Preface

Many important problems in applied science and engineering, such as the Navier–Stokes equations in fluid dynamics, the primitive equations in global climate modeling, the strain-stress equations in mechanical and material engineering, and the neutron diffusion equation in nuclear engineering contain complicated systems of nonlinear partial differential equations (PDEs). When approximated numerically on a discrete grid or mesh, such problems produce large systems of algebraic nonlinear equations, whose numerical solution may be prohibitively expensive in terms of time and storage. High-performance (parallel) computers and efficient (parallelizable) algorithms are clearly necessary.

Three classical approaches to the solution of such systems are: Newton's method, preconditioned conjugate gradients (and related Krylov-subspace acceleration techniques), and multigrid. The first two approaches require the solution of large sparse linear systems at each iteration, which are themselves often solved by multigrid. Developing robust and efficient multigrid algorithms is thus of great importance.

The original multigrid algorithm was developed for the Poisson equation in a square, discretized by finite differences on a uniform grid. For this model problem, multigrid converges rapidly, and actually solves the problem in the minimal possible time (Poisson rate).

The original multigrid algorithm uses rediscretization of the original PDE on each grid in the hierarchy of coarse grids (geometric multigrid). Unfortunately, this approach doesn't work well for more complicated problems with nonrectangular domains, nonuniform grids, variable coefficients, or nonsymmetric or indefinite coefficient matrices. In these cases, matrix-based multigrid methods are required.

Matrix-based (or matrix-dependent) multigrid is a family of methods that use the information contained in the discrete system of equations (rather than the original PDE) to construct the operators used in the multigrid linear system solver. This way, a computer code can be written such that it accepts the coefficient matrix and right-hand side of the discrete system as input and produces the numerical solution as output. The method is automatic in the sense that the above code is independent of the particular application under consideration.

Because the elements in the coefficient matrix contain all the information about the properties of the boundary-value problem and its discretization, matrix-based multigrid methods are efficient even for PDEs with variable coefficients and complicated domains. In fact, matrix-based multigrid methods are the only multigrid methods that converge well even for diffusion problems with discontinuous coefficients, even when the discontinuity lines do not align with the coarse mesh.

This book offers a new approach towards the introduction and analysis of multigrid methods from an algebraic point of view. This approach is independent of the traditional, geometric approach, which is based on rediscretizing the original PDE. Instead, it uses only the algebraic properties of the original linear system to define, analyze, and apply the multigrid iterative method. This way, multigrid methods are well embedded in the family of iterative methods for the numerical solution of large, sparse linear systems. Indeed, as is shown below, multigrid methods can actually be viewed as special cases of domain-decomposition methods.

The present edition of this book introduces the multigrid methods from a unified domain-decomposition point of view. In particular, advanced multigrid versions (such as black-box multigrid, algebraic multigrid, and semicoarsening) can all be interpreted as domain-decomposition methods. Furthermore, it introduces a new semi algebraic approach for systems of PDEs. Each chapter ends with relevant exercises.

The first three parts are introductory. The first part introduces the concept of multilevel/multiscale in many different branches in mathematics and computer science. The second part gives the required background in discretization methods, including finite differences, finite volumes, and finite elements. The third part describes iterative methods for solving the linear system resulting from this discretization. In particular, it introduces multigrid methods from a domain-decomposition point of view.

The next three parts contain the heart of the book. The discussion starts from the simplified but common case of uniform grids, proceeds to the more complicated case of locally refined grids, and concludes with the most general and difficult case of completely unstructured grids. In each of these three parts, we concentrate on a particular multigrid version that fits our framework and method of analysis, and study it in detail. We believe that this study may shed light not only on this particular version but also on other multigrid versions as well.

These three parts are ordered from simple to complex: from the simple case of rectangular uniform grids, where spectral analysis may be used to predict the convergence factors (Part IV), through the more complex case of locally refined (and, in particular, semistructured) grids, where upper bounds for the condition number are available (Part V), to the most general case of completely unstructured grids, where the notions of stability and local anisotropy motivate and guide the actual design of algebraic multilevel methods (Part VI).

The second edition also contains the mathematical background required to make the book self-contained and suitable not only for experienced researchers but also for beginners in applied science and engineering. Thanks to the introductory parts, no background in numerical analysis or multigrid methods is needed. The only prerequisites are linear algebra and calculus. The book can thus serve as a textbook in courses in numerical analysis, numerical linear algebra, scientific computing, and numerical solution of PDEs at the advanced undergraduate and graduate levels.

Acknowledgments

I wish to thank the referees for their thorough and helpful reports, and the Wingate foundation for their kind support. I wish to thank Prof. Moshe Israeli and Prof. Avram Sidi for their valuable advice in the development of the AutoMUG method, and Prof. Irad Yavneh for his valuable comments. I also wish to thank Dr. Joel Dendy and Dr. Dan Quinlan for useful discussions about semistructured grids, and Dr. Mac Hyman for his valuable support. I wish to thank Dr. Dhavide Aruliah and Prof. Uri Ascher for supplying the coefficient matrix for the Maxwell equations. Finally, I wish to thank my sons Roy and Amir for their constant patience and support.

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August 2006

Springer

http://www.springer.com/978-0-387-49764-8

Matrix-Based Multigrid
Theory and Applications
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2008, XXIV, 318 p., Hardcover
ISBN: 978-0-387-49764-8