
Preface

The purpose of this book is to give a comprehensive introduction to sequential dynamical systems (SDS). This is a class of dynamical systems defined over graphs where the dynamics arise through functional composition of local dynamics. As such, we believe that the concept and framework of SDS are important for modeling and simulation of systems where causal dependencies are intrinsic.

The book is written for mathematicians, but should be readily accessible to readers with a background in, e.g., computer science or engineering that are interested in analysis, modeling, and simulation of network dynamics. We assume the reader to be familiar with basic mathematical concepts at an undergraduate level, and we develop the additional mathematics needed.

In contrast to classical dynamical systems, the theory and analysis of SDS are based on an interplay of techniques from algebra, combinatorics, and discrete mathematics in general. To illustrate this let us take a closer look at SDS and their structure. An SDS is a triple that consists of a finite graph Y where each vertex has a state taken from a finite set K , a vertex-indexed sequence of Y -local maps $(F_{v,Y})_v$ of the form $F_v: K^n \rightarrow K^n$, and a word $w = (w_1, \dots, w_k)$ over the vertex set of Y . The associated dynamical system is the SDS-map, and it is given by the composition of the local maps $F_{v,Y}$ in the order specified by w .

SDS generalize the concept of, for example, cellular automata (CA). Major distinctions from CA include (1) SDS are considered over arbitrary graphs, (2) for SDS the local maps can be applied multiple times while with CA the rules are applied exactly once, and (3) the local maps of an SDS are applied sequentially while for CA the rules are typically applied in parallel.

Much of the classical theory of dynamical systems over, e.g., \mathbb{R}^n is based on continuity and derivatives of functions. There are notions of derivatives for the discrete case as well, but they do not play the same central role for SDS or other finite dynamical systems. On a conceptual level the theory of SDS is much more shaped by algebra and combinatorics than by the classical dynamical systems theory. This is quite natural since the main research

questions for SDS involve properties of the base graph, the local maps, and the ordering on the one hand, and the structure of the discrete phase space on the other hand. As an example, we will use Sylow's theorems to prove the existence of SDS-maps with specific phase-space properties.

To give an illustration of how SDS connects to algebra and combinatorics we consider SDS over words. For this class of SDS we have the dependency graph $G(w, Y)$ induced by the graph Y and the word w . It turns out that there is a purely combinatorial equivalence relation \sim_Y on words where equivalent words induce equivalent SDS. The equivalence classes of \sim_Y correspond uniquely to certain equivalence classes of acyclic orientations of $G(w, Y)$ (induced by a natural group action). In other words, there exists a bijection $W_k / \sim_Y \longrightarrow \bigcup_{\varphi \in \Phi} [\text{Acyc}(G(\varphi, Y)) / \sim_{\text{Fix}(\varphi)}]$, where W_k is the set of words of length k and Φ is a set of representatives with respect to the permutation action of S_k on words W_k .

The book's first two chapters are optional as far as the development of the mathematical framework is concerned. However, the reader interested in applications and modeling may find them useful as they outline and detail why SDS are oftentimes a natural modeling choice and how SDS relate to existing concepts.

In the book's first chapter we focus on presenting the main conceptual ideas for SDS. Some background material on systems that motivated and shaped SDS theory is included along with a discussion of the main ideas of the SDS framework and the questions they were originally designed to help answer.

In the second chapter we put the SDS framework into context and present other classes of discrete dynamical systems. Specifically, we discuss cellular automata, finite-state machines, and random Boolean networks.

In Chapter 3 we provide the mathematical background concepts required for the theory of SDS presented in this book. In order to keep the book self-contained, we have chosen to include some proofs. Also provided is a list of references that can be used for further studies on these topics.

In the next chapter we present the theory of SDS over permutations. That is, we restrict ourselves to the case where the words w are permutations of the vertex set of Y . In this setting the dependency graph $G(w, Y)$ is isomorphic to the base graph Y , and this simplifies many aspects significantly. We study invertible SDS, fixed points, equivalence, and SDS morphisms.

Chapter 5 contains a collection of results on SDS phase-space properties as well as results for specific classes of SDS. This includes fixed-point characterization and enumeration for SDS and CA over circulant graphs based on a deBruijn graph construction, properties of threshold-SDS, and the structure of SDS induced by the Boolean nor function.

In Chapter 6 we consider w -independent SDS. These are SDS where the associated SDS-maps have periodic points that are independent of the choice of word w . We will show that this class of SDS induces a group and that this

group encodes properties of the phase-space structures that can be generated by varying the update order w .

Chapter 7 analyzes SDS over words. Equivalence classes of acyclic orientations of the dependency graph now replace acyclic orientations of the base graph, and new symmetries in the update order w arise. We give several combinatorial results that provide an interpretation of equivalence of words and the corresponding induced SDS.

We conclude with Chapter 8, which is an outline of current and possible research directions and application areas for SDS ranging from packet routing protocols to gene-regulatory networks. In our opinion we have only started to uncover the mathematical gems of this area, and this final chapter may provide some starting points for further study.

A Guide for the Reader: The first two chapters are intended as background and motivation. A reader wishing to proceed directly to the mathematical treatment of SDS may omit these. Chapter 3 is included for reference to make the book self-contained. It can be omitted and referred to as needed in later chapters. The fourth chapter presents the core structure and results for SDS and is fundamental to all of the chapters that follow. Chapter 6 relies on results from Chapter 5, but Chapter 7 can be read directly after Chapter 4.

Each chapter comes with exercises, many of which include full solutions. The anticipated difficulty level for each problem is indicated in bold at the end of the problem text. We have ranked the problems from 1 (easy, routine) through 5 (hard, unsolved). Some of the problems are computational in the sense that some programming and use of computers may be helpful. These are marked by the additional letter ‘C’.

We thank Nils A. Baas, Chris L. Barrett, William Y. C. Chen, Anders Å. Hansson, Qing H. Hou, Reinhard Laubenbacher, Matthew Macauley, Madhav M. Marathe, and Bodo Pareigis for discussions and valuable suggestions. Special thanks to the researchers of the Center of Combinatorics at Nankai University. We also thank the students at Virginia Tech University who took the course *4984 Mathematics of Computer Simulations* — their feedback and comments plus the lecture preparations helped shape this book. Finally, we thank Vaishali Damle, Julie Park, and Springer for all their help in preparing this book.

Blacksburg, Virginia, January 2007
Tianjin, China, January 2007

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An Introduction to Sequential Dynamical Systems

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2008, XII, 248 p. 73 illus., Softcover

ISBN: 978-0-387-30654-4