operation that we present will have the property that the set of ordinals $\xi < \theta$ that can be represented as

$$\xi = [a(0)b(0)] = [a(1)b(1)] = \cdots = [a(n-1)b(n-1)],$$

for some $a < b$ in A, contains a closed and unbounded set relative to some fixed stationary set $\Gamma \subseteq \theta$ which the C-sequence C_α $(\alpha < \theta)$ avoids. Many applications however require that we know the values $[a(i)b(j)]$ when $i \neq j < n$ and when a and b run through A. It turns out that modulo taking a 'projection' $[\![\cdot\cdot]\!]$ of $[\cdot\cdot]$, (or in other words, modulo composing $[\cdot\cdot]$ with a map from θ into θ) for many of the square-bracket operations that we define in this book, the particular set

$$\{[\![a(i)b(j)]\!] : i, j < n \text{ and } i \neq j\}$$

of values will be independent of the choice of $a \neq b$ in A. This turns out to be crucial in several applications of $[\cdot\cdot]$ presented in this book. Naturally, one would also like to know whether one can define a variation on $[\cdot\cdot]$ where we would have freedom of getting arbitrary values of the form $[a(i)b(j)]$ independently of whether $i = j$ or not. It turns out that this is indeed possible for some choices of θ, though the corresponding definitions are necessarily less general as they do not apply in the case $\theta = \omega_1$ since otherwise one would be able to prove that the countable chain condition is not productive without appealing to additional axioms of set theory.[5] In case $\theta = \omega_1$, we do have symmetric binary operations with some degree

[5]Recall that the countable chain condition is a productive property under MA_{ω_1} (see [36], 41E).

of freedom in that direction (see Chapter 2), but the exact breaking point between this and what requires additional axioms of set theory has yet to be determined.

This book will also present some higher-dimensional characteristics of the walk, though in that context the full theory is yet to be developed. For example, in Chapter 10, we consider the characteristic $\tau(\alpha,\beta,\gamma)$ which to any given three ordinals $\alpha<\beta<\gamma<\theta$ assigns the place where the walk from γ to α branches from the walk from γ to β. It turns out that when C_ξ ($\xi<\theta^+$) is a square sequence, the characteristic τ can be used to 'step-up' objects living on θ to objects on its successor θ^+. For example, one application of this characteristic is found in the proof that Chang's Conjecture[6] is equivalent to a 3-dimensional Ramsey-theoretic statement saying that, for every coloring of $[\omega_2]^3$ with ω_1 colors, there is an uncountable set $B\subseteq\omega_2$ which misses at least one of the colors. The 3-dimensional characteristic $\chi(\alpha,\beta,\gamma)$ that simply measures the length of the common parts of the walks $\gamma\to\alpha$ and $\gamma\to\beta$ can be used for detecting when a subset Γ of θ admits a rich restriction of the 3-dimensional version of the oscillation mapping,

$$\mathrm{osc}:[\theta^+]^3\longrightarrow\omega,$$

defined on the basis of its 2-dimensional version as follows:

$$\mathrm{osc}(\alpha,\beta,\gamma)=\mathrm{osc}(C_{\beta_s}\setminus\alpha, C_{\gamma_t}\setminus\alpha),$$

where $s=\rho_0(\alpha,\beta)\restriction\chi(\alpha,\beta,\gamma)$ and $t=\rho_0(\alpha,\gamma)\restriction\chi(\alpha,\beta,\gamma)$. Here β_s is the member β_i of the trace of the walk $\beta=\beta_0>\cdots>\beta_l=\alpha$ whose code is the sequence s, i.e., $\rho_0(\beta_i,\beta)=s$, and similarly γ_t is the term γ_j of the walk $\gamma=\gamma_0>\cdots>\gamma_k=\alpha$ whose code is the sequence t. In Chapter 10 we show that our analysis of the 3-dimensional version of the oscillation mapping leads naturally to a square-bracket operation in that dimension, though the full analogy is yet to be completed as one still needs to determine the behavior of this operation on families of pairwise-disjoint finite subsets of θ^+ (which at the moment seems elusive).

The final section of Chapter 10 is concerned with generalizing the basic notion of walks to the context of sets of ordinals rather than ordinals themselves, with the goal of obtaining two-cardinal versions of the square-bracket operation. For example, we show that for every pair of infinite cardinals $\kappa<\lambda$ with κ regular, there is a mapping $c:[[\lambda]^\kappa]^2\to\lambda$ such that, for every cofinal subset U of $[\lambda]^\kappa$ and every $\xi\in\lambda$, there exist $x\subset y$ in U such that $c(x,y)=\xi$. Fuller analogues of the square-bracket operation are however obtained only in certain cases. For example, assuming that there is a stationary subset S of $[\lambda]^\omega$ which is equinumerous with a locally countable[7] subset of $[\lambda]^\omega$, one can define a square-bracket operation

$$[\cdot\cdot]_S:[[\lambda]^\omega]^2\to[\lambda]^\omega$$

[6] Recall that Chang's conjecture is the model-theoretic statement claiming that every model of a countable signature that has the form $(\omega_2,\omega_1,<,\ldots)$ has an uncountable elementary submodel M such that $M\cap\omega_1$ is countable.

[7] A subset $K\subseteq[\lambda]^\omega$ is locally countable, if the set $\{x\in K: x\subseteq a\}$ is countable for every $a\in[\lambda]^\omega$.

with the property that, for every cofinal subset U of $[\lambda]^\omega$, the set of all $s \in S$ that are not of the form $[xy]_S$ for some $x \subset y$ in U is not stationary in $[\lambda]^\omega$. Since under the same assumption about the stationary set S, this set can be partitioned into $|S|$ pairwise-disjoint stationary subsets, one gets the following: For every stationary subset S of $[\lambda]^\omega$ that is equinumerous with a locally countable subset of $[\lambda]^\omega$, there is a projection

$$[\![..]\!]_S : [[\lambda]^\omega]^2 \to S$$

of the square-bracket operation $[..]_S$ with the property that, for every cofinal set $U \subseteq [\lambda]^\omega$ and every $z \in S$, there exist $x \subset y$ in U such that $[\![xy]\!]_S = z$.

An interesting phenomenon that one realizes while analysing walks on ordinals is the special role of the first uncountable ordinal ω_1 in this theory. Any natural coherency requirement on the sets C_ξ $(\xi < \theta)$ that one finds in this theory is satisfiable in the case $\theta = \omega_1$. How natural the notion of walk in this context is can be seen from the fact that basically all of its characteristics lead us in one way or the other to some 'critical' structure that shows up in various rough classifications of mathematical structures. For example, any of the characteristics ρ, ρ_0, ρ_1, ρ_2 and ρ_3 of the walk that we study here lead us to the canonical linear ordering appearing on the list of five linear orderings that forms a basis for the class of all uncountable linear orderings. The first uncountable cardinal is the only cardinal on which the theory can be carried out without relying on additional axioms of set theory. The first uncountable cardinal is also the place where the theory has its deepest applications as well as its most important open problems. This special role can perhaps be best explained by the fact that many set-theoretical problems, especially those coming from other fields of mathematics, are usually concerned only about the duality between the countable and the uncountable rather than some intricate relationship between two or more uncountable cardinalities. [8] For example, consider the classical problem coming from topology asking if the hereditary separability of a given regular space X is equivalent to its dual requirement that every collection of open subsets of X has a countable subcollection with the same union.[9] It turns out that this problem has a reformulation in terms of the behavior of mappings of the form $c : [\omega_1]^2 \longrightarrow 2$ on uncountable families A of pairwise-disjoint finite subsets of ω_1. In other words, a mapping that has a certain complex behavior on uncountable families of pairwise disjoint finite sheafs would lead to an example of a regular space in which one of the implications fails. For example, one can produce a hereditary separable non-Lindelöf space assuming there is $c : [\omega_1]^2 \longrightarrow 2$ with the property that, for every uncountable family A of pairwise-disjoint finite subsets of ω_1, all of some fixed size n for every position

[8]This is of course not to say that an intricate relationship between two or more uncountable cardinalities may not be a profitable detour in the course of solving such a problem. In fact, this is one of the reasons for our attempt to develop the metric theory of ordinals without restricting ourselves to the realm of countable ordinals.

[9]In other words, every subspace of X is Lindelöf.

$i_0 < n$ and every requirement $h : n \longrightarrow 2$, there exist $a < b$ in A such that

$$c(a(i_0), b(j)) = h(j) \text{ for all } j < n.$$

On the other hand, there is a regular hereditary Lindelöf non-separable space if there is $c : [\omega_1]^2 \longrightarrow 2$ with the dual property that, for every uncountable family A of pairwise-disjoint finite subsets of ω_1, all of some fixed size n for every position $j_0 < n$ and every requirement $h : n \longrightarrow 2$, there exist $a < b$ in A such that

$$c(a(i), b(j_0)) = h(i) \text{ for all } i < n.$$

It turns out that one cannot produce c having the first property without appealing to additional axioms of set theory, but that a variation of the oscillation mapping that we reproduce in Chapter 2 of this book will give us a c with the second property and therefore a regular hereditary Lindelöf non-separable space.

One can find this kind of application of the oscillation mapping or the square-bracket operation on ω_1 in other areas of mathematics as well. For example consider the problem of finding a large subspace on which a given homogeneous polynomial $P : X \longrightarrow \mathbb{C}$ is zero, where X is some Banach space over the field $\mathbb{C}$ of complex numbers.[10] It is known that if X is infinite-dimensional, then there will always be an infinite-dimensional subspace Y of X on which P is zero, but can one find larger Y assuming X is not separable? More precisely, if X is not separable, can one find a non-separable subspace Y of X on which P vanishes? Note that a counterexample would be a polynomial $P : X \longrightarrow \mathbb{C}$ which takes all the complex values on any non-separable subspace of X. So it is natural to try appealing to the square-bracket operation for constructing such a polynomial P. It turns out that this is indeed possible and we shall reproduce it in Chapter 5 of this book for the space $X = \ell_1(\omega_1)$. In order to apply the square-bracket operation we need to take one of its convenient projections $[\![\cdot\cdot]\!]_{\mathcal{G}_0}$ with only countably many values, which one may assume to form a countable dense subset D of the unit disc of the complex plane. Then the polynomial $P : \ell_1(\omega_1) \longrightarrow \mathbb{C}$ is defined as

$$P(x) = \sum [\![\alpha\beta]\!]_{\mathcal{G}_0} x_\alpha x_\beta.$$

In order to establish that P takes all the values from $\mathbb{C}$ on any closed non-separable subspace of $\ell_1(\omega_1)$, one uses the following property of the projection $[\![\cdot\cdot]\!]_{\mathcal{G}_0}$: For every uncountable family A of pairwise-disjoint finite subsets of ω_1, all of the same size n, there is uncountable set $B \subseteq A$, an equivalence relation $\sim$ on $n = \{0, 1, \ldots, n-1\}$, and a single mapping $h : n \times n \to D$ such that

$$[\![a(i)b(j)]\!]_{\mathcal{G}_0} = h(i, j) \text{ for all } a < b \text{ in } B \text{ and } i \nsim j.$$

[10] Recall that a homogeneous polynomial of degree n on a Banach space X is a mapping $P : X \longrightarrow \mathbb{C}$ of the form $P(x) = \Phi(x, x, \ldots, x)$, where $\Phi : X^n \longrightarrow \mathbb{C}$ is a bounded symmetric n-linear form on X.

Moreover, for every uncountable $C \subseteq B$ and for every $g : n \times n \to D$ there exist $a < b$ in C such that

$$[\![a(i)b(j)]\!]_{\mathcal{G}_0} = g(i,j) \text{ for all } i,j < n \text{ with } i \sim j.$$

This particular application makes it clear how useful it is to know behavior of the square-bracket operation not only on uncountable subsets of ω_1 but also on arbitrary uncountable families A of pairwise-disjoint subsets of ω_1.

Minimal walks in general and the metric theory of ordinals in particular are applicable to problems that are not necessarily problems about uncountable structures but rather about the behavior of countable substructures of a fixed large structure. We demonstrate this in Chapter 3 where we present some applications of the metric theory of countable ordinals to classical problems from infinite-dimensional geometry, such as, for example, the distortion problem or the unconditional basic sequence problem. Given a function

$$\varrho : [\eta]^2 \longrightarrow \omega$$

that is locally finite[11] and that satisfies the two ultrametric inequalities,[12] one can use it to define *special functionals* on the vector space $c_{00}(\eta)$ of all finitely supported maps from the ordinal η into the reals. Let e_α $\alpha < \eta$ be the standard Hamel basis of $c_{00}(\eta)$. Let e^*_α $\alpha < \eta$ be the corresponding sequence of biorthogonal functionals. We say that a sequence $(E_i)_{i<n}$ of finite subsets of ω_1 is *special* if $E_i < E_j$[13] for $i < j < n$, and if

$$|E_0| = 1 \text{ and } |E_j| = \sigma_\varrho(E_0, E_1, \dots, E_{j-1}) \text{ for } 0 < j < n.$$

Here $\sigma_\varrho(E_0, E_1, \dots, E_{j-1})$ is the integer that codes in some natural way the finite metric structure induced by ϱ on the union of these sets. To a finite set $E \subseteq \eta$, we associate the vector and functional on $c_{00}(\eta)$,

$$x_E = \frac{1}{|E|^{1/2}} \sum_{\alpha \in E} e_\alpha \quad \text{and} \quad \phi_E = \frac{1}{|E|^{1/2}} \sum_{\alpha \in E} e^*_\alpha.$$

Given a special sequence $(E_i)_{i<n}$ of finite subsets of ω_1, the corresponding *special functional* is defined by $\sum_{i<n} \phi_{E_i}$. Special functionals induce a norm on $c_{00}(\eta)$, and if we let X_η denote the completion of $c_{00}(\eta)$ under this norm, the sequence e_α $(\alpha < \eta)$ will be a weakly null sequence in X_η with no infinite unconditional subsequence. More precisely, we show in Chapter 3 that, if $(E_i)_{i<\omega}$ is any infinite special sequence of finite subsets of η and, if for $i < \omega$ we let $v_i = x_{E_i}$ as defined above, then for every $n < \omega$ and every sequence $(a_i)_{i \le n} \subseteq [-1, +1]$ of scalars,

$$\max_{0 \le k \le n} \mid \sum_{i=0}^{k} a_i \mid \le \| \sum_{i=0}^{n} a_i v_i \| \le (3 + \varepsilon) \max_{0 \le k \le n} \mid \sum_{i=0}^{k} a_i \mid .$$

[11] I.e., the set $\{\xi < \alpha : \varrho(\xi, \alpha) \le n\}$ is finite for every $\alpha < \eta$ and $n < \omega$.
[12] I.e., $\varrho(\alpha, \gamma) \le \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ and $\varrho(\alpha, \beta) \le \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for all $\alpha < \beta < \gamma$.
[13] We let $E < F$ if every ordinal from E is smaller than every ordinal from F.

We have already noted that the largest ordinal η supporting such a ϱ-function is the first uncountable ordinal ω_1. It follows therefore that one can have an uncountable weakly-null sequence with no *infinite* unconditional subsequence. It follows that the function ϱ can be used to control *all* countably infinite subsets of the long weakly-null sequence e_α $(\alpha < \omega_1)$. With a considerable amount of additional work one can even build a reflexive Banach space $\mathcal{X}_{\omega_1}$ with a Schauder basis of length ω_1 with no *infinite* unconditional basic sequence. Extending these ideas further one can use ϱ to build another Banach space $\mathcal{X}^0_{\omega_1}$ with a Schauder basis of length ω_1, which on one hand keeps the distortion and all the conditional structure of the space $\mathcal{X}_{\omega_1}$ at the level of its nonseparable subspaces, but on the other hand, every infinite-dimensional closed subspace of $\mathcal{X}^0_{\omega_1}$ contains an isomorphic copy of the space c_0, or in other words separable subspaces of $\mathcal{X}^0_{\omega_1}$ do not hold any conditional structure nor do they allow their norms to be distorted.

1.2 Summary of results

Many chapters of the book can be read independently from each other. Once the basic definition of the walk is understood, the reader may choose to follow the development of a particular characteristic in various contexts. The reader inclined towards applications of the methods of this book may start by reading about a particular application and go back towards background material that is needed. In this section, we include a short summary that might help the reader in finding specific results presented in this book.

The first section of Chapter 2 is the one to be first read as it presents the notion of minimal walk along a given C-sequence and the corresponding characteristic ρ_0 that codes it. The resulting tree $T(\rho_0)$ is a tree of height ω_1 that has all of its levels countable and which admits a natural strictly increasing map into the rationals. This could be the shortest construction of such a tree found in the literature and surely is the most canonical one. This could be seen for example on the basis of the fact that if we order $T(\rho_0)$ lexicographically we obtain an uncountable linearly ordered set whose cartesian square can be covered by countably many chains. The proof of this fact depends on the development of the full lower trace of the minimal walk and the reader is advised to skip this on the first reading and go instead to the second section of Chapter 2 which presents the characteristic ρ_1 and the corresponding tree $T(\rho_1)$ for which this fact is easier to establish. The characteristic ρ_1 has its own interesting application presented in the same section. For example, we show that the tree $T(\rho_1)$ naturally leads us to an example of a homogeneous non metrizable compactum that can be represented as a weakly compact subset of some Banach space. As another application of ρ_1 presented in the same section is a functor that transfers a given graph G on the vertex set ω_1 to a graph G^* on the same vertex set such that if G^* has an uncountable clique then the vertex set ω_1 can be covered by countably many sets that are cliques of G. Moreover, we show that G^* satisfies the countable chain condition provided G

satisfies a slightly stronger version of this condition saying that for every uncountable family $\mathcal{F}$ of pairwise disjoint finite subsets of ω_1, we can find $a \neq b$ in $\mathcal{F}$ such that $\{\alpha, \beta\}$ is an edge of G for every $\alpha \in a$ and $\beta \in b$. The point of this is that essentially all known ccc graphs on ω_1 do satisfy this stronger form of the countable chain condition and therefore this in particular shows that many of the standard consequences of MA_{ω_1} are Ramsey-theoretic in nature. Recall that it is still not known if MA_{ω_1} is in fact equivalent to the statement that every ccc graph on the vertex set ω_1 contains an uncountable clique. We finish this section with a proof that unlike $T(\rho_0)$ the tree $T(\rho_1)$ does not always admit a strictly increasing map into the rationals. This amounts to measuring how much information about the minimal walk is lost when one passes from the full code $\rho_0(\alpha, \beta)$ to the maximal weight $\rho_1(\alpha, \beta)$ of the walk.

The third section of Chapter 2 presents an important theme that is going to be developed in later parts of the book, the theme of the oscillation mapping. More precisely, in Section 2.3, we present two oscillation mappings osc_0 and osc_1 corresponding to the upper and lower trace of the minimal walk, respectively. Of specific applications of these two oscillation mappings, we present a rather absolute decomposition of ω_1 into infinitely many pairwise disjoint stationary subsets, and an example of a regular hereditarily Lindelöf topological space that is not separable. In Section 2.3 we introduce two new characteristics ρ_2 and ρ_3. The characteristic ρ_2 while quite simple minded in that it counts only the number of steps of the minimal walk its full power will become apparent only in later parts of the book when dealing with walks on larger cardinals. In fact, as far as we know, ρ_2 could be the first nontrivial two-place mapping from a large cardinal θ into ω in the sense that it takes arbitrarily high value from ω on any product of two unbounded subsets of θ. Recall that large cardinals θ typically have the Ramsey-theoretic properties saying that maps $f : [\theta]^2 \longrightarrow \omega$ are constant on $[\Gamma]^2$ for large subsets $\Gamma \subseteq \theta$. The mapping ρ_3 while considerably more subtle makes sense only in the realm of countable ordinals. To a given walk from a countable ordinal β down to a smaller ordinal α, the characteristic ρ_3 assigns one of the digits 0 or 1 according to what happens on the last step of the walk. Since it is a coherent mapping it shares many properties with the characteristic ρ_1 though we shall show that, unlike $T(\rho_1)$, if we order lexicographically $T(\rho_3)$, we are not always guaranteed that the cartesian square of the corresponding uncountable linear ordering can be covered by countably many chains. On the other hand, it is true that every uncountable subset X of $T(\rho_3)$ contains an uncountable subset Y which when ordered lexicographically has the property that its cartesian square can be covered by countably many chains. This again amounts to measuring how much of the information is lost about the minimal walks by passing from the characteristic $\rho_1(\alpha, \beta)$ to the characteristic $\rho_3(\alpha, \beta)$. However, while ρ_3 looses much of the information about the minimal walk, it still gives us a rather interesting and canonical object on ω_1 that is very much reminiscent of the classical notion of a Hausdorff gap in the quotient algebra $\mathcal{P}(\omega)/\mathrm{fin}$.

In Chapter 3 we develop the metric theory of countable ordinals concentrated around the two ultrametric triangle inequalities

$$d(\alpha,\gamma) \leq \max\{d(\alpha,\beta), d(\beta,\gamma)\} \text{ and } d(\alpha,\beta) \leq \max\{d(\alpha,\gamma), d(\beta,\gamma)\},$$

whenever $\alpha < \beta < \gamma$. We have already mentioned two applications of this theory found in this chapter, a reflexive Banach space $\mathcal{X}_{\omega_1}$ with a Schauder basis of length ω_1 with no infinite unconditional basic sequence, and another Banach space $\mathcal{X}^0_{\omega_1}$ with a Schauder basis of length ω_1 that is, on one hand saturated with copies of the space c_0 but on the other hand its norm admits an arbitrarily high distortion relative to the class of nonseparable subspaces of $\mathcal{X}^0_{\omega_1}$. Chapter 3 presents also the characteristic ρ of the minimal walk which besides the two mentioned triangle inequalities has many other interesting properties that show their full power on larger cardinals than ω_1. In Chapter 3 we do present two important objects that are naturally derived from ρ, the Cohen name for a Souslin tree and a particularly canonical example of a Hausdorff gap in $\mathcal{P}(\omega)/\text{fin}$. The Souslin tree has domain ω_1 and the property that no infinite ground model set can be its chain or antichain. In Section 3.4 we study the general theory of functions $\varrho : [\omega_1]^2 \longrightarrow \omega$ that satisfy the two ultrametric triangle inequalities and that are locally finite in the sets that the set $\{\xi < \alpha : \varrho(\xi,\alpha) \leq n\}$ is finite for every $\alpha < \omega_1$ and every $n < \omega$. We show that there is a vast variety of such mappings including the universal one whose construction we reproduce in the same section.

In Chapter 4 we develop a metric theory of trees that lies behind the properties of the tree $T(\rho_1)$. The basic notion here is that of a Lipschitz map between trees, a level preserving map with the property that $\Delta(g(x), g(y)) \geq \Delta(x, y)$ for all x and y in its domain. This leads us to the notion of a Lipschitz tree, an uncountable tree T with the property that every level preserving map from an uncountable subset of T into T has a Lipschitz restriction on some uncountable subset of its domain. The main purpose of Chapter 4 is to study the class $\mathcal{C}$ of Lipschitz trees as a structure equipped with the quasi ordering $S \leq T$ defined to hold whenever there is a Lipschitz map from the tree S into the tree T, or equivalently whenever there is a strictly increasing map from S into T. We also examine the relationship between the class $\mathcal{C}$ and the larger class $\mathcal{A}$ of Aronszajn trees. Most of the theory is developed without appeal to additional axioms of set theory though in the same places we have used either MA_{ω_1} or the Proper Forcing Axiom. It turns out that Lipschitz trees share many of the properties of the tree $T(\rho_1)$ such as for example the property that when ordered lexicographically the cartesian square of the corresponding uncountable linear ordering can be covered by countably many chains. It turns out that all trees $T(\rho_0)$, $T(\rho_1)$, $T(\rho_2)$, $T(\rho_3)$, and $T(\rho)$ associated to various characteristics of walks in ω_1 are Lipschitz. It turns out also that the class $\mathcal{C}$ is totally ordered under $\leq$. In fact the chain $(\mathcal{C}, \leq)$ is discrete in the sense that every T in $\mathcal{C}$ admits a naturally defined shift $T^{(1)}$ that forms an immediate successor of T in $\mathcal{C}$. In fact, the shift $T^{(1)}$ is also an immediate successor of T even in the bigger class $\mathcal{A}$ and this is essentially equivalent to Shelah's Conjecture saying that an uncountable ordering either contains an uncountable well ordered or conversely well ordered subset, an uncountable separable ordered subset, or an uncountable subset whose cartesian square can be covered by countably many chains. It turns out that the chain $\mathcal{C}$ is not well ordered and this in particular solves an old problem of R. Laver asking if the class $\mathcal{A}$ is well quasi-ordered under a stronger ordering than $\leq$. Chapter 4 contains many other structural results about the classes $\mathcal{C}$ and $\mathcal{A}$. For example, we show that while $\mathcal{A}$ is not totally ordered under $\leq$, the chain $\mathcal{C}$ is both cofinal and coinitial in $(\mathcal{A}, \leq)$.

In Chapter 5 we introduce and study the square-bracket operation $[\alpha\beta]$ for pairs α and β of countable ordinals. The ordinal $[\alpha\beta]$ is taken from the upper trace of the minimal walk from β to α. In later parts of the book we shall see several variation of this choice of $[\alpha\beta]$ but the basic idea is always the same and it is based on the oscillation mapping osc_0 of upper traces exposed above in Section 2.3. In Chapter 5 itself we present two other variations on the square-bracket operation, one based on special Aronszajn tree and the other on the tree of all finite binary sequences. What the square-bracket operation adds to the space ω_1 of countable ordinals is a rigidity which can formally be expressed by the fact that there is a sentence of $L(Q^2)$ which has only rigid models. This is presented in Example 5.1.10 which solves a problem of Ebbinghaus and Flum who showed that every model of a sentence of $L(Q)$ has a nontrivial automorphism. In Section 5.3 we present tree geometrical application of the square-bracket operation. The first one is an example of a projective geometry of points and hyperplanes in $\mathbb{R}^n$. The second one is the example of a complex polynomial on $\ell(\omega_1)$ with no nonseparable null subspace already mentioned above. The third application of $[\cdot\cdot]$ is an example of a reflexive Banach space $\mathcal{X}$ with a Schauder basis of length ω_1 in which every operator T can be written as $\lambda I + S$, where S is an operator with separable range. While in some sense this example is subsumed by the example of the Banach space $\mathcal{X}_{\omega_1}$ from Chapter 3 we have included it as its construction uses a quite different set of ideas. In Section 5.4 we give a formal explanation of how the square-bracket operation reduces the quantification over uncountable subsets of ω_1 to that over closed and unbounded subsets of ω_1. Answering a question of W.H. Woodin, we show that there is a natural functor based on $[\cdot\cdot]$ which to every subset Γ on ω_1 associates a graph $K_\Gamma \subseteq [\omega_1]^2$ such that, modulo PFA or Woodin's axiom $(*)$, a subset Γ of ω_1 contains a closed and unbounded set if and only if K_Γ contains $[X]^2$ for some uncountable $X \subseteq \omega_1$.

In Chapter 6 we develop the characteristic ρ_0 of the minimal walk in the general context. Recall that in the context of walks on ω_1 the tree $T(\rho_0)$ that corresponds to the characteristics ρ_0 is *special* in the sense that it can be decomposed into countably many antichains. In Chapter 6 we show that a strongly inaccessible cardinal θ is not Mahlo precisely when one is able to find a C-sequence in θ for which the corresponding tree $T(\rho_0)$ *special* in the sense that it admits a regressive mapping that is not constant on any subset of $T(\rho_0)$ that cannot be covered by less than θ antichains. The idea is then used in providing a unified approach towards

Ramsey-theoretic characterizations of n-Mahlo cardinals in terms of the existence of min-homogeneous sets relative to regressive maps of the form $f : [\Gamma]^{n+2} \longrightarrow \theta$ or $f : [\Gamma]^{n+3} \longrightarrow \theta$. In the same chapter we develop the general theory of the characteristics ρ_1 and ρ_2. In particular, in Section 6.2 we develop the local version $\rho_1^\kappa : [\theta]^2 \longrightarrow \kappa$ of ρ_1 for an arbitrary regular cardinal $\kappa < \theta$. The main interest in this variation of ρ_1 is that under some very mild assumptions about θ one obtains mappings from $[\theta]^2$ into κ that have strong unboundedness properties. Similar unboundedness property of the characteristic $\rho_2 : [\theta]^2 \longrightarrow \omega$ is equivalent to the non triviality property of the C-sequence C_α $(\alpha < \theta)$ saying that there is no closed and unbounded subset C of θ such that for every $\alpha < \theta$ there is $\beta \geq \alpha$ such that $C \cap \alpha \subseteq C_\beta$. This gives a vast variety of cardinals θ for which one can find a C-sequence C_α $(\alpha < \theta)$ such that the corresponding function $\rho_2 : [\theta]^2 \longrightarrow \omega$ is strongly unbounded. This can be seen from the characterization of weakly compact cardinals given in Section 6.3 which says that a strongly inaccessible cardinal θ is weakly compact if and only if for every C-sequence C_α $(\alpha < \theta)$ there is a closed and unbounded set $C \subseteq \theta$ such that for all $\alpha < \theta$ there is $\beta \geq \alpha$ such that $C \cap \alpha = C_\beta \cap \alpha$. We finish Chapter 6 with a particular topological application of the unboundedness property of ρ_2

In Chapter 7 we study walks based on C-sequences C_α $(\alpha < \theta)$ that satisfy the coherence property saying that $C_\alpha = C_\beta \cap \alpha$ whenever α is a limit point of C_β. We call such C-sequences *square sequences.* It is this condition on the C-sequence which permits that the full theory of walks on ω_1 be lifted to the level θ that supports it. For example, we show that the full lower trace $\mathrm{F}(\alpha, \beta)$ of walks along square sequence keeps all its properties from the context of countable ordinals. As an application we show that the characteristic ρ_2 in this context has an interesting coherence property saying that $\sup_{\xi<\alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| < \infty$ for all $\alpha < \beta < \theta$. This has an interesting topological interpretation saying that the square of the sequential fan with θ edges has tightness equal to θ. Another interpretation of the same fact is the statement that the P-ideal dichotomy is incompatible with the existence of a nontrivial square sequence on any cardinal θ that is larger than ω_1. The main reason of imposing the coherence property on a given C-sequence is however motivated by attempts to develop the theory of the characteristic ρ in the general context. This is done in Section 7.2 where we also deal with local versions ρ^κ of this characteristic. A typical application of this new theory is Theorem 7.2.14 saying that if a regular uncountable cardinal $\theta \neq \omega_1$ carries a nontrivial square sequence then it also carries one for this the corresponding tree $(\theta, <^2)$ is not special. Complementing a well-known result of Laver and Shelah, this shows that the existence of a weakly compact cardinal is the exact consistence strength of the statement that all Aronszajn trees on ω_2 are special. Special square sequence is something that is more closely related to Jensen's notion of square sequences on successor cardinals and we develop the corresponding theory of walks in the following sections of Chapter 7. Particularly interesting is the case of successors of regular cardinals which we consider in Section 7.4 where we develop the theory of the corresponding characteristic ρ and the derived set-mapping $D : [\kappa^+]^2 \longrightarrow [\kappa^+]^{<\kappa}$ defined by

$$D\{\alpha, \beta\} = \{\xi \leq \min\{\alpha, \beta\} : \rho\{\xi, \alpha\} \leq \rho\{\alpha, \beta\}\}.$$

In Section 7.5 we use the set-mapping D to construct interesting forcing notions. For example, we show that if a regular cardinal κ is λ-inaccessible then there is a λ-closed κ-cc forcing notion that introduces a Souslin tree on κ^+ as well as a forcing notion that introduces a Kurepa family on κ. So, in particular, we show that if $\square_{\omega_1}$ holds then there is a property K forcing notion that introduces a Souslin tree on ω_2. In the same section we reproduce a proof that under $\square_{\omega_1}$ there is a property K forcing notion that introduces a locally compact scattered topology on ω_2 all of whose Cantor-Bendixson ranks are countable. We deal with successors of singular cardinals κ in Section 7.5. The main applications of the characteristic ρ in this context are in producing Jensen matrices $\mathrm{J}_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ of subsets of κ^+. As one application of Jensen matrices we present a construction of a cofinal Kurepa family of countable subsets of κ^+. Cofinal Kurepa families are quite useful objects. We show this by using them in constructing Bernstein decompositions of arbitrary Hausdorff spaces, or in constructing coherent families of finite-to-one maps indexed by countable subsets of κ^+.

In Chapter 8 we develop the general theory of the oscillation mapping of traces and the corresponding square-bracket operation. In fact we give three variations on the basic idea behind the definition of the square-bracket operation with a particular emphasis on the properties of the corresponding projections. Projections are something that is typically needed in applications and they do tend to impose accessibility conditions on the cardinal θ. For example, we show that if, for example, θ is the first inaccessible cardinal then there is a projection $o : [\theta]^2 \longrightarrow \omega$ of the oscillation mapping which takes all the values from ω on any set of the form $[\Gamma]^2$ for Γ unbounded subset of θ. However if one wants such a projection of the oscillation mapping that could be useful in constructing interesting complex polynomials on $\ell_1(\theta)$ or interesting bilinear mappings one needs to impose the condition that θ is not bigger than the continuum. It turns out that the basic definition of the square-bracket operation from ω_1 lifts without difficulties when ω_1 is replaced by a cardinal θ that is a successor of a regular cardinal or more generally that admits a non reflecting stationary subset. A typical application of this result is the statement that for every regular cardinal κ there is a 2-nilpotent group G of cardinality κ^+ such that every abelian subgroup of G has cardinality at most κ. To obtain a projection $[\![\cdot\cdot]\!]$ of $[\cdot\cdot]$ that would have the *generic* behaviour[14] one needs to vary the original idea using the assumption that the cardinal θ is in some sense large. We present two such variations with some topological applications of the corresponding projections. For example, we show that for every regular uncountable cardinal κ there is a topological group G of cellularity κ whose cartesian

[14]In other words, that would have the property that for every family A of θ pairwise disjoint finite subsets of θ all of some fixed size n and for every mapping $h : n \times n \longrightarrow \theta$ there exist $a < b$ in A such that $[\![a(i)b(j)]\!] = h(i, j)$ of all $i, j < n$.

square has cellularity $> \kappa$. In fact, this result is true for arbitrary uncountable cardinal κ that is not necessarily regular. This is proved in a similar fashion using the colorings defined on successors of singular cardinals exposed in Chapter 9 of this book.

In the first section of Chapter 9, we use the characteristic ρ in producing partial square sequence that can sometimes be used in places of full square sequences. In the second section of Chapter 9, we show that for a regular uncountable cardinal κ, there is a structure of the form $(\kappa^+, \kappa, R_n)_{n<\omega}$ with no elementary substructure B of size κ with $B \cap \kappa$ bounded in κ just in case there is a *strongly unbounded* function $f : [\kappa^+]^2 \longrightarrow \kappa$ satisfying the two ultrametric triangle inequalities mentioned above. This gives us a useful reformulation of the well-known model-theoretic statement known as Chang's Conjecture which is simply the statement that every structure of the form $(\omega_2, \omega_1, R_n)_{n<\omega}$ has an uncountable elementary substructure B such that $B \cap \omega_1$ is countable. In Section 9.3 we use this reformulation to show that Chang's Conjecture is also equivalent to the statement that every ccc poset forces that every mapping of the form $g : \omega_2 \times \omega_2 \longrightarrow \omega$ must be constant on some product of two infinite subsets of ω_2. This is then used in Section 9.4 in analyzing possible Ramsey-theoretic reformulations of the Continuum Hypothesis.

In Section 10.1 we develop a general stepping-up procedure that lifts structures living on a given cardinal θ to similarly behaved structure living on its successor θ^+. We manage this by just assuming the existence of a square sequence on θ^+ though we are able to step-up some complex combinatorial statements like these of Hajnal and Komjath where originally one would have expected that the higher-gap morasses are needed. As an application, we show that using $\square_{\omega_1}$ one can construct a reflexive Banach space X with a Schauder basis of length ω_2 with the property that every bounded linear operator $T : X \longrightarrow X$ can be written as a sum of an operator with a separable range and a diagonal operator with only countably many changes of eigenvalues. In Section 10.2 we show that Chang's Conjecture is equivalent to the purely Ramsey-theoretic statement saying that for every $f : [\omega_2]^2 \longrightarrow \omega_1$ there is an uncountable set $\Gamma \subseteq \omega_2$ such that $f''[\Gamma]^2 \neq \omega_1$. In Section 10.3 we introduce the 3-dimensional oscillation mapping and use it in producing a coloring $c : [\omega_2]^3 \longrightarrow \omega$ that takes all of its values from ω on the symmetric cube of any uncountable subset of ω_2.

The last section of Chapter 10 is an attempt to start the corresponding theory of two cardinal walks, an analogous theory that would work in the context of structures of the form $\mathcal{P}_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$ in place of ordinals and cardinals. In fact we were able to define a notion of walk in this context that is sufficiently rich for giving us the analogue of the square-bracket operation in this context. For example, we show that for every pair $\kappa < \lambda$ of cardinals with κ regular there is a projection $[\![\cdot\cdot]\!] : [[\lambda]^\kappa]^2 \longrightarrow \lambda$ of the square-bracket operation in the context of $[\lambda]^\kappa$ (or more formally, in the context of $\mathcal{P}_{\kappa^+}(\lambda)$) such that for every cofinal subset U of $[\lambda]^\kappa$ and every $\xi \in \lambda$ there exist $x \subset y$ in U such that $[\![xy]\!] = \xi$. A particularly interesting case of this result is when $\kappa = \omega$ which allows a further elaboration. For example, we show that if there is a stationary subset of $[\lambda]^\omega$ that is equinumerous with a locally countable subset of $[\lambda]^\omega$ then there is a variation of the square-bracket operation $[\cdot\cdot]_S : [[\lambda]^\omega]^2 \longrightarrow [\lambda]^\omega$ with properties quite analogous to those of the square-bracket operation of ω_1 (which is really the case $\kappa = \omega$ and $\lambda = \omega_1$) in the sense that for every cofinal subset U of $[\lambda]^\omega$ the set of all $s \in S$ that are not of the form $[xy]_S$ for $x \subset y$ in U is not stationary in $[\lambda]^\omega$. One of the points of this variation of the square bracket operation is that the set S can be split into $|S|$ pairwise disjoint stationary sets, and since quite frequently S has cardinality θ that is bigger than λ, we can define projections $[\![\cdot\cdot]\!]_S : [[\lambda]^\kappa]^2 \longrightarrow \theta$ that take more than λ colors (more precisely, $\theta = |S|$ colors) on any cofinal subset U of $[\lambda]^\omega$.

1.3 Prerequisites and notation

We have tried to keep the prerequisites needed for mastering the material of this book to a minimum. Though no specific training in set theory is necessary, the reader should be familiar with the notion of ordinal and recursive definitions and inductive proofs on them. This all can be found in most of the introductory texts on the subject such as, for example, the newer ones [51], [64] and [55], or the older text [62] which has particularly detailed expositions of the recursive definitions over ordinals. By comparing, the reader will notice that we are using standard notation in essentially complete agreement with these textbooks. In these sources the reader will find all the operations and properties of ordinals that we will use, but if more complete treatment is needed the reader may also consult one of the specialized texts like [7]. We shall also look at ordinals with their natural topology induced by order. It is in this context that we refer to ‘closed’ subsets of a particular ordinal θ. The notions of ‘unbounded’ and ‘cofinality’ in this context refer of course to the natural ordering of θ. The combinatorially inclined readers will notice that we are adopting the Erdős–Rado notation for the symmetric powers $[S]^\theta$, the collections of all subsets of the set S of cardinality θ. Of special interest are of course the finite symmetric powers $[S]^k$ and the mappings defined on them because of the clear connection that this book has with Ramsey theory (see [40], [33] and [133]). In fact most of our characteristics of walks are mappings with domains equal to symmetric squares $[\theta]^2$ of some ordinals θ. A pair $\{\alpha, \beta\} \in [\theta]^2$ is usually assumed to be written such that $\alpha < \beta$. In other words, sometimes it is convenient to identify the symmetric square $[\theta]^2$ with the set

$$\{(\alpha, \beta) \in \theta^2 : \alpha < \beta\}$$

of ordered pairs. In this way a given characteristic $f : [\theta]^2 \longrightarrow \eta$ of the walk can be identified with the sequence $f_\beta : \beta \longrightarrow \eta$ ($\beta < \theta$) of fiber mappings defined by

$$f_\beta(\alpha) = f(\alpha, \beta).$$

Also we can more clearly express when the given characteristic f is 'coherent' in some way by referring to the coherence between the corresponding fiber mappings. Moreover, this identification allows us to use the notation $f(\alpha,\beta)$ instead of the more cumbersome notation $f(\{\alpha,\beta\})$. Basically all our characteristics

$$f : [\theta]^2 \longrightarrow \eta$$

of walks are defined recursively in the sense that $f(\alpha,\beta)$ is defined on the basis of the values $f(\alpha',\beta')$ on lexicographically smaller pairs (α,β). All the recursive definitions require us to specify the boundary values $f(\alpha,\alpha)$ which are typically taken to be constant values such us 0 or $\emptyset$ depending on the context. It is for this reason that sometimes we implicitly assume that the diagonal $\{(\alpha,\alpha) : \alpha < \theta\}$ is a part of the domain of f.

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