Chapter 5

Polyconvex, quasiconvex and rank one convex functions

5.1 Introduction

We now turn our attention to the vectorial case. Recall that we are considering integrals of the form

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx$$

where

- $\Omega \subset \mathbb{R}^n$ is an open set;
- $u : \Omega \to \mathbb{R}^N$ and hence $\nabla u \in \mathbb{R}^{N\times n}$;
- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, is a Carathéodory function.

While in Part I we were essentially concerned with the scalar case ($N = 1$ or $n = 1$), we now deal with the vectorial case ($N, n > 1$). The convexity of $\xi \to f(x, u, \xi)$ played the central role in the scalar case ($N = 1$ or $n = 1$), see Chapter 3. In the vectorial case, it is still a sufficient condition to ensure weak lower semicontinuity of I in $W^{1,p}(\Omega; \mathbb{R}^N)$; it is, however, far from being a necessary one. Such a condition is the so-called *quasiconvexity* introduced by Morrey. It turns out (see Chapter 8) that

$$f \text{ quasiconvex } \Leftrightarrow\ I \text{ weakly lower semicontinuous.}$$

Since the notion of quasiconvexity is not a pointwise condition, it is hard to verify if a given function f is quasiconvex. Therefore one is led to introduce a slightly weaker condition, known as *rank one convexity,* that is equivalent to the *ellipticity* of the Euler-Lagrange system of equations associated to the functional I. We also define a stronger condition, called *polyconvexity*, that naturally arises when we try to generalize the notions of duality for convex functions to the vectorial context. One can relate all these definitions through the following diagram

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank one convex.}$$

We should again emphasize that in the scalar case all these notions are equivalent to the usual convexity condition.

The definitions and main properties of these generalized notions of convexity are discussed in Section 5.2.

In Section 5.3, we give several examples. In particular we show that all the reverse implications are false.

Finally, in an appendix (Section 5.4), we gather certain elementary properties of determinants.

5.2 Definitions and main properties

5.2.1 Definitions and notations

Recall that, if $\xi \in \mathbb{R}^{N\times n}$, we write

$$\xi = \begin{pmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^N & \cdots & \xi_n^N \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} = (\xi_1, \cdots, \xi_n) = \left(\xi_\alpha^i\right)_{1\le\alpha\le n}^{1\le i\le N}.$$

In particular if $u : \mathbb{R}^n \to \mathbb{R}^N$ we write

$$\nabla u = \begin{pmatrix} \dfrac{\partial u^1}{\partial x_1} & \cdots & \dfrac{\partial u^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial u^N}{\partial x_1} & \cdots & \dfrac{\partial u^N}{\partial x_n} \end{pmatrix}.$$

We may now define all the notions introduced above.

Definition 5.1 *(i) A function* $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ *is said to be* rank one convex *if*

$$f(\lambda\xi + (1-\lambda)\eta) \le \lambda f(\xi) + (1-\lambda) f(\eta)$$

for every $\lambda \in [0,1]$, $\xi, \eta \in \mathbb{R}^{N\times n}$ *with* $\operatorname{rank}\{\xi - \eta\} \le 1$.

(ii) A Borel measurable and locally bounded function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ *is said to be* quasiconvex *if*

$$f(\xi) \le \frac{1}{\operatorname{meas} D} \int_D f(\xi + \nabla\varphi(x))\, dx$$

for every bounded open set $D \subset \mathbb{R}^n$, *for every* $\xi \in \mathbb{R}^{N \times n}$ *and for every* $\varphi \in W_0^{1,\infty}\left(D; \mathbb{R}^N\right)$.

(iii) A function $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ *is said to be* polyconvex *if there exists* $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\}$ *convex, such that*

$$f(\xi) = F(T(\xi)),$$

where $T : \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau(n,N)}$ *is such that*

$$T(\xi) := (\xi, \operatorname{adj}_2 \xi, \cdots, \operatorname{adj}_{n\wedge N} \xi).$$

In the preceding definition, $\operatorname{adj}_s \xi$ *stands for the matrix of all* $s \times s$ *minors of the matrix* $\xi \in \mathbb{R}^{N\times n}$, $2 \le s \le n \wedge N = \min\{n, N\}$ *and*

$$\tau(n, N) := \sum_{s=1}^{n\wedge N} \sigma(s), \quad \text{where} \quad \sigma(s) := \binom{N}{s}\binom{n}{s} = \frac{N!n!}{(s!)^2 (N-s)!(n-s)!}.$$

(iv) A function $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ *is said to be* separately convex*, or* convex in each variable*, if the function*

$$x_i \to f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_m) \quad \text{is convex for every } i = 1, \cdots, m,$$

for every fixed $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m) \in \mathbb{R}^{m-1}$.

(v) A function f *is called* polyaffine, quasiaffine *or* rank one affine *if* f *and* $-f$ *are, respectively, polyconvex, quasiconvex or rank one convex.*

Remark 5.2 (i) The concepts were introduced by Morrey [453], but the terminology is that of Ball [53]; note, however, that Ball calls quasiaffine functions null Lagrangians.

(ii) If we adopt the tensorial notation, the notion of rank one convexity can be read as follows: the function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $\varphi = \varphi(t)$, defined by

$$\varphi(t) := f(\xi + ta \otimes b)$$

is convex for every $\xi \in \mathbb{R}^{N\times n}$ and for every $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$, where we have denoted by

$$a \otimes b = \left(a^i b_\alpha\right)_{1\le\alpha\le n}^{1\le i\le N}.$$

(iii) It is easily seen that in the definition of quasiconvexity, one can replace the set of test functions $W_0^{1,\infty}$ by $C_0^\infty\left(D; \mathbb{R}^N\right)$.

(iv) We will see in Proposition 5.11 that if in the definition of quasiconvexity the inequality holds for one bounded open set D, it holds for any such set.

(v) We did not give a definition of quasiconvex functions f that may take the value $+\infty$, contrary to polyconvexity and rank one convexity. There have been such definitions given, for example by Ball-Murat [65] and Dacorogna-Fusco [186] (see also Wagner [594]), in the case where f is allowed to take the value $+\infty$. However, although such definitions have been shown to be necessary for weak lower semicontinuity, it has not been proved that they were sufficient and this seems to be a difficult problem. The notion of quasiconvexity being useful only as an equivalent to weak lower semicontinuity we have disregarded the extension to the case $\mathbb{R}\cup\{+\infty\}$; while those of polyconvexity and rank one convexity will be shown to be useful.

(vi) We have gathered in Section 5.4 some elementary facts about determinants and adj_s of matrices. Note that in the case $N = n = 2$, the notion of polyconvexity can be read as follows

$$\begin{cases} \sigma(1) = 4,\ \sigma(2) = 1,\ \tau(n, N) = \tau(2, 2) = 5, \\ \quad T(\xi) = (\xi, \det \xi),\ f(\xi) = F(\xi, \det \xi). \end{cases}$$

(vii) In the definition of polyconvexity of a given function f, the associated function F (i.e. $f(\xi) = F(T(\xi))$) in general is not unique. For example, let $N = n = 2$,

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix}$$

and

$$\begin{aligned} f(\xi) = |\xi|^2 &= \left(\xi_1^1\right)^2 + \left(\xi_1^2\right)^2 + \left(\xi_2^1\right)^2 + \left(\xi_2^2\right)^2 \\ &= \left(\xi_1^1 - \xi_2^2\right)^2 + \left(\xi_2^1 + \xi_1^2\right)^2 + 2\det \xi. \end{aligned}$$

Let $F_1, F_2 : \mathbb{R}^5 \to \mathbb{R}$ be defined by

$$F_1(\xi, a) := |\xi|^2 \quad \text{and} \quad F_2(\xi, a) := \left(\xi_1^1 - \xi_2^2\right)^2 + \left(\xi_2^1 + \xi_1^2\right)^2 + 2a.$$

Then F_1 and F_2 are convex, $F_1 \neq F_2$ and

$$f(\xi) = F_1(T(\xi)) = F_1(\xi, \det \xi) = F_2(T(\xi)) = F_2(\xi, \det \xi).$$

We will see, after Theorem 5.6, that using either Carathéodory theorem or the separation theorem one can privilege one among the numerous functions F.

(viii) The notion of separate convexity does not play any direct role in the calculus of variations. However it can serve as a model for better understanding of the more difficult notion of rank one convexity.

(ix) We will see (see Theorem 5.20) that the notions of polyaffine, quasiaffine or rank one affine are equivalent. Therefore the first and third concepts will not be used anymore. $\diamondsuit$

5.2.2 Main properties

In Section 5.3, we give several examples of polyconvex, quasiconvex and rank one convex functions, but before that we show the relationship between these notions. The following result is essentially due to Morrey [453], [455].

Theorem 5.3 *(i) Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$. Then*

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex.}$$

If $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$, then

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ rank one convex.}$$

(ii) If $N = 1$ or $n = 1$, then all these notions are equivalent.

(iii) If $f \in C^2\left(\mathbb{R}^{N\times n}\right)$, then rank one convexity is equivalent to Legendre-Hadamard *condition (or* ellipticity *condition)*

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f(\xi)}{\partial \xi_\alpha^i \partial \xi_\beta^j} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq 0$$

for every $\lambda \in \mathbb{R}^N, \mu \in \mathbb{R}^n, \xi = \left(\xi_\alpha^i\right)_{1\leq\alpha\leq n}^{1\leq i\leq N} \in \mathbb{R}^{N\times n}$.

(iv) If $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is convex, polyconvex, quasiconvex or rank one convex, then f is locally Lipschitz.

Remark 5.4 (i) We will show later that all the counter implications are false.

- The fact that

$$f \text{ polyconvex} \not\Leftarrow f \text{ convex}$$

is elementary. For example, when $N = n = 2$, the function

$$f(\xi) := \det \xi$$

is polyconvex but not convex.

- We will see several examples (with $N, n \geq 2$), notably in Sections 5.3.2, 5.3.8 and 5.3.9, of quasiconvex functions that are not polyconvex so that we have

$$f \text{ quasiconvex} \not\Leftarrow f \text{ polyconvex.}$$

However, there are no elementary examples of this fact.

- The result that

$$f \text{ rank one convex} \not\Leftarrow f \text{ quasiconvex}$$

is the fundamental example of Sverak (see Section 5.3.7), which is valid for $n \geq 2$ and $N \geq 3$. However it is still an open problem to know whether f rank one convex implies f quasiconvex, when $N = 2$ (so, in particular, the case $N = n = 2$ is open).

(ii) The Legendre-Hadamard condition is the usual inequality required for the Euler-Lagrange system of equations and is known in this case as ellipticity (see Agmon-Douglis-Nirenberg [7]).

(iii) It is straightforward to see that

$$f \text{ rank one convex } \Rightarrow f \text{ separately convex.}$$

However, the reverse implication is false, as the following example shows. Let $N = n = 2$ and

$$f(\xi) := \xi_1^1 \xi_2^1 .$$

This function is clearly separately convex but not rank one convex. ♢

Before proceeding with the proof of the theorem, we give a lemma involving some elementary properties of the determinants.

Lemma 5.5 *Let $\xi \in \mathbb{R}^{N\times n}$ and $T(\xi)$ be defined as above.*

(i) For every $\xi, \eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}\{\xi - \eta\} \le 1$ and for every $\lambda \in [0,1]$, the following identity holds:

$$T(\lambda\xi + (1-\lambda)\eta) = \lambda T(\xi) + (1-\lambda) T(\eta) .$$

(ii) For every $D \subset \mathbb{R}^n$ a bounded open set, $\xi \in \mathbb{R}^{N\times n}$, $\varphi \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)$, the following result is valid:

$$T(\xi) = \frac{1}{\operatorname{meas} D} \int_D T(\xi + \nabla\varphi(x))\, dx.$$

Proof. The proof is elementary and can be found in Proposition 5.65 and Theorem 8.35. We give here, for the sake of illustration, the proof in the case $N = n = 2$. We then have

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix}$$

and

$$T(\xi) = (\xi, \det\xi) = \left(\xi_1^1,\, \xi_2^1,\, \xi_1^2,\, \xi_2^2,\, \xi_1^1\xi_2^2 - \xi_2^1\xi_1^2\right) .$$

(i) Since $\operatorname{rank}\{\xi - \eta\} \le 1$, there exist $a, b \in \mathbb{R}^2$ such that

$$\eta = \xi + a \otimes b = \begin{pmatrix} \xi_1^1 + a^1 b_1 & \xi_2^1 + a^1 b_2 \\ \xi_1^2 + a^2 b_1 & \xi_2^2 + a^2 b_2 \end{pmatrix}.$$

We therefore get that

$$\begin{aligned} \det(\lambda\xi + (1-\lambda)\eta) &= \det(\xi + (1-\lambda)\, a\otimes b) \\ &= \lambda \det\xi + (1-\lambda)\det\eta. \end{aligned}$$

We then deduce that, whenever $\operatorname{rank}\{\xi - \eta\} \le 1$,

$$\begin{aligned} T(\lambda\xi + (1-\lambda)\eta) &= (\lambda\xi + (1-\lambda)\eta, \det(\lambda\xi + (1-\lambda)\eta)) \\ &= \lambda T(\xi) + (1-\lambda) T(\eta) . \end{aligned}$$

(ii) The proof is similar to the preceding one. Note first that if $\varphi \in C^2\left(D;\mathbb{R}^2\right)$, then

$$\det\nabla\varphi = \frac{\partial\varphi^1}{\partial x_1}\frac{\partial\varphi^2}{\partial x_2} - \frac{\partial\varphi^1}{\partial x_2}\frac{\partial\varphi^2}{\partial x_1} = \frac{\partial}{\partial x_1}(\varphi^1\frac{\partial\varphi^2}{\partial x_2}) - \frac{\partial}{\partial x_2}(\varphi^1\frac{\partial\varphi^2}{\partial x_1}).$$

Integrating by part the above identity, we have that, if $\varphi \in C_0^2\left(D;\mathbb{R}^2\right)$, then

$$\begin{aligned}\det\xi\ \operatorname{meas} D &= \int_D [\det\xi + \xi_1^1\frac{\partial\varphi^2}{\partial x_2} + \xi_2^2\frac{\partial\varphi^1}{\partial x_1} - \xi_2^1\frac{\partial\varphi^2}{\partial x_1} - \xi_1^2\frac{\partial\varphi^1}{\partial x_2} + \det\nabla\varphi]dx \\ &= \int_D \det\left(\xi + \nabla\varphi\left(x\right)\right)dx.\end{aligned}$$

By density, the above identity holds also if $\varphi \in W_0^{1,\infty}\left(D;\mathbb{R}^2\right)$. We then deduce that for every $\varphi \in W_0^{1,\infty}\left(D;\mathbb{R}^2\right)$, we must have

$$\begin{aligned}T\left(\xi\right)\operatorname{meas} D &= (\int_D \left(\xi + \nabla\varphi\left(x\right)\right)dx, \int_D \det\left(\xi + \nabla\varphi\left(x\right)\right)dx) \\ &= \int_D T\left(\xi + \nabla\varphi\left(x\right)\right)dx.\end{aligned}$$

This concludes the proof of the lemma. ■

We may now proceed with the proof of Theorem 5.3.

Proof. *Part 1*: f convex $\Rightarrow$ f polyconvex. This implication is trivial.

Part 2: f polyconvex $\Rightarrow$ f quasiconvex. Since f is polyconvex, there exists $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R}$ convex, such that

$$f\left(\xi\right) = F\left(T\left(\xi\right)\right).$$

Using Lemma 5.5 and Jensen inequality we obtain

$$\begin{aligned}&\frac{1}{\operatorname{meas} D}\int_D f\left(\xi + \nabla\varphi\left(x\right)\right)dx = \frac{1}{\operatorname{meas} D}\int_D F\left(T\left(\xi + \nabla\varphi\left(x\right)\right)\right)dx \\ &\geq F(\frac{1}{\operatorname{meas} D}\int_D T\left(\xi + \nabla\varphi\left(x\right)\right)dx) = F\left(T\left(\xi\right)\right) = f\left(\xi\right),\end{aligned}$$

for every bounded open set $D \subset \mathbb{R}^n$, for every $\xi \in \mathbb{R}^{N\times n}$ and for every $\varphi \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)$. The inequality is precisely the definition of quasiconvexity.

Part 3: f quasiconvex $\Rightarrow$ f rank one convex. The proof is similar to that of Theorem 3.13 of Chapter 3. Recall that we want to show that

$$f\left(\lambda\xi + \left(1-\lambda\right)\eta\right) \leq \lambda f\left(\xi\right) + \left(1-\lambda\right)f\left(\eta\right)$$

for every $\lambda \in \left[0,1\right]$, $\xi,\eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}\left\{\xi-\eta\right\} \leq 1$. To achieve this goal we let $\epsilon > 0$ and we apply Lemma 3.11. We therefore find disjoint open sets

$D_\xi\,, D_\eta \subset D$ and $\varphi \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)$ such that

$$\begin{cases} |\mathrm{meas}\, D_\xi - \lambda\, \mathrm{meas}\, D| \le \epsilon, \;\; |\mathrm{meas}\, D_\eta - (1-\lambda)\, \mathrm{meas}\, D| \le \epsilon \\ \nabla\varphi\,(x) = \begin{cases} (1-\lambda)\,(\xi-\eta) & \text{if } x \in D_\xi \\ -\lambda\,(\xi-\eta) & \text{if } x \in D_\eta \end{cases} \\ \|\nabla\varphi\|_{L^\infty} \le \gamma \end{cases}$$

where $\gamma = \gamma\,(\xi,\eta,D)$ is a constant independent of ϵ. We may then use the quasiconvexity of f to get

$$\begin{aligned} &\int_D f\,(\lambda\xi + (1-\lambda)\,\eta + \nabla\varphi\,(x))\,dx \\ &= \int_{D_\xi} f\,(\xi)\,dx + \int_{D_\eta} f\,(\eta)\,dx + \int_{D-(D_\xi\cup D_\eta)} f\,(\lambda\xi + (1-\lambda)\,\eta + \nabla\varphi\,(x))\,dx \\ &\ge f\,(\lambda\xi + (1-\lambda)\,\eta)\,\mathrm{meas}\, D. \end{aligned}$$

Using the properties of the function φ and the fact that ϵ is arbitrary, we have indeed obtained that f is rank one convex.

Part 4. If we now consider the case where $f : \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{+\infty\}$, the first implication: f convex $\Rightarrow$ f polyconvex is still trivial. The implication f polyconvex $\Rightarrow$ f rank one convex is also elementary if we use Lemma 5.5. Indeed since f is polyconvex, there exists $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R}\cup\{+\infty\}$ convex so that

$$f\,(\xi) = F\,(T\,(\xi))\,.$$

Let $\lambda \in [0,1]$, $\xi,\eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}\{\xi-\eta\} \le 1$, then, using Lemma 5.5, we get

$$\begin{aligned} f\,(\lambda\xi + (1-\lambda)\,\eta) &= F\,(T\,(\lambda\xi + (1-\lambda)\,\eta)) = F\,(\lambda T\,(\xi) + (1-\lambda)\,T\,(\eta)) \\ &\le \lambda F\,(T\,(\xi)) + (1-\lambda)\,F\,(T\,(\eta)) = \lambda f\,(\xi) + (1-\lambda)\,f\,(\eta) \end{aligned}$$

which is precisely the rank one convexity of f.

(ii) The second statement of the theorem, asserting that if $N = 1$ or $n = 1$, then all the notions are equivalent, is trivial.

(iii) We now assume that f is C^2 and rank one convex, that is

$$\varphi\,(t) := f\,(\xi + t\lambda\otimes\mu)$$

is convex in $t \in \mathbb{R}$ for every $\xi \in \mathbb{R}^{N\times n}$ and for every $\lambda \in \mathbb{R}^N$, $\mu \in \mathbb{R}^n$. Since φ is also C^2, we obtain immediately Legendre-Hadamard condition, by computing $\varphi''\,(t)$ and using the convexity of φ.

(iv) The last part of Theorem 5.3 is an immediate consequence of Theorem 2.31 of Chapter 2, since a rank one convex function is evidently separately convex. ■

5.2.3 Further properties of polyconvex functions

We now give different characterizations of polyconvex functions that are based on Carathéodory theorem and separation theorems. The next result is due to Dacorogna [177] and [179], following earlier results of Ball [53].

We first recall the notation that for any integer I

$$\Lambda_I := \{\lambda = (\lambda_1, \cdots, \lambda_I) : \lambda_i \geq 0 \text{ and } \textstyle\sum_{i=1}^{I} \lambda_i = 1\}.$$

Theorem 5.6 Part 1. *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$, then the following two statements are equivalent:*

(i) f is polyconvex;

(ii) the next two properties hold:

• *there exists a convex function $c : \mathbb{R}^\tau \to \mathbb{R} \cup \{+\infty\}$, where $\tau = \tau(n, N)$, such that*

$$f(\xi) \geq c(T(\xi)) \text{ for every } \xi \in \mathbb{R}^{N\times n}; \tag{5.1}$$

• *for every $\xi_i \in \mathbb{R}^{N\times n}$, $\lambda \in \Lambda_{\tau+1}$, satisfying*

$$\textstyle\sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = T(\sum_{i=1}^{\tau+1} \lambda_i \xi_i), \tag{5.2}$$

then

$$f(\textstyle\sum_{i=1}^{\tau+1} \lambda_i \xi_i) \leq \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i). \tag{5.3}$$

Part 2. *If (ii) is satisfied and if $F : \mathbb{R}^\tau \to \mathbb{R} \cup \{+\infty\}$ is defined by*

$$F(X) := \inf\{\textstyle\sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) : \lambda \in \Lambda_{\tau+1}, \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = X\}, \tag{5.4}$$

then F is convex and

$$f(\xi) = F(T(\xi)) \text{ for every } \xi \in \mathbb{R}^{N\times n}. \tag{5.5}$$

Moreover, for every $X \in \mathbb{R}^\tau$,

$$\begin{aligned} F(X) = \quad & \sup\{G(X) : G : \mathbb{R}^\tau \to \mathbb{R} \cup \{+\infty\} \text{ convex} \\ & \text{and } f(\xi) = G(T(\xi)),\ \forall \xi \in \mathbb{R}^{N\times n}\}. \end{aligned}$$

Part 3. *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$, i.e. f takes only finite values. Then the following conditions are equivalent:*

(i) f is polyconvex;

(iii) for every $\xi \in \mathbb{R}^{N\times n}$, there exists $\beta = \beta(\xi) \in \mathbb{R}^\tau$ such that

$$f(\eta) \geq f(\xi) + \langle \beta(\xi); T(\eta) - T(\xi) \rangle \tag{5.6}$$

for every $\eta \in \mathbb{R}^{N\times n}$ and where $\langle \cdot;\cdot \rangle$ denotes the scalar product in $\mathbb{R}^\tau$.

Part 4. *If (iii) is satisfied, then the function*

$$h(X) := \sup_{\xi \in \mathbb{R}^{N\times n}} \{\langle \beta(\xi); X - T(\xi) \rangle + f(\xi)\} \tag{5.7}$$

is convex, takes only finite values and satisfies

$$f(\xi) = h(T(\xi)) \text{ for every } \xi \in \mathbb{R}^{N\times n}. \tag{5.8}$$

Example 5.7 Let $N = n = 2$. Then (5.3) and (5.2) become

$$\begin{cases} f(\sum_{i=1}^{6} \lambda_i \xi_i) \leq \sum_{i=1}^{6} \lambda_i f(\xi_i), \\ \sum_{i=1}^{6} \lambda_i \det(\xi_i) = \det(\sum_{i=1}^{6} \lambda_i \xi_i) \end{cases}$$

and (5.6) is read

$$f(\eta) \geq f(\xi) + \langle \gamma(\xi); \eta - \xi \rangle + \delta(\xi)(\det \eta - \det \xi)$$

where $\gamma(\xi) \in \mathbb{R}^{2\times 2}$ and $\delta(\xi) \in \mathbb{R}$. ♢

Remark 5.8 (i) The above theorem is a direct adaptation of Carathéodory theorem and the separation theorems for polyconvex functions.

(ii) The condition (5.1) in the theorem implies that F defined in (5.4) does not take the value $-\infty$.

(iii) The theorem is important for the following reasons.

- It gives an *intrinsic* definition of polyconvexity, in the sense that it is not given in terms of convexity properties of an associated function F.

- As already mentioned in the definition of the polyconvexity of a given function f, the associated convex function F is not unique. Equation (5.4) allows us to privilege one such function F. A similar remark can be done using (5.7), as was also observed by Kohn and Strang [373], [374].

- If $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ (i.e. f takes only finite values), then F defined by (5.4) also takes finite values.

(iv) In view of the above remark we can conclude that if f takes only finite values then (i), (ii) and (iii) of Theorem 5.6 are equivalent.

(v) Some other properties of polyconvex functions in the cases $N = n = 2$ or $N = n = 3$ are given by Aubert [39]. ♢

Proof. We follow here the proof of Dacorogna [177], [179], inspired by earlier considerations by Ball [53], which were based on results of Busemann-Ewald-Shephard [110] and Busemann-Shephard [111].

Parts 1 and 2. **(i) ⇒ (ii)**. Since f is polyconvex, there exists $F : \mathbb{R}^\tau \to \mathbb{R} \cup \{+\infty\}$, $\tau = \tau(n, N)$, convex such that

$$f(\xi) = F(T(\xi)). \tag{5.9}$$

The existence of a function c is trivial, just choose $c = F$. The convexity of F coupled with (5.2) gives immediately (5.3).

(ii) ⇒ (i). Assume that (5.3) holds for every (λ_i, ξ_i), $1 \leq i \leq \tau + 1$, satisfying (5.2). We wish to show that there exists $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\}$ convex satisfying (5.9). Let $I \geq \tau + 1$ ($\tau = \tau(n, N)$) be an integer and for $X \in \mathbb{R}^\tau$ define

$$F_I(X) := \inf\{ \sum_{i=1}^{I} \lambda_i f(\xi_i) : \lambda \in \Lambda_I,\ \sum_{i=1}^{I} \lambda_i T(\xi_i) = X \}. \tag{5.10}$$

We will show that F_I satisfies (5.9) and that one can choose $I = \tau + 1$, without loss of generality, establishing hence (5.4). The proof is divided into four steps.

Step 1. We first show that F_I is well defined.

Step 2. We next prove that I can be taken to be $\tau + 1$ in (5.10) without loss of generality and we therefore denote F_I by F (satisfying then (5.4)).

Step 3. We then show that F is convex.

Step 4. We finally establish that F satisfies (5.5).

We now proceed with the details of these four steps.

Step 1. Let us start by showing that F_I is well defined. To do this we must see that given $X \in \mathbb{R}^{\tau(n,N)}$ and $I \geq \tau + 1$, then there exist $\lambda \in \Lambda_I$ and ξ_i such that $\sum \lambda_i T(\xi_i) = X$. In view of Carathéodory theorem, this is equivalent to showing that

$$\operatorname{co} T\left(\mathbb{R}^{N\times n}\right) = \mathbb{R}^{\tau(n,N)}, \tag{5.11}$$

where $\operatorname{co} M$ denotes the convex hull of M and

$$T\left(\mathbb{R}^{N\times n}\right) = \left\{X \in \mathbb{R}^{\tau(n,N)} : \text{ there exists } \xi \in \mathbb{R}^{N\times n} \text{ with } T(\xi) = X\right\}.$$

In order to establish (5.11), we proceed by contradiction. Assume that

$$\operatorname{co}\left(T\left(\mathbb{R}^{N\times n}\right)\right) \neq \mathbb{R}^{\tau}.$$

Then from the separation theorems (see Corollary 2.11), there exist $0 \neq \alpha \in \mathbb{R}^{\tau}$, $\beta \in \mathbb{R}$, such that

$$\operatorname{co}\left(T\left(\mathbb{R}^{N\times n}\right)\right) \subset V := \{X \in \mathbb{R}^{\tau} : \langle\alpha; X\rangle \leq \beta\} \tag{5.12}$$

where $\langle\cdot;\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{\tau}$, $\tau = \tau(n, N)$. Recall from the definition of polyconvexity that

$$\tau(n, N) = \sum_{s=1}^{n\wedge N} \sigma(s)$$

where $\sigma(s) = \binom{N}{s}\binom{n}{s}$. We then let for $X \in \mathbb{R}^{\tau(n,N)}$

$$X = (X_1, X_2, \cdots, X_{n\wedge N}) \in \mathbb{R}^{\sigma(1)} \times \mathbb{R}^{\sigma(2)} \times \cdots \times \mathbb{R}^{\sigma(n\wedge N)} = \mathbb{R}^{\tau(n,N)}$$

and similarly for $\alpha \in \mathbb{R}^{\tau}$. We may then write

$$\langle\alpha; X\rangle = \sum_{s=1}^{n\wedge N} \langle\alpha_s; X_s\rangle .$$

Since $\alpha \neq 0$, there exists $t \in \{1, \cdots, n \wedge N\}$ such that $\alpha_t \neq 0$ while $\alpha_s = 0$ if $s < t$ (if $\alpha_1 \neq 0$, then take $t = 1$). We now show that (5.12) leads to a contradiction and therefore (5.11) holds. Let $\xi \in \mathbb{R}^{N\times n}$ and therefore

$$T(\xi) = (\xi, \operatorname{adj}_2 \xi, \cdots, \operatorname{adj}_{n\wedge N} \xi) \in T\left(\mathbb{R}^{N\times n}\right) \subset \operatorname{co} T\left(\mathbb{R}^{N\times n}\right).$$

We choose $\xi \in \mathbb{R}^{N\times n}$ such that

$$\langle \alpha; T(\xi) \rangle = \langle \alpha_t; \operatorname{adj}_t \xi \rangle \neq 0.$$

This is possible by choosing $(N-t)$ lines of ξ to be zero vectors of $\mathbb{R}^n$ and choosing the other t lines of ξ so that $\langle \alpha_t; \operatorname{adj}_t \xi \rangle$ is non-zero.

Let $\lambda \in \mathbb{R}$ be arbitrary and multiply any of the t non zero lines of ξ by λ. Denote the obtained matrix by η. We then have $T(\eta) \in T\left(\mathbb{R}^{N\times n}\right) \subset \operatorname{co} T\left(\mathbb{R}^{N\times n}\right)$ and

$$\langle \alpha; T(\eta) \rangle = \langle \alpha_t; \operatorname{adj}_t \eta \rangle = \lambda \langle \alpha_t; \operatorname{adj}_t \xi \rangle = \lambda \langle \alpha; T(\xi) \rangle .$$

Using (5.12), we deduce that $T(\xi), T(\eta) \in V$, i.e.

$$\begin{cases} \langle \alpha; T(\xi) \rangle \leq \beta \\ \langle \alpha; T(\eta) \rangle = \lambda \langle \alpha; T(\xi) \rangle \leq \beta. \end{cases}$$

The arbitrariness of λ and the fact that $\langle \alpha; T(\xi) \rangle \neq 0$ lead immediately to a contradiction. This completes Step 1.

Step 2. We now want to show that in (5.10) we can take $I = \tau + 1$. This is done as in Theorem 2.13.

So let $X \in \mathbb{R}^\tau$, $\xi_i \in \mathbb{R}^{N\times n}$ and $\lambda \in \Lambda_I$ be such that

$$X = \sum_{i=1}^{I} \lambda_i T(\xi_i).$$

We first prove that there is no loss of generality if we choose $I = \tau + 2$. Define

$$T(\operatorname{epi} f) := \{(T(\xi), a) \in \mathbb{R}^\tau \times \mathbb{R} : f(\xi) \leq a\} \subset \mathbb{R}^{\tau+1}.$$

We then trivially have that $(T(\xi_i), f(\xi_i)) \in T(\operatorname{epi} f)$ and if $\lambda \in \Lambda_I$, we get

$$\left(X, \textstyle\sum_{i=1}^{I} \lambda_i f(\xi_i)\right) = \textstyle\sum_{i=1}^{I} \lambda_i \left(T(\xi_i), f(\xi_i)\right) \in \operatorname{co} T(\operatorname{epi} f).$$

Using Carathéodory theorem, we find that in (5.10) we can take $I = \tau + 2$. It now remains to reduce I from $\tau + 2$ to $\tau + 1$ and this is done as in Theorem 2.35. We show that given $X,\ T(\xi_i) \in \mathbb{R}^\tau$, $1 \leq i \leq \tau + 2$, $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ and $\alpha \in \Lambda_{\tau+2}$ with

$$\sum_{i=1}^{\tau+2} \alpha_i T(\xi_i) = X, \tag{5.13}$$

then there exist $\beta \in \Lambda_{\tau+2}$ such that at least one of the $\beta_i = 0$ (meaning, upon relabeling, that $\beta \in \Lambda_{\tau+1}$) and

$$\sum_{i=1}^{\tau+2} \beta_i f(\xi_i) \leq \sum_{i=1}^{\tau+2} \alpha_i f(\xi_i) \quad \text{with} \quad \sum_{i=1}^{\tau+2} \beta_i T(\xi_i) = X. \tag{5.14}$$

It is clear that (5.14) will imply Step 2. Assume all $\alpha_i > 0$ in (5.13) and (5.14), otherwise (5.14) would be trivial. Since from (5.13), we have

$$X \in \operatorname{co}\{T(\xi_1), \cdots, T(\xi_{\tau+2})\} \subset \mathbb{R}^\tau,$$

it results, from Carathéodory theorem, that there exists $\widetilde{\alpha} \in \Lambda_{\tau+2}$ with at least one of the $\widetilde{\alpha}_i = 0$ such that

$$\sum_{i=1}^{\tau+2} \widetilde{\alpha}_i T(\xi_i) = X.$$

We may assume without loss of generality that

$$\sum_{i=1}^{\tau+2} \widetilde{\alpha}_i f(\xi_i) > \sum_{i=1}^{\tau+2} \alpha_i f(\xi_i), \tag{5.15}$$

otherwise choosing $\beta_i = \widetilde{\alpha}_i$ we would immediately obtain (5.14). We then let

$$J := \{i \in \{1, \cdots, \tau+2\} : \alpha_i - \widetilde{\alpha}_i < 0\}.$$

Observe that $J \neq \emptyset$, since otherwise $\alpha_i \geq \widetilde{\alpha}_i \geq 0$ for every $1 \leq i \leq \tau+2$ and since at least one of the $\widetilde{\alpha}_i = 0$, we would have a contradiction with $\sum_{i=1}^{\tau+2} \alpha_i = \sum_{i=1}^{\tau+2} \widetilde{\alpha}_i = 1$ and the fact that $\alpha_i > 0$ for every i. We then define

$$\lambda := \min_{i \in J}\{\frac{\alpha_i}{\widetilde{\alpha}_i - \alpha_i}\}$$

and we have clearly $\lambda > 0$. Finally let

$$\beta_i := \alpha_i + \lambda(\alpha_i - \widetilde{\alpha}_i), \ 1 \leq i \leq \tau+2.$$

We therefore have

$$\beta_i \geq 0, \ \sum_{i=1}^{\tau+2} \beta_i = 1, \text{ at least one of the } \beta_i = 0, \ \sum_{i=1}^{\tau+2} \beta_i T(\xi_i) = X$$

and from (5.15)

$$\begin{aligned} \sum_{i=1}^{\tau+2} \beta_i f(\xi_i) &= \sum_{i=1}^{\tau+2} \alpha_i f(\xi_i) + \lambda \sum_{i=1}^{\tau+2} (\alpha_i - \widetilde{\alpha}_i) f(\xi_i) \\ &\leq \sum_{i=1}^{\tau+2} \alpha_i f(\xi_i). \end{aligned}$$

We have therefore obtained (5.14) and this concludes Step 2. Since I can be taken to be $\tau+1$, we will then denote F_I by F (i.e. (5.10) can be replaced by (5.4)).

Step 3. We now show F is convex. Let $\lambda \in [0,1]$, $X, Y \in \mathbb{R}^{\tau}$. We want to prove that

$$\lambda F(X) + (1-\lambda) F(Y) \geq F(\lambda X (1-\lambda) Y).$$

Fix $\epsilon > 0$. From (5.4) we deduce that there exist λ, $\mu \in \Lambda_{\tau+1}$ and ξ_i, $\eta_i \in \mathbb{R}^{N\times n}$ such that

$$\lambda F(X) + (1-\lambda) F(Y) + \epsilon \geq \lambda \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) + (1-\lambda) \sum_{i=1}^{\tau+1} \mu_i f(\eta_i), \tag{5.16}$$

with

$$\sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = X, \ \sum_{i=1}^{\tau+1} \mu_i T(\eta_i) = Y. \tag{5.17}$$

For $1 \leq i \leq \tau+1$, let

$$\begin{cases} \widetilde{\lambda}_i = \lambda \lambda_i & C_i = \xi_i \\ \widetilde{\lambda}_{i+\tau+1} = (1-\lambda)\mu_i & C_{i+\tau+1} = \eta_i . \end{cases}$$

Then (5.16) and (5.17) can be rewritten as

$$\lambda F(X) + (1-\lambda) F(Y) + \epsilon \geq \sum_{i=1}^{2\tau+2} \widetilde{\lambda}_i f(C_i) \tag{5.18}$$

with $\widetilde{\lambda} \in \Lambda_{2\tau+2}$ and

$$\sum_{i=1}^{2\tau+2} \widetilde{\lambda}_i T(C_i) = \lambda X + (1-\lambda) Y. \tag{5.19}$$

Taking the infimum in the right hand side of (5.18) over all $\widetilde{\lambda}_i$, C_i satisfying (5.19), using (5.10) and Step 2 we have

$$\lambda F(X) + (1-\lambda) F(Y) + \epsilon \geq F(\lambda X + (1-\lambda) Y);$$

$\epsilon > 0$ being arbitrary, we have indeed established the convexity of F.

Step 4. It now remains to show (5.5), i.e.

$$f(\xi) = F(T(\xi))$$

where F satisfies (5.4), namely

$$F(X) = \inf\{ \textstyle\sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) : \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = X \}.$$

We have just shown that F is convex. Choosing $X = T(\xi)$ we have from (5.3), (5.2) and (5.11) that the infimum in (5.4) is attained precisely by $f(\xi)$, hence (5.5) holds. The fact that F is the supremum over all convex functions G satisfying

$$f(\xi) = G(T(\xi)) \text{ for every } \xi \in \mathbb{R}^{N\times n},$$

follows at once from (5.4). This concludes Part 2.

Parts 3 and 4. **(i) $\Rightarrow$ (iii)**. Since f is polyconvex and finite we may use Parts 1 and 2 to find $F : \mathbb{R}^\tau \to \mathbb{R}$ convex and finite satisfying (see (5.4))

$$\begin{cases} f(\xi) = F(T(\xi)) \\ F(X) := \inf\{ \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) : \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = X \}. \end{cases}$$

Since F is convex and finite, it is continuous and therefore (see Corollary 2.51 of Chapter 2), for each $X \in \mathbb{R}^\tau$, there exists $\gamma(X) \in \mathbb{R}^\tau$ such that

$$F(Y) \geq F(X) + \langle \gamma(X); Y - X \rangle$$

for all $Y \in \mathbb{R}^\tau$. Choosing $Y = T(\eta)$, $X = T(\xi)$, $\beta(\xi) = \gamma(T(\xi))$, we get (5.6), namely

$$f(\eta) \geq f(\xi) + \langle \beta(\xi); T(\eta) - T(\xi) \rangle .$$

(iii) $\Rightarrow$ (i). We define h as in (5.7), namely

$$h(X) := \sup_{\xi \in \mathbb{R}^{N\times n}} \{ \langle \beta(\xi); X - T(\xi) \rangle + f(\xi) \} .$$

The function h, being a supremum of affine functions, is convex. If $X = T(\eta)$ then (5.6) ensures that the supremum in (5.7) is attained by $f(\eta)$ and therefore we have

$$f(\eta) = h(T(\eta))$$

as claimed. Moreover, h takes only finite values, since by Part 2 we have $h \leq F$, where F is as in (5.4). ■

We now obtain as a corollary that a polyconvex function with subquadratic growth must be convex. This is in striking contrast with quasiconvex and rank one convex functions as was established by Sverak [549] (see Theorem 5.54) and later by Gangbo [300] in an indirect way; see also Section 5.3.10. We also prove that a polyconvex function cannot have an arbitrary bound from below, contrary to quasiconvex and rank one convex functions (see Section 5.3.8).

Corollary 5.9 *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be polyconvex.*

(i) If there exist $\alpha \geq 0$ and $0 \leq p < 2$ such that

$$f(\xi) \leq \alpha (1 + |\xi|^p) \text{ for every } \xi \in \mathbb{R}^{N\times n},$$

then f is convex.

(ii) There exists $\gamma \geq 0$ such that

$$f(\xi) \geq -\gamma (1 + |\xi|^{n\wedge N}) \text{ for every } \xi \in \mathbb{R}^{N\times n}.$$

Proof. **(i)** Since f is polyconvex and finite, we can find, for every $\xi \in \mathbb{R}^{N\times n}$, according to Theorem 5.6 (iii), $\beta = \beta(\xi) \in \mathbb{R}^{\tau}$ such that

$$f(\eta) \geq f(\xi) + \langle \beta(\xi); T(\eta) - T(\xi) \rangle, \text{ for every } \eta \in \mathbb{R}^{N\times n}. \tag{5.20}$$

Using the growth condition on f, we find that

$$f(\xi) + \langle \beta(\xi); T(\eta) - T(\xi) \rangle \leq f(\eta) \leq \alpha\left(1 + |\eta|^p\right), \text{ for every } \eta \in \mathbb{R}^{N\times n}. \tag{5.21}$$

We can also rewrite it as

$$f(\xi) + \langle \beta(\xi); T(\eta) - T(\xi) \rangle = f(\xi) + \langle \beta_1(\xi); \eta - \xi \rangle + \sum_{s=2}^{n\wedge N} \langle \beta_s(\xi); \operatorname{adj}_s \eta - \operatorname{adj}_s \xi \rangle$$

and hence, for every $\eta \in \mathbb{R}^{N\times n}$,

$$g(\xi) + \langle \beta_1(\xi); \eta \rangle + \sum_{s=2}^{n\wedge N} \langle \beta_s(\xi); \operatorname{adj}_s \eta \rangle \leq \alpha\left(1 + |\eta|^p\right) \tag{5.22}$$

where

$$g(\xi) := f(\xi) - \langle \beta_1(\xi); \xi \rangle - \sum_{s=2}^{n\wedge N} \langle \beta_s(\xi); \operatorname{adj}_s \xi \rangle .$$

Replacing η by $t\eta$, with $t \in \mathbb{R}$, in (5.22) we get

$$g(\xi) + t\langle \beta_1(\xi); \eta \rangle + \sum_{s=2}^{n\wedge N} t^s \langle \beta_s(\xi); \operatorname{adj}_s \eta \rangle \leq \alpha\left(1 + |t|^p |\eta|^p\right).$$

Letting $t \to \infty$, using the fact that η is arbitrary and $p < 2$, we obtain that $\beta_s(\xi) = 0$ for every $s = 2, \cdots, n\wedge N$. Returning to (5.21) we find that, for every $\xi \in \mathbb{R}^{N\times n}$,

$$f(\xi) + \langle \beta_1(\xi); \eta - \xi \rangle \leq f(\eta), \text{ for every } \eta \in \mathbb{R}^{N\times n}$$

which implies that f is convex. Indeed we have that, for $\lambda \in [0,1]$,

$$\begin{array}{rcl} f(\xi) & \geq & f(\lambda\xi + (1-\lambda)\eta) + \langle \xi - (\lambda\xi + (1-\lambda)\eta); \beta_1(\lambda\xi + (1-\lambda)\eta) \rangle \\ f(\eta) & \geq & f(\lambda\xi + (1-\lambda)\eta) + \langle \eta - (\lambda\xi + (1-\lambda)\eta); \beta_1(\lambda\xi + (1-\lambda)\eta) \rangle . \end{array}$$

Multiplying the first equation by λ and the second by $(1-\lambda)$ and adding them we obtain the convexity of f.

(ii) The second part of the corollary follows at once from (5.20). More precisely, we have from (5.20) that, for every $\xi \in \mathbb{R}^{N\times n}$,

$$f(\xi) \geq f(0) + \langle \beta(0); T(\xi) \rangle \geq -\gamma\left(1 + |\xi|^{n\wedge N}\right)$$

for an appropriate $\gamma = \gamma(f(0), \beta(0))$. ■

Another direct consequence of Theorem 5.6 is that we can easily construct (see Dacorogna [177]) rank one convex functions that are not polyconvex. We will see more sophisticated examples in the next sections.

Let $N = n = 2$, $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^{2\times 2}$ and $\lambda_1, \lambda_2, \lambda_3 \in (0,1)$ be such that

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \sum_{i=1}^{3} \lambda_i \det \xi_i = \det\left(\sum_{i=1}^{3} \lambda_i \xi_i \right) \\ \det(\xi_1 - \xi_2) \neq 0, \ \det(\xi_1 - \xi_3) \neq 0, \ \det(\xi_2 - \xi_3) \neq 0. \end{cases}$$

For example we can choose $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$ and

$$\xi_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \ \xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \ \xi_3 = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}.$$

We then define $f : \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ as

$$f(\xi) := \begin{cases} 0 & \text{if } \xi = \xi_1, \xi_2, \xi_3 \\ +\infty & \text{otherwise.} \end{cases}$$

Proposition 5.10 *f is rank one convex but not polyconvex.*

Proof. *Part 1.* To show that f is rank one convex, we have to prove that

$$f(\lambda \xi + (1-\lambda)\eta) \le \lambda f(\xi) + (1-\lambda) f(\eta) \tag{5.23}$$

for every $\lambda \in [0,1]$ and every $\xi, \eta \in \mathbb{R}^{2\times 2}$ such that $\operatorname{rank}\{\xi - \eta\} \le 1$. Three cases can happen.

Case 1. $\xi \neq \xi_i$ or $\eta \neq \xi_i$ for every $i = 1, 2, 3$, then $f(\xi) = +\infty$ or $f(\eta) = +\infty$ and therefore (5.23) is trivially satisfied.

Case 2. $\xi = \xi_i$ and $\eta = \xi_j$ with $i \neq j$. This case is impossible, since by construction $\operatorname{rank}\{\xi_i - \xi_j\} = 2$ if $i \neq j$.

Case 3. $\xi = \eta = \xi_i$, then (5.23) is trivially satisfied.

Part 2. It now remains to show that f is not polyconvex. We proceed by contradiction. If f were polyconvex, we should have, using Theorem 5.6 and the construction of $(\lambda_i, \xi_i)_{1\le i\le 3}$, that

$$f\left(\sum_{i=1}^{3} \lambda_i \xi_i \right) \le \sum_{i=1}^{3} \lambda_i f(\xi_i).$$

This is however impossible since the left hand side takes the value $+\infty$ while the right hand side is 0. ■

5.2.4 Further properties of quasiconvex functions

We first show that if in the definition of quasiconvexity the inequality holds for one bounded open set, it holds for any such set.

Proposition 5.11 *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be Borel measurable and locally bounded. Let $D \subset \mathbb{R}^n$ be a bounded open set and let the inequality*

$$f(\xi) \operatorname{meas} D \leq \int_D f(\xi + \nabla\varphi(x))\, dx \tag{5.24}$$

hold for every $\xi \in \mathbb{R}^{N\times n}$ and for every $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^N)$. Then the inequality

$$f(\xi) \operatorname{meas} E \leq \int_E f(\xi + \nabla\psi(x))\, dx \tag{5.25}$$

holds for every bounded open set $E \subset \mathbb{R}^n$, for every $\xi \in \mathbb{R}^{N\times n}$ and for every $\psi \in W_0^{1,\infty}(E;\mathbb{R}^N)$.

Proof. We wish to show (5.25) assuming that (5.24) holds. So let $\psi \in W_0^{1,\infty}(E;\mathbb{R}^N)$ be given and choose first $a > 0$ sufficiently large so that

$$E \subset Q_a := (-a, a)^n$$

Define next

$$v(x) := \begin{cases} \psi(x) & \text{if } x \in E \\ 0 & \text{if } x \in Q_a - E \end{cases}$$

so that $v \in W_0^{1,\infty}(Q_a;\mathbb{R}^N)$.

Let then $x_0 \in D$ and choose ν sufficiently large so that

$$x_0 + \frac{1}{\nu} Q_a = x_0 + \left(-\frac{a}{\nu}, \frac{a}{\nu}\right)^n \subset D.$$

Define next

$$\varphi(x) := \begin{cases} \frac{1}{\nu} v(\nu(x - x_0)) & \text{if } x \in x_0 + \frac{1}{\nu} Q_a \\ 0 & \text{if } x \in D - [x_0 + \frac{1}{\nu} Q_a]. \end{cases}$$

Observe that $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^N)$ and

$$\begin{aligned}
&\int_D f(\xi + \nabla\varphi(x))\, dx \\
&= f(\xi) \operatorname{meas}(D - [x_0 + \frac{1}{\nu} Q_a]) + \int_{[x_0 + \frac{1}{\nu} Q_a]} f(\xi + \nabla v(\nu(x - x_0)))\, dx \\
&= f(\xi) [\operatorname{meas}(D) - \frac{\operatorname{meas} Q_a}{\nu^n}] + \frac{1}{\nu^n} \int_{Q_a} f(\xi + \nabla v(y))\, dy \\
&= f(\xi) [\operatorname{meas}(D) - \frac{\operatorname{meas} Q_a}{\nu^n} + \frac{\operatorname{meas}(Q_a - E)}{\nu^n}] + \frac{1}{\nu^n} \int_E f(\xi + \nabla\psi(y))\, dy.
\end{aligned}$$

Appealing to (5.24), we deduce that

$$f(\xi) \operatorname{meas}(D) \leq f(\xi) [\operatorname{meas}(D) - \frac{\operatorname{meas} E}{\nu^n}] + \frac{1}{\nu^n} \int_E f(\xi + \nabla\psi(y))\, dy$$

which is equivalent to the claim, namely (5.25). ■

In some examples (such as Sverak example in Section 5.3.7), it might be more convenient to replace the set of test functions $W_0^{1,\infty}$ by the set of periodic functions.

Notation 5.12 For $D := (0,1)^n$, we let

$$W_{per}^{1,\infty}\left(D;\mathbb{R}^N\right) := \left\{u \in W^{1,\infty}\left(\mathbb{R}^n;\mathbb{R}^N\right) : u\left(x+e_i\right) = u\left(x\right),\ i = 1,\cdots,n\right\}$$

where $\{e_1,\cdots,e_n\}$ is the standard orthonormal basis. ♢

We therefore have the following.

Proposition 5.13 *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be Borel measurable and locally bounded. The following two statements are then equivalent:*

(i) f is quasiconvex;

(ii) for $D = (0,1)^n$, the inequality

$$f\left(\xi\right) \leq \int_D f\left(\xi + \nabla\psi\left(x\right)\right) dx \tag{5.26}$$

holds for every $\xi \in \mathbb{R}^{N\times n}$ and for every $\psi \in W_{per}^{1,\infty}\left(D;\mathbb{R}^N\right)$.

Proof. **(ii) ⇒ (i).** This follows at once from Proposition 5.11 and the fact that

$$W_0^{1,\infty}\left(D;\mathbb{R}^N\right) \subset W_{per}^{1,\infty}\left(D;\mathbb{R}^N\right).$$

(i) ⇒ (ii). Let $\psi \in W_{per}^{1,\infty}\left(D;\mathbb{R}^N\right)$ and observe first that if $\nu \in \mathbb{N}$ and if

$$\psi_\nu\left(x\right) := \frac{1}{\nu}\psi\left(\nu x\right)$$

then, from the periodicity of ψ, we get

$$\int_D f\left(\xi + \nabla\psi_\nu\left(x\right)\right) dx = \frac{1}{\nu^n}\int_{\nu D} f\left(\xi + \nabla\psi\left(y\right)\right) dy = \int_D f\left(\xi + \nabla\psi\left(x\right)\right) dx. \tag{5.27}$$

Choose then $\eta_\nu \in C_0^\infty\left(D\right)$ such that $0 \leq \eta_\nu \leq 1$ in D,

$$\eta_\nu \equiv 1 \text{ on } D_\nu := \left(\frac{1}{\nu}, 1 - \frac{1}{\nu}\right)^n \quad \text{and} \quad \left\|\nabla\eta_\nu\right\|_{L^\infty} \leq c_1\nu$$

where $c_1 > 0$ is a constant independent of ν.

Let then

$$\varphi_\nu\left(x\right) := \eta_\nu\left(x\right)\psi_\nu\left(x\right)$$

and observe that $\varphi_\nu \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)$ and

$$\begin{aligned}\left\|\nabla\varphi_\nu - \nabla\psi_\nu\right\|_{L^\infty} &= \left\|\left(\eta_\nu - 1\right)\nabla\psi_\nu + \nabla\eta_\nu \otimes \psi_\nu\right\|_{L^\infty} \\ &\leq c_2\left\|\psi\right\|_{W^{1,\infty}}\end{aligned}$$

where $c_2 > 0$ is a constant, independent of ν. Since the function f is locally bounded we can find $c_3 > 0$, independent of ν, so that

$$\|f(\xi + \nabla\psi_\nu) - f(\xi + \nabla\varphi_\nu)\|_{L^\infty} \le c_3 \,.$$

Appealing to the quasiconvexity of f, to (5.27) and to the preceding observations, we find

$$\begin{aligned}
\int_D f(\xi + \nabla\psi(x))\,dx &= \int_D f(\xi + \nabla\varphi_\nu(x))\,dx \\
&\quad + \int_D [f(\xi + \nabla\psi_\nu(x)) - f(\xi + \nabla\varphi_\nu(x))]\,dx \\
&= \int_D f(\xi + \nabla\varphi_\nu(x))\,dx \\
&\quad + \int_{D-D_\nu} [f(\xi + \nabla\psi_\nu(x)) - f(\xi + \nabla\varphi_\nu(x))]\,dx \\
&\ge f(\xi) - c_3 \operatorname{meas}(D - D_\nu)\,.
\end{aligned}$$

Letting $\nu \to \infty$ we have indeed obtained (5.26), as wished. ■

5.2.5 Further properties of rank one convex functions

There is no known equivalent to Theorem 5.6 for rank one convex functions. We, nevertheless, give here a characterization of rank one convex functions that is in the same spirit as Part 1 of Theorem 5.6, but much weaker. It will turn out to be useful in Chapter 6.

To characterize rank one convex functions, we give a property of matrices $\xi_i \in \mathbb{R}^{N\times n}$ that will play the same role as (5.2) of Theorem 5.6 for polyconvex functions. We follow here the presentation of Dacorogna [176] and [179].

We also recall that for any integer I

$$\Lambda_I := \{\lambda = (\lambda_1, \cdots, \lambda_I) : \lambda_i \ge 0 \quad \text{and} \quad \textstyle\sum_{i=1}^I \lambda_i = 1\}.$$

Definition 5.14 *Let I be an integer and $\lambda \in \Lambda_I$. Let $\xi_i \in \mathbb{R}^{N\times n}$, $1 \le i \le I$. We say that $(\lambda_i, \xi_i)_{1\le i\le I}$ satisfy (H_I) if*

(i) when $I = 2$, then $\operatorname{rank}\{\xi_1 - \xi_2\} \le 1$;

(ii) when $I > 2$, then, up to a permutation, $\operatorname{rank}\{\xi_1 - \xi_2\} \le 1$ *and if, for every $2 \le i \le I-1$, we define*

$$\begin{cases} \mu_1 = \lambda_1 + \lambda_2 & \eta_1 = \dfrac{\lambda_1\xi_1 + \lambda_2\xi_2}{\lambda_1 + \lambda_2} \\ \mu_i = \lambda_{i+1} & \eta_i = \xi_{i+1} \end{cases}$$

then $(\mu_i, \eta_i)_{1\le i\le I-1}$ satisfy (H_{I-1}).

Example 5.15 (a) When $I=2$, $\lambda\in\Lambda_2$, then (λ_1,ξ_1), (λ_2,ξ_2) satisfy (H_2) if and only if

$$\operatorname{rank}\{\xi_1-\xi_2\}\le 1.$$

(b) When $I=3$, $\lambda\in\Lambda_3$, then $(\lambda_i,\xi_i)_{1\le i\le 3}$ satisfy (H_3) if, up to a permutation,

$$\begin{cases}\operatorname{rank}\{\xi_1-\xi_2\}\le 1\\ \operatorname{rank}\{\xi_3-\dfrac{\lambda_1\xi_1+\lambda_2\xi_2}{\lambda_1+\lambda_2}\}\le 1.\end{cases}$$

(c) When $I=4$, $\lambda\in\Lambda_4$, then $(\lambda_i,\xi_i)_{1\le i\le 4}$ satisfy (H_4) if, up to a permutation, either one of the conditions

$$\begin{cases}\operatorname{rank}\{\xi_1-\xi_2\}\le 1,\ \operatorname{rank}\{\xi_3-\dfrac{\lambda_1\xi_1+\lambda_2\xi_2}{\lambda_1+\lambda_2}\}\le 1\\ \operatorname{rank}\{\xi_4-\dfrac{\lambda_1\xi_1+\lambda_2\xi_2+\lambda_3\xi_3}{\lambda_1+\lambda_2+\lambda_3}\}\le 1\end{cases}$$

or

$$\begin{cases}\operatorname{rank}\{\xi_1-\xi_2\}\le 1,\ \operatorname{rank}\{\xi_3-\xi_4\}\le 1\\ \operatorname{rank}\{\dfrac{\lambda_1\xi_1+\lambda_2\xi_2}{\lambda_1+\lambda_2}-\dfrac{\lambda_3\xi_3+\lambda_4\xi_4}{\lambda_3+\lambda_4}\}\le 1\end{cases}$$

holds. ♢

Proposition 5.16 *Let $f:\mathbb{R}^{N\times n}\to\mathbb{R}\cup\{+\infty\}$, then the following two conditions are equivalent.*

(i) f is rank one convex.

(ii) The expression

$$f(\textstyle\sum_{i=1}^{I}\lambda_i\xi_i)\le\sum_{i=1}^{I}\lambda_i f(\xi_i) \tag{5.28}$$

holds whenever $(\lambda_i,\xi_i)_{1\le i\le I}$ satisfy (H_I).

Proof. **(ii) $\Rightarrow$ (i)**. This is trivial since it suffices to choose $I=2$ in (5.28).

(i) $\Rightarrow$ (ii). We establish (5.28) by induction. By definition of rank one convexity, (5.28) holds for $I=2$; assume therefore that the proposition is true for $I-1$. Observe that

$$\sum_{i=1}^{I}\lambda_i f(\xi_i)=(\lambda_1+\lambda_2)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}f(\xi_1)+\frac{\lambda_2}{\lambda_1+\lambda_2}f(\xi_2)\right)+\sum_{i=3}^{I}\lambda_i f(\xi_i).$$

If we now use the rank one convexity of f and the hypothesis (H_I) we get

$$(\lambda_1+\lambda_2)\,f\left(\frac{\lambda_1\xi_1+\lambda_2\xi_2}{\lambda_1+\lambda_2}\right)+\sum_{i=3}^{I}\lambda_i f(\xi_i)\le\sum_{i=1}^{I}\lambda_i f(\xi_i).$$

Using again the rank one convexity of f, hypothesis (H_I) and the hypothesis of induction, we have indeed established (5.28). ■

The above result is much weaker than Theorem 5.6 in the sense that one cannot fix an upper bound on I. Two simple examples show that the situation is intrinsically more complicated for rank one convex functions.. The first one has been established in Dacorogna [176], [179].

Example 5.17 Let $N = n = 2$,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & -2 \\ 1/2 & 0 \end{pmatrix},\ D = \begin{pmatrix} -1/4 & 4 \\ 0 & 4 \end{pmatrix},$$

and

$$\begin{cases} \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1/5 \\ \xi_1 = A,\ \xi_2 = B,\ \xi_3 = C,\ \xi_4 = D,\ \xi_5 = A. \end{cases}$$

It is then easy to see that $(\lambda_i, \xi_i)_{1 \le i \le 5}$ satisfy (H_5) since

$$\begin{cases} \det(\xi_1 - \xi_2) = 0 \\ \det\{\xi_3 - \dfrac{\lambda_1\xi_1 + \lambda_2\xi_2}{\lambda_1 + \lambda_2}\} = 0 \\ \det\{\xi_4 - \dfrac{\lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3}{\lambda_1 + \lambda_2 + \lambda_3}\} = 0 \\ \det\{\xi_5 - \dfrac{\lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3 + \lambda_4\xi_4}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}\} = 0. \end{cases}$$

However, if we combine together ξ_1 and ξ_5 and if we consider

$$\begin{cases} \mu_1 = \lambda_1 + \lambda_5 = 2/5,\ \mu_2 = \mu_3 = \mu_4 = 1/5 \\ \eta_1 = A,\ \eta_2 = B,\ \eta_3 = C,\ \eta_4 = D \end{cases}$$

then it is easy to see that $(\mu_i, \eta_i)_{1 \le i \le 4}$ do not satisfy (H_4). In other words, if we use Proposition 5.16, we have the surprising result that if $f : \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ is rank one convex then

$$f(\frac{2}{5}A + \frac{1}{5}B + \frac{1}{5}C + \frac{1}{5}D) \le \frac{2}{5}f(A) + \frac{1}{5}f(B) + \frac{1}{5}f(C) + \frac{1}{5}f(D)$$

i.e.

$$f\left(\sum_{i=1}^4 \mu_i\eta_i\right) \le \sum_{i=1}^4 \mu_i f(\eta_i) \tag{5.29}$$

even though $(\mu_i, \eta_i)_{1 \le i \le 4}$ do not satisfy (H_4). In order to show (5.28), we have to write the inequality (with $(\lambda_i, \xi_i)_{1 \le i \le 5}$) as

$$\begin{aligned} & f(\frac{1}{5}A + \frac{1}{5}B + \frac{1}{5}C + \frac{1}{5}D + \frac{1}{5}A) \\ \le\ & \frac{1}{5}f(A) + \frac{1}{5}f(B) + \frac{1}{5}f(C) + \frac{1}{5}f(D) + \frac{1}{5}f(A). \end{aligned} \qquad \diamondsuit$$

The next example is even more striking and has been given by Casadio Tarabusi [127]. A similar example has also been found by Aumann-Hart [50] and Tartar [571].

Example 5.18 Let $N = n = 2$ and (see Figure 5.1)

$$\begin{cases} \xi_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \; \xi_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \xi_3 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; \xi_4 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \\ \lambda_1 = \dfrac{8}{15}, \; \lambda_2 = \dfrac{4}{15}, \; \lambda_3 = \dfrac{2}{15}, \; \lambda_4 = \dfrac{1}{15}. \end{cases}$$

Observe that $\lambda \in \Lambda_4$ and

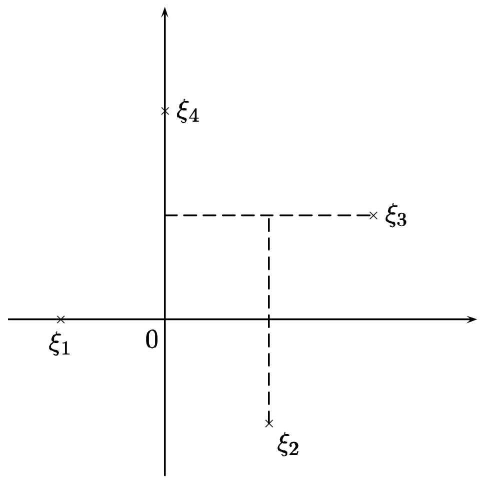

Figure 5.1: The matrices $\xi_1, \xi_2, \xi_3, \xi_4$

$$\operatorname{rank}\{\xi_i - \xi_j\} = 2, \text{ if } i \neq j.$$

Let

$$\begin{cases} \eta_1 = \xi_1, \; \eta_2 = \xi_2, \; \eta_3 = \xi_3, \; \eta_4 = \xi_4, \; \eta_5 = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \displaystyle\sum_{i=1}^{4} \lambda_i \xi_i \\ \mu_1 = \dfrac{8}{16}, \; \mu_2 = \dfrac{4}{16}, \; \mu_3 = \dfrac{2}{16}, \; \mu_4 = \dfrac{1}{16}, \; \mu_5 = \dfrac{1}{16}. \end{cases}$$

Observe that $(\mu_i, \eta_i)_{1 \leq i \leq 5}$ satisfy (H_5), since

$$\begin{cases} \det(\eta_4 - \eta_5) = 0 \\ \det\{\eta_3 - \dfrac{\mu_4\eta_4 + \mu_5\eta_5}{\mu_4 + \mu_5}\} = 0 \\ \det\{\eta_2 - \dfrac{\mu_3\eta_3 + \mu_4\eta_4 + \mu_5\eta_5}{\mu_3 + \mu_4 + \mu_5}\} = 0 \\ \det\{\eta_1 - \dfrac{\mu_2\eta_2 + \mu_3\eta_3 + \mu_4\eta_4 + \mu_5\eta_5}{\mu_2 + \mu_3 + \mu_4 + \mu_5}\} = 0. \end{cases}$$

Therefore, using Proposition 5.16, we obtain for every $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$

$$f(0) = f(\textstyle\sum_{i=1}^{5} \mu_i \eta_i) \le \sum_{i=1}^{5} \mu_i f(\eta_i);$$

which means that

$$16 f(0) \le 8 f(\xi_1) + 4 f(\xi_2) + 2 f(\xi_3) + f(\xi_4) + f(\eta_5). \tag{5.30}$$

Noting that $\eta_5 = 0$ and dividing the above inequality by 15, we have that

$$f(0) = f(\textstyle\sum_{i=1}^{4} \lambda_i \xi_i) \le \sum_{i=1}^{4} \lambda_i f(\xi_i). \tag{5.31}$$

We have therefore obtained the inequality (5.31) of rank one convexity even though none of the $\xi_i - \xi_j$ differs by rank one. ♢

Remark 5.19 An interesting point should be emphasized if one compares the two examples, namely the inequalities (5.29) and (5.31) of rank one convexity. The first one deals with any rank one convex function $f : \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$, while in the second one we have to restrict our analysis to functions $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ (i.e. that are finite everywhere), since we subtract $f(0)$ from both sides in the inequality (5.30).

Indeed, the inequality (5.31) does not hold if we allow the function f to take the value $+\infty$ as the following example shows. Let

$$f(\xi) = \chi_{\{\xi_1,\xi_2,\xi_3,\xi_4\}}(\xi) = \begin{cases} 0 & \text{if } \xi \in \{\xi_1, \xi_2, \xi_3, \xi_4\} \\ +\infty & \text{otherwise.} \end{cases}$$

This function is clearly rank one convex, since $\operatorname{rank}\{\xi_i - \xi_j\} = 2$ for $i \ne j$. Therefore

$$\textstyle\sum_{i=1}^{4} \lambda_i f(\xi_i) = 0 < f(\sum_{i=1}^{4} \lambda_i \xi_i) = f(0) = +\infty. \qquad ♢$$

5.3 Examples

We have seen in Section 5.2 the definitions and the relations between the notions of convexity, polyconvexity, quasiconvexity and rank one convexity. We now discuss several examples, the most important being the following.

i) We start in Section 5.3.1 with the complete characterization of the *quasi-affine* functions (i.e. the functions f such that f and $-f$ are quasiconvex) by showing that they are linear combinations of *minors* of the matrix ∇u.

ii) In Section 5.3.2 we study the case of *quadratic* functions f. The main result being that rank one convexity and quasiconvexity are equivalent. Note that the quadratic case is important in the sense that it leads to associated *linear* Euler-Lagrange equations. Therefore, in the linear case, the ellipticity of the Euler-Lagrange equations corresponds exactly to the quasiconvexity of the integrand and thus, anticipating the results of Chapter 8, to the weak lower semicontinuity of the associated variational problem.

iii) The third important result is considered in Sections 5.3.3 and 5.3.4. We study functions invariant under rotations, notably those depending on singular values. We characterize their convexity and polyconvexity.

iv) In Section 5.3.7, we present the celebrated example of Sverak that provides, in dimensions $N \geq 3$ and $n \geq 2$, an example of a rank one convex function that is not quasiconvex.

v) In Section 5.3.8, we consider the example of Alibert-Dacorogna-Marcellini, which is valid when $N = n = 2$. It characterizes for a homogeneous polynomial of degree four the different notions of convexity encountered in Section 5.2.

5.3.1 Quasiaffine functions

We start with a result established by Ball [53], that is an extension of results of Edelen [255], Ericksen [265] and Rund [520]. It characterizes completely the quasiaffine functions (see also Anderson-Duchamp [27], Ball-Curie-Olver [59], Sivaloganathan [541] and Vasilenko [588]). We follow here the proof of Dacorogna [179].

Theorem 5.20 *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$. The following conditions are equivalent.*

(i) f is quasiaffine.

(ii) f is rank one affine, meaning that f and $-f$ are rank one convex, i.e.

$$f\left(\lambda\xi + (1-\lambda)\eta\right) = \lambda f\left(\xi\right) + (1-\lambda) f\left(\eta\right)$$

for every $\lambda \in [0,1]$, $\xi, \eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}\{\xi - \eta\} \leq 1$.

(ii') The function $f \in C^1$ and for every $\xi \in \mathbb{R}^{N\times n}$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$,

$$f\left(\xi + a \otimes b\right) = f\left(\xi\right) + \left\langle \nabla f\left(\xi\right); a \otimes b\right\rangle,$$

where $\langle \cdot ; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{N\times n}$.

(iii) f is polyaffine, i.e. f and $-f$ are polyconvex.

(iii') There exists $\beta \in \mathbb{R}^{\tau(n,N)}$ such that

$$f\left(\xi\right) = f\left(0\right) + \left\langle \beta; T\left(\xi\right)\right\rangle$$

for every $\xi \in \mathbb{R}^{N\times n}$ and where $\langle \cdot ; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{\tau(n,N)}$ and T is as in Definition 5.1.

Example 5.21 (i) If $N = n = 2$, then the theorem asserts that the only quasiaffine functions are of the type

$$f\left(\xi\right) = f\left(0\right) + \left\langle \beta; \xi\right\rangle + \gamma \det \xi.$$

In particular the only fully non-linear quasiaffine function is $\det \xi$.

(ii) More generally if $n, N > 1$, then the only non-linear quasiaffine functions are linear combinations of the $s \times s$ minors of the matrix $\xi \in \mathbb{R}^{N\times n}$, where $2 \leq s \leq n \wedge N = \min\{n, N\}$. ♢

Before proceeding with the proof of the theorem, we mention two corollaries. The first one is a straightforward combination of Theorems 5.20 and 8.35.

Corollary 5.22 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be quasiaffine. Let $v \in u + W_0^{1,p}(\Omega)$, with $p \geq n \wedge N$, then*

$$\int_\Omega f(\nabla u(x))\, dx = \int_\Omega f(\nabla v(x))\, dx.$$

The second one was established by Dacorogna-Ribeiro [212] and we will use it in Theorems 6.24 and 7.47.

Corollary 5.23 *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be quasiaffine.*

(i) If f is locally constant, then it is constant.

(ii) If f has a local extremum, then it is constant.

Proof. (Corollary 5.23). **(i)** We show that if f is locally constant around a point $\xi \in \mathbb{R}^{N\times n}$ then f is constant everywhere, establishing the result. So assume that there exists $\epsilon > 0$ such that

$$f(\xi + v) = f(\xi),\ \forall\, v \in \mathbb{R}^{N\times n} \text{ with } \left|v_j^i\right| \leq \epsilon \tag{5.32}$$

and let us show that

$$f(\xi + w) = f(\xi),\ \forall\, w \in \mathbb{R}^{N\times n}. \tag{5.33}$$

The procedure consists in working component by component. We start to show that for every $w_1^1 \in \mathbb{R}$ and $\left|v_j^i\right| \leq \epsilon$ we have (denoting by $\left\{e^1, \cdots, e^N\right\}$ and $\{e_1, \cdots, e_n\}$ the standard basis of $\mathbb{R}^N$ and $\mathbb{R}^n$ respectively)

$$f(\xi + w_1^1 e^1 \otimes e_1 + \textstyle\sum_{(i,j)\neq(1,1)} v_j^i e^i \otimes e_j) = f(\xi + w_1^1 e^1 \otimes e_1) = f(\xi). \tag{5.34}$$

Indeed if $\left|w_1^1\right| \leq \epsilon$ this is nothing else than (5.32) so we may assume that $\left|w_1^1\right| > \epsilon$ and use the fact that f is quasiaffine, to deduce that

$$\begin{aligned} & f(\xi + \frac{\epsilon w_1^1}{|w_1^1|} e^1 \otimes e_1 + \textstyle\sum_{(i,j)\neq(1,1)} v_j^i e^i \otimes e_j) \\ & \quad = \frac{\epsilon}{|w_1^1|} f(\xi + w_1^1 e^1 \otimes e_1 + \textstyle\sum_{(i,j)\neq(1,1)} v_j^i e^i \otimes e_j) \\ & \quad +(1 - \frac{\epsilon}{|w_1^1|}) f(\xi + \textstyle\sum_{(i,j)\neq(1,1)} v_j^i e^i \otimes e_j). \end{aligned}$$

Therefore appealing to (5.32) and to the preceding identity we have indeed established (5.34). Proceeding iteratively in a similar manner with the other components $(w_2^1, w_3^1, \cdots)$ we have indeed obtained (5.33) and thus the proof of (i) is complete.

(ii) We now show that if ξ is a local extremum point of f, then f is constant in a neighborhood of ξ and thus applying (i) we have the result.

Assume that ξ is a local minimum point of f (the case of a local maximizer being handled similarly). We therefore have that there exists $\epsilon > 0$ so that

$$f(\xi) \leq f(\xi + v), \text{ for every } v \in \mathbb{R}^{N \times n} \text{ so that } \left| v_j^i \right| \leq \epsilon. \tag{5.35}$$

Let us show that this implies that

$$f(\xi) = f(\xi + v), \text{ for every } v \in \mathbb{R}^{N \times n} \text{ so that } \left| v_j^i \right| \leq \epsilon. \tag{5.36}$$

We write

$$v = \sum_{i=1}^{N} \sum_{j=1}^{n} v_j^i e^i \otimes e_j$$

and observe that, since f is quasiaffine,

$$f(\xi) = \frac{1}{2} f(\xi + v_1^1 e^1 \otimes e_1) + \frac{1}{2} f(\xi - v_1^1 e^1 \otimes e_1)$$

and since (5.35) is satisfied we deduce that

$$f(\xi \pm v_1^1 e^1 \otimes e_1) = f(\xi), \ \left| v_1^1 \right| \leq \epsilon. \tag{5.37}$$

We next write, using again the fact that f is quasiaffine,

$$f(\xi + v_1^1 e^1 \otimes e_1) = \frac{1}{2} f(\xi + v_1^1 e^1 \otimes e_1 + v_2^1 e^1 \otimes e_2) + \frac{1}{2} f(\xi + v_1^1 e^1 \otimes e_1 - v_2^1 e^1 \otimes e_2)$$

and since (5.35) and (5.37) hold, we deduce that

$$f(\xi + v_1^1 e^1 \otimes e_1 \pm v_2^1 e^1 \otimes e_2) = f(\xi + v_1^1 e^1 \otimes e_1) = f(\xi), \ \left| v_1^1 \right|, \left| v_2^1 \right| \leq \epsilon.$$

Iterating the procedure we have indeed established (5.36). Appealing to (i), we have therefore proved the corollary. ■

We should mention that some of the results of Theorem 5.20 will be proved in a more straightforward way in Sections 5.4 and 8.5. Indeed, the implication *(iii')* $\Rightarrow$ *(ii)* can also be found in Proposition 5.65, while the implication *(iii')* $\Rightarrow$ *(i)* is also established in Theorem 8.35.

We now turn to the proof of Theorem 5.20.

Proof. **(i) $\Rightarrow$ (ii).** This implication follows immediately from Theorem 5.3.

(ii') $\Rightarrow$ (ii). This case is trivial.

(ii) $\Rightarrow$ (ii'). We fix $\xi \in \mathbb{R}^{N \times n}$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$ and let for $t \in [0, 1]$

$$\varphi(t) := f(\xi + t a \otimes b).$$

Since f is rank one affine then φ is affine and thus $\varphi \in C^1$ and

$$\varphi(t) = \varphi(0) + t \varphi'(0).$$

Since $\varphi \in C^1$, then, obviously, $f \in C^1$ and the result immediately follows from the above identity.

(iii') $\Rightarrow$ (iii). This implication follows from the definition of polyconvexity.

(iii) $\Rightarrow$ (i). The result follows from Theorem 5.3.

(ii') $\Rightarrow$ (iii'). This is the only non trivial implication. So recall that

$$\xi = \begin{pmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^N & \cdots & \xi_n^N \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} = (\xi_1, \cdots, \xi_n) .$$

Assume also that f is such that

$$f(\xi + a \otimes b) - f(\xi) = \langle \nabla f(\xi) ; a \otimes b \rangle , \tag{5.38}$$

for every $\xi \in \mathbb{R}^{N \times n}$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$. We wish to show that there exists $\beta \in \mathbb{R}^{\tau(n,N)}$ such that

$$f(\xi) - f(0) = \langle \beta ; T(\xi) \rangle , \text{ for every } \xi \in \mathbb{R}^{N \times n}. \tag{5.39}$$

In the sequel we assume that $n \geq N$, otherwise we reverse the roles of n and N. We then proceed by induction on N.

Step 1. $N = 1$. Since $N = 1$, (5.38) can be read as

$$f(\xi + \eta) - f(\xi) = \langle \nabla f(\xi) ; \eta \rangle$$

for every $\xi, \eta \in \mathbb{R}^n$. It is then trivial to see that the above identity implies that f is affine and therefore if we choose $\beta = \nabla f(0)$, we have immediately (5.39).

Step 2. $N = 2$. This step is unnecessary but we prove it for the sake of illustration. Let

$$\xi = \begin{pmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \xi_1^2 & \cdots & \xi_n^2 \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = (\xi_1, \cdots, \xi_n)$$

and for $a \in \mathbb{R}^2$, $b \in \mathbb{R}^n$

$$a \otimes b = \begin{pmatrix} a^1 b \\ a^2 b \end{pmatrix} = \begin{pmatrix} a^1 b_1 & \cdots & a^1 b_n \\ a^2 b_1 & \cdots & a^2 b_n \end{pmatrix} .$$

We want to show that if f is rank one affine, i.e.

$$f(\xi + a \otimes b) - f(\xi) = \langle \nabla f(\xi) ; a \otimes b \rangle$$

then there exists $\beta \in \mathbb{R}^{\tau(n,2)}$ such that

$$f(\xi) = f(0) + \langle \beta ; T(\xi) \rangle$$

where

$$T(\xi) = (\xi, \operatorname{adj}_2 \xi) \in \mathbb{R}^{2\times n} \times \mathbb{R}^{\binom{n}{2}} = \mathbb{R}^{\tau(n,2)}.$$

For the notations concerning $\operatorname{adj}_2 \xi$, see Section 5.4. But note that, up to a sign and the ordering, an element of the matrix $\operatorname{adj}_2 \xi$ is essentially $\det(\xi_k, \xi_l)$, $1 \le k < l \le n$. We then fix ξ^2 and choose $a = e^1 = (1,0)$ in (5.38) and define

$$g\left(\xi^1\right) := f\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix}.$$

Thus the function

$$t \to g\left(\xi^1 + tb\right) = f\begin{pmatrix}\xi^1 + tb\\ \xi^2\end{pmatrix}$$

is affine and we may then use Step 1 to find $\gamma = \gamma\left(\xi^2\right) \in \mathbb{R}^n$ such that

$$g\left(\xi^1\right) = g(0) + \left\langle \gamma\left(\xi^2\right); \xi^1\right\rangle = f\begin{pmatrix}0\\ \xi^2\end{pmatrix} + \left\langle \gamma\left(\xi^2\right); \xi^1\right\rangle.$$

Repeating the argument when $\xi^1 = 0$ for $f\binom{0}{\xi^2}$, we have

$$f\begin{pmatrix}0\\ \xi^2\end{pmatrix} = f(0) + \left\langle \beta^2; \xi^2\right\rangle.$$

Combining the above two identities, we obtain

$$f\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix} = f(0) + \left\langle \beta^2; \xi^2\right\rangle + \left\langle \gamma\left(\xi^2\right); \xi^1\right\rangle. \tag{5.40}$$

Since f is rank one affine, it is affine (when ξ^1 is fixed) with respect to ξ^2 and therefore $\gamma\left(\xi^2\right) = \left(\gamma_1\left(\xi^2\right), \cdots, \gamma_n\left(\xi^2\right)\right)$ is affine and hence there exist $\beta^1 = \left(\beta^1_1, \cdots, \beta^1_n\right) \in \mathbb{R}^n$, $\delta_1, \cdots, \delta_n \in \mathbb{R}^n$ such that

$$\gamma_l\left(\xi^2\right) = \beta^1_l + \left\langle \delta_l; \xi^2\right\rangle,\ l = 1, \cdots, n.$$

Returning to (5.40), we therefore get

$$f\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix} = f(0) + \left\langle \beta^1; \xi^1\right\rangle + \left\langle \beta^2; \xi^2\right\rangle + \sum_{l=1}^{n} \xi^1_l \left\langle \delta_l; \xi^2\right\rangle$$

or in other words

$$f\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix} = f(0) + \left\langle \beta^1; \xi^1\right\rangle + \left\langle \beta^2; \xi^2\right\rangle + \sum_{l=1}^{n}\sum_{\alpha=1}^{n} \delta_{l\alpha}\xi^1_l\xi^2_\alpha. \tag{5.41}$$

Since f is rank one affine we have from (5.41) that if

$$h(\xi) := \sum_{l=1}^{n}\sum_{\alpha=1}^{n} \delta_{l\alpha}\xi^1_l\xi^2_\alpha$$

then h is rank one affine and therefore using Lemma 5.24 we must have

$$\delta_{l\alpha} = -\delta_{\alpha l}\,.$$

Thus there exists $\epsilon \in \mathbb{R}^{\binom{n}{2}}$ such that

$$h(\xi) = \sum_{1 \leq l < \alpha \leq n} \delta_{l\alpha} \left(\xi_l^1 \xi_\alpha^2 - \xi_\alpha^1 \xi_l^2\right) = \langle \epsilon; \operatorname{adj}_2 \xi \rangle\,.$$

Combining (5.41) with the above identity, we deduce (5.39) and this concludes Step 2.

Step N. We now proceed with the general case. Assume that we have proved the theorem for every $l < N$. Fixing $\xi^2, \cdots, \xi^N$ and using the fact that f is rank one affine, then f is affine in ξ^1, for $\xi^2, \cdots, \xi^N$ fixed. Therefore there exist

$$\psi \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} = (\psi_1, \cdots, \psi_n) \in \mathbb{R}^n \quad \text{and} \quad \chi \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \in \mathbb{R},$$

such that

$$f(\xi) = \langle \psi \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix}; \xi^1 \rangle + \chi \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix}. \tag{5.42}$$

Using the hypothesis of induction and proceeding as in Step 2 we find that

$$\begin{cases} \chi \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} = f(0) + \langle \beta^0;\, T \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \rangle \\ \psi_l \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} = \beta_l + \langle \gamma_l;\, T \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \rangle,\ l = 1, \cdots, n \end{cases} \tag{5.43}$$

for some $\beta^0, \gamma_1, \cdots, \gamma_n \in \mathbb{R}^{\tau(n,N-1)}$ and $\beta^1 = (\beta_1, \cdots, \beta_n) \in \mathbb{R}^n$. Combining (5.42) and (5.43) we have that

$$f(\xi) = f(0) + \langle \beta^0;\, T \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \rangle + \langle \beta^1; \xi^1 \rangle + \sum_{l=1}^{n} \xi_l^1 \langle \gamma_l;\, T \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \rangle$$

which can be rewritten as

$$\begin{aligned} f(\xi) = {} & f(0) + \left\langle \beta^0;\ T \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right\rangle + \left\langle \beta^1; \xi^1 \right\rangle \\ & + \sum_{s=1}^{N-1} \sum_{l=1}^{n} \sum_{\alpha=1}^{\binom{n}{s}} \sum_{i=1}^{\binom{N-1}{s}} \gamma_{l\alpha}^{is} \xi_l^1 \left(\operatorname{adj}_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\alpha^i . \end{aligned} \tag{5.44}$$

Letting

$$h(\xi) := \sum_{s=1}^{N-1} h_s(\xi) \quad \text{where} \quad h_s(\xi) := \sum_{l=1}^{n} \sum_{\alpha=1}^{\binom{n}{s}} \sum_{i=1}^{\binom{N-1}{s}} \gamma_{l\alpha}^{is} \xi_l^1 \left(\operatorname{adj}_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\alpha^i$$

we deduce from the fact that f is rank one affine and from (5.44) that h is rank one affine. Since h is rank one affine, we deduce that so is h_s. Indeed let us first show this for h_1. Write

$$h_1(\xi) = \sum_{i=2}^{N} h_1^i(\xi) \quad \text{where} \quad h_1^i(\xi) := \sum_{l=1}^{n} \sum_{\alpha=1}^{n} \gamma_{l\alpha}^{i1} \xi_l^1 \xi_\alpha^i .$$

By first choosing $\xi^3 = \cdots = \xi^N = 0$, we obtain that h_1^2 is rank one affine (since then $h = h_1^2$); iterating this process we find that all the h_1^i are rank one affine and thus h_1 is rank one affine. We then infer that so is $h - h_1$. With the same reasoning, we get that all the h_s, $1 \le s \le N-1$, are rank one affine.

We may then use Lemma 5.24 to deduce that there exist

$$\delta_\beta^{js} \in \mathbb{R},\ 1 \le s \le N-1,\ 1 \le \beta \le \tbinom{n}{s+1},\ 1 \le j \le \tbinom{N}{s+1}$$

such that

$$h(\xi) = \sum_{s=1}^{N-1} \sum_{\beta=1}^{\binom{n}{s+1}} \sum_{j=1}^{\binom{N}{s+1}} \delta_\beta^{js} \left(\operatorname{adj}_{s+1} \xi \right)_\beta^j .$$

Combining (5.44) and the above identity, we have indeed found $\beta \in \mathbb{R}^{\tau(n,N)}$ such that

$$f(\xi) = f(0) + \langle \beta; T(\xi) \rangle ,$$

which is the claimed result. ■

In the above proof we have used the following lemma.

Lemma 5.24 *Let $n \geq N$ and $\xi \in \mathbb{R}^{N\times n}$,*

$$\xi = \begin{pmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^N & \cdots & \xi_n^N \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} = (\xi_1, \cdots, \xi_n) .$$

For $1 \leq s \leq N-1$, let

$$g(\xi) := \sum_{l=1}^{n} \sum_{\alpha=1}^{\binom{n}{s}} \sum_{i=1}^{\binom{N-1}{s}} \gamma_{l\alpha}^{i} \xi_l^1 \left(\operatorname{adj}_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right)_{\alpha}^{i} .$$

If g is rank one affine, meaning that

$$g(\xi + a \otimes b) = g(\xi) + \langle \nabla g(\xi); a \otimes b \rangle ,$$

then there exist $\delta_\beta^j \in \mathbb{R}$, $1 \leq \beta \leq \binom{n}{s+1}$, $1 \leq j \leq \binom{N}{s+1}$ such that

$$g(\xi) = \sum_{\beta=1}^{\binom{n}{s+1}} \sum_{j=1}^{\binom{N}{s+1}} \delta_\beta^j \left(\operatorname{adj}_{s+1} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} \right)_{\beta}^{j} = \left\langle \delta; \operatorname{adj}_{s+1} \xi \right\rangle .$$

Proof. *Part 1.* We start, for the sake of illustration, with the case $N = 2$, therefore $s = 1$ and

$$g(\xi) = \sum_{l=1}^{n} \sum_{\alpha=1}^{n} \gamma_{l\alpha} \xi_l^1 \xi_\alpha^2 .$$

Since g is rank one affine and quadratic then

$$\frac{d^2}{dt^2} g(\xi + ta \otimes b) = g(a \otimes b) = \sum_{l,\alpha=1}^{n} \gamma_{l\alpha} a^1 a^2 b_l b_\alpha = 0,$$

for every $a = \left(a^1, a^2\right) \in \mathbb{R}^2$, $b = (b_1, \cdots, b_n) \in \mathbb{R}^n$. We therefore immediately deduce that $\gamma_{l\alpha} = -\gamma_{\alpha l}$ and hence

$$\begin{aligned} g(\xi) &= \sum_{1 \leq l < \alpha \leq n} \gamma_{l\alpha} \left(\xi_l^1 \xi_\alpha^2 - \xi_\alpha^1 \xi_l^2 \right) = \sum_{1 \leq l < \alpha \leq n} \gamma_{l\alpha} \det \begin{pmatrix} \xi_l^1 & \xi_\alpha^1 \\ \xi_l^2 & \xi_\alpha^2 \end{pmatrix} \\ &= \sum_{\beta=1}^{\binom{n}{2}} \delta_\beta \left(\operatorname{adj}_2 \xi \right)_\beta = \langle \delta; \operatorname{adj}_2 \xi \rangle , \end{aligned}$$

since $\operatorname{adj}_2 \xi$ is a vector of $\mathbb{R}^{\binom{n}{2}}$ composed of elements of the form $\det(\xi_l, \xi_\alpha)$, $1 \leq l < \alpha \leq n$ and therefore δ_β is essentially $\gamma_{l\alpha}$ with the appropriate sign.

Part 2. We now proceed with the general case. Let

$$g(\xi) = \sum_{i=1}^{\binom{N-1}{s}} g^i(\xi) \quad \text{where} \quad g^i(\xi) := \sum_{l=1}^{n} \sum_{\alpha=1}^{\binom{n}{s}} \gamma_{l\alpha}^i \xi_l^1 \left(\operatorname{adj}_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\alpha^i .$$

As in the theorem, it is easy to see that g is rank one affine if and only if g^i is rank one affine. Therefore it is enough to prove, the stronger version, that for every i, $1 \leq i \leq \binom{N-1}{s}$ there exists j, $1 \leq j \leq \binom{N}{s+1}$, and $\delta_\beta^j \in \mathbb{R}$, so that if

$$g^i(\xi) := \sum_{l=1}^{n} \sum_{\alpha=1}^{\binom{n}{s}} \gamma_{l\alpha}^i \xi_l^1 \left(\operatorname{adj}_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\alpha^i$$

is rank one affine, then

$$g^i(\xi) = \sum_{\beta=1}^{\binom{n}{s+1}} \delta_\beta^j \left(\operatorname{adj}_{s+1} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\beta^j .$$

It is clear that the above identities imply the lemma. We should draw the attention that all the δ_β^j corresponding to

$$\left(\operatorname{adj}_{s+1} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\beta^j$$

which do not contain the row ξ^1 are chosen to be 0.

For notational convenience, we show the above result only when $i = \binom{N-1}{s}$, the general case being handled similarly. So let $i = \binom{N-1}{s}$, which corresponds to $j = \binom{N}{s+1}$ and we therefore have

$$\left(\operatorname{adj}_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\alpha^i = (-1)^{i+1} \left(\operatorname{adj}_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^{s+1} \end{pmatrix} \right)_\alpha , \; 1 \leq \alpha \leq \binom{n}{s} .$$

We also, from now on, drop the indices i and j and write, to simplify the notations, $\gamma_{l\alpha}^i = (-1)^{i+1} \gamma_{l\alpha}$ in this case. We therefore have to show that if

$$g(\xi) := \sum_{l=1}^{n} \sum_{\alpha=1}^{\binom{n}{s}} \gamma_{l\alpha} \xi_l^1 \left(\operatorname{adj}_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^{s+1} \end{pmatrix} \right)_\alpha$$

is rank one affine then there exists $\delta_\beta \in \mathbb{R}$, $1 \leq \beta \leq \binom{n}{s+1}$, such that

$$g(\xi) = \sum_{\beta=1}^{\binom{n}{s+1}} \delta_\beta \left(\operatorname{adj}_{s+1} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^{s+1} \end{pmatrix} \right)_\beta . \tag{5.45}$$

Recall that for given α, $1 \leq \alpha \leq \binom{n}{s}$, there exists a unique s-tuple $(\lambda_1, \lambda_2, \cdots, \lambda_s)$ with $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_s \leq n$, such that

$$\left(\operatorname{adj}_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^{s+1} \end{pmatrix} \right)_\alpha = (-1)^{1+\alpha} \det \begin{pmatrix} \xi^2_{\lambda_1} & \cdots & \xi^2_{\lambda_s} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_s} \end{pmatrix} . \tag{5.46}$$

We now fix an arbitrary $(s+1)$-tuple $(\lambda_1, \cdots, \lambda_{s+1})$, where $1 \leq \lambda_1 < \cdots < \lambda_{s+1} \leq n$ and we denote by β the associate integer (as in (5.46)), more precisely

$$\left(\operatorname{adj}_s \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^{s+1} \end{pmatrix} \right)_\beta = (-1)^{1+\beta} \det \begin{pmatrix} \xi^1_{\lambda_1} & \cdots & \xi^1_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{pmatrix} .$$

Note that there are $\binom{n}{s+1}$ such $(s+1)$-tuple. Denote by α_1 the integer corresponding (as in (5.46)) to the s-tuple $(\lambda_1, \cdots, \lambda_s)$, by α_k the integer corresponding to the s-tuple $(\lambda_1, \cdots, \lambda_{k-1}, \lambda_{k+1}, \cdots ., \lambda_{s+1})$, $2 \leq k \leq s$ and by α_{s+1} the integer corresponding to the s-tuple $(\lambda_2, \cdots, \lambda_{s+1})$. Finally let

$$\begin{aligned} X_\beta(\xi) := & \sum_{l_1=1}^{n} (-1)^{1+\alpha_1} \gamma_{l_1 \alpha_1} \xi^1_{l_1} \det \begin{pmatrix} \xi^2_{\lambda_1} & \cdots & \xi^2_{\lambda_s} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_s} \end{pmatrix} \\ & + \sum_{k=2}^{s} \sum_{l_k=1}^{n} (-1)^{1+\alpha_k} \gamma_{l_k \alpha_k} \xi^1_{l_k} \\ & \det \begin{pmatrix} \xi^2_{\lambda_1} & \cdots & \xi^2_{\lambda_{k-1}} & \xi^2_{\lambda_{k+1}} & \cdots & \xi^2_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_{k-1}} & \xi^{s+1}_{\lambda_{k+1}} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{pmatrix} \\ & + \sum_{l_{s+1}=1}^{n} (-1)^{1+\alpha_{s+1}} \gamma_{l_{s+1} \alpha_{s+1}} \xi^1_{l_{s+1}} \det \begin{pmatrix} \xi^2_{\lambda_2} & \cdots & \xi^2_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_2} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{pmatrix} . \end{aligned} \tag{5.47}$$

We then obviously have that

$$g(\xi) = \sum_{\beta=1}^{\binom{n}{s+1}} X_\beta(\xi).$$

Since g is rank one affine, then so is X_β . Therefore in order to show (5.45) it is then sufficient to find $\delta_\beta \in \mathbb{R}$, $1 \leq \beta \leq \binom{n}{s+1}$ such that

$$X_\beta(\xi) = \delta_\beta \det \begin{pmatrix} \xi^1_{\lambda_1} & \cdots & \xi^1_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{pmatrix}. \tag{5.48}$$

To deduce the claim we will use the fact that the function $t \to X_\beta(\xi + ta \otimes b)$ is affine for every $\xi \in \mathbb{R}^{N\times n}$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$. We will always choose

$$a^1 = a^2 = 1 \ \text{ and } \ a^3 = \cdots = a^N = 0$$

and we will make several different choices of $\xi \in \mathbb{R}^{N\times n}$ and $b \in \mathbb{R}^n$.

1) We first choose $\xi_{\lambda_1} = \xi_{\lambda_{s+1}}$, meaning that

$$\xi_{\lambda_1} = \begin{pmatrix} \xi^2_{\lambda_1} \\ \vdots \\ \xi^{s+1}_{\lambda_1} \end{pmatrix} = \xi_{\lambda_{s+1}} = \begin{pmatrix} \xi^2_{\lambda_{s+1}} \\ \vdots \\ \xi^{s+1}_{\lambda_{s+1}} \end{pmatrix}. \tag{5.49}$$

For such a choice of ξ, we have

$$\begin{aligned} X_\beta(\xi) = \ & \sum_{l_1=1}^{n} (-1)^{1+\alpha_1} \gamma_{l_1\alpha_1} \xi^1_{l_1} \det \begin{pmatrix} \xi^2_{\lambda_1} & \cdots & \xi^2_{\lambda_s} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_s} \end{pmatrix} \\ & + \sum_{l_{s+1}=1}^{n} (-1)^{1+\alpha_{s+1}} \gamma_{l_{s+1}\alpha_{s+1}} \xi^1_{l_{s+1}} \det \begin{pmatrix} \xi^2_{\lambda_2} & \cdots & \xi^2_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_2} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{pmatrix}. \end{aligned}$$

We then let

$$b_l = 0 \ \text{ if } \ l = \lambda_2, \cdots, \lambda_s. \tag{5.50}$$

Using the fact that the function $t \to X_\beta(\xi + ta \otimes b)$ is affine, we deduce that the coefficient of the term in t^2 must be 0 for every above choices of ξ and b.

We thus obtain that

$$\left[\sum_{l_1=1}^{n}(-1)^{1+\alpha_1}\gamma_{l_1\alpha_1}b_{l_1}b_{\lambda_1}+\sum_{l_{s+1}=1}^{n}(-1)^{1+\alpha_{s+1}}(-1)^{s+1}\gamma_{l_{s+1}\alpha_{s+1}}b_{l_{s+1}}b_{\lambda_{s+1}}\right]$$
$$\det\begin{pmatrix}\xi_{\lambda_2}^3 & \cdots & \xi_{\lambda_s}^3\\ \vdots & \ddots & \vdots\\ \xi_{\lambda_2}^{s+1} & \cdots & \xi_{\lambda_s}^{s+1}\end{pmatrix}=0.$$

Since $\xi\in\mathbb{R}^{N\times n}$ and $b\in\mathbb{R}^n$ are arbitrary, letting aside (5.49) and (5.50), we find that

$$\begin{cases}\gamma_{l_1\alpha_1}=0 \text{ if } l_1\neq\lambda_{s+1} \text{ and } \gamma_{l_{s+1}\alpha_{s+1}}=0 \text{ if } l_{s+1}\neq\lambda_1\\ (-1)^{1+\alpha_{s+1}}\gamma_{\lambda_1\alpha_{s+1}}=(-1)^{s+1+\alpha_1}\gamma_{\lambda_{s+1}\alpha_1}\,.\end{cases}\tag{5.51}$$

2) We proceed in a similar manner with the other coefficients, namely we let, if $2\leq k\leq s$,

$$\xi_{\lambda_k}=\xi_{\lambda_{s+1}} \text{ and } b_l=0 \text{ if } l=\lambda_1,\cdots,\lambda_{k-1},\lambda_{k+1},\cdots,\lambda_s\,.\tag{5.52}$$

We then use the fact that the function $t\to X_\beta(\xi+ta\otimes b)$ is affine and thus the coefficient of the term in t^2 must be 0 for every ξ and b as in (5.52). We therefore get that

$$\left[\sum_{l_1=1}^{n}(-1)^{1+\alpha_1}\gamma_{l_1\alpha_1}b_{l_1}(-1)^{k+1}b_{\lambda_k}+\sum_{l_k=1}^{n}(-1)^{1+\alpha_k}\gamma_{l_k\alpha_k}b_{l_k}(-1)^{s+1}b_{\lambda_{s+1}}\right]$$
$$\det\begin{pmatrix}\xi_{\lambda_1}^3 & \cdots & \xi_{\lambda_{k-1}}^3 & \xi_{\lambda_{k+1}}^3 & \cdots & \xi_{\lambda_s}^3\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \xi_{\lambda_1}^{s+1} & \cdots & \xi_{\lambda_{k-1}}^{s+1} & \xi_{\lambda_{k+1}}^{s+1} & \cdots & \xi_{\lambda_s}^{s+1}\end{pmatrix}=0.$$

As above we can then deduce that, for every $2\leq k\leq s$,

$$\begin{cases}\gamma_{l_1\alpha_1}=0 \text{ if } l_1\neq\lambda_{s+1} \text{ and } \gamma_{l_k\alpha_k}=0 \text{ if } l_k\neq\lambda_k\\ (-1)^{1+\alpha_k}\gamma_{\lambda_k\alpha_k}=(-1)^{s+k+\alpha_1}\gamma_{\lambda_{s+1}\alpha_1}\,.\end{cases}\tag{5.53}$$

Combining (5.47), (5.51) and (5.53), we have

$$\begin{aligned} X_\beta(\xi) = \; & (-1)^{1+\alpha_1} \gamma_{\lambda_{s+1}\alpha_1} \xi^1_{\lambda_{s+1}} \det \begin{pmatrix} \xi^2_{\lambda_1} & \cdots & \xi^2_{\lambda_s} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_s} \end{pmatrix} \\ & + \sum_{k=2}^{s} (-1)^{1+\alpha_1} (-1)^{s+k+1} \gamma_{\lambda_{s+1}\alpha_1} \xi^1_{\lambda_k} \\ & \quad \det \begin{pmatrix} \xi^2_{\lambda_1} & \cdots & \xi^2_{\lambda_{k-1}} & \xi^2_{\lambda_{k+1}} & \cdots & \xi^2_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_{k-1}} & \xi^{s+1}_{\lambda_{k+1}} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{pmatrix} \\ & + (-1)^s (-1)^{1+\alpha_1} \gamma_{\lambda_{s+1}\alpha_1} \xi^1_{\lambda_1} \det \begin{pmatrix} \xi^2_{\lambda_2} & \cdots & \xi^2_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_2} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{pmatrix}. \end{aligned}$$

Letting, in the above computation,

$$\delta_\beta := (-1)^{s+1+\alpha_1} \gamma_{\lambda_{s+1}\alpha_1}$$

we have indeed obtained (5.48). This completes the proof of the lemma. ■

5.3.2 Quadratic case

We now turn our attention to the case where f is quadratic. This case is of particular interest since the associated Euler-Lagrange equations are linear. It has therefore received much attention. Let us first mention the theorem.

Theorem 5.25 *Let M be a symmetric matrix in $\mathbb{R}^{(N\times n)\times(N\times n)}$. Let*

$$f(\xi) := \langle M\xi; \xi\rangle,$$

where $\xi \in \mathbb{R}^{N\times n}$ and $\langle\cdot;\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{N\times n}$. The following statements then hold.

(i) f is rank one convex if and only if f is quasiconvex.

(ii) If $N = 2$ or $n = 2$, then

$$f \text{ polyconvex} \Leftrightarrow f \text{ quasiconvex} \Leftrightarrow f \text{ rank one convex}.$$

(iii) If $N, n \geq 3$, then in general

$$f \text{ rank one convex} \not\Rightarrow f \text{ polyconvex}.$$

Remark 5.26 (i) The proof of (i) of Theorem 5.25 was given by Van Hove [585], [586], although it was implicitly known earlier.

(ii) The second part of the theorem has received considerable attention. The question was raised in 1937 by Bliss and received a progressive answer through the works of Albert [9], Hestenes-MacShane [338], MacShane [411], Marcellini [422], Reid [506], Serre [530] and Terpstra [575]. The proof of (ii) of Theorem 5.25 relies on an algebraic lemma whose importance is summarized in Uhlig [582].

(iii) A counterexample to the third part of the theorem was given by Terpstra [575] and later by Serre [530] (see also Ball [56]).

(iv) Note also that even if $N = n = 2$ and f is quadratic, then in general

$$f \text{ polyconvex } \not\Rightarrow f \text{ convex},$$

as the trivial example $f(\xi) = \det \xi$ shows. ♢

Before proceeding with the proof of the theorem we mention two simple facts that are summarized in the next lemmas.

Lemma 5.27 *Let M be a symmetric matrix in $\mathbb{R}^{(N\times n)\times(N\times n)}$ and let*

$$f(\xi) := \langle M\xi;\xi\rangle .$$

Then the following results hold.

(i) f is convex if and only if

$$f(\xi) \geq 0$$

for every $\xi \in \mathbb{R}^{N\times n}$.

(ii) f is polyconvex if and only if there exists $\alpha \in \mathbb{R}^{\sigma(2)}$ such that

$$f(\xi) \geq \langle \alpha; \operatorname{adj}_2 \xi\rangle$$

for every $\xi \in \mathbb{R}^{N\times n}$ and where $\langle\cdot;\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{\sigma(2)}$ and $\sigma(2) = \binom{N}{2}\binom{n}{2}$.

(iii) f is quasiconvex if and only if

$$\int_D f(\nabla\varphi(x))\, dx \geq 0$$

for every bounded open set $D \subset \mathbb{R}^n$ and for every $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^N)$.

(iv) f is rank one convex if and only if

$$f(a\otimes b) \geq 0$$

for every $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$.

Proof. (Lemma 5.27). Parts (i), (iii) and (iv) are trivial. The fact that

$$f(\xi) \geq \langle \alpha; \operatorname{adj}_2 \xi \rangle \tag{5.54}$$

implies that f is polyconvex follows immediately from the following observation.

Let

$$g(\xi) := f(\xi) - \langle \alpha; \operatorname{adj}_2 \xi \rangle$$

then by (5.54) and (i) of the lemma, we deduce that g is convex. Thus $f(\xi) = g(\xi) + \langle \alpha; \operatorname{adj}_2 \xi \rangle$ is polyconvex.

Assume now that f is polyconvex. We wish to show that (5.54) holds for some $\alpha \in \mathbb{R}^{\sigma(2)}$. Using Theorem 5.6, bearing in mind that $f(0) = 0$, we find that there exists $\beta = (\beta_{\sigma(1)}, \beta_{\sigma(2)}, \cdots, \beta_{\sigma(n \wedge N)}) \in \mathbb{R}^{\tau(n,N)}$ such that

$$f(\xi) \geq \langle \beta; T(\xi) \rangle = \sum_{s=1}^{n \wedge N} \langle \beta_{\sigma(s)}; \operatorname{adj}_s \xi \rangle .$$

Multiplying ξ by $\epsilon > 0$, we get

$$f(\epsilon\xi) = \epsilon^2 f(\xi) \geq \epsilon \langle \beta_{\sigma(1)}; \xi \rangle + \epsilon^2 \langle \beta_{\sigma(2)}; \operatorname{adj}_2 \xi \rangle + O(\epsilon^3) . \tag{5.55}$$

Dividing by ϵ and letting $\epsilon \to 0$, we obtain

$$\langle \beta_{\sigma(1)}; \xi \rangle \leq 0$$

for every $\xi \in \mathbb{R}^{N \times n}$, thus $\beta_{\sigma(1)} = 0$. Returning to (5.55), dividing by ϵ^2 and letting $\epsilon \to 0$ we have indeed obtained (5.54) with $\alpha = \beta_{\sigma(2)}$. ■

The second important point that we wish to mention is the following lemma concerning Fourier transforms for which the proof is straightforward.

Lemma 5.28 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ be extended by $\varphi \equiv 0$ outside of Ω. Define for $\xi \in \mathbb{R}^n$*

$$\widehat{\varphi^\alpha}(\xi) := \int_{\mathbb{R}^n} \varphi^\alpha(x) e^{-2\pi i \langle \xi; x \rangle} dx, \ 1 \leq \alpha \leq N.$$

Then

$$\widehat{\nabla\varphi} = 2\pi i \left(\widehat{\varphi^\alpha} \xi_j \right)_{1 \leq j \leq n}^{1 \leq \alpha \leq N} = 2\pi i \, \widehat{\varphi} \otimes \xi,$$

in particular $\operatorname{rank}\{\operatorname{Re}(\widehat{\nabla\varphi})\}, \operatorname{rank}\{\operatorname{Im}(\widehat{\nabla\varphi})\} \leq 1$.

Remark 5.29 Lemma 5.28 explains in a way other than that of Theorem 5.3 why matrices of rank one play such an important role in quasiconvex analysis.◇

We now proceed with the proof of Theorem 5.25.

Proof. (i) Recall that

$$f(\xi) = \langle M\xi; \xi \rangle .$$

Theorem 5.3 implies that if f is quasiconvex then f is rank one convex. We now prove the converse. By Lemma 5.27 we have to show that

$$\int_{\Omega} \langle M\nabla\varphi(x);\nabla\varphi(x)\rangle\,dx \geq 0 \tag{5.56}$$

for every bounded open set Ω, for every $\varphi \in W_0^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$ (we will set $\varphi \equiv 0$ outside of Ω), knowing that

$$f(a\otimes b) = \langle Ma\otimes b; a\otimes b\rangle \geq 0. \tag{5.57}$$

We then use Plancherel formula (we write $\overline{\xi}$ for the complex conjugate of ξ) to get

$$\begin{aligned}\int_{\Omega} \langle M\nabla\varphi(x);\nabla\varphi(x)\rangle\,dx &= \int_{\mathbb{R}^n} \langle M\nabla\varphi(x);\nabla\varphi(x)\rangle\,dx \\ &= \int_{\mathbb{R}^n} \langle\, M\widehat{\nabla\varphi}(\xi);\overline{\widehat{\nabla\varphi}}(\xi)\,\rangle\,d\xi.\end{aligned} \tag{5.58}$$

Using Lemma 5.28 and (5.57) in (5.58), we obtain (5.56).

(ii) We do not prove this result and we refer to the above bibliography.

(iii) We now want to show that if $N = n = 3$, then there exists f rank one convex which is not polyconvex. We give here an example due to Serre [530]. Let

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 \end{pmatrix}$$

and let

$$\begin{aligned} f(\xi) := \ & \left(\xi_1^1 - \xi_2^3 - \xi_3^2\right)^2 + \left(\xi_2^1 - \xi_1^3 + \xi_3^1\right)^2 \\ & + \left(\xi_1^2 - \xi_1^3 - \xi_3^1\right)^2 + \left(\xi_2^2\right)^2 + \left(\xi_3^3\right)^2. \end{aligned}$$

We divide the proof into two steps.

Step 1. We first show that there exists $\epsilon > 0$ such that

$$f(a\otimes b) - \epsilon\,|a\otimes b|^2 \geq 0 \tag{5.59}$$

for every $a, b \in \mathbb{R}^3$ and where $|\xi|^2 := \langle\xi;\xi\rangle$ denotes the Euclidean norm. Lemma 5.27 will then ensure that

$$g(\xi) = f(\xi) - \epsilon\,|\xi|^2 \tag{5.60}$$

is rank one convex. In Step 2 we then prove that this g is not polyconvex and this will end the proof of the theorem. We first let

$$\epsilon_0 := \inf\left\{f(a\otimes b) : a, b \in \mathbb{R}^3,\ |a\otimes b| = 1\right\}. \tag{5.61}$$

Then, since $f \geq 0$, we have $\epsilon_0 \geq 0$. In order to prove (5.59) it is sufficient to prove that $\epsilon_0 > 0$. We proceed by contradiction and assume that $\epsilon_0 = 0$. Observe that in (5.61) the minimum is attained and therefore there exist $a, b \in \mathbb{R}^3$ such that

$$f\left(a \otimes b\right) = \epsilon_0 = 0 \quad \text{and} \quad |a \otimes b| = 1. \tag{5.62}$$

Recall that

$$a \otimes b = \begin{pmatrix} a^1 b_1 & a^1 b_2 & a^1 b_3 \\ a^2 b_1 & a^2 b_2 & a^2 b_3 \\ a^3 b_1 & a^3 b_2 & a^3 b_3 \end{pmatrix},$$

therefore the first equation of (5.62) becomes

$$\begin{cases} a^1 b_1 = a^2 b_3 + a^3 b_2 \\ a^1 b_2 = a^3 b_1 - a^1 b_3 \\ a^2 b_1 = a^1 b_3 + a^3 b_1 \\ \qquad a^2 b_2 = 0 \\ \qquad a^3 b_3 = 0. \end{cases} \tag{5.63}$$

We then show that (5.63) is in contradiction with the fact that $|a \otimes b| = 1$. To do so, we carefully examine (5.63) and separate the discussion in several cases.

Case 1. $a^2 = a^3 = 0$ (cf. the two last equations of (5.63)), then (5.63) becomes

$$\begin{cases} a^2 = a^3 = 0 \\ a^1 b_1 = a^1 b_3 = 0 \\ a^1 b_2 = -a^1 b_3 \,. \end{cases} \tag{5.64}$$

Case 1a. $a^1 = 0$, therefore $a^1 = a^2 = a^3 = 0$ and hence $|a \otimes b| = 0$, contradiction.

Case 1b. $b_1 = 0$, hence from (5.64), $a^1 b_3 = 0$ and thus $a^1 b_2 = 0$. We then also conclude that $|a \otimes b| = 0$ and this is a contradiction.

Case 2. $a^2 = b_3 = 0$ (cf. the two last equations of (5.63)), then (5.63) becomes

$$\begin{cases} a^2 = b_3 = 0 \\ a^1 b_1 = a^3 b_2 \\ a^1 b_2 = a^3 b_1 \\ \quad a^3 b_1 = 0. \end{cases}$$

Case 2a. $a^3 = 0$, then $a^1 b_1 = a^1 b_2 = 0$ and therefore $|a \otimes b| = 0$, contradiction.

Case 2b. $b_1 = 0$, then $a^3 b_2 = a^1 b_2 = 0$ and therefore $|a \otimes b| = 0$, contradiction.

Similarly for the case $a^3 = b_2 = 0$ and $b_2 = b_3 = 0$. Thus $\epsilon_0 > 0$ and hence Step 1, i.e. g defined by (5.60), is rank one convex for every $0 < \epsilon \leq \epsilon_0$.

Step 2. We now show that g is not polyconvex. In view of Lemma 5.27 it is sufficient to show that for every $\alpha \in \mathbb{R}^{3\times 3}$, there exists $\xi \in \mathbb{R}^{3\times 3}$ such that

$$g(\xi) + \langle \alpha; \operatorname{adj}_2 \xi \rangle < 0.$$

We prove that the above inequality holds for matrices ξ of the following form

$$\xi := \begin{pmatrix} b+d & c-a & a \\ c+a & 0 & b \\ c & d & 0 \end{pmatrix}.$$

For such matrices we have $f(\xi) = 0$ and therefore

$$\begin{aligned} g(\xi) &= -\epsilon |\xi|^2 \\ &= -\epsilon [(b+d)^2 + (c-a)^2 + a^2 + (c+a)^2 + b^2 + c^2 + d^2] \end{aligned}$$

and

$$\operatorname{adj}_2 \xi = \begin{pmatrix} -bd & bc & cd + ad \\ ad & -ac & -\left(bd + d^2 - c^2 + ac\right) \\ bc - ab & ac + a^2 - b^2 - bd & a^2 - c^2 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \langle \alpha; \operatorname{adj}_2 \xi \rangle = {} & -\alpha_1 bd + \alpha_2 bc + \alpha_3 (cd + ad) \\ & + \alpha_4 ad - \alpha_5 ac - \alpha_6 \left(bd + d^2 - c^2 + ac\right) \\ & + \alpha_7 (bc - ab) + \alpha_8 \left(ac + a^2 - b^2 - bd\right) + \alpha_9 \left(a^2 - c^2\right). \end{aligned}$$

As in Step 1 we consider several cases.

Case 1. If $\alpha_8 > 0$, then take $a = c = d = 0$ and $b \neq 0$, to get

$$\begin{aligned} g(\xi) + \langle \alpha; \operatorname{adj}_2 \xi \rangle &= -\epsilon |\xi|^2 + \langle \alpha; \operatorname{adj}_2 \xi \rangle \\ &= -\epsilon \left(2b^2\right) - \alpha_8 b^2 < 0. \end{aligned}$$

Case 2. If $\alpha_6 > 0$, then take $a = b = c = 0$ and $d \neq 0$, to get

$$g(\xi) + \langle \alpha; \operatorname{adj}_2 \xi \rangle = -\epsilon \left(2d^2\right) - \alpha_6 d^2 < 0.$$

We therefore can assume that $\alpha_8 \leq 0$ and $\alpha_6 \leq 0$.

Case 3. If $\alpha_9 - \alpha_6 > 0$ $(\alpha_8 \leq 0,\ \alpha_6 \leq 0)$, then take $a = b = d = 0$ and $c \neq 0$ to get

$$g(\xi) + \langle \alpha; \operatorname{adj}_2 \xi \rangle = -\epsilon \left(3c^2\right) + (\alpha_6 - \alpha_9) c^2 < 0.$$

We therefore assume $\alpha_8 \leq 0$, $\alpha_6 \leq 0$ and $\alpha_9 - \alpha_6 \leq 0$. From these three inequalities we deduce that $\alpha_8 + \alpha_9 \leq 0$, and then taking $b = c = d = 0$ and $a \neq 0$, we get

$$g(\xi) + \langle \alpha; \operatorname{adj}_2 \xi \rangle = -\epsilon \left(3a^2\right) + (\alpha_8 + \alpha_9) a^2 < 0.$$

And this concludes the proof of the theorem. ■

5.3.3 Convexity of $SO(n) \times SO(n)$ and $O(N) \times O(n)$ invariant functions

We now discuss the different notions of convexity for functions having some symmetries and follow the presentation of Dacorogna-Maréchal [204].

Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ and let $\Gamma_1 \subset \mathbb{R}^{N \times N}$ be a subgroup of $GL(N)$ (the set of invertible matrices) and $\Gamma_2 \subset \mathbb{R}^{n \times n}$ be a subgroup of $GL(n)$. Assume that f is $\Gamma_1 \times \Gamma_2$-invariant, meaning that

$$f(U\xi V) = f(\xi), \ \forall U \in \Gamma_1, \ \forall V \in \Gamma_2.$$

We will be concerned with groups Γ that are either $O(n)$ (the set of orthogonal matrices) or $SO(n)$ (the set of special orthogonal matrices); see Chapter 13 for precise definitions.

We start with some notation and we refer to Chapter 13 for more details. In the whole of this section, we assume that $N \geq n$, but all the results can be carried in a straightforward way to the case where $N \leq n$.

Notation 5.30 (i) Let $N \geq n$ and $\xi \in \mathbb{R}^{N \times n}$. The *singular values* of ξ, denoted by

$$0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi),$$

are defined to be the square root of the eigenvalues of the symmetric and positive semidefinite matrix $\xi^t \xi \in \mathbb{R}^{n \times n}$. A similar definition holds when $N \leq n$. We let

$$\lambda(\xi) = (\lambda_1(\xi), \cdots, \lambda_n(\xi)).$$

(ii) When $N = n$, we denote by

$$0 \leq \mu_1(\xi) \leq \cdots \leq \mu_n(\xi),$$

the *signed singular values* of $\xi \in \mathbb{R}^{n \times n}$; they are defined as

$$\mu_1(\xi) = \lambda_1(\xi) \operatorname{sign}(\det \xi) \quad \text{and} \quad \mu_j(\xi) = \lambda_j(\xi), \ j = 2, \cdots, n.$$

We let

$$\mu(\xi) = (\mu_1(\xi), \cdots, \mu_n(\xi)).$$

(iii) We denote, for every integer $m \geq 1$:

- $\Pi(m)$ the subgroup of $O(m)$ that consists of the matrices having exactly one nonzero entry per row and per column, moreover each entry belongs to $\{-1, 1\}$;

- $\Pi_e(m)$ the subgroup of $\Pi(m)$ that consists of the matrices having an even number of entries equal to -1;

- $S(m)$ the subgroup of $\Pi_e(m)$ of all permutation matrices.

We therefore have

$$S(m) \quad\subset\quad \Pi_e(m) \quad\subset\quad \Pi(m) \quad\subset\quad O(m) \quad\subset\quad GL(m)\,.$$

(iv) We let $\mathbb{R}_d^{N\times n}$ be the subspace of $\mathbb{R}^{N\times n}$ consisting of *diagonal matrices*, meaning that

$$\xi \in \mathbb{R}_d^{N\times n} \;\Rightarrow\; \xi_j^i = 0 \text{ if } i \neq j.$$

(v) For a vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we denote by $\operatorname{diag}_{N\times n}$ (when $N = n$ we simply write diag) the matrix $\xi \in \mathbb{R}_d^{N\times n}$ such that

$$\xi_i^i = x_i\,. \qquad\diamondsuit$$

We start with some simple observations. The first proposition is an immediate consequence of the singular values decomposition theorem (see Theorem 13.3).

Proposition 5.31 *(i) Let $f\colon \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\}$. Then f is $SO(n) \times SO(n)$-invariant if and only if f satisfies*

$$f = f \circ \operatorname{diag} \circ \mu,$$

and

$$g := f \circ \operatorname{diag}$$

is then the unique $\Pi_e(n)$-invariant function such that $f = g \circ \mu$.

(ii) Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$, where $N \geq n$. Then f is $O(N) \times O(n)$-invariant if and only if f satisfies

$$f = f \circ \operatorname{diag}_{N\times n} \circ \lambda,$$

and

$$g := f \circ \operatorname{diag}_{N\times n}$$

is then the unique $\Pi(n)$-invariant function such that $f = g \circ \lambda$.

It is clear that, if $N = n$, the notions of $O(N) \times O(n)$, $SO(N) \times O(n)$ and $O(N) \times SO(n)$-invariance coincide but differ from that of $SO(N) \times SO(n)$-invariance. However, if $N \neq n$, all four notions coincide as we now show.

Proposition 5.32 *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$, where $N > n$. Then the following are equivalent:*

(i) f is $O(N) \times O(n)$-invariant;

(ii) f is $SO(N) \times SO(n)$-invariant.

Proof. Obviously, we need only prove that (ii) implies (i). We will see that, if f is $SO(N) \times SO(n)$-invariant, then

$$f = f \circ \operatorname{diag}_{N\times n} \circ \lambda. \tag{5.65}$$

The conclusion will then follow from Proposition 5.31.

Let $\xi \in \mathbb{R}^{N\times n}$. By the singular values decomposition theorem (Theorem 13.3), there exist $U \in O(N)$, $V \in O(n)$ such that

$$\xi = U\Lambda V^t, \quad \text{where } \Lambda := \operatorname{diag}_{N\times n}(\lambda_1(\xi), \cdots, \lambda_n(\xi)).$$

So we have to consider several cases. First of all let us introduce the following notation. If $m \geq 1$ is an integer, we let

$$H_m := \operatorname{diag}(-1, 1, \cdots, 1) \in \mathbb{R}^{m\times m} \quad \text{and} \quad K_m := \operatorname{diag}(1, \cdots, 1, -1) \in \mathbb{R}^{m\times m}.$$

- If $U \in SO(N)$ and $V \in SO(n)$, then, from (ii) the conclusion follows, namely

$$f(\xi) = f(\Lambda) = (f \circ \operatorname{diag}_{N\times n} \circ \lambda)(\xi).$$

- If $U \in O(N) - SO(N)$ and $V \in O(n) - SO(n)$, we may write $\Lambda = H_N \Lambda H_n$, so that

$$U\Lambda V^t = (UH_N)\Lambda(VH_n)^t$$

with $UH_N \in SO(N)$ and $VH_n \in SO(n)$. Thus (5.65) holds by (ii).

- If $U \in O(N) - SO(N)$ and $V \in SO(n)$, we may write $\Lambda = K_N \Lambda$, so that

$$U\Lambda V^t = (UK_N)\Lambda V^t$$

with $UK_N \in SO(N)$. Equation (5.65) then follows from (ii).

- If $U \in SO(N)$ and $V \in O(n) - SO(n)$, we may write $\Lambda = H_N K_N \Lambda H_n$, so that

$$U\Lambda V^t = (UH_N K_N)\Lambda(VH_n)^t,$$

with $UH_N K_N \in SO(N)$ and $VH_n \in SO(n)$. Thus (5.65) holds.

We have therefore shown the claim, namely that $f = f \circ \operatorname{diag}_{N\times n} \circ \lambda$. ■

The main result concerns the convexity of such functions.

Theorem 5.33 ***(A)*** *Let* $f : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\}$ *be* $SO(n) \times SO(n)$*-invariant,* $f \not\equiv +\infty$*, and let* $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be the unique* $\Pi_e(n)$*-invariant function such that*

$$f = g \circ \mu.$$

Then the following are equivalent:

(i) f *is lower semicontinuous and convex;*

(ii) the restriction of f *to* $\mathbb{R}_d^{n\times n}$*, the subspace of* $\mathbb{R}^{n\times n}$ *of diagonal matrices, is lower semicontinuous and convex;*

(iii) g *is lower semicontinuous and convex.*

(B) *Let $N > n$, let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{+\infty\}$ be $SO(N)\times SO(n)$-invariant or, equivalently, $O(N)\times O(n)$-invariant, $f \not\equiv +\infty$, and let $g\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ to be the unique $\Pi(n)$-invariant function such that*

$$f = g\circ\lambda.$$

Then the following are equivalent:

(i) f is lower semicontinuous and convex;

(ii) the restriction of f to $\mathbb{R}_d^{N\times n}$, the subspace of $\mathbb{R}^{N\times n}$ of diagonal matrices, is lower semicontinuous and convex;

(iii) g is lower semicontinuous and convex.

Remark 5.34 (i) We discuss now the history of this theorem first in the case where $N = n$ and in the $O(n)\times O(n)$-invariant case. The result was established by Ball [53], Hill [341] and Thompson-Freede [577]; see also Dacorogna-Marcellini [202] and Le Dret [397]. In elasticity, an $O(n)\times O(n)$-invariant function is called *isotropic.*

(ii) The case $N = n$ and $SO(n)\times SO(n)$-invariant, was first established by Dacorogna-Koshigoe [192] in the case $n = 2$, and later by Vincent [589] when $n \geq 3$, as a consequence of the convexity theorem of Kostant [377]. A different proof, inspired by Rosakis [516] and based on the notion of signed singular values and a generalized Von Neumann inequality (see Theorem 13.10), was given by Dacorogna-Maréchal [204]. In this last paper, the case $N \neq n$ was also handled. ♢

Proof. **(A)** The fact that (i) implies (ii) is clear. The fact that (ii) implies (iii) results immediately from the equality $g = f\circ \operatorname{diag}$. Finally, suppose that (iii) holds. Then $g^{**} = g$, and Theorem 6.17 (i) implies that

$$f^{**} = g^{**}\circ\mu = g\circ\mu = f,$$

which shows that f is lower semicontinuous and convex.

(B) The fact that (i) implies (ii) is clear. The fact that (ii) implies (iii) results immediately from the equality $g = f\circ \operatorname{diag}_{N\times n}$. Finally, suppose that (iii) holds. Theorem 6.17 (ii) then implies that

$$f^{**} = g^{**}\circ\lambda = g\circ\lambda = f,$$

which shows that f is lower semicontinuous and convex. ■

In the case of $O(n)\times O(n)$-invariant functions, the analogous statement can be derived in several ways from the above results and we do not discuss the details.

Corollary 5.35 *Let $f\colon \mathbb{R}^{n\times n} \to \mathbb{R}\cup\{+\infty\}$ be $O(n)\times O(n)$-invariant, $f \not\equiv +\infty$, and let $g\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be the unique $\Pi(n)$-invariant function such that*

$$f = g\circ\lambda.$$

Then the following are equivalent:

(i) f is lower semicontinuous and convex;

(ii) the restriction of f to $\mathbb{R}_d^{n\times n}$ is lower semicontinuous and convex;

(iii) g is lower semicontinuous and convex.

Remark 5.36 As a convex $\Pi(n)$-invariant function, the function g appearing in Theorem 5.33 (B) or in Corollary 5.35 must be such that each function

$$x_k \to g(x_1, \cdots, x_n), \quad k = 1, \cdots, n$$

is non-decreasing on $\mathbb{R}_+$. We now prove this only when $k = 1$, the other cases being handled similarly. As a matter of fact, for all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ with $x_1 \geq 0$,

$$g(0, x_2, \cdots, x_n) \leq \frac{1}{2} g(-x_1, x_2, \cdots, x_n) + \frac{1}{2} g(x_1, x_2, \cdots, x_n) = g(x),$$

and if $z > 0$, we see, using the above inequality, that

$$\begin{aligned} g(x) &\leq \frac{x_1}{x_1+z} g(x_1+z, x_2, \cdots, x_n) + \frac{z}{x_1+z} g(0, x_2, \cdots, x_n) \\ &\leq \frac{x_1}{x_1+z} g(x_1+z, x_2, \cdots, x_n) + \frac{z}{x_1+z} g(x_1+z, x_2, \cdots, x_n) \\ &= g(x_1+z, x_2, \cdots, x_n). \end{aligned}$$

Thus $x_1 \to g(x_1, \cdots, x_n)$ is non-decreasing on $\mathbb{R}_+$. ♢

We now give a simple corollary, which follows from Theorem 5.33 and in a more direct way from Theorem 13.10. It will be used in Theorems 5.39, 5.43 and 7.43.

Corollary 5.37 *Let $\xi \in \mathbb{R}^{n\times n}$ and*

$$0 \leq b_1 \leq \cdots \leq b_n .$$

The functions

$$f_\nu(\xi) = \sum_{i=\nu}^{n} b_i \lambda_i(\xi)$$

are convex for every $\nu = 1, \cdots, n$.

If $|b_1| \leq b_2 \leq \cdots \leq b_n$, then the following functions are also convex

$$g_\nu(\xi) = \sum_{i=\nu}^{n} b_i \mu_i(\xi), \ \nu = 1, \cdots, n.$$

5.3.4 Polyconvexity and rank one convexity of $SO(n) \times SO(n)$ and $O(N) \times O(n)$ invariant functions

We now discuss the polyconvexity and rank one convexity of functions having the symmetries considered in the previous section. We first discuss the case of a $O(N) \times O(n)$-invariant function and then the $SO(2) \times SO(2)$-invariant case. We also assume, as in the previous section, that $N \geq n$, but all the results immediately extend to the case where $N \leq n$.

We start with some notation.

Notation 5.38 Let $N \geq n$.

(i) We let

$$\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \cdots, n\},$$

$$K_+^n := \{x \in \mathbb{R}^n : 0 \leq x_1 \leq \cdots \leq x_n\}.$$

In particular, when $n = 1$, $K_+ = \mathbb{R}_+$.

(ii) For $X \in \mathbb{R}^{\binom{N}{s} \times \binom{n}{s}}$, $1 \leq s \leq n-1$, we denote by $\Lambda^s(X) \in K_+^{\binom{n}{s}}$ its singular values. In particular, when $s = 1$, we have

$$\Lambda^1(\xi) = (\lambda_1(\xi), \cdots, \lambda_n(\xi)).$$

In the notation of Section 5.3.3 we have $\Lambda^1(\xi) = \lambda(\xi)$.

(iii) For every $x \in K_+^n$, we adopt the following notation.

- If $s = 2$, we let

$$\operatorname{adj}_2 x \in K_+^{\binom{n}{2}}$$

the vector in $\mathbb{R}^{\binom{n}{2}}$ composed of every $x_i x_j$ with $i < j$ rearranged in an increasing way (for example if $n = 3$ then $\operatorname{adj}_2 x = (x_1x_2, x_1x_3, x_2x_3)$). Note that, unless $n = 2, 3$, the ordering of $\operatorname{adj}_2 x$ depends on x itself. For example, if $n = 4$, then for some x we can have

$$\operatorname{adj}_2 x = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$$

and for others

$$\operatorname{adj}_2 x = (x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_3x_4).$$

- Similarly, if $2 < s < n$, we let

$$\operatorname{adj}_s x \in K_+^{\binom{n}{s}}$$

to be the vector in $\mathbb{R}^{\binom{n}{s}}$ composed of every $x_{i_1} \cdots x_{i_s}$, $i_1 < \cdots < i_s$ rearranged in an increasing way.

- Finally, when $s = n$, we denote by either of the following symbols

$$\operatorname{adj}_n x = \det x = \prod_{i=1}^{n} x_i \, .$$

Note that with these notations we have for every $\xi \in \mathbb{R}^{N \times n}$ and every $1 \leq s \leq n$ that

$$\Lambda^s \left(\operatorname{adj}_s \xi \right) = \operatorname{adj}_s \Lambda^1 \left(\xi \right) . \qquad \diamondsuit$$

The next theorem is stated, for the convenience of the reader, first when $N = n = 2$, then when $N = n = 3$ and finally in the general case $N \geq n$.

Theorem 5.39 *Let $N \geq n$,*

$$0 \leq \lambda_1 \left(\xi \right) \leq \cdots \leq \lambda_n \left(\xi \right) ,$$

be the singular values of $\xi \in \mathbb{R}^{N \times n}$. Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ and $g : \mathbb{R}_+^n \to \mathbb{R}$ be such that

$$f \left(\xi \right) = g \left(\lambda_1 \left(\xi \right), \cdots, \lambda_n \left(\xi \right) \right) .$$

(i) Let $N = n = 2$. Assume that there exists

$$G : \mathbb{R}_+^2 \times \mathbb{R}_+ \to \mathbb{R}, \; G = G \left(x, \delta \right) = G \left(x_1, x_2, \delta \right) ,$$

convex, non-decreasing in each variable, symmetric with respect to the first two variables, meaning that

$$G \left(x_2, x_1, \delta \right) = G \left(x_1, x_2, \delta \right) ,$$

and such that

$$g \left(x_1, x_2 \right) = G \left(x_1, x_2, x_1 x_2 \right) ,$$

then f is polyconvex.

(ii) Let $N = n = 3$. Assume that there exists

$$G : \mathbb{R}_+^3 \times \mathbb{R}_+^3 \times \mathbb{R}_+ \to \mathbb{R}$$

$$G = G \left(x, y, \delta \right) = G \left(x_1, x_2, x_3, y_1, y_2, y_3, \delta \right)$$

convex, non-decreasing in each variable and symmetric in the variables x and y separately, meaning that for every permutation P and P' of three elements

$$G \left(P x, P' y, \delta \right) = G \left(x, y, \delta \right) ,$$

and such that

$$g \left(x_1, x_2, x_3 \right) = G \left(x_1, x_2, x_3, x_1 x_2, x_1 x_3, x_2 x_3, x_1 x_2 x_3 \right) .$$

Then f is polyconvex.

(iii) General case: $N \geq n$. Assume that there exists

$$G : \mathbb{R}_+^n \times \mathbb{R}_+^{\binom{n}{2}} \times \cdots \times \mathbb{R}_+^{\binom{n}{n-1}} \times \mathbb{R}_+ \to \mathbb{R}$$

$$G = G(z) = G\left(z^1, z^2, \cdots, z^{n-1}, z^n\right)$$

convex, non-decreasing in each variable and symmetric in each of the variables z^i separately, i.e., for every permutation P_i of $\binom{n}{i}$ elements

$$G\left(P_1\Lambda^1, P_2\Lambda^2, \cdots, P_{n-1}\Lambda^{n-1}, \Lambda^n\right) = G\left(\Lambda^1, \Lambda^2, \cdots, \Lambda^{n-1}, \Lambda^n\right)$$

and such that

$$g(x) = G\left(x, \operatorname{adj}_2 x, \cdots, \operatorname{adj}_{n-1} x, \operatorname{adj}_n x\right).$$

Then f is polyconvex.

Remark 5.40 (i) The above result is due to Ball [53] when $N = n = 2$ and $N = n = 3$ and to Dacorogna-Marcellini [202] when $N = n$. Here we follow this last proof. A different approach, more in the spirit of Section 5.3.3, has been given by Dacorogna-Maréchal [205]. One can also consult Mielke [443].

(ii) The above sufficient condition is in some sense also necessary, once we have taken care of the appropriate symmetries implied by the fact that f depends only on singular values. For example, since the function f does not see changes of signs of the determinant, then G should not see it either (and the function F, defined in the proof, as well). This will be achieved in Theorem 5.43 when $N = n = 2$. $\diamondsuit$

Proof. We first proceed, just for the sake of better understanding the proof, with the case $N = n = 2$.

Case: $N = n = 2$. We divide the proof into two steps.

Step 1. We start with the following preliminary observation. Since G is convex over $\mathbb{R}_+^2 \times \mathbb{R}_+$ we have (cf. Corollary 2.51)

$$G(x, \delta) = \sup_{\substack{b_0, b_2 \in \mathbb{R} \\ b_1 \in \mathbb{R}^2}} \left\{ \begin{array}{c} b_0 + \langle b_1; x\rangle + b_2\delta : \\ b_0 + \langle b_1; y\rangle + b_2\epsilon \leq G(y, \epsilon),\ \forall (y, \epsilon) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \end{array} \right\}.$$

It is easy to see (cf. below) that since $x \in K_+^2$ and $\delta \geq 0$ and since G is non decreasing in each variable and symmetric in the x variable, there is no loss of generality in considering the supremum only on $b_2 \geq 0$ and $b_1 \in K_+^2$. Hence, for every $(x,\delta)\in K_+^2\times\mathbb{R}_+$, we have

$$G(x,\delta)=\sup_{\substack{b_0\in\mathbb{R}\\ b_2\geq 0\\ b_1\in K_+^2}}\left\{\begin{array}{c} b_0+\langle b_1;x\rangle+b_2\delta: \\ b_0+\langle b_1;y\rangle+b_2\epsilon\leq G(y,\epsilon),\ \forall(y,\epsilon)\in K_+^2\times\mathbb{R}_+\end{array}\right\}.$$

Let us now prove that we can indeed restrict the supremum to $(b_1,b_2)\in K_+^2\times\mathbb{R}_+$. Define

$$L(b_0,b_1,b_2,x,\delta):=b_0+\langle b_1;x\rangle+b_2\delta.$$

1) Assume first that we have $b_2<0$ and

$$L(b_0,b_1,b_2,y,\epsilon)\leq G(y,\epsilon),\ \forall(y,\epsilon)\in K_+^2\times\mathbb{R}_+$$

and let us show that we can increase the value by considering $b_2=0$. Indeed, since $\delta\geq 0$, we surely have

$$L(b_0,b_1,b_2,x,\delta)\leq L(b_0,b_1,0,x,\delta)$$

and moreover, since G is non decreasing in the variable ϵ,

$$\begin{aligned} L(b_0,b_1,0,y,\epsilon) &= L(b_0,b_1,b_2,y,0)\leq G(y,0)\\ &\leq G(y,\epsilon),\ \forall(y,\epsilon)\in K_+^2\times\mathbb{R}_+ .\end{aligned}$$

We have therefore shown that the supremum can be restricted to $b_2\geq 0$.

2) A completely analogous argument shows that we can also restrict our attention to $b_1\in\mathbb{R}_+^2$. Once this is achieved, we can further consider only $b_1\in K_+^2$, since x itself belongs to K_+^2 and G is symmetric with respect to the two first variables.

Step 2. Let $F:\mathbb{R}^{2\times 2}\times\mathbb{R}\to\mathbb{R}$ be defined by

$$F(\xi,\delta):=G\left(\Lambda^1(\xi),|\delta|\right)=G(\lambda_1(\xi),\lambda_2(\xi),|\delta|).$$

Observe that

$$\begin{aligned} F(\xi,\det\xi) &= G(\lambda_1(\xi),\lambda_2(\xi),\lambda_1(\xi)\lambda_2(\xi))\\ &= g(\lambda_1(\xi),\lambda_2(\xi))=f(\xi).\end{aligned}$$

Hence if we prove that F is convex, we will have established that f is polyconvex. We have by Step 1 that, for every $(x,\delta)\in K_+^2\times\mathbb{R}_+$,

$$G(x,\delta)=\sup_{\substack{b_0\in\mathbb{R}\\ b_2\geq 0\\ b_1\in K_+^2}}\left\{\begin{array}{c} b_0+\langle b_1;x\rangle+b_2\delta: \\ b_0+\langle b_1;y\rangle+b_2\epsilon\leq G(y,\epsilon),\ \forall(y,\epsilon)\in K_+^2\times\mathbb{R}_+\end{array}\right\}.$$

Since for every $y \in K_{+}^{2}$, we can find $\eta \in \mathbb{R}^{2\times 2}$ so that

$$\Lambda^{1}(\eta) = y$$

(just choose $\eta = \operatorname{diag}(y_1, y_2)$), we deduce that

$$F(\xi,\delta) = \sup_{\substack{b_0 \in \mathbb{R},\ b_2 \geq 0 \\ b_1 \in K_{+}^{2}}} \left\{ \begin{array}{c} b_0 + \left\langle b_1; \Lambda^{1}(\xi)\right\rangle + b_2 |\delta| : \\ b_0 + \left\langle b_1; \Lambda^{1}(\eta)\right\rangle + b_2 |\epsilon| \leq F(\eta,\epsilon), \\ \forall (\eta,\epsilon) \in \mathbb{R}^{2\times 2}\times\mathbb{R} \end{array} \right\}.$$

Since the function $(\eta,\epsilon) \to b_0 + \left\langle b_1; \Lambda^{1}(\eta)\right\rangle + b_2|\epsilon|$ is convex (by Corollary 5.37 and since $b_2 \geq 0$ and $b_1 \in K_{+}^{2}$), we deduce that F is convex. The proof, in the case $N = n = 2$, is therefore complete.

General case: $N \geq n$. Recall first the notations of Sections 5.2 and 5.4. Let

$$\tau(n,N) := \sum_{s=1}^{n} \binom{N}{s}\binom{n}{s}$$

and $T : \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau(n,N)}$ be such that

$$T(\xi) := (\xi, \operatorname{adj}_2 \xi, \cdots, \operatorname{adj}_n \xi)$$

where

$$\mathbb{R}^{\tau(n,N)} := \mathbb{R}^{N\times n} \times \mathbb{R}^{\binom{N}{2}\times\binom{n}{2}} \times \cdots \times \mathbb{R}^{\binom{N}{n-1}\times\binom{n}{n-1}} \times \mathbb{R}^{\binom{N}{n}}.$$

For $X = \left(X^1, X^2, \cdots, X^{n-1}, X^n\right) \in \mathbb{R}^{\tau(n,N)}$ we denote by

$$\Lambda(X) := \left(\Lambda^{1}\left(X^{1}\right), \Lambda^{2}\left(X^{2}\right), \cdots, \Lambda^{n-1}\left(X^{n-1}\right), \Lambda^{n}\left(X^{n}\right)\right) \in K_{+}^{\theta(n)}$$

where

$$K_{+}^{\theta(n)} := K_{+}^{n} \times K_{+}^{\binom{n}{2}} \times \cdots \times K_{+}^{\binom{n}{n-1}} \times K_{+}.$$

Finally define $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R}$ by

$$F(X) := G(\Lambda(X)).$$

Observe that, for $\xi \in \mathbb{R}^{N\times n}$,

$$\begin{aligned} F(T(\xi)) &= G(\Lambda(T(\xi))) \\ &= G\left(\Lambda^{1}(\xi), \Lambda^{2}(\operatorname{adj}_2 \xi), \cdots, \Lambda^{n-1}\left(\operatorname{adj}_{n-1}\xi\right), \Lambda^{n}(\operatorname{adj}_n \xi)\right) \\ &= G\left(\Lambda^{1}(\xi), \operatorname{adj}_2 \Lambda^{1}(\xi), \cdots, \operatorname{adj}_{n-1}\Lambda^{1}(\xi), \operatorname{adj}_n \Lambda^{1}(\xi)\right) \\ &= g\left(\Lambda^{1}(\xi)\right) = g(\lambda_1(\xi), \cdots, \lambda_n(\xi)) = f(\xi). \end{aligned}$$

Hence to prove the polyconvexity of f it remains only to prove the convexity of F. We then use the convexity of G to deduce, for every $z = \left(z^1, \cdots, z^n\right) \in K_{+}^{\theta(n)}$, that

$$G(z) = \sup_{b_0,\, b_\nu \in \mathbb{R}^{\binom{n}{\nu}}} \left\{ \begin{array}{c} b_0 + \sum_{\nu=1}^{n} \langle b_\nu ; z^\nu \rangle : \\ b_0 + \sum_{\nu=1}^{n} \langle b_\nu ; y^\nu \rangle \le G(y)\,,\ \forall y \in \mathbb{R}_+^{\theta(n)} \end{array} \right\}.$$

The facts that G is non decreasing in each variable and symmetric in each of the variables but the last one, that $z^\nu \in K_+^{\binom{n}{\nu}}$, for every $\nu = 1, \cdots, n$, allow (as in Step 1 of the case where $N = n = 2$) to restrict the above supremum to

$$G(z) = \sup_{b_0 \in \mathbb{R},\ b_\nu \in K_+^{\binom{n}{\nu}}} \left\{ \begin{array}{c} b_0 + \sum_{\nu=1}^{n} \langle b_\nu ; z^\nu \rangle : \\ b_0 + \sum_{\nu=1}^{n} \langle b_\nu ; y^\nu \rangle \le G(y)\,,\ \forall y \in K_+^{\theta(n)} \end{array} \right\}.$$

Since for every $y^\nu \in K_+^{\binom{n}{\nu}}$ and every $\nu = 1, \cdots, n$, we can find $\eta^\nu \in \mathbb{R}^{\binom{N}{\nu} \times \binom{n}{\nu}}$ so that

$$\Lambda^\nu(\eta^\nu) = y^\nu$$

(just choose η^ν a diagonal matrix with the appropriate entries), we obtain that for every $X = (X^1, \cdots, , X^n) \in \mathbb{R}^{\tau(n,N)}$,

$$\begin{aligned} F(X) &= G(\Lambda(X)) \\ &= \sup_{b_0 \in \mathbb{R},\ b_\nu \in K_+^{\binom{n}{\nu}}} \left\{ \begin{array}{c} b_0 + \sum_{\nu=1}^{n} \langle b_\nu ; \Lambda^\nu(X^\nu) \rangle : \\ b_0 + \sum_{\nu=1}^{n} \langle b_\nu ; \Lambda^\nu(\eta^\nu) \rangle \le F(\eta)\,,\ \forall \eta \in \mathbb{R}^{\tau(n,N)} \end{array} \right\} \end{aligned}$$

Observe that since $b_\nu \in K_+^{\binom{n}{\nu}}$ for $\nu = 1, \cdots, n$, we have that the function

$$\eta = (\eta^1, \cdots, \eta^n) \in \mathbb{R}^{\tau(n,N)} \to b_0 + \sum_{\nu=1}^{n} \langle b_\nu ; \Lambda^\nu(\eta^\nu) \rangle$$

is convex (cf. Corollary 5.37) and hence F is convex. Thus the function f is polyconvex and this achieves the proof of the theorem. ■

The next example will turn out, in the subsequent chapters, to be useful.

Example 5.41 Let $\xi \in \mathbb{R}^{n \times n}$, then the functions

$$f_\nu(\xi) := \prod_{i=\nu}^{n} \lambda_i(\xi)$$

are polyconvex for every $\nu = 1, \cdots, n$. The proof follows from the theorem, but it can be seen in a more straightforward way from the following argument. For $1 \le s \le n$, the function

$$X \in \mathbb{R}^{\binom{n}{s} \times \binom{n}{s}} \to \lambda_{\binom{n}{s}}(X)$$

is convex, according to Corollary 5.37. Hence the function

$$\xi \to \lambda_{\binom{n}{s}}(\mathrm{adj}_s\, \xi)$$

is polyconvex. Since

$$\lambda_{\binom{n}{s}}\left(\operatorname{adj}_s \xi\right)=\prod_{i=n-s+1}^{n} \lambda_i\left(\xi\right),$$

we have the claim. $\diamondsuit$

We now turn our attention to the $SO(2) \times SO(2)$-invariant case and give here a theorem due to Dacorogna-Koshigoe [192], which shows, in particular, that at least when $N=n=2$, the sufficient condition of Theorem 5.39 is also necessary. We here follow the proof of Dacorogna-Maréchal [205]; but let us first introduce the following definition of polyconvexity for vectors.

Definition 5.42 *A function $g: \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ is said to be* polyconvex *if there exists $G: \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ convex such that*

$$g\left(x_1, x_2\right)=G\left(x_1, x_2, x_1 x_2\right).$$

There is of course a similar definition for polyconvex functions over $\mathbb{R}^n$ (for details see [205]), but we will not need this extension here.

In the next theorem we use the notations of Section 5.3.3.

Theorem 5.43 *Let $f: \mathbb{R}^{2\times 2} \to \mathbb{R}$ be $SO(2) \times SO(2)$-invariant and let $g: \mathbb{R}^2 \to \mathbb{R}$ be the unique $\Pi_e(2)$-invariant function such that*

$$f=g \circ \mu.$$

The following statements are all equivalent.

(i) f is polyconvex.

(ii) g is polyconvex.

(iii) For every $(a_i, b_i) \in \mathbb{R}^2$, $t_i \geq 0$, $i=1,2,3,4$ with

$$\textstyle\sum_{i=1}^4 t_i=1 \quad \text{and} \quad \sum_{i=1}^4 t_i a_i b_i=\left(\sum_{i=1}^4 t_i a_i\right)\left(\sum_{i=1}^4 t_i b_i\right)$$

the following inequality holds

$$\textstyle g\left(\sum_{i=1}^4 t_i\left(a_i, b_i\right)\right) \leq \sum_{i=1}^4 t_i g\left(a_i, b_i\right).$$

In particular, if $G: \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$G(a, b, \delta):=\inf \left\{\begin{array}{c}\sum_{i=1}^4 t_i g\left(a_i, b_i\right): \\ \sum_{i=1}^4 t_i\left(a_i, b_i, a_i b_i\right)=(a, b, \delta) \text{ and } \sum_{i=1}^4 t_i=1\end{array}\right\},$$

then G is well defined. Moreover if g satisfies the above condition, then G is convex and

$$g(a, b)=G(a, b, a b)$$

for every $(a, b) \in \mathbb{R}^2$.

(iv) For every $(a, b) \in \mathbb{R}^2$, *there exists* $\beta = \beta(a, b) \in \mathbb{R}^3$ *such that*

$$g(x, y) \geq g(a, b) + \langle \beta(a, b) ; (x, y, xy) - (a, b, ab) \rangle$$

for every $(x, y) \in \mathbb{R}^2$ *and where* $\langle \cdot ; \cdot \rangle$ *denotes the scalar product in* $\mathbb{R}^3$.

Remark 5.44 (i) The equivalence between (i) and (ii) can be restated as:

$$f|_{\mathbb{R}_d^{2\times 2}} \text{ is polyconvex } \Leftrightarrow f \text{ is polyconvex,}$$

where $\mathbb{R}_d^{2\times 2}$ is the subspace of diagonal matrices of $\mathbb{R}^{2\times 2}$ and $f|_{\mathbb{R}_d^{2\times 2}}$ is the restriction of f to this subspace.

(ii) The same result holds if $f \colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is $O(2) \times O(2)$-invariant and $g \colon \mathbb{R}^2 \to \mathbb{R}$ is the unique $\Pi(2)$-invariant function such that

$$f = g \circ \lambda.$$

(iii) The result can be, in part, extended to the case where $f \colon \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$, see Dacorogna-Maréchal [205] for details.

(iv) We recall that when we say that a function $g \colon \mathbb{R}^2 \to \mathbb{R}$ is $\Pi_e(2)$-invariant we mean that, for every $x_1, x_2 \in \mathbb{R}$,

$$g(x_1, x_2) = g(x_2, x_1) = g(-x_1, -x_2) = g(-x_2, -x_1). \qquad \diamondsuit$$

Proof. The equivalence between (ii), (iii) and (iv) is proved in exactly the same way as the one of Theorem 5.6 and we will therefore omit the proof.

(i) $\Rightarrow$ **(ii)**. Since f is polyconvex, we can find a convex function

$$F \colon \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$$

so that

$$f(\xi) = F(\xi, \det \xi).$$

Let $(x_1, x_2, \delta) \in \mathbb{R}^3$ and let

$$G(x_1, x_2, \delta) := F(\xi, \delta)$$

where $\xi = \operatorname{diag}(x_1, x_2) \in \mathbb{R}^{2\times 2}$. Observe that $G : \mathbb{R}^3 \to \mathbb{R}$ is convex and, since g is $\Pi_e(2)$-invariant, we have

$$g(x_1, x_2) = G(x_1, x_2, x_1 x_2).$$

Thus g is polyconvex.

(ii) $\Rightarrow$ **(i)**. We divide the proof into two steps.

Step 1. Since g is polyconvex, we can find $G:\mathbb{R}^3\to\mathbb{R}$ convex such that

$$g\left(x_1,x_2\right)=G\left(x_1,x_2,x_1x_2\right).$$

In general the function $\left(x_1,x_2\right)\to G\left(x_1,x_2,\delta\right)$ is not $\Pi_e\left(2\right)$-invariant, although g is. To remedy to this difficulty, we let $H:\mathbb{R}^3\to\mathbb{R}$ be defined by

$$\begin{aligned}H\left(x_1,x_2,\delta\right):=\frac{1}{4}&\left[G\left(x_1,x_2,\delta\right)+G\left(x_2,x_1,\delta\right)+G\left(-x_1,-x_2,\delta\right)\right.\\&\left.+G\left(-x_2,-x_1,\delta\right)\right].\end{aligned}$$

The function H is convex and furthermore $\left(x_1,x_2\right)\to H\left(x_1,x_2,\delta\right)$ is $\Pi_e\left(2\right)$-invariant. Moreover, since g is $\Pi_e\left(2\right)$-invariant we also have

$$g\left(x_1,x_2\right)=H\left(x_1,x_2,x_1x_2\right).$$

We then define, for $\xi\in\mathbb{R}^{2\times2}$,

$$F\left(\xi,\delta\right):=H\left(\mu_1\left(\xi\right),\mu_2\left(\xi\right),\delta\right).$$

Since we clearly have

$$f\left(\xi\right)=F\left(\xi,\det\xi\right),$$

we will deduce the claim, namely that f is polyconvex, once we will have shown that $F\colon\mathbb{R}^{2\times2}\times\mathbb{R}\to\mathbb{R}$ is convex.

This is done in a completely analogous manner to the one of Theorem 5.39. Indeed since H is convex over $\mathbb{R}^3$ we have (cf. Corollary 2.51)

$$H\left(x_1,x_2,\delta\right)=\sup_{b_0,b_1,b_2,b_3\in\mathbb{R}}\left\{\begin{array}{c}b_0+b_1x_1+b_2x_2+b_3\delta:\\ b_0+b_1y_1+b_2y_2+b_3\epsilon\le H\left(y_1,y_2,\epsilon\right),\\ \forall\left(y_1,y_2,\epsilon\right)\in\mathbb{R}^3\end{array}\right\}.$$

It is easy to see (cf. Step 2 below) that, if $\left|x_1\right|\le x_2$, we have

$$H\left(x_1,x_2,\delta\right)=\sup_{\substack{b_0,\,b_3\in\mathbb{R}\\ \left|b_1\right|\le b_2}}\left\{\begin{array}{c}b_0+b_1x_1+b_2x_2+b_3\delta:\\ b_0+b_1y_1+b_2y_2+b_3\epsilon\le H\left(y_1,y_2,\epsilon\right),\\ \text{for every }\left|y_1\right|\le y_2\text{ and }\epsilon\in\mathbb{R}\end{array}\right\}. \tag{5.66}$$

since $\left(x_1,x_2\right)\to H\left(x_1,x_2,\delta\right)$ is $\Pi_e\left(2\right)$-invariant.

Since for every $\left|y_1\right|\le y_2$, we can find $\eta\in\mathbb{R}^{2\times2}$ so that

$$\mu_1\left(\eta\right)=y_1\quad\text{and}\quad\mu_2\left(\eta\right)=y_2$$

(just choose $\eta=\operatorname{diag}\left(y_1,y_2\right)$), we deduce that

$$F\left(\xi,\delta\right)=\sup_{\substack{b_0,\,b_3\in\mathbb{R}\\ \left|b_1\right|\le b_2}}\left\{\begin{array}{c}b_0+b_1\mu_1\left(\xi\right)+b_2\mu_2\left(\xi\right)+b_3\delta:\\ b_0+b_1\mu_1\left(\eta\right)+b_2\mu_2\left(\eta\right)+b_3\epsilon\le F\left(\eta,\epsilon\right),\\ \forall\left(\eta,\epsilon\right)\in\mathbb{R}^{2\times2}\times\mathbb{R}\end{array}\right\}.$$

Since $|b_1| \leq b_2$, we find that the function

$$(\eta, \epsilon) \to b_0 + b_1\mu_1(\eta) + b_2\mu_2(\eta) + b_3\epsilon$$

is convex (by Corollary 5.37) and we thus deduce that F is convex. The proof is therefore complete.

Step 2. Let us now prove that (5.66) holds. So let, for $|x_1| \leq x_2$ and $b_0, b_1, b_2, b_3, \delta \in \mathbb{R}$,

$$L(b_1, b_2, x_1, x_2, \delta) := b_0 + b_1x_1 + b_2x_2 + b_3\delta$$

(we do not denote in L the dependence on b_0, b_3, since they will not change in the following computations) be such that

$$L(b_1, b_2, y_1, y_2, \epsilon) \leq H(y_1, y_2, \epsilon), \ \forall (y_1, y_2, \epsilon) \in \mathbb{R}^3. \tag{5.67}$$

The claim (5.66) will follow, if we can find $|c_1| \leq c_2$ so that

$$L(b_1, b_2, x_1, x_2, \delta) \leq L(c_1, c_2, x_1, x_2, \delta) \tag{5.68}$$

while

$$L(c_1, c_2, y_1, y_2, \epsilon) \leq H(y_1, y_2, \epsilon), \ \forall (y_1, y_2, \epsilon) \in \mathbb{R}^3. \tag{5.69}$$

This is done as follows. Let

$$\sigma(b_1, b_2) := \begin{cases} 1 & \text{if } b_1b_2 > 0 \\ 0 & \text{if } b_1b_2 = 0 \\ -1 & \text{if } b_1b_2 < 0. \end{cases}$$

Let τ be a permutation of $\{1, 2\}$ such that

$$\left|b_{\tau(1)}\right| \leq \left|b_{\tau(2)}\right|$$

and

$$c_1 := \sigma(b_1, b_2)\left|b_{\tau(1)}\right| \quad \text{and} \quad c_2 := \left|b_{\tau(2)}\right|.$$

According to Proposition 13.9, the inequality (5.68) is satisfied. Observe that, for every $y_1, y_2 \in \mathbb{R}$,

$$c_1y_1 + c_2y_2 = \begin{cases} b_1y_1 + b_2y_2 & \text{if } b_2 \geq |b_1| \\ -b_1y_1 - b_2y_2 & \text{if } -b_2 \geq |b_1| \\ b_2y_1 + b_1y_2 & \text{if } b_1 \geq |b_2| \\ -b_2y_1 - b_1y_2 & \text{if } -b_1 \geq |b_2|. \end{cases}$$

This implies that

$$\begin{aligned} L(c_1, c_2, y_1, y_2, \epsilon) \leq \ & \max\{L(b_1, b_2, y_1, y_2, \epsilon), L(b_1, b_2, -y_1, -y_2, \epsilon), \\ & L(b_1, b_2, y_2, y_1, \epsilon), L(b_1, b_2, -y_2, -y_1, \epsilon)\}. \end{aligned}$$

Since (5.67) holds and $(x_1, x_2) \to H(x_1, x_2, \delta)$ is $\Pi_e(2)$-invariant, we get (5.69) and hence the claim (5.66) is established. ■

Having discussed the convexity and the polyconvexity of $SO(2) \times SO(2)$ or $O(N) \times O(n)$-invariant functions, one would be tempted to think that similar results exist for rank one and quasiconvex functions. This is not the case as was first observed by Dacorogna-Koshigoe [192] (see Example 5.45) for rank one convex functions. Later Müller [463] showed the same result for quasiconvex functions.

Example 5.45 The examples are based on computations of Dacorogna-Douchet-Gangbo-Rappaz in [185]. In both examples, $N = n = 2$ and $b \geq 0$.

(i) Let $\alpha > 2 + \sqrt{2}$ and

$$f_{\alpha,b}(\xi) = |\xi|^{2\alpha} - 2^{\alpha-1} b \,|\det \xi|^{\alpha}.$$

(ii) Let $\alpha > (9 + 5\sqrt{5})/4$ and

$$f_{\alpha,b}(\xi) = |\xi|^{2\alpha} \left(|\xi|^2 - 2b \det \xi \right).$$

Note that both functions are $SO(2) \times SO(2)$-invariant. In both cases, there exist $b_2 < b_1$ (for the precise values of b_1, b_2 see [185]) such that

$$f_{\alpha,b} \text{ is rank one convex } \Leftrightarrow \; b \leq b_2\,,$$

$$f_{\alpha,b}|_{\mathbb{R}^{2\times 2}_d} \text{ is rank one convex } \Leftrightarrow \; b \leq b_1\,. \qquad \diamondsuit$$

We finally conclude this section by mentioning other results on rank one convexity of $O(n) \times O(n)$-invariant functions. As seen in Proposition 5.31, any such function is necessarily of the form

$$f(\xi) = g(\lambda_1(\xi), \cdots, \lambda_n(\xi))$$

where $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ are the singular values of the matrix $\xi \in \mathbb{R}^{n\times n}$. Assuming that the function f is twice differentiable, it is therefore natural to ask conditions on the derivatives of g that ensure the rank one convexity of the function f. This was achieved by Knowles-Sternberg [371] when $n = 2$ and then in various different ways by Aubert [41], Aubert-Tahraoui [48], Ball [55], Dacorogna-Marcellini [202] and Davies [223]. When $n = 3$, Aubert-Tahraoui in [47] gave also some necessary conditions and, although in a slightly different context, necessary and sufficient conditions were derived by Simpson-Spector [540] (see also Zee-Sternberg [613]). In the case of general n, certain results exist but are less explicit; see Dacorogna [182] and Silhavy [536].

5.3.5 Functions depending on a quasiaffine function

The following theorem was established in Dacorogna [173].

Theorem 5.46 *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$, $\Phi : \mathbb{R}^{N\times n} \to \mathbb{R}$ be quasiaffine but not identically constant and $g : \mathbb{R} \to \mathbb{R}$ be such that*

$$f(\xi) = g(\Phi(\xi))$$

(in particular, if $N = n$, one can take $\Phi(\xi) = \det \xi$). Then

$$f \text{ polyconvex } \Leftrightarrow f \text{ quasiconvex } \Leftrightarrow f \text{ rank one convex } \Leftrightarrow g \text{ convex.}$$

Proof. The implications

$$g \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank one convex}$$

follow immediately from Theorem 5.3. It therefore remains to show that

$$f \text{ rank one convex } \Rightarrow g \text{ convex.}$$

We want to prove that for $t \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, then

$$g(t\alpha + (1-t)\beta) \leq tg(\alpha) + (1-t)g(\beta)$$

provided f is rank one convex. Following Theorem 5.20, we have that

$$\Phi(\xi) = a_0 + \langle a; T(\xi)\rangle = a_0 + \langle a_1; \xi\rangle + \sum_{j=2}^{n\wedge N} \left\langle a_j; \operatorname{adj}_j \xi \right\rangle,$$

where $a_0 \in \mathbb{R}$, $a_1 \in \mathbb{R}^{N\times n}$ and $a_j \in \mathbb{R}^{\sigma(j)}$ where $\sigma(j) = \binom{N}{j}\binom{n}{j}$. Since Φ is not identically constant, then at least one of the a_j , $1 \leq j \leq n \wedge N$ is not zero. Let s be such that $a_s \neq 0$ but $a_{s-1} = a_{s-2} = \cdots = a_1 = 0$ (if $a_1 \neq 0$, we then take $s = 1$). Since $a_s \neq 0$ $\left(\in \mathbb{R}^{\sigma(s)}\right)$ we have that at least one of the components of $a_s = (a_s^1, \cdots, a_s^{\sigma(s)})$ is non-zero. For notational convenience, we take $a_s^{\sigma(s)} \neq 0$.

First choose $\eta \in \mathbb{R}^{N \times n}$ in the following way

$$\eta = \begin{pmatrix} \eta_1^1 & \cdots & \eta_s^1 & \eta_{s+1}^1 & \cdots & \eta_n^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_1^s & \cdots & \eta_s^s & \eta_{s+1}^s & \cdots & \eta_n^s \\ \eta_1^{s+1} & \cdots & \eta_s^{s+1} & \eta_{s+1}^{s+1} & \cdots & \eta_n^{s+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_1^N & \cdots & \eta_s^N & \eta_{s+1}^N & \cdots & \eta_n^N \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha - a_0}{a_s^{\sigma(s)}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

More precisely, we take all components to be zero but the following ones:

$$\eta_1^1 = \frac{\alpha - a_0}{a_s^{\sigma(s)}},\ \eta_i^i = 1 \text{ for } 2 \leq i \leq s.$$

We next choose $\lambda \in \mathbb{R}^{N \times n}$ in exactly the same manner except that we replace the first component by $(\beta - a_0)/a_s^{\sigma(s)}$. We then immediately have

$$\begin{cases} \Phi(\eta) = \alpha,\ \Phi(\lambda) = \beta \\ \quad \operatorname{rank}\{\eta - \lambda\} \leq 1 \end{cases}$$

since $a_j = 0$ if $j < s$,

$$\operatorname{adj}_s \eta = (0, \cdots, 0, \frac{\alpha - a_0}{a_s^{\sigma(s)}}) \quad \text{and} \quad \operatorname{adj}_s \lambda = (0, \cdots, 0, \frac{\beta - a_0}{a_s^{\sigma(s)}})$$

and $\operatorname{adj}_j \eta = \operatorname{adj}_j \lambda = 0$ if $j \geq s+1$.

We also clearly have from Theorem 5.20 that

$$\Phi(t\eta + (1-t)\lambda) = t\alpha + (1-t)\beta.$$

Using the rank one convexity of f and the above construction we get

$$\begin{aligned} g(t\alpha + (1-t)\beta) &= g(\Phi(t\eta + (1-t)\lambda)) = f(t\eta + (1-t)\lambda) \\ &\leq tf(\eta) + (1-t)f(\lambda) = tg(\Phi(\eta)) + (1-t)g(\Phi(\lambda)) \\ &= tg(\alpha) + (1-t)g(\beta) \end{aligned}$$

which is the desired result. ■

5.3.6 The area type case

The next result is due to Morrey [453], but we follow a different proof, established in Dacorogna [171].

Theorem 5.47 *Let $N = n + 1$ and for $\xi \in \mathbb{R}^{(n+1)\times n}$ let*

$$\operatorname{adj}_n \xi = (\det \widehat{\xi}^1, -\det \widehat{\xi}^2, \cdots, (-1)^{k+1} \det \widehat{\xi}^k, \cdots, (-1)^{n+2} \det \widehat{\xi}^{n+1}),$$

where $\widehat{\xi}^k$ is the $n \times n$ matrix obtained from ξ by suppressing the kth row. Let $f : \mathbb{R}^{(n+1)\times n} \to \mathbb{R}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that

$$f(\xi) = g(\operatorname{adj}_n \xi).$$

Then

$$f \text{ polyconvex} \Leftrightarrow f \text{ quasiconvex} \Leftrightarrow f \text{ rank one convex} \Leftrightarrow g \text{ convex.}$$

Remark 5.48 It is clear that if $u : \mathbb{R}^n \to \mathbb{R}^{n+1}$, then $\operatorname{adj}_n \nabla u$ represents the normal to the surface

$$\{u(x) : x \in \mathbb{R}^n\}.$$

In the case $n = 2$, $u(x_1, x_2) = (u^1, u^2, u^3)$ we have

$$\operatorname{adj}_2 \nabla u = \begin{pmatrix} \frac{\partial u^2}{\partial x_1}\frac{\partial u^3}{\partial x_2} - \frac{\partial u^2}{\partial x_2}\frac{\partial u^3}{\partial x_1} \\ \frac{\partial u^3}{\partial x_1}\frac{\partial u^1}{\partial x_2} - \frac{\partial u^1}{\partial x_1}\frac{\partial u^3}{\partial x_2} \\ \frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2} - \frac{\partial u^1}{\partial x_2}\frac{\partial u^2}{\partial x_1} \end{pmatrix}. \qquad \diamondsuit$$

Before proceeding with the proof of the theorem, we mention an algebraic lemma, stronger than needed, that will be fully used in Section 6.6.4. We will prove the lemma, established in Dacorogna [171], after the proof of Theorem 5.47.

Lemma 5.49 *Let $0 < t < 1$, $a, b \in \mathbb{R}^{n+1}$ and $\xi \in \mathbb{R}^{(n+1)\times n}$ be such that*

$$\operatorname{adj}_n \xi = ta + (1-t)b \neq 0.$$

Then there exist $\alpha, \beta \in \mathbb{R}^{(n+1)\times n}$ such that

$$\begin{cases} \xi = t\alpha + (1-t)\beta \\ \operatorname{adj}_n \alpha = a, \ \operatorname{adj}_n \beta = b \\ \operatorname{rank}\{\alpha - \beta\} \leq 1. \end{cases}$$

Proof. (Theorem 5.47). The implications

$$g \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex}$$

follow immediately from Theorem 5.3.

It therefore remains to show that

$$f \text{ rank one convex } \Rightarrow \ g \text{ convex.}$$

We let $t \in (0,1)$, $a, b \in \mathbb{R}^{n+1}$ and we wish to show that

$$g\,(ta + (1-t)\,b) \leq tg\,(a) + (1-t)\,g\,(b) \tag{5.70}$$

provided f is rank one convex and $f\,(\xi) = g\,(\operatorname{adj}_n \xi)$. We divide the proof into two cases.

Case 1: $ta + (1-t)\,b \neq 0$. We let

$$c := ta + (1-t)\,b = \left(c^1, \cdots, c^{n+1}\right) \in \mathbb{R}^{n+1}.$$

Since $c \neq 0$, we assume, for notational convenience, that $c^1 \neq 0$ (the general case is handled similarly). We then let

$$\xi := \begin{pmatrix} \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_n^2 \\ \xi_1^3 & \xi_2^3 & \cdots & \xi_n^3 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{n+1} & \xi_2^{n+1} & \cdots & \xi_n^{n+1} \end{pmatrix} = \begin{pmatrix} -c^2 & -\dfrac{c^3}{c^1} & \cdots & -\dfrac{c^{n+1}}{c^1} \\ c^1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It is then easy to see that

$$\operatorname{adj}_n \xi = c = ta + (1-t)\,b \neq 0.$$

We may now apply Lemma 5.49 to get $\alpha, \beta \in \mathbb{R}^{(n+1)\times n}$ such that

$$\begin{cases} \xi = t\alpha + (1-t)\,\beta \\ \operatorname{adj}_n \alpha = a, \ \ \operatorname{adj}_n \beta = b \\ \operatorname{rank}\{\alpha - \beta\} \leq 1. \end{cases}$$

Returning to (5.70), using the rank one convexity of f, we obtain

$$\begin{aligned} g\,(ta + (1-t)\,b) &= g\,(\operatorname{adj}_n \xi) = f\,(\xi) = f\,(t\alpha + (1-t)\,\beta) \\ &\leq tf\,(\alpha) + (1-t)\,f\,(\beta) = tg\,(a) + (1-t)\,g\,(b)\,, \end{aligned}$$

which is precisely the result.

Case 2: $ta + (1-t)\,b = 0$. Observe first that the rank one convexity of f implies that f is continuous (cf. Theorem 5.3), thus from $f\,(\xi) = g\,(\operatorname{adj}_n \xi)$ we deduce that g is continuous. Therefore using Case 1 for $\widetilde{a} = a + (\epsilon, 0, \cdots, 0)$ and $\widetilde{b} = b + (\epsilon, 0, \cdots, 0)$ where $\epsilon > 0$ is arbitrary, we deduce (5.70) by continuity of g. ■

We now conclude this section by proving Lemma 5.49.

Proof. We give here a different proof than the one in Dacorogna [171] or [179]. We decompose the proof into two steps.

Step 1. We start by assuming that $\xi \in \mathbb{R}^{(n+1)\times n}$ has the following special form

$$\xi = \operatorname{diag}_{(n+1)\times n}(x_1, \cdots, x_n) = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \\ 0 & \cdots & 0 \end{pmatrix}$$

with $x_1, \cdots, x_n \in \mathbb{R}$, all different from 0, and thus

$$\operatorname{adj}_n \xi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^n x_1 \cdots x_n \end{pmatrix} = ta + (1-t)\, b \neq 0.$$

We next observe that for every $\lambda \in \mathbb{R}^{n+1}$ and $\mu \in \mathbb{R}^n$ we have

$$\operatorname{adj}_n(\xi + \lambda \otimes \mu) = \operatorname{adj}_n \xi + \left\langle \operatorname{adj}_{n-1} \xi; \lambda \otimes \mu \right\rangle$$

where

$$\left\langle \operatorname{adj}_{n-1} \xi; \lambda \otimes \mu \right\rangle = (-1)^n \begin{pmatrix} -\lambda^{n+1}\mu_1 \prod_{j\neq 1} x_j \\ \vdots \\ -\lambda^{n+1}\mu_n \prod_{j\neq n} x_j \\ \sum_{s=1}^n \left[\lambda^s \mu_s \prod_{j\neq s} x_j \right] \end{pmatrix}.$$

We then search for $\alpha, \beta \in \mathbb{R}^{(n+1)\times n}$ of the form

$$\begin{cases} \alpha = \xi + (1-t)\, \lambda \otimes \mu \\ \beta = \xi - t\lambda \otimes \mu \end{cases}$$

where $\lambda \in \mathbb{R}^{n+1}$ and $\mu \in \mathbb{R}^n$ are to be determined. We therefore immediately deduce that

$$\xi = t\alpha + (1-t)\,\beta \quad \text{and} \quad \operatorname{rank}\{\alpha - \beta\} \leq 1.$$

We next observe that

$$\begin{aligned} \operatorname{adj}_n \alpha &= \operatorname{adj}_n \xi + (1-t) \left\langle \operatorname{adj}_{n-1} \xi; \lambda \otimes \mu \right\rangle \\ \operatorname{adj}_n \beta &= \operatorname{adj}_n \xi - t \left\langle \operatorname{adj}_{n-1} \xi; \lambda \otimes \mu \right\rangle. \end{aligned}$$

Thus the equations $\operatorname{adj}_n \alpha = a$ and $\operatorname{adj}_n \beta = b$ reduce to the single system of equations

$$\left\langle \operatorname{adj}_{n-1} \xi; \lambda \otimes \mu \right\rangle = a - b := c \tag{5.71}$$

that we solve by considering two cases.

Case 1: $c^1 = \cdots = c^n = 0$. We then choose

$$\lambda^1 = 1,\ \lambda^2 = \cdots = \lambda^{n+1} = 0,\ \mu_2 = \cdots = \mu_n = 0$$

and

$$\mu_1 = (-1)^n \frac{c^{n+1}}{\prod_{j=2}^n x_j}$$

so as to satisfy (5.71).

Case 2: there exists $k \in \{1, \cdots, n\}$ with $c^k \neq 0$. Equation (5.71) is then satisfied if we choose

$$\mu_i = (-1)^{n+1} \frac{c^i}{\prod_{j \neq i} x_j},\ i = 1, \cdots, n$$

and $\lambda^i = 0$ whenever $i \neq k, n+1$, $\lambda^{n+1} = 1$ and

$$\lambda^k = (-1)^n \frac{c^{n+1}}{\mu_k \prod_{j \neq k} x_j} = \frac{-c^{n+1}}{c^k}.$$

Step 2. We now reduce the general case $\xi \in \mathbb{R}^{(n+1)\times n}$ to the special form of the previous step by using the singular values decomposition theorem (cf. Theorem 13.3). We can indeed find $R \in O(n+1)$, $Q \in SO(n)$ and $x_1, \cdots, x_n \in \mathbb{R}$ so that

$$\widetilde{\xi} := R\xi Q = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \\ 0 & \cdots & 0 \end{pmatrix}.$$

Using Proposition 5.66, and noting that $\operatorname{adj}_n Q = \det Q = 1$, we find that

$$\operatorname{adj}_n \widetilde{\xi} = \operatorname{adj}_n R \operatorname{adj}_n \xi \neq 0.$$

Observing that $\operatorname{adj}_n R \in O(n+1)$ (by Proposition 5.66), we set

$$\widetilde{a} := \operatorname{adj}_n R\, a \quad \text{and} \quad \widetilde{b} := \operatorname{adj}_n R\, b$$

and we can find, from Step 1, $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}^{(n+1)\times n}$ such that

$$\begin{cases} \widetilde{\xi} = t\widetilde{\alpha} + (1-t)\widetilde{\beta} \\ \operatorname{adj}_n \widetilde{\alpha} = \widetilde{a},\ \operatorname{adj}_n \widetilde{\beta} = \widetilde{b} \\ \operatorname{rank}\{\widetilde{\alpha} - \widetilde{\beta}\} \leq 1. \end{cases}$$

Setting

$$\alpha := R^t\, \widetilde{\alpha}\, Q^t \quad \text{and} \quad \beta := R^t\, \widetilde{\beta}\, Q^t$$

we have indeed obtained the claim of the lemma. ■

5.3.7 The example of Sverak

We now turn to an example of a rank one convex function that is not quasiconvex. This fundamental result was obtained by Sverak [551] when $N \geq 3$ and $n \geq 2$ and we follow his presentation here. The question of extending Sverak example to the case where $n \geq N = 2$ is still open.

Theorem 5.50 *Let $N \geq 3$ and $n \geq 2$. Then there exists $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ rank one convex but not quasiconvex.*

Proof. The proof is divided into four steps.

Step 1. Assume that we have already constructed a rank one convex function $g : \mathbb{R}^{3\times 2} \to \mathbb{R}$, that is not quasiconvex. In particular (appealing to Proposition 5.13), there exists $\eta \in \mathbb{R}^{3\times 2}$ and $\psi \in W_{per}^{1,\infty}\left(D_2; \mathbb{R}^3\right)$, where $D_2 = (0,1)^2$ such that

$$\int_{D_2} g\left(\eta + \nabla\psi\left(x\right)\right) dx < g\left(\eta\right).$$

Then define $\pi : \mathbb{R}^{N\times n} \to \mathbb{R}^{3\times 2}$ to be

$$\pi\left(\xi\right) = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \\ \xi_1^3 & \xi_2^3 \end{pmatrix}, \text{ for } \xi \in \mathbb{R}^{N\times n}.$$

Finally, let

$$f\left(\xi\right) = g\left(\pi\left(\xi\right)\right).$$

This function is clearly rank one convex, since g is. It is also not quasiconvex, since choosing any $\xi \in \mathbb{R}^{N\times n}$ so that $\pi\left(\xi\right) = \eta$, $D_n = (0,1)^n$ and

$$\varphi^i\left(x_1, \cdots, x_n\right) := \begin{cases} \psi^i\left(x_1, x_2\right) & \text{if } i = 1, 2, 3 \\ 0 & \text{if not} \end{cases}$$

we get that $\varphi \in W_{per}^{1,\infty}\left(D_n; \mathbb{R}^N\right)$ and

$$\int_{D_n} f\left(\xi + \nabla\varphi\left(x\right)\right) dx < f\left(\xi\right).$$

Step 2. In view of Step 1, it is therefore sufficient to prove the theorem for functions $f : \mathbb{R}^{3\times 2} \to \mathbb{R}$. We first let

$$L := \{\xi \in \mathbb{R}^{3\times 2} : \xi = \begin{pmatrix} x & 0 \\ 0 & y \\ z & z \end{pmatrix} \text{ where } x, y, z \in \mathbb{R}\}$$

and $P : \mathbb{R}^{3\times 2} \to L$ be defined by

$$P\left(\xi\right) := \begin{pmatrix} \xi_1^1 & 0 \\ 0 & \xi_2^2 \\ \left(\xi_1^3 + \xi_2^3\right)/2 & \left(\xi_1^3 + \xi_2^3\right)/2 \end{pmatrix}.$$

We next let $g : L \to \mathbb{R}$ be defined by

$$g\begin{pmatrix} x & 0 \\ 0 & y \\ z & z \end{pmatrix} = -xyz.$$

Finally, for $\epsilon, \gamma \geq 0$ let the function $f_{\epsilon,\gamma} : \mathbb{R}^{3\times 2} \to \mathbb{R}$ be such that

$$f_{\epsilon,\gamma}(\xi) := g(P(\xi)) + \epsilon|\xi|^2 + \epsilon|\xi|^4 + \gamma|\xi - P(\xi)|^2.$$

We claim that we can find ϵ and γ so that $f_{\epsilon,\gamma}$ is rank one convex (see Step 4) but not quasiconvex (see Step 3), giving the desired claim.

Step 3. Choose $\xi = 0$ and

$$\varphi(x_1, x_2) = \frac{1}{2\pi}\begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_2 \\ \sin 2\pi(x_1 + x_2) \end{pmatrix}.$$

Observe that $\varphi \in W_{per}^{1,\infty}(D; \mathbb{R}^3)$, where $D = (0,1)^2$ and $\nabla\varphi \in L$ (hence $P(\nabla\varphi) = \nabla\varphi$). Moreover,

$$\int_D g(\nabla\varphi(x))\,dx = -\int_0^1\int_0^1 (\cos 2\pi x_1)^2(\cos 2\pi x_2)^2\,dx_1 dx_2 < 0.$$

Therefore (see Proposition 5.13), for every $\epsilon \geq 0$ sufficiently small and for every $\gamma \geq 0$, the function $f_{\epsilon,\gamma}$ is not quasiconvex.

Step 4. We now show that for every $\epsilon > 0$, we can find $\gamma = \gamma(\epsilon) > 0$ so that $f_{\epsilon,\gamma}$ is rank one convex. This is equivalent to showing that the Legendre-Hadamard condition is satisfied, namely

$$L_f(\xi,\eta) := \frac{d^2}{dt^2}\left[f_{\epsilon,\gamma}(\xi + t\eta)\right]\Big|_{t=0} \geq 0,\ \forall \xi, \eta \in \mathbb{R}^{3\times 2} \text{ with } \operatorname{rank}\{\eta\} = 1. \tag{5.72}$$

Letting

$$L_g(\xi,\eta) := \frac{d^2}{dt^2}\left[g(P(\xi + t\eta))\right]\Big|_{t=0}$$

we find

$$L_f(\xi,\eta) = L_g(\xi,\eta) + 2\epsilon|\eta|^2 + 4\epsilon|\xi|^2|\eta|^2 + 8\epsilon(\langle\xi;\eta\rangle)^2 + 2\gamma|\eta - P(\eta)|^2.$$

We show (5.72) in two substeps.

Step 4'. Observe that since g is a homogeneous polynomial of degree three, we can find $c > 0$ such that

$$L_g(\xi,\eta) \geq -c|\xi||\eta|^2.$$

We therefore deduce that

$$L_f(\xi,\eta) \geq (-c + 4\epsilon|\xi|)|\xi||\eta|^2$$

and thus (5.72) holds for every $\eta \in \mathbb{R}^{3\times 2}$ (independently of the fact that $\operatorname{rank}\{\eta\} = 1$) and for every $\xi \in \mathbb{R}^{3\times 2}$ that satisfies

$$|\xi| \geq \frac{c}{4\epsilon}\,.$$

Step 4". It therefore remains to show (5.72) in the compact set

$$K := \left\{(\xi,\eta) \in \mathbb{R}^{3\times 2} \times \mathbb{R}^{3\times 2} : |\xi| \leq \frac{c}{4\epsilon},\ |\eta| = 1,\ \operatorname{rank}\{\eta\} = 1\right\}$$

in view of Step 4' and of the fact that $L_f(\xi,\eta)$ is homogeneous of degree two in the variable η.

Moreover, we also find that

$$L_f(\xi,\eta) \geq H(\xi,\eta,\gamma) := L_g(\xi,\eta) + 2\epsilon|\eta|^2 + 2\gamma|\eta - P(\eta)|^2$$

and therefore (5.72) will follow if we can show that for every $\epsilon > 0$ we can find $\gamma = \gamma(\epsilon)$ so that $H \geq 0$ on K.

Assume, for the sake of contradiction, that this is not the case. We can then find $\gamma_\nu \to \infty$, $(\xi_\nu,\eta_\nu) \in K$ so that

$$L_g(\xi_\nu,\eta_\nu) + 2\epsilon \leq L_g(\xi_\nu,\eta_\nu) + 2\epsilon + 2\gamma_\nu|\eta_\nu - P(\eta_\nu)|^2 < 0.$$

Since K is compact, we have up to a subsequence (still labeled (ξ_ν,η_ν)) that

$$(\xi_\nu,\eta_\nu) \to (\xi,\eta) \in K,\ L_g(\xi,\eta) + 2\epsilon \leq 0 \text{ and } P(\eta) = \eta.$$

However we have $\epsilon > 0$ and, by construction,

$$L_g(\xi,\eta) \equiv 0,\ \forall\,\xi,\eta \in \mathbb{R}^{3\times 2} \text{ with } P(\eta) = \eta \text{ and } \operatorname{rank}\{\eta\} = 1.$$

This leads to the desired contradiction and therefore the theorem holds. ■

5.3.8 The example of Alibert-Dacorogna-Marcellini

We now turn our attention to an example where $N = n = 2$. It involves a homogeneous polynomial of degree four. We characterize, with the help of one single real parameter, the different notions of convexity.

Theorem 5.51 *Let $\gamma \in \mathbb{R}$ and let $f_\gamma : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined as*

$$f_\gamma(\xi) = |\xi|^2\,(\,|\xi|^2 - 2\gamma\det\xi\,).$$

Then

$$\begin{array}{rcl} f_\gamma \textit{ is convex} & \Leftrightarrow & |\gamma| \le \gamma_c = \dfrac{2}{3}\sqrt{2}, \\ f_\gamma \textit{ is polyconvex} & \Leftrightarrow & |\gamma| \le \gamma_p = 1, \\ f_\gamma \textit{ is quasiconvex} & \Leftrightarrow & |\gamma| \le \gamma_q \quad \textit{and} \quad \gamma_q > 1, \\ f_\gamma \textit{ is rank one convex} & \Leftrightarrow & |\gamma| \le \gamma_r = \dfrac{2}{\sqrt{3}}\,. \end{array}$$

We now make some comments about this theorem.

(i) The last result and the fact that if f_γ is polyconvex, then $|\gamma| \le 1$, were established by Dacorogna-Marcellini [193]. All the other results were first proved in Alibert-Dacorogna [14]. The most interesting fact is the third one.

(ii) The example also provides a quasiconvex function that is not polyconvex (such an example was already seen in Theorem 5.25 when $N, n \ge 3$; see also when $n = N = 2$, Theorem 5.54 and Sverak [552]).

(iii) The problem of knowing if $\gamma_q = 2/\sqrt{3}$ is still open. If this is not the case (meaning that $\gamma_q < 2/\sqrt{3}$), then this would provide a rank one convex function that is not quasiconvex, giving a final answer to this long standing question. However many numerical evidences tend to indicate that $\gamma_q = 2/\sqrt{3}$, see Dacorogna-Douchet-Gangbo-Rappaz [185], Dacorogna-Haeberly [191] and Gremaud [321].

(iv) The polyconvexity of the function

$$f_1(\xi) = |\xi|^2 \left(|\xi|^2 - 2\det\xi \right)$$

has, since the work of Alibert-Dacorogna [14], been reproved notably by Iwaniec-Lutoborski [353]. Hartwig [335] also proved this fact exhibiting a convex function $F : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$, namely

$$F(\xi, \delta) = \begin{cases} [\,|\xi|^2 + 2\det\xi - 2\delta\,][\,|\xi|^2 + 2\det\xi - 4\delta\,] & \text{if } |\xi|^2 + 2\det\xi \ge 4\delta \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$f_1(\xi) = F(\xi, \det\xi)\,.$$

We now proceed with the proof of the theorem. But before that we want to observe that in all four statements we can restrict our attention to the case where $\gamma \ge 0$. Indeed, observe first that

$$f_\gamma(Q\xi) = f_{-\gamma}(\xi) \text{ for every } \xi \in \mathbb{R}^{2\times 2} \text{ and } Q \in O(2) \text{ with } \det Q = -1.$$

This easily implies that f_γ is convex (respectively polyconvex, quasiconvex, rank one convex) if and only if $f_{-\gamma}$ is convex (respectively polyconvex, quasiconvex, rank one convex). Hence, we may assume throughout, without loss of generality, that $\gamma \ge 0$.

We first start with the statement on the convexity of f_γ.

Proof. (Theorem 5.51: Convexity). We have to show that

$$f_\gamma \text{ is convex } \Leftrightarrow \gamma \leq \gamma_c = \frac{2}{3}\sqrt{2}.$$

This result was first proved by Alibert-Dacorogna, but we give here the proof based on Dacorogna-Maréchal [204].

According to Theorem 5.33, it is sufficient to verify the claim only on diagonal matrices. So let

$$g(x, y) := \left(x^2 + y^2\right)\left[\left(x^2 + y^2\right) - 2\gamma xy\right].$$

The Hessian of g is therefore given by

$$\nabla^2 g(x, y) = \begin{pmatrix} 4\left(x^2 + y^2\right) + 8x^2 - 12\gamma xy & 8xy - 6\gamma\left(x^2 + y^2\right) \\ 8xy - 6\gamma\left(x^2 + y^2\right) & 4\left(x^2 + y^2\right) + 8y^2 - 12\gamma xy \end{pmatrix}.$$

Setting

$$x = r\cos\frac{\theta}{2},\ y = r\sin\frac{\theta}{2}$$

we find that

$$\nabla^2 g(x, y) = 2r^2 \begin{pmatrix} 4 + 2\cos\theta - 3\gamma\sin\theta & 2\sin\theta - 3\gamma \\ 2\sin\theta - 3\gamma & 4 - 2\cos\theta - 3\gamma\sin\theta \end{pmatrix}.$$

The function g is therefore convex if and only if the trace and the determinant of $\nabla^2 g$ are non negative. This is true if and only if

$$4 - 3\gamma\sin\theta \geq 0,$$

$$12 - 9\gamma^2 - 12\gamma\sin\theta + 9\gamma^2\sin^2\theta \geq 0.$$

Step 1: ($\Leftarrow$). We first consider the case where $\gamma \leq \gamma_c = 2\sqrt{2}/3$. This immediately implies that the first inequality holds. Since the discriminant of the polynomial (in $\sin\theta$) that appears in the second inequality is given by

$$\Delta = 36\gamma^2\left(9\gamma^2 - 8\right) \leq 0,$$

we have indeed obtained the claim.

Step 2: ($\Rightarrow$). We now show that if f_γ is convex, then $\gamma \leq \gamma_c$. We prove the result by contradiction and write for a certain $t > 1$

$$\gamma = t\gamma_c = \frac{2}{3}\sqrt{2}\,t.$$

The polynomial that appears in the second inequality is then transformed into

$$12 - 8t^2 - 8\sqrt{2}\,t\sin\theta + 8t^2\sin^2\theta.$$

Observe that the minimum of this polynomial (in $\sin\theta$) is attained at

$$\sin\theta = \frac{1}{\sqrt{2}\,t}$$

and its value is then

$$8\left(1-t^2\right) < 0.$$

This is the desired contradiction. ■

We now discuss the rank one convexity of f_γ.

Proof. (Theorem 5.51: Rank one convexity). We have to show that

$$f_\gamma \text{ is rank one convex } \Leftrightarrow\ 0 \le \gamma \le \gamma_r = 2/\sqrt{3}$$

and this was established first by Dacorogna-Marcellini [193].

We start with some notations and with the computation of the second variation.

Notation. To every $\xi \in \mathbb{R}^{2\times 2}$ we associate $\widetilde{\xi} \in \mathbb{R}^{2\times 2}$ in the following way

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix},\ \widetilde{\xi} = \begin{pmatrix} \xi_2^2 & -\xi_1^2 \\ -\xi_2^1 & \xi_1^1 \end{pmatrix}.$$

We immediately observe that

$$\begin{cases} |\xi| = |\,\widetilde{\xi}\,|\,,\ \det\xi = \det\widetilde{\xi} \\ \langle\, \xi; \widetilde{\eta}\,\rangle = \langle\, \widetilde{\xi}; \eta\,\rangle\,,\ \langle\, \xi; \widetilde{\xi}\,\rangle = 2\det\xi \\ \det(\xi+\eta) = \det\xi + \langle\, \widetilde{\xi}; \eta\,\rangle + \det\eta \end{cases}$$

where $\langle\cdot;\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{2\times 2}$. We also have that

$$\frac{\partial}{\partial \xi_j^i}(\det\xi) = \widetilde{\xi}_j^i,\ \text{ i.e. }\ \nabla(\det\xi) = \widetilde{\xi}.$$

(Note that in the notations of Section 5.4 $\widetilde{\xi} = \operatorname{adj}_1\xi$).

It will be convenient to decompose any matrix in its "*conformal*" and "*anti-conformal*" parts, which are given by

$$\xi^+ := \frac{1}{2}(\xi + \widetilde{\xi}\,),\ \xi^- := \frac{1}{2}(\xi - \widetilde{\xi}\,).$$

We find the following relations. For $\xi, \eta \in \mathbb{R}^{2\times 2}$ we have

$$2\det\xi^+ = \left|\xi^+\right|^2 \quad\text{and}\quad 2\det\xi^- = -\left|\xi^-\right|^2$$

$$|\xi|^2 = \left|\xi^+\right|^2 + \left|\xi^-\right|^2 \quad\text{and}\quad 2\det\xi = \left|\xi^+\right|^2 - \left|\xi^-\right|^2 = 2\det\xi^+ + 2\det\xi^-$$

$$\langle\, \xi;\eta\,\rangle = \langle\, \xi^+;\eta^+\,\rangle + \langle\, \xi^-;\eta^-\,\rangle \quad\text{and}\quad \langle\, \xi^+;\eta^-\,\rangle = \langle\, \xi^-;\eta^+\,\rangle = 0$$

$$|\xi|^2 - 2\det\xi = 2\left|\xi^-\right|^2 \quad \text{and} \quad |\xi|^2 + 2\det\xi = 2\left|\xi^+\right|^2 .$$

Second variation. We next compute the second variation of f_γ

$$\psi_\gamma(\xi,\eta) := \sum_{i,j=1}^{2}\sum_{\alpha,\beta=1}^{2} \frac{\partial^2 f_\gamma}{\partial\xi_\alpha^i \partial\xi_\beta^j}\,\eta_\alpha^i \eta_\beta^j .$$

We first calculate ∇f and find

$$\frac{\partial f_\gamma}{\partial \xi_\alpha^i} = 4|\xi|^2 \xi_\alpha^i - 4\gamma(\det\xi)\,\xi_\alpha^i - 2\gamma|\xi|^2 \widetilde{\xi}_\alpha^i .$$

We then deduce that, for $1 \le i,\, j,\, \alpha,\, \beta \le 2$,

$$\begin{aligned}\frac{\partial^2 f_\gamma}{\partial\xi_\alpha^i \partial\xi_\beta^j} &= 8\xi_\alpha^i \xi_\beta^j + 4|\xi|^2 \delta^{ij}\delta_{\alpha\beta} - 4\gamma\xi_\alpha^i \widetilde{\xi}_\beta^j \\ &\quad -4\gamma(\det\xi)\,\delta^{ij}\delta_{\alpha\beta} - 4\gamma\widetilde{\xi}_\alpha^i \xi_\beta^j - 2\gamma|\xi|^2 \widetilde{\delta}^{ij}\widetilde{\delta}_{\alpha\beta} ,\end{aligned}$$

where

$$\delta^{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} , \quad \widetilde{\delta}^{ij} = \begin{cases} (-1)^j & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

and similarly for $\delta_{\alpha\beta}$ and $\widetilde{\delta}_{\alpha\beta}$. We therefore have that, if

$$\psi_\gamma(\xi,\eta) = \sum_{i,j,\alpha,\beta=1}^{2} \frac{\partial^2 f_\gamma}{\partial\xi_\alpha^i \partial\xi_\beta^j}\,\eta_\alpha^i \eta_\beta^j ,$$

then

$$\begin{aligned}\psi_\gamma(\xi,\eta) = \;& 8\left(\langle\xi;\eta\rangle\right)^2 + 4|\xi|^2|\eta|^2 - 8\gamma\langle\xi;\eta\rangle\langle\widetilde{\xi};\eta\rangle \\ & -4\gamma|\eta|^2\det\xi - 4\gamma|\xi|^2\det\eta.\end{aligned} \tag{5.73}$$

In terms of the above decomposition we have

$$\begin{aligned}\tfrac{1}{4}\psi_\gamma(\xi,\eta) = \;& 2(1-\gamma)\langle\xi^+;\eta^+\rangle^2 + 4\langle\xi^+;\eta^+\rangle\langle\xi^-;\eta^-\rangle \\ & +2(1+\gamma)\langle\xi^-;\eta^-\rangle^2 + (1-\gamma)|\xi^+|^2|\eta^+|^2 \\ & +|\xi^+|^2|\eta^-|^2 + |\xi^-|^2|\eta^+|^2 + (1+\gamma)|\xi^-|^2|\eta^-|^2 .\end{aligned} \tag{5.74}$$

Step 1: ($\Leftarrow$). We first show that if $\gamma \le 2/\sqrt{3}$, then f_γ is rank one convex. This is equivalent to showing (see Theorem 5.3) that the Legendre-Hadamard condition holds, i.e.,

$$\psi_\gamma(\xi,\eta) \ge 0, \text{ for every } \xi,\eta \in \mathbb{R}^{2\times 2} \text{ with } \det\eta = 0. \tag{5.75}$$

Using (5.74) and the fact that $\det\eta = 0$ if and only if $|\eta^+|^2 = |\eta^-|^2$, we immediately obtain

$$\begin{aligned}\tfrac{1}{4}\psi_\gamma(\xi,\eta) = \; & [\,(4-3\gamma)\langle\, \xi^+;\eta^+ \,\rangle^2 + 4\langle\, \xi^+;\eta^+ \,\rangle\langle\, \xi^-;\eta^- \,\rangle \\ & + (4+3\gamma)\langle\, \xi^-;\eta^- \,\rangle^2\,] \\ & +[\,(2-\gamma)(|\xi^+|^2|\eta^+|^2 - \langle\, \xi^+;\eta^+ \,\rangle^2) \\ & + (2+\gamma)(|\xi^-|^2|\eta^-|^2 - \langle\, \xi^-;\eta^- \,\rangle^2)\,].\end{aligned}$$

Since $\gamma \le 2/\sqrt{3} \le 2$, we deduce that the term in the second bracket is non-negative. The discriminant of the term in the first bracket is

$$\Delta = 4\,[4-(4-3\gamma)(4+3\gamma)]$$

and is non-positive if $\gamma \le 2/\sqrt{3}$. Therefore

$$\psi_\gamma(\xi,\eta) \ge 0, \text{ for every } \gamma \le \frac{2}{\sqrt{3}},$$

as claimed and the proof of Step 1 is complete.

Step 2: $(\Rightarrow)$. We now prove that if f_γ is rank one convex, then $\gamma \le 2/\sqrt{3}$. In order to show the result, we prove that if $\gamma > 2/\sqrt{3}$, then f_γ is not rank one convex, which is equivalent (see (5.75)) to showing that there exist $\xi_\gamma, \eta_\gamma \in \mathbb{R}^{2\times 2}$ with $\det\eta_\gamma = 0$ such that $\psi_\gamma(\xi_\gamma,\eta_\gamma) < 0$. This is easily done. Choose

$$\xi_\gamma = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \; \eta_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with a defined below. A direct computation gives

$$\frac{1}{4}\psi_\gamma(\xi_\gamma,\eta_\gamma) = 3a^2 - 3\gamma a + 1.$$

If the discriminant $\Delta = 9\gamma^2 - 12$ is positive, and this happens if $\gamma > 2/\sqrt{3}$, we can then choose a so that $\psi_\gamma(\xi_\gamma,\eta_\gamma) < 0$, as wished.

This concludes the study of the rank one convexity of the function f_γ. ■

We next turn our attention to the polyconvexity of f_γ.

Proof. (Theorem 5.51: Polyconvexity). We have to prove that

$$f_\gamma \text{ is polyconvex } \Leftrightarrow \; 0 \le \gamma \le \gamma_p = 1.$$

Step 1: $(\Rightarrow)$. We first show that if f_γ is polyconvex, then $0 \le \gamma \le 1$. Using Corollary 5.9, we can find $c \ge 0$ such that

$$f_\gamma(\xi) \ge -c(1+|\xi|^2) \text{ for every } \xi \in \mathbb{R}^{2\times 2}.$$

In particular the inequality holds for

$$\xi = t\,I, \quad t \in \mathbb{R}.$$

We therefore find that

$$f_\gamma(\xi) = 4(1-\gamma)\,t^4 \ge -c(1+2t^2).$$

Dividing both sides by t^4 and letting $t \to \infty$, we find that

$$1-\gamma \ge 0,$$

as wished.

Step 2: ($\Leftarrow$). We start with a preliminary step.

Step 2'. We show that if f_γ is polyconvex, then f_β is polyconvex for every $0 \le \beta \le \gamma$. We have to prove, according to Theorem 5.6, that

$$f_\beta(\xi) \le \sum_{i=1}^{6} \lambda_i f_\beta(\xi_i)$$

whenever ξ, $\xi_i \in \mathbb{R}^{2\times 2}$, $\lambda \in \Lambda_6$, satisfy

$$\xi = \sum_{i=1}^{6} \lambda_i \xi_i\,,\ \sum_{i=1}^{6} \lambda_i \det \xi_i = \det \xi.$$

We consider two cases.

Case 1. Assume that

$$\sum_{i=1}^{6} \lambda_i \left|\xi_i\right|^2 \det \xi_i \le \left|\xi\right|^2 \det \xi.$$

Then the claim is trivial since, recalling that $\beta \ge 0$ and observing that the function $\xi \to \left|\xi\right|^4$ is convex,

$$f_\beta(\xi) = \left|\xi\right|^4 - 2\beta\left|\xi\right|^2 \det\xi \le \sum_{i=1}^{6} \lambda_i [\,\left|\xi_i\right|^4 - 2\beta\left|\xi_i\right|^2 \det\xi_i\,] = \sum_{i=1}^{6} \lambda_i f_\beta(\xi_i)\,.$$

Case 2. Assume now that

$$\sum_{i=1}^{6} \lambda_i \left|\xi_i\right|^2 \det \xi_i \ge \left|\xi\right|^2 \det \xi.$$

Then the claim follows from the observation

$$f_\beta(\xi) = f_\gamma(\xi) - 2(\beta-\gamma)\left|\xi\right|^2 \det\xi,$$

from the hypothesis $0 \le \beta \le \gamma$ and from the polyconvexity of f_γ.

This achieves the proof of Step 2'.

Step 2". It therefore remains to show that

$$f_1(\xi) = \left|\xi\right|^2 (\,\left|\xi\right|^2 - 2\det\xi\,)$$

is polyconvex and the proof will be complete. As we already mentioned, there are three proofs of the preceding fact: the original one of Alibert-Dacorogna, the one of Hartwig and that of Iwaniec-Lutoborski, which is in the same spirit as the one of Alibert-Dacorogna but slightly simpler, and we will follow here this last one. We will show that, for every $\xi, \eta \in \mathbb{R}^{2\times 2}$,

$$f_1(\eta) \geq f_1(\xi) + 4(|\xi|^2 - \det \xi) \langle \xi ; \eta - \xi \rangle - 2 |\xi|^2 [\det \eta - \det \xi] .$$

This last inequality, combined with Theorem 5.6, gives that f_1 is polyconvex.

In order to show the inequality, it is sufficient (see Theorem 5.43 and the remark following it) to verify it on diagonal matrices, so we will set

$$\xi = \operatorname{diag}(a, b) \quad \text{and} \quad \eta = \operatorname{diag}(x, y) .$$

We therefore have to prove that

$$\begin{aligned}(x-y)^2 \left(x^2+y^2\right) \geq\ & (a-b)^2 \left(a^2+b^2\right) \\ & + 4 \left(a^2+b^2-ab\right) [a(x-a) + b(y-b)] \\ & - 2 \left(a^2+b^2\right) (xy - ab) .\end{aligned}$$

This can be rewritten, setting $X = x - a$ and $Y = y - b$, as

$$\alpha X^2 - 2\beta XY + \gamma Y^2 \geq 0 \tag{5.76}$$

where

$$\begin{aligned}\alpha &= (x-y+a)^2 + a^2 + (a-b)^2 \\ \beta &= (a-b)(x-y+a-b) \\ \gamma &= (x-y-b)^2 + b^2 + (a-b)^2 .\end{aligned}$$

The inequality (5.76), and thus the polyconvexity of f_1, follows from the fact that α, $\gamma \geq 0$ and from

$$\begin{aligned}\Delta &= \alpha\gamma - \beta^2 \\ &= [\, a^2 + b^2 - (x-y)(a-b)\,]^2 \\ &\quad + (x-y+a-b)^2 [\,(x-y)^2 + (a-b)^2\,] \\ &\geq 0.\end{aligned}$$

This concludes the claim for the polyconvexity. ■

We finally show the statement on quasiconvexity. It is clearly the most difficult to prove and we will first start with the following result, proved by Alibert-Dacorogna [14], which is a consequence of regularity results for Laplace equation. We will use it twice: once when $\xi = 0$ and $p = 4$, in the proof of Theorem 5.51, and the second time when $\xi = 0$ and $1 < p < 2$ in Theorem 5.54. The statement with $\xi \neq 0$ and $p = 4$ is just a curiosity.

Theorem 5.52 *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^2$ be a bounded open set. Then there exists $\epsilon = \epsilon(\Omega, p) > 0$ such that*

$$\int_\Omega [\,|\nabla\varphi(x)|^2 \pm 2\det(\nabla\varphi(x))\,]^{p/2} dx \geq \epsilon \int_\Omega |\nabla\varphi(x)|^p \, dx \tag{5.77}$$

for every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$.
Moreover, when $p = 4$, the inequality

$$\begin{aligned} \int_\Omega [\,|\xi + \nabla\varphi(x)|^2 & \pm 2\det(\xi + \nabla\varphi(x))\,]^2 dx \\ & \geq (|\xi|^2 \pm 2\det\xi)^2 \operatorname{meas}\Omega + \epsilon \int_\Omega |\nabla\varphi(x)|^4 \, dx \end{aligned} \tag{5.78}$$

holds for every $\xi \in \mathbb{R}^{2\times 2}$ and every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$.

The result (5.77) is clearly non-trivial, except when $p = 2$ (in this case we can take $\epsilon = 1$ and equality, instead of inequality, holds). Observe also that the inequality (5.77) shows that the functional on the left-hand side of (5.77) is coercive in $W_0^{1,p}(\Omega, \mathbb{R}^2)$, even though the integrand is not coercive (not even up to a quasiaffine function, which here can be at most quadratic).

Proof. (Theorem 5.52). We prove (5.77) and (5.78) only for the minus sign, the proof being identical for the plus sign. For this purpose we adapt an idea of Sverak [552].

Step 1. We first prove the result for $\xi = 0$ and $1 < p < \infty$. We start with an algebraic relation. We clearly have that there exists a constant $\alpha = \alpha(p)$ such that for every $\xi \in \mathbb{R}^{2\times 2}$

$$\begin{aligned} \left[|\xi|^2 - 2\det\xi\right]^{p/2} &= \left[\left(\xi_1^1 - \xi_2^2\right)^2 + \left(\xi_2^1 + \xi_1^2\right)^2\right]^{p/2} \\ &\geq \alpha\left[\left|\xi_1^1 - \xi_2^2\right|^p + \left|\xi_2^1 + \xi_1^2\right|^p\right]. \end{aligned}$$

We now turn to the claim and note that it is sufficient to prove the claim for $\varphi = \left(\varphi^1, \varphi^2\right) \in C_0^\infty\left(\Omega, \mathbb{R}^2\right)$, the general result being obtained by density. We also extend the function outside Ω by setting $\varphi \equiv 0$ there. Then denoting $\partial\varphi^j / \partial x_i$ by $\partial_i \varphi^j$, $i, j \in \{1, 2\}$, we have from the above algebraic relation

$$\begin{aligned} & \int_\Omega [\,|\nabla\varphi(x)|^2 - 2\det(\nabla\varphi(x))\,]^{p/2} dx \\ \geq \quad & \alpha \int_\Omega [\,\left|\partial_1\varphi^1(x) - \partial_2\varphi^2(x)\right|^p + \left|\partial_2\varphi^1(x) + \partial_1\varphi^2(x)\right|^p\,] dx. \end{aligned}$$

The classical regularity results for Cauchy-Riemann equations (see, for example, Proposition 4 on page 60 in Stein [543]) leads to the existence of a constant $\beta > 0$ such that

$$\|\nabla\varphi\|_{L^p}^p \leq \beta \int_\Omega [\,\left|\partial_1\varphi^1(x) - \partial_2\varphi^2(x)\right|^p + \left|\partial_2\varphi^1(x) + \partial_1\varphi^2(x)\right|^p\,] dx.$$

Choosing $\epsilon \leq \alpha/\beta$, we have (5.77).

Step 2. We now prove the general case, where ξ is not necessarily 0 but $p = 4$. We start with the following algebraic observation

$$[\langle \xi - \widetilde{\xi}; \eta \rangle]^2 \leq |\xi - \widetilde{\xi}|^2 |\eta|^2 = 2[|\xi|^2 - 2\det\xi]|\eta|^2 . \tag{5.79}$$

We next compute

$$\begin{aligned}
& [|\xi + \nabla\varphi|^2 - 2\det(\xi + \nabla\varphi)]^2 \\
= \; & [|\xi|^2 - 2\det\xi + 2\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle + |\nabla\varphi|^2 - 2\det(\nabla\varphi)]^2 \\
= \; & [|\xi|^2 - 2\det\xi]^2 + 4[\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle]^2 + [|\nabla\varphi|^2 - 2\det(\nabla\varphi)]^2 \\
& +4[|\xi|^2 - 2\det\xi][\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle - \det(\nabla\varphi)] \\
& +2[|\xi|^2 - 2\det\xi]|\nabla\varphi|^2 + 4[|\nabla\varphi|^2 - 2\det(\nabla\varphi)]\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle .
\end{aligned}$$

Using (5.79), we obtain

$$\begin{aligned}
& [|\xi + \nabla\varphi|^2 - 2\det(\xi + \nabla\varphi)]^2 \\
\geq \; & [|\xi|^2 - 2\det\xi]^2 + 5[\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle]^2 + [|\nabla\varphi|^2 - 2\det(\nabla\varphi)]^2 \\
& +4[|\xi|^2 - 2\det\xi][\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle - \det(\nabla\varphi)] \\
& +4[|\nabla\varphi|^2 - 2\det(\nabla\varphi)]\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle .
\end{aligned}$$

Noticing that

$$\begin{aligned}
0 \; \leq \; & 5[\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle]^2 \\
& +4[|\nabla\varphi|^2 - 2\det(\nabla\varphi)]\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle + \frac{4}{5}[|\nabla\varphi|^2 - 2\det(\nabla\varphi)]^2
\end{aligned}$$

we deduce that

$$\begin{aligned}
[|\xi + \nabla\varphi|^2 - 2\det(\xi + \nabla\varphi)]^2 \; & \geq [|\xi|^2 - 2\det\xi]^2 + \tfrac{1}{5}[|\nabla\varphi|^2 - 2\det(\nabla\varphi)]^2 \\
& \quad +4[|\xi|^2 - 2\det\xi][\langle \xi - \widetilde{\xi}; \nabla\varphi \rangle - \det(\nabla\varphi)].
\end{aligned}$$

We then integrate the above inequality, bearing in mind that $\varphi = 0$ on $\partial\Omega$, and we find

$$\begin{aligned}
\int_\Omega [|\xi + \nabla\varphi|^2 - 2\det(\xi + \nabla\varphi)]^2 dx \; \geq \; & [|\xi|^2 - 2\det\xi]^2 \operatorname{meas}\Omega \\
& +\frac{1}{5}\int_\Omega [|\nabla\varphi|^2 - 2\det(\nabla\varphi)]^2 dx.
\end{aligned}$$

Using Step 1, with $p = 4$, we find that

$$\int_\Omega [|\xi + \nabla\varphi|^2 - 2\det(\xi + \nabla\varphi)]^2 dx \geq [|\xi|^2 - 2\det\xi]^2 \operatorname{meas}\Omega + \frac{\alpha}{5\beta}\int_\Omega |\nabla\varphi|^4 dx.$$

Choosing $\epsilon = \alpha/5\beta$, we have indeed established (5.78) and thus the theorem is proved. ■

We now continue with the proof of the main theorem.

Proof. (Theorem 5.51: Quasiconvexity). We have to establish that

$$f_\gamma \text{ is quasiconvex } \Leftrightarrow \gamma \leq \gamma_q \text{ and } \gamma_q > 1.$$

In the first step, we prove the existence of a γ_q with the above property; this is the easy part of the proof. The difficult part, which will be dealt with in Step 2, is to show that $\gamma_q > 1$.

Step 1: *Existence of* γ_q . We start by showing that if f_γ is quasiconvex, then f_β is quasiconvex for every $0 \leq \beta \leq \gamma$. Let

$$I_\gamma(\xi, \varphi) := \int_\Omega [f_\gamma(\xi + \nabla\varphi(x)) - f_\gamma(\xi)]\, dx$$

for every $\xi \in \mathbb{R}^{2\times 2}$ and every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^2)$. We have to show that $I_\gamma(\xi, \varphi) \geq 0$ implies $I_\beta(\xi, \varphi) \geq 0$. We have to deal with two cases.

Case 1. If

$$\int_\Omega [\,|\xi + \nabla\varphi(x)|^2 \det(\xi + \nabla\varphi(x)) - |\xi|^2 \det \xi\,] dx \leq 0,$$

then the claim is trivial using the convexity of $\xi \to |\xi|^4$ and the fact that $\beta \geq 0$.

Case 2. If

$$\int_\Omega [\,|\xi + \nabla\varphi(x)|^2 \det(\xi + \nabla\varphi(x)) - |\xi|^2 \det \xi\,] dx \geq 0,$$

we observe that

$$\begin{aligned} I_\beta(\xi, \varphi) \quad - \quad & I_\gamma(\xi, \varphi) \\ &= 2(\gamma - \beta) \int_\Omega [\,|\xi + \nabla\varphi(x)|^2 \det(\xi + \nabla\varphi(x)) - |\xi|^2 \det \xi\,] dx \geq 0, \end{aligned}$$

as wished.

We may now define γ_q by taking the largest γ such that f_γ is quasiconvex. It exists because of the preceding observation and from the fact that

$$1 = \gamma_p \leq \gamma_q \leq \gamma_r = \frac{2}{\sqrt{3}}$$

and this completes Step 1.

Step 2: $\gamma_q > 1$. We therefore have to show that there exists $\alpha > 0$ small enough, so that if $\gamma = 1 + \alpha$, then f_γ is quasiconvex. We start with a preliminary result.

Step 2'. We prove the quasiconvexity of f_γ at 0 for $\gamma = 1 + \alpha$ with $\alpha > 0$ small enough. We have to prove that

$$\int_\Omega f_\gamma(\nabla\varphi(x))\, dx \geq 0$$

for every $\varphi \in W_0^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$ and for some $\alpha > 0$. Observe first the following algebraic inequality (we use the fact that $|\xi|^2 \geq 2\det\xi$), valid for any $\xi \in \mathbb{R}^{2\times 2}$,

$$\begin{aligned} f_\gamma(\xi) &= |\xi|^4 - 2(1+\alpha)|\xi|^2\det\xi \\ &= \frac{1}{2}[\,|\xi|^4 - 4|\xi|^2\det\xi + 4(\det\xi)^2\,] \\ &\quad + \frac{1}{2}[\,|\xi|^4 - 4(\det\xi)^2\,] - 2\alpha|\xi|^2\det\xi \\ &\geq \frac{1}{2}[\,|\xi|^2 - 2\det\xi\,]^2 - \alpha|\xi|^4 . \end{aligned}$$

We then integrate and use Theorem 5.52 to get

$$\int_\Omega f_\gamma(\nabla\varphi(x))\,dx \geq (\epsilon-\alpha)\int_\Omega |\nabla\varphi(x)|^4\,dx. \tag{5.80}$$

Choosing $0 \leq \alpha \leq \epsilon$, we have indeed obtained the result.

Step 2". We now proceed with the general case. We already know that $\gamma_q \geq \gamma_p = 1$, so we will assume throughout this step that $\gamma \geq 1$ and we will set $\alpha = \gamma - 1$.

Expanding f_γ, keeping in mind its special structure, we find

$$\begin{aligned} f_\gamma(\xi+\eta) &= f_\gamma(\xi) + \langle\, \nabla f_\gamma(\xi);\eta\,\rangle + \frac{1}{2}\langle\, \nabla^2 f_\gamma(\xi)\,\eta;\eta\,\rangle \\ &\quad + \langle\, \nabla f_\gamma(\eta);\xi\,\rangle + f_\gamma(\eta)\,. \end{aligned}$$

Recall that $\langle\, \nabla^2 f_\gamma(\xi)\,\eta;\eta\,\rangle$ is given by (5.73). We rewrite this as

$$f_\gamma(\xi+\eta) - f_\gamma(\xi) = A_\gamma(\xi,\eta) + B_\gamma(\xi,\eta) + C_\gamma(\xi,\eta) + D_\gamma(\eta) + E_\gamma(\eta) \tag{5.81}$$

where

$$A_\gamma(\xi,\eta) := \langle\, \nabla f_\gamma(\xi);\eta\,\rangle - 2\gamma|\xi|^2\det\eta$$

$$\begin{aligned} B_\gamma(\xi,\eta) &:= \tfrac{1}{2}\langle\, \nabla^2 f_\gamma(\xi)\,\eta;\eta\,\rangle + 2\gamma|\xi|^2\det\eta \\ &= 4(\langle\xi;\eta\rangle)^2 + 2|\xi|^2|\eta|^2 - 4\gamma\langle\xi;\eta\rangle\langle\,\widetilde{\xi};\eta\rangle - 2\gamma|\eta|^2\det\xi \\ C_\gamma(\xi,\eta) &:= \langle\, \nabla f_\gamma(\eta);\xi\,\rangle \\ &= 4\langle\,\xi;\eta\,\rangle|\eta|^2 - 4\gamma\langle\,\xi;\eta\,\rangle\det\eta - 2\gamma\langle\,\widetilde{\xi};\eta\,\rangle|\eta|^2 \end{aligned}$$

$$D_\gamma(\eta) := (1-\epsilon)\,f_1(\eta) + \frac{\epsilon^2}{2}|\eta|^4$$

$$\begin{aligned} E_\gamma(\eta) &:= \epsilon f_1(\eta) - 2(\gamma-1)|\eta|^2\det\eta - \tfrac{\epsilon^2}{2}|\eta|^4 \\ &\geq \epsilon f_1(\eta) - (\alpha + \tfrac{\epsilon^2}{2})|\eta|^4 . \end{aligned}$$

Observe that

$$D_\gamma(\eta) + E_\gamma(\eta) = f_\gamma(\eta)\,.$$

From Step 2' (applying (5.80) with $\gamma = 1$ and hence $\alpha = 0$), we have that, for every $\varphi \in W_0^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$,

$$\int_\Omega E_\gamma\left(\nabla\varphi\left(x\right)\right)dx \geq \left[\,\epsilon^2 - (\alpha + \frac{\epsilon^2}{2})\,\right]\int_\Omega \left|\nabla\varphi\left(x\right)\right|^4 dx$$

which for $\alpha > 0$ sufficiently small with respect to ϵ^2 leads to

$$\int_\Omega E_\gamma\left(\nabla\varphi\left(x\right)\right)dx \geq 0. \tag{5.82}$$

We also have that for $\epsilon > 0$ and $\alpha > 0$ even smaller (see Lemma 5.53)

$$\sigma_{\epsilon,\alpha}\left(\xi,\eta\right) = B_\gamma\left(\xi,\eta\right) + C_\gamma\left(\xi,\eta\right) + D_\gamma\left(\eta\right) \geq 0 \tag{5.83}$$

for every $\xi, \eta \in \mathbb{R}^{2\times 2}$.

We are now in a position to conclude by combining (5.81), (5.82) and (5.83). We therefore have, for every $\xi \in \mathbb{R}^{2\times 2}$, $\varphi \in W_0^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$,

$$\int_\Omega \left[f_\gamma\left(\xi + \nabla\varphi\left(x\right)\right) - f_\gamma\left(\xi\right)\right]dx \geq \int_\Omega A_\gamma\left(\xi, \nabla\varphi\left(x\right)\right)dx = 0.$$

This is the desired claim. ■

The above proof relied on the following algebraic lemma.

Lemma 5.53 *Let*

$$\sigma_{\epsilon,\alpha}\left(\xi,\eta\right) = B_\gamma\left(\xi,\eta\right) + C_\gamma\left(\xi,\eta\right) + D_\gamma\left(\eta\right)$$

where $\gamma = 1 + \alpha$ *and*

$$B_\gamma\left(\xi,\eta\right) = 4\left(\langle\xi;\eta\rangle\right)^2 + 2\left|\xi\right|^2\left|\eta\right|^2 - 4\gamma\langle\xi;\eta\rangle\langle\,\widetilde{\xi};\eta\rangle - 2\gamma\left|\eta\right|^2\det\xi$$

$$C_\gamma\left(\xi,\eta\right) = 4\langle\,\xi;\eta\,\rangle\left|\eta\right|^2 - 4\gamma\langle\,\xi;\eta\,\rangle\det\eta - 2\gamma\langle\,\widetilde{\xi};\eta\,\rangle\left|\eta\right|^2$$

$$D_\gamma\left(\eta\right) = (1-\epsilon)\left[\,\left|\eta\right|^4 - 2\left|\eta\right|^2\det\eta\,\right] + \frac{\epsilon^2}{2}\left|\eta\right|^4.$$

For every $\epsilon > 0$ *sufficiently small, there exists* $\alpha_0 = \alpha_0\left(\epsilon\right) > 0$ *such that if* $0 \leq \alpha \leq \alpha_0$, *then*

$$\sigma_{\epsilon,\alpha}\left(\xi,\eta\right) \geq 0, \text{ for every } \xi, \eta \in \mathbb{R}^{2\times 2}.$$

Proof. The idea of the proof is to show that, for every $\epsilon > 0$ sufficiently small, there exists $\alpha_0 = \alpha_0\left(\epsilon\right) > 0$ such that if $0 \leq \alpha \leq \alpha_0$, then

$$\xi \to \sigma_{\epsilon,\alpha}\left(\xi,\eta\right)$$

is a strictly convex polynomial of degree two for every $\eta \in \mathbb{R}^{2\times 2}$. In Step 2 we prove that by choosing both ϵ sufficiently small and $\alpha_0(\epsilon)$ even smaller (uniformly with respect to η), then

$$\sigma_{\epsilon,\alpha}\left(\overline{\xi},\eta\right) \geq 0$$

at the unique minimum point $\overline{\xi} = \overline{\xi}(\eta)$.

Step 1. We first show that for $\alpha = \gamma - 1 > 0$ sufficiently small

$$B_\gamma(\xi,\eta) \geq \frac{11 - 9\gamma^2}{6} |\xi|^2 |\eta|^2 \text{ for every } \xi,\eta \in \mathbb{R}^{2\times 2}. \tag{5.84}$$

The case $\xi = 0$ or $\eta = 0$ being trivial, we can assume because of the homogeneity of B_γ that

$$|\xi| = |\eta| = 1.$$

Moreover, since $\widetilde{Q\xi R} = Q\widetilde{\xi}R$ for every $Q, R \in SO(2)$, we have

$$B_\gamma(\xi, Q\eta R) = B_\gamma\left(Q^t\xi R^t,\eta\right)$$

and thus it is enough to prove (5.84) for matrices ξ and η of the form (according to Theorem 13.3)

$$\xi = \begin{pmatrix} \cos\theta\cos A & \sin A\cos B \\ \sin A\sin B & \sin\theta\cos A \end{pmatrix} \text{ and } \eta = \begin{pmatrix} \cos\varphi & 0 \\ 0 & \sin\varphi \end{pmatrix}.$$

We therefore find

$$\begin{aligned} B_\gamma(\xi,\eta) \;=\;& 2 + \gamma\sin(2B)\sin^2 A + \\ & \left[4\cos^2(\theta-\varphi) - 4\gamma\cos(\theta-\varphi)\sin(\theta+\varphi) - \gamma\sin(2\theta)\right]\cos^2 A. \end{aligned}$$

Since $\sin(2B) \geq -1$, we find that

$$\begin{aligned} & B_\gamma(\xi,\eta) \geq 2 - \gamma + \\ & \quad \left[\gamma + 4\cos^2(\theta-\varphi) - 4\gamma\cos(\theta-\varphi)\sin(\theta+\varphi) - \gamma\sin(2\theta)\right]\cos^2 A. \end{aligned}$$

Since $\gamma > 1$ is sufficiently close to 1 and we want to minimize $B_\gamma(\xi,\eta)$, we have to choose $\cos^2 A = 1$. We can thus write

$$B_\gamma(\xi,\eta) \geq 2 + 4\cos^2(\theta-\varphi) - 4\gamma\cos(\theta-\varphi)\sin(\theta+\varphi) - \gamma\sin(2\theta)$$

or, writing $a = 2\theta$ and $b = 2\varphi$,

$$B_\gamma(\xi,\eta) \geq g(a,b) := 4 + 2\cos(a-b) - 3\gamma\sin a - 2\gamma\sin b. \tag{5.85}$$

We easily have that

$$\nabla g(a,b) = 0 \;\Leftrightarrow\; \cos b = -\frac{3}{2}\cos a = \frac{1}{\gamma}\sin(a-b). \tag{5.86}$$

We can next write that

$$g(a,b) \geq \min\{g(a,b) : \nabla g(a,b) = 0\} \tag{5.87}$$

and therefore two cases can happen.

Case 1: $\cos a = \cos b = \sin(a-b) = 0$. At such a point (recalling that γ is sufficiently close to 1) we have

$$g(a,b) \geq 6 - 5\gamma. \tag{5.88}$$

Case 2: $\cos a \neq 0$ and $\cos b \neq 0$. From (5.86), we find

$$\cos b = -\frac{3}{2}\cos a \quad \text{and} \quad \sin b = \frac{3}{2}(\gamma - \sin a).$$

We hence deduce that

$$\frac{4}{9} = \frac{4}{9}\cos^2 b + \frac{4}{9}\sin^2 b = \gamma^2 + 1 - 2\gamma \sin a.$$

Therefore at such a point (a,b) we have

$$\begin{aligned} g(a,b) &= 4 + 2\cos a \cos b + 2\sin a \sin b - 3\gamma \sin a - 2\gamma \sin b \\ &= 1 - 3\gamma^2 + 3\gamma \sin a = \frac{11 - 9\gamma^2}{6}. \end{aligned}$$

Combining (5.85), (5.87), (5.88) and the above identity, we have indeed obtained (5.84).

Step 2. We now prove that by choosing both ϵ sufficiently small and $\alpha_0(\epsilon)$ even smaller (uniformly with respect to η), then

$$\sigma_{\epsilon,\alpha}(\xi,\eta) \geq 0 \text{ for every } \xi, \eta \in \mathbb{R}^{2\times 2}.$$

We start by observing that

$$\sigma_{\epsilon,\alpha}(\xi,0) = 0 \text{ for every } \xi \in \mathbb{R}^{2\times 2}.$$

So from now on we will assume that $\eta \neq 0$ and is fixed. From Step 1, we see that the function

$$\xi \to \sigma_{\epsilon,\alpha}(\xi,\eta)$$

has a unique minimum, which satisfies

$$\nabla_\xi \sigma_{\epsilon,\alpha}(\xi,\eta) = 0;$$

i.e.

$$\begin{aligned} 4|\eta|^2\eta - 4\gamma(\det\eta)\eta - 2\gamma|\eta|^2\widetilde{\eta} + 4|\eta|^2\xi - 2\gamma|\eta|^2\widetilde{\xi} & \\ +8\langle \xi;\eta \rangle \eta - 4\gamma \langle \xi;\eta \rangle \widetilde{\eta} - 4\gamma \langle \widetilde{\xi};\eta \rangle \eta &= 0. \end{aligned} \tag{5.89}$$

We now multiply (5.89) first by ξ, then by η and finally by $\widetilde{\eta}$ to get

$$\begin{aligned} 2\langle \xi;\eta \rangle & (|\eta|^2 - \gamma \det \eta) - \gamma \langle \widetilde{\xi};\eta \rangle |\eta|^2 \\ = & \; 2|\xi|^2 |\eta|^2 - 2\gamma |\eta|^2 \det \xi + 4(\langle \xi;\eta \rangle)^2 - 4\gamma \langle \xi;\eta \rangle \langle \widetilde{\xi};\eta \rangle \\ & +4\langle \xi;\eta \rangle |\eta|^2 - 4\gamma \langle \xi;\eta \rangle \det \eta - 2\gamma \langle \widetilde{\xi};\eta \rangle |\eta|^2 \end{aligned} \tag{5.90}$$

$$\begin{aligned} -\gamma \langle \widetilde{\xi};\eta \rangle |\eta|^2 & \\ = & -\tfrac{2}{3}|\eta|^4 + \tfrac{4}{3}\gamma |\eta|^2 \det \eta - 2\langle \xi;\eta \rangle |\eta|^2 + \tfrac{4}{3}\gamma \langle \xi;\eta \rangle \det \eta \end{aligned} \tag{5.91}$$

$$\begin{aligned} 2\langle \widetilde{\xi};\eta \rangle & (|\eta|^2 - 2\gamma \det \eta) \\ = & \langle \xi;\eta \rangle (3\gamma |\eta|^2 - 8\det \eta) + \gamma |\eta|^4 - 4|\eta|^2 \det \eta + 4\gamma (\det \eta)^2 . \end{aligned} \tag{5.92}$$

We next combine (5.89) to (5.92) to show that $\sigma_{\epsilon,\alpha} \geq 0$ at a stationary point provided $\alpha = \gamma - 1$ and ϵ are small enough. Combining (5.91) and (5.92), so as to eliminate $\langle \widetilde{\xi};\eta \rangle$, we find that

$$\begin{aligned} \langle \xi;\eta \rangle & [3(4-3\gamma^2)|\eta|^4 - 8\gamma |\eta|^2 \det \eta + 16\gamma^2 (\det \eta)^2] \\ = & -|\eta|^2 [(4-3\gamma^2)|\eta|^4 - 4\gamma |\eta|^2 \det \eta + 4\gamma^2 (\det \eta)^2]. \end{aligned} \tag{5.93}$$

We now use (5.90), (5.91) and (5.93) to compute $\sigma_{\epsilon,\alpha}$ at the minimum point. First appeal to (5.90) to obtain

$$\begin{aligned} \sigma_{\epsilon,\alpha} = & \; 2\langle \xi;\eta \rangle (|\eta|^2 - \gamma \det \eta) - \gamma \langle \widetilde{\xi};\eta \rangle |\eta|^2 \\ & +(1-\epsilon+\frac{\epsilon^2}{2})|\eta|^4 - 2(1-\epsilon)|\eta|^2 \det \eta. \end{aligned}$$

Replacing the second term, with the help of (5.91), we find

$$\sigma_{\epsilon,\alpha} = -\tfrac{2}{3}\gamma \langle \xi;\eta \rangle \det \eta + (\frac{1}{3} - \epsilon + \frac{\epsilon^2}{2})|\eta|^4 - 2\left(1-\epsilon-\tfrac{2}{3}\gamma\right)|\eta|^2 \det \eta.$$

Inserting (5.93) in the above identity, we obtain

$$\begin{aligned} \frac{3\sigma_{\epsilon,\alpha}}{|\eta|^2} & [3(4-3\gamma^2)|\eta|^4 - 8\gamma |\eta|^2 \det \eta + 16\gamma^2 (\det \eta)^2] \\ = & [(1-3\epsilon+\tfrac{3\epsilon^2}{2})|\eta|^2 - 2(3-3\epsilon-2\gamma)\det \eta] \\ & \times [3(4-3\gamma^2)|\eta|^4 - 8\gamma |\eta|^2 \det \eta + 16\gamma^2 (\det \eta)^2] \\ & +2\gamma \det \eta [(4-3\gamma^2)|\eta|^4 - 4\gamma |\eta|^2 \det \eta + 4\gamma^2 (\det \eta)^2]. \end{aligned}$$

Setting

$$t = |\eta| \quad \text{and} \quad \delta = 2\det(\eta/|\eta|) \quad (\Rightarrow |\delta| \leq 1),$$

we get

$$\begin{aligned}\frac{3\sigma_{\epsilon,\alpha}}{t^4} & \left[3\left(4-3\gamma^2\right)-4\gamma\delta+4\gamma^2\delta^2\right] \\ = & \left[\left(1-3\epsilon+\tfrac{3}{2}\epsilon^2\right)-\left(3-3\epsilon-2\gamma\right)\delta\right]\left[3\left(4-3\gamma^2\right)-4\gamma\delta+4\gamma^2\delta^2\right] \\ & +\gamma\delta\left[\left(4-3\gamma^2\right)-2\gamma\delta+\gamma^2\delta^2\right].\end{aligned}$$

Letting $\alpha=\gamma-1\geq 0$ and using the fact that $|\delta|\leq 1$, we find the following three estimates for α small enough

$$\begin{aligned}\left[3\left(4-3\gamma^2\right)\right. & - \left.4\gamma\delta+4\gamma^2\delta^2\right] \\ & = \left[3\left(1-6\alpha-3\alpha^2\right)-4\left(1+\alpha\right)\delta+4\left(1+\alpha\right)^2\delta^2\right] \\ & \leq \left[3-4\delta+4\delta^2\right]+1\leq 12\end{aligned}$$

$$\begin{aligned}\left[\left(1-3\epsilon+\tfrac{3}{2}\epsilon^2\right)-\left(3-3\epsilon-2\gamma\right)\delta\right] & \left[3\left(4-3\gamma^2\right)-4\gamma\delta+4\gamma^2\delta^2\right] \\ = & \tfrac{3}{2}\epsilon^2\left[2+\left(1-2\delta\right)^2\right] \\ & +\left(1-3\epsilon\right)\left(1-\delta\right)\left[3-4\delta+4\delta^2\right]+O_\delta\left(\alpha\right)\end{aligned}$$

$$\begin{aligned}\gamma\delta & \left[\left(4-3\gamma^2\right)-2\gamma\delta+\gamma^2\delta^2\right] \\ & = \left(1+\alpha\right)\delta\left[\left(1-6\alpha-3\alpha^2\right)-2\left(1+\alpha\right)\delta+\left(1+\alpha\right)^2\delta^2\right] \\ & = \delta\left(1-\delta\right)^2+O_\delta\left(\alpha\right)\end{aligned}$$

where $O_\delta\left(\alpha\right)$ stands for a term that goes to 0 as α tends to 0 uniformly for $|\delta|\leq 1$.

Combining these three estimates, we find for ϵ sufficiently small, since $|\delta|\leq 1$,

$$\frac{36\sigma_{\epsilon,\alpha}}{t^4}\geq 3\epsilon^2+3\left(1-\delta\right)\left[1-\delta+\delta^2-\epsilon\left(3-4\delta+4\delta^2\right)\right]+O_\delta\left(\alpha\right)\geq 3\epsilon^2+O_\delta\left(\alpha\right).$$

Choosing $\alpha << \epsilon$ (recalling that ϵ is small), we get the result; i.e.

$$\sigma_{\epsilon,\alpha}\left(\xi,\eta\right)\geq 0,\ \text{for every }\xi,\eta\in\mathbb{R}^{2\times 2}.$$

This concludes the proof of the lemma. ■

5.3.9 Quasiconvex functions with subquadratic growth.

We have seen in Corollary 5.9 that a polyconvex function having a subquadratic growth, must be convex. This, however, is not the case for quasiconvex and rank one convex functions. We now give such an example, following Sverak [549] (for the case $p=1$, see Theorem 5.55).

Theorem 5.54 *Let $1<p<2$. Then there exists a function $f:\mathbb{R}^{2\times 2}\to\mathbb{R}$ quasiconvex, non-convex and satisfying*

$$0\leq f\left(\xi\right)\leq\gamma\left(1+|\xi|^p\right),\ \forall\,\xi\in\mathbb{R}^{2\times 2}$$

and where γ is a positive constant.

Proof. We start with the following easily established algebraic inequality valid for any $\xi \in \mathbb{R}^{2\times 2}$

$$\min\{|\xi - I|^2, |\xi + I|^2\} \geq \frac{1}{2}[\,|\xi|^2 - 2\det\xi\,] \geq 0. \tag{5.94}$$

We next define

$$g(\xi) := \min\{|\xi - I|^p, |\xi + I|^p\}.$$

Anticipating on the definition and properties of the quasiconvex envelope given in Chapter 6 (see Theorem 6.9), we let

$$f := Qg$$

and we claim that f has all the desired properties. By definition it is quasiconvex and satisfies the growth condition, we therefore only need to show that it is not convex. This will be proved, once shown that

$$f(0) = Qg(0) > 0, \tag{5.95}$$

since clearly

$$Cg(0) = 0$$

where Cg denotes the convex envelope of g.

Assume for the sake of contradiction that

$$Qg(0) = 0$$

and use Theorem 6.9 to find a sequence $\varphi^\nu \in W_0^{1,\infty}(D;\mathbb{R}^2)$, here $D \subset \mathbb{R}^2$ is a bounded open set with $\operatorname{meas} D = 1$, such that

$$0 = Qg(0) \geq -\frac{1}{\nu} + \int_D g(\nabla\varphi^\nu(x))\,dx. \tag{5.96}$$

Invoking (5.94), we can deduce from the above inequality that

$$\frac{1}{\nu} \geq 2^{-p/2}\int_D \left[|\nabla\varphi^\nu(x)|^2 - 2\det(\nabla\varphi^\nu(x))\right]^{p/2} dx.$$

The estimate of Theorem 5.52 then implies that

$$\varphi^\nu \to 0 \ \text{ in } \ W^{1,p}(D;\mathbb{R}^2).$$

This therefore leads to

$$\lim_{\nu\to\infty}\int_D g(\nabla\varphi^\nu(x))\,dx = 2^{p/2},$$

contradicting (5.96). We have therefore proved (5.95) and the theorem follows. ■

5.3.10 The case of homogeneous functions of degree one

We would now like to discuss the convexity properties of homogeneous functions of degree one, $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ and we have the following theorem.

Theorem 5.55 *Let $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be positively homogeneous of degree one, namely*

$$f(t\xi) = tf(\xi) \text{ for every } t \geq 0 \text{ and every } \xi \in \mathbb{R}^{2\times 2}. \tag{5.97}$$

The following three properties hold.

(i) f is polyconvex if and only if it is convex.

(ii) If f is $SO(2) \times SO(2)$-invariant, in the sense that

$$f(\xi) = f(Q\xi R) \text{ for every } Q, R \in SO(2),$$

then f is rank one convex if and only if it is convex.

(iii) The function

$$f(\xi) = \begin{cases} 7|\xi| + \frac{3(\xi_1^1)^2 + 2\xi_1^1\xi_2^2 + 3(\xi_2^2)^2 + 4\xi_2^1\xi_1^2}{|\xi|} & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases}$$

is rank one convex but not convex.

Remark 5.56 (i) The first statement follows at once from Corollary 5.9.

(ii) The second assertion has been proved by Dacorogna [181] and the last one is a particular case of the study undertaken by Dacorogna-Haeberly [190].

(iii) Müller [461] (see also Zhang [618]) produced, in an indirect way similar to that of Theorem 5.54, an example of a quasiconvex function satisfying (5.97) and that is not convex.

(iv) It is not presently known if the function given in (iii) of the theorem is quasiconvex. Numerical evidences given in Dacorogna-Haeberly [191] tend to indicate that it is quasiconvex. ◊

Before proceeding with the proof we need the following elementary lemma established in Dacorogna [181], for a different proof see Dacorogna-Maréchal [206]. The lemma is false if either the function is not everywhere finite or in dimensions 3 and higher, see [206] for details. Note that in dimension 4, the function given in Theorem 5.55, being rank one convex, is separately convex but not convex.

Lemma 5.57 *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be positively homogeneous of degree one and separately convex (meaning that $x \to g(x,y)$ and $y \to g(x,y)$ are both convex). Then g is convex.*

Proof. (Lemma 5.57). Since g is homogeneous of degree one, it is clear that g is convex if and only if

$$g(x_1 + x_2, y_1 + y_2) \leq g(x_1, y_1) + g(x_2, y_2). \tag{5.98}$$

We consider two cases.

Case 1: $x_1 x_2 \geq 0$ or $y_1 y_2 \geq 0$. Since the hypothesis $x_1 x_2 \geq 0$ is handled similarly to $y_1 y_2 \geq 0$, we will assume that this last one holds. Since g is separately convex it is continuous (cf. Theorem 2.31) and hence it is enough to prove the result for $y_1 y_2 > 0$. Observe then that

$$\sigma := \frac{y_1 + y_2}{|y_1 + y_2|} = \frac{y_1}{|y_1|} = \frac{y_2}{|y_2|} \in \{\pm 1\}\,.$$

We therefore have, using the convexity of g with respect to the first variable,

$$\begin{aligned} g\left(x_1 + x_2, y_1 + y_2\right) &= |y_1 + y_2|\, g\left(\frac{|y_1|}{|y_1 + y_2|}\frac{x_1}{|y_1|} + \frac{|y_2|}{|y_1 + y_2|}\frac{x_2}{|y_2|}, \sigma\right) \\ &\leq |y_1|\, g\left(\frac{x_1}{|y_1|}, \sigma\right) + |y_2|\, g\left(\frac{x_2}{|y_2|}, \sigma\right) = g\left(x_1, y_1\right) + g\left(x_2, y_2\right) \end{aligned}$$

as wished.

Case 2: $x_1 x_2 < 0$ and $y_1 y_2 < 0$. This case is more involved than the previous one and we divide the proof into two steps.

Step 1. We first show that

$$g\left(x_1 + x_2, 0\right) \leq g\left(x_1, y\right) + g\left(x_2, -y\right),\ \forall y \in \mathbb{R}. \tag{5.99}$$

Since $x_1 x_2 < 0$, we have either

$$x_1\left(x_1 + x_2\right) \geq 0 \quad \text{or} \quad x_2\left(x_1 + x_2\right) \geq 0.$$

Without loss of generality (otherwise exchange the roles of (x_1, y) with that of $(x_2, -y)$), we will assume that

$$x_1\left(x_1 + x_2\right) \geq 0. \tag{5.100}$$

We then choose $\epsilon > 0$ sufficiently small and let

$$a := \frac{x_1 + (1 - 2\epsilon)\, x_2}{(1 - \epsilon)} \quad \text{and} \quad \mu := \frac{1 - 2\epsilon}{1 - \epsilon}.$$

Observe that

$$\begin{cases} -2\epsilon\mu + 2\left(1 - \mu\right)\left(1 - 2\epsilon\right) = 0 \\ \mu\left(x_1 + x_2\right) + 2\left(1 - \mu\right) x_1 = a \\ 2\epsilon x_2 + \left(1 - \epsilon\right) a = x_1 + x_2\,. \end{cases}$$

Appealing to Case 1, since $(-y)\,.0 \geq 0$, we find

$$\begin{aligned} g\left(x_1 + x_2, -2\epsilon y\right) &= g\left(2\epsilon x_2 + (1 - \epsilon)\, a, 2\epsilon\, (-y) + (1 - \epsilon)\, 0\right) \\ &\leq 2\epsilon g\left(x_2, -y\right) + \left(1 - \epsilon\right) g\left(a, 0\right). \end{aligned}$$

Since (5.100) holds, we also have from Case 1

$$\begin{aligned} g\left(a, 0\right) &= g\left(\mu\left(x_1 + x_2\right) + 2\left(1 - \mu\right) x_1, \mu\left(-2\epsilon y\right) + 2\left(1 - \mu\right)\left(1 - 2\epsilon\right) y\right) \\ &\leq \mu g\left(x_1 + x_2, -2\epsilon y\right) + \left(1 - \mu\right) g\left(2 x_1, 2\left(1 - 2\epsilon\right) y\right) \\ &= \mu g\left(x_1 + x_2, -2\epsilon y\right) + 2\left(1 - \mu\right) g\left(x_1, \left(1 - 2\epsilon\right) y\right). \end{aligned}$$

Combining the last two inequalities, we find

$$\begin{aligned} g\left(x_1+x_2,-2\epsilon y\right) &\leq 2\epsilon g\left(x_2,-y\right)+\left(1-2\epsilon\right) g\left(x_1+x_2,-2\epsilon y\right) \\ &\quad +2\epsilon g\left(x_1,\left(1-2\epsilon\right) y\right) \end{aligned}$$

or, in other words,

$$2\epsilon g\left(x_1+x_2,-2\epsilon y\right) \leq 2\epsilon g\left(x_2,-y\right)+2\epsilon g\left(x_1,\left(1-2\epsilon\right) y\right).$$

Dividing by 2ϵ and letting ϵ tend to 0, using the continuity of g, we have indeed obtained (5.99).

Step 2. We now prove (5.98). Observe that the hypothesis $y_1 y_2<0$ implies

$$\frac{y_1+y_2}{y_1} \geq 0 \quad \text{or} \quad \frac{y_1+y_2}{y_2} \geq 0.$$

We will assume that the first possibility happens, the second one being handled similarly.

We can therefore write,

$$g\left(x_1+x_2, y_1+y_2\right)=g(\frac{y_1+y_2}{y_1} x_1+x_2-\frac{y_2}{y_1} x_1, \frac{y_1+y_2}{y_1} y_1+0).$$

Since $\left(y_1+y_2\right) \cdot 0 \geq 0$, we can apply Case 1 and get

$$g\left(x_1+x_2, y_1+y_2\right) \leq \frac{y_1+y_2}{y_1} g\left(x_1, y_1\right)+g(x_2-\frac{y_2}{y_1} x_1, 0). \tag{5.101}$$

We also have, invoking Step 1,

$$\begin{aligned} g(x_2-\frac{y_2}{y_1} x_1, 0) &\leq g\left(x_2, y_2\right)+g(-\frac{y_2}{y_1} x_1,-\frac{y_2}{y_1} y_1) \\ &= g\left(x_2, y_2\right)-\frac{y_2}{y_1} g\left(x_1, y_1\right). \end{aligned}$$

Combining the above inequality and (5.101), we obtain (5.98) and thus the lemma. ■

We now proceed with the proof of the theorem.

Proof. (Theorem 5.55). **(i)** As already mentioned the proof of the first part immediately follows from Corollary 5.9.

(ii) The implication f convex $\Rightarrow$ f rank one convex, being always true, we need only prove the reverse one. According to Theorem 5.33, it is sufficient to show that f is convex on diagonal matrices. Therefore let

$$g\left(x_1, x_2\right):=f\left(\begin{array}{cc} x_1 & 0 \\ 0 & x_2 \end{array}\right)$$

and observe that the rank one convexity of f implies the separate convexity of g. Lemma 5.57 gives immediately the claim.

(iii) We first discuss the fact that f is non convex. We let

$$\xi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \eta = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and for $t \in \mathbb{R}$, we define

$$t \to \varphi(t) := f(\xi + t\eta) = 5\left(1+t^2\right)^{1/2} + 6\left(1+t^2\right)^{-1/2}.$$

A direct computation shows that

$$\varphi''(t) = \left(17t^2 - 1\right)\left(1+t^2\right)^{-5/2}$$

and hence $\varphi''(0) = -1 < 0$, which implies that f is non convex.

It therefore remains only to show that f is rank one convex. We divide the proof of this fact into three steps.

Step 1. The rank one convexity of f is equivalent to showing that for every fixed $\xi \in \mathbb{R}^{2\times 2}$, $a, b \in \mathbb{R}^2$ the function

$$t \to \varphi_{\xi,a,b}(t) := f(\xi + ta \otimes b)$$

is convex in $t \in \mathbb{R}$.

Since $f(\xi) \geq 0$, we have that if there exists $\alpha \in \mathbb{R}$ such that

$$\xi = \alpha\, a \otimes b,$$

then

$$f(\xi + ta \otimes b) = f((\alpha + t)\, a \otimes b) = |\alpha + t|\, f(a \otimes b)$$

and thus $\varphi_{\xi,a,b}$ is convex in t. From now on we may therefore assume that ξ is not parallel to $a \otimes b$. The function $\varphi_{\xi,a,b}$ is then twice continuously differentiable and its convexity is therefore equivalent to the Legendre-Hadamard condition, namely

$$\langle\, \nabla^2 f(\xi)\, a \otimes b; a \otimes b \,\rangle \geq 0 \tag{5.102}$$

for every $\xi \in \mathbb{R}^{2\times 2}$, $a, b \in \mathbb{R}^2$ with ξ not parallel to $a \otimes b$.

Step 2. We now compute the Hessian of f. It will be more convenient, in the present analysis, to identify $\mathbb{R}^{2\times 2}$ with $\mathbb{R}^4$ and, therefore, a matrix ξ will be written as a vector $(\xi_1, \xi_2, \xi_3, \xi_4)$. We then let

$$\langle \xi; \eta \rangle = \sum_{i=1}^{4} \xi_i \eta_i,\ |\xi|^2 = \langle \xi; \xi \rangle,\ \det \xi = \xi_1\xi_4 - \xi_2\xi_3\,.$$

Letting

$$M = \begin{pmatrix} 9 & 0 & 0 & 1 \\ 0 & 6 & 2 & 0 \\ 0 & 2 & 6 & 0 \\ 1 & 0 & 0 & 9 \end{pmatrix}$$

we can rewrite f, when $\xi \neq 0$, as

$$f(\xi) = |\xi| + \frac{\langle M\xi;\xi \rangle}{|\xi|}.$$

Computing the Hessian of f, when $\xi \neq 0$, we first find, for $\alpha = 1, 2, 3, 4$, that

$$\begin{aligned}\frac{\partial f(\xi)}{\partial \xi_\alpha} &= \frac{\xi_\alpha}{|\xi|} + \frac{2|\xi|(M\xi)_\alpha - \langle M\xi;\xi\rangle \dfrac{\xi_\alpha}{|\xi|}}{|\xi|^2} \\ &= \frac{\xi_\alpha}{|\xi|} + \frac{2|\xi|^2 (M\xi)_\alpha - \langle M\xi;\xi\rangle \xi_\alpha}{|\xi|^3}\end{aligned}$$

and thus

$$\begin{aligned}\frac{\partial^2 f(\xi)}{\partial \xi_\alpha \partial \xi_\beta} &= \frac{\delta_{\alpha\beta}}{|\xi|} - \frac{\xi_\alpha \xi_\beta}{|\xi|^3} + \frac{1}{|\xi|^6}\{-3|\xi|\xi_\beta [2|\xi|^2 (M\xi)_\alpha - \langle M\xi;\xi\rangle \xi_\alpha] \\ &\quad + [4(M\xi)_\alpha \xi_\beta + 2|\xi|^2 M_{\alpha\beta} - \langle M\xi;\xi\rangle \delta_{\alpha\beta} - 2(M\xi)_\beta \xi_\alpha] |\xi|^3\},\end{aligned}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

Since the quadratic form $\langle \nabla^2 f(\xi)\lambda;\lambda \rangle$ is homogeneous of degree -1 in ξ and 2 in λ, we only need to consider the case where $|\xi| = |\lambda| = 1$. We hence get that

$$\begin{aligned}\sum_{\alpha,\beta=1}^{4} \frac{\partial^2 f(\xi)}{\partial \xi_\alpha \partial \xi_\beta}\lambda_\alpha \lambda_\beta &= 1 - (\langle \xi;\lambda\rangle)^2 - 4\langle M\xi;\lambda\rangle \langle \xi,\lambda\rangle + 2\langle M\lambda;\lambda\rangle \\ &\quad - \langle M\xi;\xi\rangle + 3\langle M\xi;\xi\rangle (\langle \xi;\lambda\rangle)^2.\end{aligned}$$

We can still transform this expression into a more amenable one, by choosing a vector $\eta \in \mathbb{R}^4$ and $\theta \in \mathbb{R}$ so that

$$\lambda = \xi\cos\theta + \eta\sin\theta, \text{ with } |\eta| = 1 \text{ and } \langle \xi;\eta\rangle = 0.$$

We therefore obtain that

$$\langle \xi;\lambda\rangle = \cos\theta, \quad \langle M\xi;\lambda\rangle = \langle M\xi;\xi\rangle\cos\theta + \langle M\xi;\eta\rangle \sin\theta$$

$$\langle M\lambda;\lambda\rangle = \langle M\xi;\xi\rangle \cos^2\theta + 2\langle M\xi;\eta\rangle \cos\theta\sin\theta + \langle M\eta;\eta\rangle \sin^2\theta.$$

Returning to the quadratic form we therefore find that

$$\langle \nabla^2 f(\xi)\lambda;\lambda \rangle = [1 + 2\langle M\eta;\eta\rangle - \langle M\xi;\xi\rangle] \sin^2\theta.$$

Hence (5.102) is equivalent to showing that

$$1 + 2\langle M\eta;\eta\rangle - \langle M\xi;\xi\rangle \geq 0 \tag{5.103}$$

for every $\xi, \eta \in \mathbb{R}^4$ and $\theta \in \mathbb{R}$ satisfying

$$|\xi| = |\eta| = 1, \ \langle \xi; \eta \rangle = 0 \ \text{ and } \ \det(\xi \cos\theta + \eta \sin\theta) = 0. \tag{5.104}$$

Step 3. It therefore remains to show (5.103) whenever (5.104) holds. We start by observing that the matrix M has eigenvalues

$$\mu_1 = 4 \leq \mu_2 = \mu_3 = 8 \leq \mu_4 = 10$$

and corresponding orthonormal eigenvectors

$$\varphi_1 = \tfrac{1}{\sqrt{2}}(0,1,-1,0) \quad \varphi_2 = \tfrac{1}{\sqrt{2}}(0,1,1,0)$$
$$\varphi_3 = \tfrac{1}{\sqrt{2}}(1,0,0,-1) \quad \varphi_4 = \tfrac{1}{\sqrt{2}}(1,0,0,1).$$

Note that

$$\det\varphi_1 = \det\varphi_4 = -\det\varphi_2 = -\det\varphi_3 = \tfrac{1}{2}.$$

Expanding the vectors $\xi, \eta \in \mathbb{R}^4$ in this basis we have

$$\xi = \sum_{i=1}^{4} \xi_i \varphi_i\,, \quad \eta = \sum_{i=1}^{4} \eta_i \varphi_i\,,$$

and from now on ξ_i and η_i will always denote the components of ξ and η in this new basis and in particular we find that

$$\det\xi = \frac{1}{2}(\xi_1^2 + \xi_4^2 - \xi_2^2 - \xi_3^2).$$

Moreover, (5.103) is equivalent to showing that

$$2\langle M\eta, \eta\rangle - \langle M\xi, \xi\rangle = \sum_{i=1}^{4} \mu_i \left(2\eta_i^2 - \xi_i^2\right) \geq -1. \tag{5.105}$$

Moreover, (5.104) can then be rewritten as

$$|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = |\eta|^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = 1,$$

$$\langle \xi; \eta \rangle = 0 \ \Leftrightarrow \ \xi_1\eta_1 + \xi_4\eta_4 = -(\xi_2\eta_2 + \xi_3\eta_3)\,,$$

with

$$\begin{aligned} &\det(\xi\cos\theta + \eta\sin\theta) = 0 \\ &\Leftrightarrow \left(\xi_1^2 + \xi_4^2 - \xi_2^2 - \xi_3^2\right)\cos^2\theta + \left(\eta_1^2 + \eta_4^2 - \eta_2^2 - \eta_3^2\right)\sin^2\theta \\ &\quad + 2\left(\xi_1\eta_1 + \xi_4\eta_4 - \xi_2\eta_2 - \xi_3\eta_3\right)\cos\theta\sin\theta = 0. \end{aligned}$$

We now argue by contradiction and assume that (5.105) does not hold, meaning that we can find $\xi, \eta \in \mathbb{R}^4$ and $\theta \in \mathbb{R}$ as above and so that

$$\sum_{i=1}^{4} \mu_i \left(2\eta_i^2 - \xi_i^2\right) < -1.$$

Observing that

$$2\mu_1 |\eta|^2 - \frac{\mu_3 + \mu_4}{2} |\xi|^2 = -1,$$

we can rewrite the above inequality as

$$12\eta_4^2 + 5\xi_1^2 + 8\left(\eta_2^2 + \eta_3^2\right) < \xi_4^2 - \xi_2^2 - \xi_3^2 \,. \tag{5.106}$$

Similarly, writing

$$\sum_{i=1}^{4} \mu_i \left(2\eta_i^2 - \xi_i^2\right) < -1 < 2 = (\mu_1 + \mu_2) |\eta|^2 - \mu_4 |\xi|^2$$

we find that

$$8\eta_4^2 + 6\xi_1^2 + 2\left(\xi_2^2 + \xi_3^2\right) < 4\left(\eta_1^2 - \eta_2^2 - \eta_3^2\right). \tag{5.107}$$

From (5.106) and (5.107), we deduce that

$$8\left(\eta_2^2 + \eta_3^2\right) < \xi_4^2 - \xi_2^2 - \xi_3^2 \text{ and } \frac{1}{2}\left(\xi_2^2 + \xi_3^2\right) < \eta_1^2 - \eta_2^2 - \eta_3^2 \,.$$

Inserting these inequalities in the identity $\det(\xi\cos\theta + \eta\sin\theta) = 0$ and also using the fact that $\xi_1\eta_1 + \xi_4\eta_4 = -(\xi_2\eta_2 + \xi_3\eta_3)$ leads to the desired contradiction, namely

$$\begin{aligned}
0 &= \left(\xi_1^2 + \xi_4^2 - \xi_2^2 - \xi_3^2\right)\cos^2\theta + \left(\eta_1^2 + \eta_4^2 - \eta_2^2 - \eta_3^2\right)\sin^2\theta \\
&\quad +2\left(\xi_1\eta_1 + \xi_4\eta_4 - \xi_2\eta_2 - \xi_3\eta_3\right)\cos\theta\sin\theta \\
&> \left[\xi_1^2 + 8\left(\eta_2^2 + \eta_3^2\right)\right]\cos^2\theta + \left[\,\eta_4^2 + \frac{1}{2}\left(\xi_2^2 + \xi_3^2\right)\right]\sin^2\theta \\
&\quad -4\left(\xi_2\eta_2 + \xi_3\eta_3\right)\cos\theta\sin\theta \\
&\geq 8\left(\eta_2^2 + \eta_3^2\right)\cos^2\theta + \frac{1}{2}\left(\xi_2^2 + \xi_3^2\right)\sin^2\theta - 4\left|(\xi_2\eta_2 + \xi_3\eta_3)\cos\theta\sin\theta\right| \\
&\geq \frac{1}{2}\left[4|\cos\theta|\sqrt{\eta_2^2 + \eta_3^2} - |\sin\theta|\sqrt{\xi_2^2 + \xi_3^2}\right]^2 \geq 0.
\end{aligned}$$

This concludes the proof of the theorem. ■

5.3.11 Some more examples

We now give some more examples.

Theorem 5.58 *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ and let $|.|$ denote the Euclidean norm, namely, for $\xi \in \mathbb{R}^{N\times n}$, we let*

$$|\xi| := \left(\sum_{\alpha=1}^{n} \sum_{i=1}^{N} \left(\xi_\alpha^i\right)^2\right)^{1/2}.$$

(i) Let $g : \mathbb{R}_+ \to \mathbb{R}$ *be such that*

$$f(\xi) = g(|\xi|).$$

Then

$$\begin{aligned} f \text{ convex} \quad &\Leftrightarrow \quad f \text{ polyconvex} \Leftrightarrow f \text{ quasiconvex} \Leftrightarrow f \text{ rank one convex} \\ &\Leftrightarrow \quad g \text{ convex and } g(0) = \inf\{g(x) : x \geq 0\}. \end{aligned}$$

(ii) Let $N = n$, $1 \leq \alpha < 2n$, $h : \mathbb{R} \to \mathbb{R}$ *be such that*

$$f(\xi) = |\xi|^\alpha + h(\det \xi).$$

Then

$$f \text{ polyconvex} \Leftrightarrow f \text{ quasiconvex} \Leftrightarrow f \text{ rank one convex} \Leftrightarrow h \text{ convex.}$$

(iii) Let $N = n$, $p > 0$, $1 \leq s \leq n - 1$ *and*

$$f(\xi) = \begin{cases} \left(\frac{|\operatorname{adj}_s \xi|^{n/s}}{\det \xi}\right)^p & \text{if } \det \xi > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$f \text{ polyconvex} \Leftrightarrow f \text{ rank one convex} \Leftrightarrow p \geq \frac{s}{n-s}.$$

Remark 5.59 (i) The result (i) was established by Dacorogna [176].

(ii) Case (ii) was proved by Ball-Murat [65]. Note that the hypothesis $\alpha < 2n$ cannot be dropped in general. Indeed, if $n = 2$ and $\alpha = 4$, then

$$f(\xi) = |\xi|^4 - 2(\det \xi)^2$$

is even convex.

(iii) Case (iii) is interesting in elasticity for slightly compressible materials and was established by Charrier-Dacorogna-Hanouzet-Laborde [144]. It was then generalized by Dacorogna-Maréchal [206]. ♢

Proof. (i) Let $\xi \in \mathbb{R}^{N \times n}$ and

$$f(\xi) = g(|\xi|).$$

In view of Theorem 5.3, it remains to show that

$$f \text{ rank one convex} \Rightarrow g \text{ convex and } g(0) = \inf\{g(x) : x \geq 0\}$$

which will be proved in Step 1 and

$$g \text{ convex and } g(0) = \inf\{g(x) : x \geq 0\} \Rightarrow f \text{ convex}$$

which we will establish in Step 2.

Step 1. Let $x > 0$ and define $\xi \in \mathbb{R}^{N \times n}$ to be such that

$$\xi_1^1 = x \quad \text{and} \quad \xi_j^i = 0 \text{ if } (i,j) \neq (1,1).$$

We then deduce that

$$g(0) = f\left(\frac{\xi - \xi}{2}\right) \leq \frac{1}{2} f(\xi) + \frac{1}{2} f(-\xi) = g(x)$$

as wished.

Let us now show that g is convex. Let $\lambda \in [0,1]$, $\alpha, \beta \geq 0$. Define $\xi, \eta \in \mathbb{R}^{N \times n}$ by

$$\xi_1^1 = \alpha,\ \eta_1^1 = \beta \quad \text{and} \quad \xi_j^i = \eta_j^i = 0 \text{ if } (i,j) \neq (1,1).$$

Observing that rank $\{\xi - \eta\} \leq 1$ and using the rank one convexity of f we get

$$\begin{aligned} g(\lambda\alpha + (1-\lambda)\beta) &= f(\lambda\xi + (1-\lambda)\eta) \\ &\leq \lambda f(\xi) + (1-\lambda) f(\eta) = \lambda g(|\alpha|) + (1-\lambda) g(|\beta|) \\ &= \lambda g(\alpha) + (1-\lambda) g(\beta) \end{aligned}$$

which is indeed the claimed convexity inequality.

Step 2. Note that since g is convex and

$$g(0) = \inf\{g(x) : x \geq 0\},$$

then g is non decreasing on $\mathbb{R}_+$.

We now want to show that g convex $\Rightarrow$ f convex. This is immediate since

$$\begin{aligned} f(\lambda\xi + (1-\lambda)\eta) &= g(|\lambda\xi + (1-\lambda)\eta|) \leq g(\lambda|\xi| + (1-\lambda)|\eta|) \\ &\leq \lambda g(|\xi|) + (1-\lambda) g(|\eta|) = \lambda f(\xi) + (1-\lambda) f(\eta) \end{aligned}$$

and this achieves the proof of the third part of the theorem.

(ii) Let $n = N$, $\xi \in \mathbb{R}^{n \times n}$, $1 \leq \alpha < 2n$ and

$$f(\xi) = |\xi|^\alpha + h(\det\xi).$$

It follows from Theorem 5.3 that it only remains to prove that

$$f \text{ rank one convex } \Rightarrow h \text{ convex.}$$

Let $\lambda \in (0,1)$, $a, b \in \mathbb{R}$, we want to show that

$$h(\lambda a + (1-\lambda) b) \leq \lambda h(a) + (1-\lambda) h(b). \tag{5.108}$$

We will assume, with no loss of generality, that $a \neq b$ and $a \neq 0$. Let $\epsilon \neq 0$ with $\epsilon(b-a) > 0$ and

$$\xi := \operatorname{diag}\left(\frac{a\epsilon}{b-a}, \left(\frac{b-a}{\epsilon}\right)^{\frac{1}{n-1}}, \cdots, \left(\frac{b-a}{\epsilon}\right)^{\frac{1}{n-1}}\right) \in \mathbb{R}^{n \times n}.$$

It is then easy to see that, letting $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$,

$$\begin{cases} \det \xi = a, \ \ \det (\xi + \epsilon\, e_1 \otimes e_1) = b \\ \det (\xi + (1-\lambda)\, \epsilon\, e_1 \otimes e_1) = \lambda a + (1-\lambda)\, b. \end{cases}$$

Since f is rank one convex, we have

$$\begin{aligned} & \left|\xi + (1-\lambda)\, \epsilon\, e_1 \otimes e_1\right|^{\alpha} + h\left(\lambda a + (1-\lambda)\, b\right) \\ = \ & f\left(\lambda \xi + (1-\lambda)\left(\xi + \epsilon\, e_1 \otimes e_1\right)\right) \\ \leq \ & \lambda f(\xi) + (1-\lambda)\, f\left(\xi + \epsilon\, e_1 \otimes e_1\right) \\ = \ & \lambda \left|\xi\right|^{\alpha} + (1-\lambda)\left|\xi + \epsilon\, e_1 \otimes e_1\right|^{\alpha} + \lambda h(a) + (1-\lambda)\, h(b). \end{aligned} \tag{5.109}$$

Observe that

$$\begin{aligned} & \lambda \left|\xi\right|^{\alpha} + (1-\lambda)\left|\xi + \epsilon\ (e_1 \otimes e_1)\right|^{\alpha} - \left|\xi + (1-\lambda)\, \epsilon\ (e_1 \otimes e_1)\right|^{\alpha} \\ = \ & \lambda [\, (\frac{a\epsilon}{b-a})^2 + (n-1)\, (\frac{b-a}{\epsilon})^{\frac{2}{n-1}}\,]^{\alpha/2} \\ & + (1-\lambda)\, [\, (\frac{a\epsilon}{b-a} + \epsilon)^2 + (n-1)\, (\frac{b-a}{\epsilon})^{\frac{2}{n-1}}\,]^{\alpha/2} \\ & - [\, (\frac{a\epsilon}{b-a} + (1-\lambda)\, \epsilon)^2 + (n-1)\, (\frac{b-a}{\epsilon})^{\frac{2}{n-1}}\,]^{\alpha/2} \\ = \ & O(\epsilon^{\frac{2n-\alpha}{n-1}}) \end{aligned}$$

where $O(t)$ stands for a term that goes to 0 as $t \to 0$. It is clear that if $1 \leq \alpha < 2n$, then the right hand side in the above identity tends to zero as $\epsilon \to 0$. Thus combining (5.109) and the above identity, as $\epsilon \to 0$, we have indeed obtained (5.108), i.e. that h is convex.

(iii) We decompose the proof into two steps.

Step 1: $p \geq \dfrac{s}{n-s} \ \Rightarrow\ f$ polyconvex. Define first $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$h(x, \delta) := \begin{cases} x^{np/s} \delta^{-p} & \text{if } x, \delta > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

It is then easy to see that h is convex if and only if $p \geq \dfrac{s}{n-s}$. We then let, for $1 \leq s \leq n-1$, $F : \mathbb{R}^{\binom{n}{s} \times \binom{n}{s}} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(\eta, \delta) := h(|\eta|, \delta).$$

Then from the convexity of h and from the fact that $x \to h(x, \delta)$ is non decreasing in $\mathbb{R}_+$, we deduce that F is convex. Observing that

$$f(\xi) = F(\operatorname{adj}_s \xi, \det \xi)$$

we immediately obtain the polyconvexity of f from the fact that $p \geq \dfrac{s}{n-s}$.

Step 2: f rank one convex $\Rightarrow p \geq \dfrac{s}{n-s}$. Let $\xi \in \mathbb{R}^{n \times n}$, $a, b \in \mathbb{R}^n$ be such that

$$\det(\xi + ta \otimes b) > 0, \text{ for every } t > 0.$$

Then the rank one convexity of f implies that

$$t \to \varphi(t) := f(\xi + ta \otimes b) = \left(\frac{|\operatorname{adj}_s(\xi + ta \otimes b)|^{n/s}}{\det(\xi + ta \otimes b)} \right)^p$$

is convex. We next simplify the notations by letting $\lambda_1, \cdots, \lambda_5$ be such that

$$\begin{cases} |\operatorname{adj}_s(\xi + ta \otimes b)|^2 = \lambda_1^2 t^2 + \lambda_2 t + \lambda_3^2 \\ \det(\xi + ta \otimes b) = \lambda_4 t + \lambda_5 . \end{cases}$$

Such $\lambda_1, \cdots, \lambda_5$ exist since

$$t \to \operatorname{adj}_s(\xi + ta \otimes b) \quad \text{and} \quad t \to \det(\xi + ta \otimes b)$$

are linear functions (cf. Proposition 5.65). Combining the above notation with the definition of φ, we find

$$\varphi(t) = \left(\lambda_1^2 t^2 + \lambda_2 t + \lambda_3^2\right)^{\frac{np}{2s}} (\lambda_4 t + \lambda_5)^{-p}.$$

After an elementary computation we obtain

$$\begin{aligned} \varphi''(t) &= \left(\lambda_1^2 t^2 + \lambda_2 t + \lambda_3^2\right)^{\frac{np}{2s}-2} (\lambda_4 t + \lambda_5)^{-p-2} \\ &\quad \times [\, \lambda_1^4 \lambda_4^2 t^4 \frac{p}{s^2} (n-s)^2 (p - \frac{s}{n-s}) + O(t^3) \,]. \end{aligned}$$

Since φ is convex for $t > 0$ we must have $p \geq \dfrac{s}{n-s}$. ■

5.4 Appendix: some basic properties of determinants

In the whole of Chapter 5, we have seen the importance of *determinants* in quasiconvex analysis. We gather in this appendix some well known algebraic properties of determinants. In the first part, we carefully introduce the notation for the minors $\operatorname{adj}_s \xi$ of a given matrix ξ.

We first introduce some notation. Let $n \in \mathbb{N}$ (the set of positive integers) and let $1 \leq s \leq n$. We define

$$I_s^n := \{(\alpha_1, \cdots, \alpha_s) \in \mathbb{N}^s : 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leq n\}.$$

We call the elements of I_s^n increasing s-tuples. The number of elements of I_s^n is then

$$\operatorname{card} I_s^n = \binom{n}{s} = \frac{n!}{s!\,(n-s)!}.$$

We next endow I_s^n with the following ordering relation:

$$\alpha = (\alpha_1, \cdots, \alpha_s) \succ (\beta_1, \cdots, \beta_s) = \beta$$

if and only if

$$\alpha_k < \beta_k\,,$$

where k is the largest integer less than or equal to s such that $\alpha_k \neq \beta_k$ and $\alpha_l = \beta_l$ for every $l > k$. (This is the inverse of the lexicographical order when read backward.)

Example 5.60 (i) $n = 4$, $s = 2$. Then

$$(1,2) \succ (1,3) \succ (2,3) \succ (1,4) \succ (2,4) \succ (3,4)\,.$$

(ii) $n = 5$, $s = 3$. Then

$$\begin{aligned}(1,2,3) &\succ (1,2,4) \succ (1,3,4) \succ (2,3,4) \succ (1,2,5)\\ &\succ (1,3,5) \succ (2,3,5) \succ (1,4,5) \succ (2,4,5) \succ (3,4,5)\,.\end{aligned}$$

(iii) $s = n - 1$. Then

$$(1, \cdots, n-1) \succ \cdots \succ (1, \cdots, k-1, k+1, \cdots, n) \succ \cdots \succ (2, \cdots, n)\,. \qquad \diamondsuit$$

We then define the map φ_s^n

$$\varphi_s^n : \{1, 2, 3, \cdots, \tbinom{n}{s}\} \to I_s^n$$

as the only bijection that respects the order defined above.

Example 5.61 (i) $n = 4$, $s = 2$. Then

$$\begin{aligned}\varphi_2^4(1) &= (3,4)\,,\ \varphi_2^4(2) = (2,4)\,,\ \varphi_2^4(3) = (1,4)\,,\\ \varphi_2^4(4) &= (2,3)\,,\ \varphi_2^4(5) = (1,3)\,,\ \varphi_2^4(6) = (1,2)\,.\end{aligned}$$

(ii) $s = n - 1$. Then

$$\begin{aligned}\varphi_{n-1}^n(1) &= (2, \cdots, n)\\ \varphi_{n-1}^n(k) &= (1, \cdots, k-1, k+1, \cdots, n)\\ \varphi_{n-1}^n(n) &= (1, \cdots, n-1)\,. \qquad \diamondsuit\end{aligned}$$

We are now in a position to define, for a given matrix $\xi \in \mathbb{R}^{N \times n}$, the *adjugate* matrix of order s, $1 \leq s \leq n \wedge N = \min\{n, N\}$,

$$\operatorname{adj}_s \xi \in \mathbb{R}^{\binom{N}{s} \times \binom{n}{s}}.$$

Let $\xi \in \mathbb{R}^{N \times n}$ be such that

$$\xi = \begin{pmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^N & \cdots & \xi_n^N \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} = (\xi_1, \cdots, \xi_n)\,.$$

We define $\operatorname{adj}_s \xi$ to be the following matrix in $\mathbb{R}^{\binom{N}{s} \times \binom{n}{s}}$:

$$\begin{aligned} \operatorname{adj}_s \xi &= \begin{pmatrix} (\operatorname{adj}_s \xi)_1^1 & \cdots & (\operatorname{adj}_s \xi)_{\binom{n}{s}}^1 \\ \vdots & \ddots & \vdots \\ (\operatorname{adj}_s \xi)_1^{\binom{N}{s}} & \cdots & (\operatorname{adj}_s \xi)_{\binom{n}{s}}^{\binom{N}{s}} \end{pmatrix} \in \mathbb{R}^{\binom{N}{s} \times \binom{n}{s}} \\ &= \begin{pmatrix} (\operatorname{adj}_s \xi)^1 \\ \vdots \\ (\operatorname{adj}_s \xi)^{\binom{N}{s}} \end{pmatrix} = \left((\operatorname{adj}_s \xi)_1, \cdots, (\operatorname{adj}_s \xi)_{\binom{n}{s}} \right), \end{aligned}$$

where

$$(\operatorname{adj}_s \xi)_\alpha^i = (-1)^{i+\alpha} \det \begin{pmatrix} \xi_{\alpha_1}^{i_1} & \cdots & \xi_{\alpha_s}^{i_1} \\ \vdots & \ddots & \vdots \\ \xi_{\alpha_1}^{i_s} & \cdots & \xi_{\alpha_s}^{i_s} \end{pmatrix}$$

and $(i_1, \cdots, i_s)$, $(\alpha_1, \cdots, \alpha_s)$ are the s-tuples corresponding to i and α by the bijections φ_s^N and φ_s^n, meaning that

$$\varphi_s^N(i) = (i_1, \cdots, i_s) \quad \text{and} \quad \varphi_s^n(\alpha) = (\alpha_1, \cdots, \alpha_s)\,.$$

Notation 5.62 We sometimes, as in examples (iv) and (vii) below, denote by

$$\widehat{\xi}_{\alpha_1, \cdots, \alpha_l}^{i_1, \cdots, i_k}$$

the $(N-k) \times (n-l)$ matrix obtained from $\xi \in \mathbb{R}^{N \times n}$ by suppressing the k rows $i_1, \cdots, i_k$ and the l columns $\alpha_1, \cdots, \alpha_l$. ♢

Example 5.63 (i) $N = n = 2$, $s = 1$. Let

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix}.$$

Then

$$I_s^n = I_s^N = \{1, 2\}$$

and the bijection $\varphi_1^2 : \{1,2\} \to \{2,1\}$. Hence

$$\operatorname{adj}_1 \xi = \begin{pmatrix} (\operatorname{adj}_1 \xi)_1^1 & (\operatorname{adj}_1 \xi)_2^1 \\ (\operatorname{adj}_1 \xi)_1^2 & (\operatorname{adj}_1 \xi)_2^2 \end{pmatrix} = \begin{pmatrix} \xi_2^2 & -\xi_1^2 \\ -\xi_2^1 & \xi_1^1 \end{pmatrix}.$$

(note that $\operatorname{adj}_1 \xi$ is exactly $\widetilde{\xi}$ defined in Theorem 5.51 above).

(ii) $N = n = s = 2$. Then

$$I_s^n = I_s^N = \{(1,2)\}$$

and $\varphi_2^2(1) = (1,2)$. Hence

$$\operatorname{adj}_2 \xi = \det \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix} = \det \xi.$$

(iii) $N = 3,\ s = n = 2$. Then

$$I_s^n = I_2^2 = \{(1,2)\}$$

and $\varphi_2^2(1) = (1,2)$, while

$$I_s^N = I_2^3 = \{(1,2)\,;(1,3)\,;(2;3)\}$$

and $\varphi_2^3(1) = (2,3)$, $\varphi_2^3(2) = (1,3)$, $\varphi_2^3(3) = (1,2)$. Therefore, if

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \\ \xi_1^3 & \xi_2^3 \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = (\xi_1, \xi_2),$$

then

$$\operatorname{adj}_2 \xi = \begin{pmatrix} (\operatorname{adj}_2 \xi)_1^1 \\ (\operatorname{adj}_2 \xi)_1^2 \\ (\operatorname{adj}_2 \xi)_1^3 \end{pmatrix} = \begin{pmatrix} \det \begin{pmatrix} \xi_1^2 & \xi_2^2 \\ \xi_1^3 & \xi_2^3 \end{pmatrix} \\ -\det \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^3 & \xi_2^3 \end{pmatrix} \\ \det \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix} \end{pmatrix}.$$

(iv) $N = n+1,\ s = n$. We let

$$\xi = \begin{pmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^{n+1} & \cdots & \xi_n^{n+1} \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^{n+1} \end{pmatrix} = (\xi_1, \cdots, \xi_n).$$

Then

$$\operatorname{adj}_n \xi = \begin{pmatrix} (\operatorname{adj}_n \xi)_1^1 \\ \vdots \\ (\operatorname{adj}_n \xi)_1^{n+1} \end{pmatrix} = \begin{pmatrix} \det \begin{pmatrix} \xi_1^2 & \cdots & \xi_n^2 \\ \vdots & \ddots & \vdots \\ \xi_1^{n+1} & \cdots & \xi_n^{n+1} \end{pmatrix} \\ \vdots \\ (-1)^{n+2} \det \begin{pmatrix} \xi_1^1 & \cdots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^n & \cdots & \xi_n^n \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \det \widehat{\xi}^{\,1} \\ \vdots \\ (-1)^{n+2} \det \widehat{\xi}^{\,n+1} \end{pmatrix}$$

where $\widehat{\xi}^{\,k}$ denotes the $n \times n$ matrix obtained by suppressing the k th row in the matrix ξ.

(v) $N = n = s = 3$. Then $I_3^3 = \{(1, 2, 3)\}$ and therefore

$$\operatorname{adj}_3 \xi = \det \xi.$$

(vi) $N = n = 3,\ s = 2$. Then

$$\operatorname{adj}_2 \xi = \begin{pmatrix} \det \begin{pmatrix} \xi_2^2 & \xi_3^2 \\ \xi_2^3 & \xi_3^3 \end{pmatrix} & -\det \begin{pmatrix} \xi_1^2 & \xi_3^2 \\ \xi_1^3 & \xi_3^3 \end{pmatrix} & \det \begin{pmatrix} \xi_1^2 & \xi_2^2 \\ \xi_1^3 & \xi_2^3 \end{pmatrix} \\ -\det \begin{pmatrix} \xi_2^1 & \xi_3^1 \\ \xi_2^3 & \xi_3^3 \end{pmatrix} & \det \begin{pmatrix} \xi_1^1 & \xi_3^1 \\ \xi_1^3 & \xi_3^3 \end{pmatrix} & -\det \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^3 & \xi_2^3 \end{pmatrix} \\ \det \begin{pmatrix} \xi_2^1 & \xi_3^1 \\ \xi_2^2 & \xi_3^2 \end{pmatrix} & -\det \begin{pmatrix} \xi_1^1 & \xi_3^1 \\ \xi_1^2 & \xi_3^2 \end{pmatrix} & \det \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix} \end{pmatrix}.$$

The above expression is the usual transpose of the matrix of *cofactors*.

(vii) $N = n$ and $s = n - 1$. Then

$$\operatorname{adj}_{n-1} \xi \in \mathbb{R}^{n \times n}$$

and

$$\left(\operatorname{adj}_{n-1} \xi\right)_\alpha^i = (-1)^{i+\alpha} \det(\widehat{\xi}_{\alpha}^{\,i})$$

where $\widehat{\xi}_{\alpha}^{\,i}$ is the $(n-1) \times (n-1)$ matrix obtained from $\xi \in \mathbb{R}^{n \times n}$ by suppressing the i th row and the α th column. ♢

Remark 5.64 Note that one can write the rows of $\operatorname{adj}_s \xi$ as

$$\left(\operatorname{adj}_s \xi\right)^i = (-1)^{i+1} \operatorname{adj}_s \begin{pmatrix} \xi^{i_1} \\ \vdots \\ \xi^{i_s} \end{pmatrix}, \ 1 \leq i \leq \tbinom{N}{s},$$

where $(i_1, \cdots, i_s) = \varphi_s^N(i)$ is the s-tuple associated to the integer i. So, in particular,

$$\left(\operatorname{adj}_s \xi\right)^1 = \operatorname{adj}_s \begin{pmatrix} \xi^{N-s+1} \\ \xi^{N-s+2} \\ \vdots \\ \xi^{N-1} \\ \xi^N \end{pmatrix}, \cdots, \left(\operatorname{adj}_s \xi\right)^{\binom{N}{s}} = (-1)^{\binom{N}{s}+1} \operatorname{adj}_s \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^{s-1} \\ \xi^s \end{pmatrix}.$$

A similar remark applies to the columns of $\operatorname{adj}_s \xi$. $\diamondsuit$

We now give some elementary properties of determinants.

Proposition 5.65 *Let $\xi \in \mathbb{R}^{N \times n}$.*

(i) If $N = n$, then, for every $\xi \in \mathbb{R}^{n \times n}$,

$$\left\langle \xi^\mu ; \left(\operatorname{adj}_{n-1} \xi\right)^\nu \right\rangle = \left\langle \xi_\mu ; \left(\operatorname{adj}_{n-1} \xi\right)_\nu \right\rangle = \delta_{\mu\nu} \det \xi, \ \mu, \nu = 1, 2, \cdots, n,$$

where $\langle \cdot ; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$ and $\delta_{\mu\nu}$ denotes the Kronecker symbol.

(ii) If $N = n$, then, for every $\xi \in \mathbb{R}^{n \times n}$,

$$\xi \left(\operatorname{adj}_{n-1} \xi\right)^t = \det \xi \cdot I$$

where I is the identity matrix in $\mathbb{R}^{n \times n}$ and ξ^t denotes the transpose of the matrix ξ. In particular if $\det \xi \neq 0$, then

$$\xi^{-1} = \frac{1}{\det \xi} \left(\operatorname{adj}_{n-1} \xi\right)^t.$$

(iii) If $N = n+1$, then, for every $\xi \in \mathbb{R}^{(n+1) \times n}$,

$$\left\langle \xi_\nu ; \operatorname{adj}_n \xi \right\rangle = 0, \ \nu = 1, \cdots, n,$$

where $\langle \cdot ; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{n+1}$.

(iv) If $N = n-1$, then, for every $\xi \in \mathbb{R}^{(n-1) \times n}$,

$$\left\langle \xi^\nu ; \operatorname{adj}_{n-1} \xi \right\rangle = 0, \ \nu = 1, \cdots, n-1,$$

where $\langle \cdot ; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

(v) If $N = n$, then, for every $\xi \in \mathbb{R}^{n\times n}$,

$$\frac{\partial}{\partial \xi_\alpha^i}(\det \xi) = \left(\operatorname{adj}_{n-1}\xi\right)_\alpha^i\ ,\ 1 \le i,\ \alpha \le n = N.$$

(vi) Denote

$$T(\xi) = (\xi, \operatorname{adj}_2 \xi, \cdots, \operatorname{adj}_{n\wedge N}\xi) \in \mathbb{R}^{\tau(n,N)}$$

where $n \wedge N = \min\{n, N\}$ and

$$\tau(n,N) = \sum_{s=1}^{n\wedge N}\sigma(s) = \sum_{s=1}^{n\wedge N}\tbinom{n}{s}\tbinom{N}{s}.$$

Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^N$. Define

$$a \otimes b = \left(a^i b_\alpha\right)_{1\le\alpha\le n}^{1\le i\le N} \in \mathbb{R}^{N\times n}.$$

Let $t \in [0,1]$, then, for every $\xi \in \mathbb{R}^{N\times n}$,

$$T(\xi + (1-t)\,a\otimes b) = tT(\xi) + (1-t)\,T(\xi + a\otimes b)\,.$$

Proof. **(i)** The case $\mu = \nu$ is just the way a determinant is computed, by expanding it along the νth row or the νth column. When $\mu \neq \nu$, then both $\left\langle \xi^\mu; \left(\operatorname{adj}_{n-1}\xi\right)^\nu\right\rangle$ and $\left\langle \xi_\mu; \left(\operatorname{adj}_{n-1}\xi\right)_\nu\right\rangle$ are again determinants of $n\times n$ matrices, but the first matrix has twice the row ξ^μ and the second has twice the column ξ_μ. Thus both determinants are equal to 0, as claimed.

(ii) This follows at once from (i).

(iii) Let $N = n+1$ and $\nu \in \{1, \cdots, n\}$. We have to show that

$$\langle \xi_\nu; \operatorname{adj}_n \xi\rangle = 0.$$

Define the matrix $\eta = [\xi_\nu; \xi] \in \mathbb{R}^{(n+1)\times(n+1)}$ (recall that $\xi \in \mathbb{R}^{(n+1)\times n}$). Then $\eta_1 = \eta_{\nu+1}$ and therefore $\det \eta = 0$. Using (i), we obtain

$$0 = \det \eta = \langle \eta_1; (\operatorname{adj}_n \eta)_1\rangle = \langle \xi_\nu; \operatorname{adj}_n \xi\rangle\,.$$

(iv) This is established exactly as (iii).

(v) This is a direct consequence of (i).

(vi) We divide the proof into three steps.

Step 1. The result is equivalent to

$$\operatorname{adj}_s(\xi + (1-t)\,a\otimes b) = t\operatorname{adj}_s\xi + (1-t)\operatorname{adj}_s(\xi + a\otimes b)$$

for every $1 \le s \le n\wedge N$. In terms of components this is equivalent to

$$\begin{aligned}&(\operatorname{adj}_s(\xi + (1-t)\,a\otimes b))_\alpha^i\\ &\qquad = t\left(\operatorname{adj}_s\xi\right)_\alpha^i + (1-t)\left(\operatorname{adj}_s(\xi + a\otimes b)\right)_\alpha^i\ ,\end{aligned} \tag{5.110}$$

$1 \leq i \leq \binom{N}{s}$, $1 \leq \alpha \leq \binom{n}{s}$. Recall that

$$(\operatorname{adj}_s \xi)_\alpha^i = (-1)^{i+\alpha} \det \begin{pmatrix} \xi_{\alpha_1}^{i_1} & \cdots & \xi_{\alpha_s}^{i_1} \\ \vdots & \ddots & \vdots \\ \xi_{\alpha_1}^{i_s} & \cdots & \xi_{\alpha_s}^{i_s} \end{pmatrix}.$$

By abuse of notation, let

$$\xi = \begin{pmatrix} \xi_{\alpha_1}^{i_1} & \cdots & \xi_{\alpha_s}^{i_1} \\ \vdots & \ddots & \vdots \\ \xi_{\alpha_1}^{i_s} & \cdots & \xi_{\alpha_s}^{i_s} \end{pmatrix}, \; a \otimes b = \begin{pmatrix} a^{i_1} b_{\alpha_1} & \cdots & a^{i_1} b_{\alpha_s} \\ \vdots & \ddots & \vdots \\ a^{i_s} b_{\alpha_1} & \cdots & a^{i_s} b_{\alpha_s} \end{pmatrix}.$$

Therefore (5.110) is equivalent to showing that, for every $\xi \in \mathbb{R}^{s \times s}$, $a, b \in \mathbb{R}^s$, $t \in [0,1]$,

$$\det (\xi + (1-t)\, a \otimes b) = t \det \xi + (1-t) \det (\xi + a \otimes b)\,. \tag{5.111}$$

This is a standard property of determinants that we prove in the two steps below.

Step 2. We start by proving (5.111) when

$$a = b = e^1 = e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^s.$$

Note that, for every $x \in \mathbb{R}$, we have

$$\left(\xi + x e^1 \otimes e_1\right)^1 = \xi^1 + x e^1 \quad \text{and} \quad \left(\operatorname{adj}_{s-1}\left(\xi + x e^1 \otimes e_1\right)\right)^1 = \left(\operatorname{adj}_{s-1} \xi\right)^1.$$

The first identity is obvious and the second one follows since the components of $\left(\operatorname{adj}_{s-1} \xi\right)^1$ are given by determinants where the first row of ξ does not appear.

We can therefore apply (i) to find

$$\begin{aligned} & \det\left(\xi + (1-t)\, e^1 \otimes e_1\right) \\ = \;& \left\langle \left(\xi + (1-t)\, e^1 \otimes e_1\right)^1 ; \left(\operatorname{adj}_{s-1}\left(\xi + (1-t)\, e^1 \otimes e_1\right)\right)^1 \right\rangle \\ = \;& \left\langle \xi^1 + (1-t)\, e^1 ; \left(\operatorname{adj}_{s-1} \xi\right)^1 \right\rangle \\ = \;& t \left\langle \xi^1 ; \left(\operatorname{adj}_{s-1} \xi\right)^1 \right\rangle + (1-t) \left\langle \xi^1 + e^1 ; \left(\operatorname{adj}_{s-1} \xi\right)^1 \right\rangle \\ = \;& t \left\langle \xi^1 ; \left(\operatorname{adj}_{s-1} \xi\right)^1 \right\rangle \\ & + (1-t) \left\langle \xi^1 + e^1 ; \left(\operatorname{adj}_{s-1}\left(\xi + e^1 \otimes e_1\right)\right)^1 \right\rangle \\ = \;& t \det \xi + (1-t) \det\left(\xi + e^1 \otimes e_1\right) \end{aligned}$$

which is the claim of Step 2.

Step 3. The general statement (5.111) follows at once from Step 2 and Theorem 13.3. Indeed, we can find $R, Q \in O(s)$ such that

$$R\left(e^1 \otimes e_1\right) Q = a \otimes b.$$

We therefore find, using Step 2,

$$\begin{aligned}
\det(\xi + (1-t)\, a \otimes b) &= \det\left(R\left(R^t \xi Q^t + (1-t)\, e^1 \otimes e_1\right) Q\right) \\
&= \det R \,\det\left(R^t \xi Q^t + (1-t)\, e^1 \otimes e_1\right)\, \det Q \\
&= t \det R \det\left(R^t \xi Q^t\right) \det Q \\
&\quad + (1-t) \det R \det\left(R^t \xi Q^t + e^1 \otimes e_1\right) \det Q \\
&= t \det \xi + (1-t) \det\left(\xi + R\left(e^1 \otimes e_1\right) Q\right) \\
&= t \det \xi + (1-t) \det(\xi + a \otimes b)
\end{aligned}$$

which is the claim. ■

We also have the following useful result (see Buttazzo-Dacorogna-Gangbo [113] and Dacorogna-Maréchal [205]).

Proposition 5.66 *(i) Let $\xi \in \mathbb{R}^{N \times n}$, $\eta \in \mathbb{R}^{n \times m}$ and*

$$1 \leq s \leq N \wedge n \wedge m := \min\{N, n, m\}.$$

Then

$$\operatorname{adj}_s(\xi\eta) = \operatorname{adj}_s \xi \ \operatorname{adj}_s \eta.$$

(ii) Let $\xi \in \mathbb{R}^{N \times n}$ and $1 \leq s \leq N \wedge n$, then

$$\operatorname{adj}_s\left(\xi^t\right) = \left(\operatorname{adj}_s \xi\right)^t.$$

(iii) If $N = n$ and $R \in O(n)$ (respectively $R \in SO(n)$), then

$$\operatorname{adj}_s R \in O\left(\tbinom{n}{s}\right) \quad \textit{(respectively } \operatorname{adj}_s R \in SO\left(\tbinom{n}{s}\right) \textit{)}.$$

(iv) If $N = n$ and $\xi \in \mathbb{R}^{n \times n}$ is invertible, then $\operatorname{adj}_s \xi \in \mathbb{R}^{\binom{n}{s} \times \binom{n}{s}}$ is invertible and

$$\left(\operatorname{adj}_s \xi\right)^{-1} = \operatorname{adj}_s\left(\xi^{-1}\right).$$

(v) If $N = n$ and if $R \in SO(n)$, then

$$\operatorname{adj}_{n-1} R = R.$$

Proof. (i) We have to prove that

$$\left(\operatorname{adj}_s(\xi\eta)\right)^i_j = \left(\operatorname{adj}_s \xi \operatorname{adj}_s \eta\right)^i_j$$

for every $1 \leq i \leq \binom{N}{s}$, $1 \leq j \leq \binom{m}{s}$. To simplify the notation, we will write

$$\alpha := \varphi_s^N\,,\ \beta := \varphi_s^n\,,\ \gamma := \varphi_s^m\,.$$

Let the s-tuples corresponding to i and j (and later k) be given by

$$\alpha\,(i) = (i_1, \cdots, i_s)\,,\ \beta\,(k) = (k_1, \cdots, k_s)\,,\ \gamma\,(j) = (j_1, \cdots, j_s)\,.$$

For a matrix $\theta \in \mathbb{R}^{N\times m}$, we let

$$\theta_{\gamma(j)}^{\alpha(i)} := \begin{pmatrix} \theta_{j_1}^{i_1} & \cdots & \theta_{j_s}^{i_1} \\ \vdots & \ddots & \vdots \\ \theta_{j_1}^{i_s} & \cdots & \theta_{j_s}^{i_s} \end{pmatrix} \in \mathbb{R}^{s\times s}$$

and, for $1 \leq \nu \leq m$,

$$(\,\theta_{\gamma(j)}^{\alpha(i)}\,)_\nu := \begin{pmatrix} \theta_\nu^{i_1} \\ \vdots \\ \theta_\nu^{i_s} \end{pmatrix} \in \mathbb{R}^s.$$

For $1 \leq p, q \leq s$, we have that

$$(\,(\,\xi\eta_{\gamma(j)}^{\alpha(i)}\,)_p^q = (\,\xi\eta\,)_{j_p}^{i_q} = \sum_{\nu=1}^n \xi_\nu^{i_q}\eta_{j_p}^\nu\,.$$

In other words, the pth column vector of the matrix is given by

$$\begin{aligned} (\,(\xi\eta)_{\gamma(j)}^{\alpha(i)}\,)_p \quad &= \begin{pmatrix} (\,(\,\xi\eta\,)_{\gamma(j)}^{\alpha(i)}\,)_p^1 \\ \vdots \\ (\,(\,\xi\eta\,)_{\gamma(j)}^{\alpha(i)}\,)_p^s \end{pmatrix} = \begin{pmatrix} \sum_{\nu=1}^n \xi_\nu^{i_1}\eta_{j_p}^\nu \\ \vdots \\ \sum_{\nu=1}^n \xi_\nu^{i_s}\eta_{j_p}^\nu \end{pmatrix} \\ &= \sum_{\nu=1}^n \eta_{j_p}^\nu \begin{pmatrix} \xi_\nu^{i_1} \\ \vdots \\ \xi_\nu^{i_s} \end{pmatrix} = \sum_{\nu=1}^n \eta_{j_p}^\nu (\,\xi^{\alpha(i)}\,)_\nu. \end{aligned}$$

We therefore have, by definition of adj_s, that

$$\begin{aligned} &(\operatorname{adj}_s(\xi\eta))_j^i \\ =\;& (-1)^{i+j}\det(\,(\xi\eta)_{\gamma(j)}^{\alpha(i)}\,) \\ =\;& (-1)^{i+j}\det(\,(\,(\xi\eta)_{\gamma(j)}^{\alpha(i)}\,)_1\,, \cdots, (\,(\xi\eta)_{\gamma(j)}^{\alpha(i)}\,)_s\,) \\ =\;& (-1)^{i+j}\det(\textstyle\sum_{\nu=1}^n \eta_{j_1}^\nu (\,\xi^{\alpha(i)}\,)_\nu\,, \cdots, \sum_{\nu=1}^n \eta_{j_s}^\nu (\,\xi^{\alpha(i)}\,)_\nu\,) \\ =\;& (-1)^{i+j}\det(\textstyle\sum_{\nu_1=1}^n \eta_{j_1}^{\nu_1} (\,\xi^{\alpha(i)}\,)_{\nu_1}\,, \cdots, \sum_{\nu_s=1}^n \eta_{j_s}^{\nu_s} (\,\xi^{\alpha(i)}\,)_{\nu_s}\,) \\ =\;& (-1)^{i+j} \sum_{\nu_1,\cdots,\nu_s=1}^n \eta_{j_1}^{\nu_1}\cdots\eta_{j_s}^{\nu_s}\det(\,(\,\xi^{\alpha(i)}\,)_{\nu_1}\,, \cdots, (\,\xi^{\alpha(i)}\,)_{\nu_s}\,). \end{aligned}$$

Now, if $\nu_p = \nu_q$ for two distinct integers $p, q \in \{1, \cdots, s\}$, we clearly have

$$\det(\,(\,\xi^{\alpha(i)}\,)_{\nu_1}\,, \cdots, (\,\xi^{\alpha(i)}\,)_{\nu_s}\,) = 0.$$

Thus, writing $F_{n,s}$ for all s-tuples $(\nu_1, \cdots, \nu_s)$ in $\{1, \cdots, n\}^s$ such that the ν_p are pairwise distinct, we find

$$(\mathrm{adj}_s\,(\xi\eta))^i_j = (-1)^{i+j} \sum_{(\nu_1,\cdots,\nu_s)\in F_{n,s}} \eta^{\nu_1}_{j_1} \cdots \eta^{\nu_s}_{j_s} \det(\,(\,\xi^{\alpha(i)}\,)_{\nu_1}\,, \cdots, (\,\xi^{\alpha(i)}\,)_{\nu_s}\,). \tag{5.112}$$

On the other hand we can write

$$\begin{aligned}
(\mathrm{adj}_s\,\xi\,\mathrm{adj}_s\,\eta)^i_j &= \sum_{k=1}^{\binom{n}{s}} (\mathrm{adj}_s\,\xi)^i_k\,(\mathrm{adj}_s\,\eta)^k_j \\
&= \sum_{k=1}^{\binom{n}{s}} (-1)^{i+k} \det(\,\xi^{\alpha(i)}_{\beta(k)}\,)\,(-1)^{k+j} \det(\,\eta^{\beta(k)}_{\gamma(j)}\,) \\
&= (-1)^{i+j} \sum_{k=1}^{\binom{n}{s}} \det(\,\xi^{\alpha(i)}_{\beta(k)}\eta^{\beta(k)}_{\gamma(j)}\,).
\end{aligned}$$

Since, for $1 \le p, q, r \le s$,

$$(\,\xi^{\alpha(i)}_{\beta(k)}\,)^q_p = \xi^{i_q}_{k_p} \quad \text{and} \quad (\,\eta^{\beta(k)}_{\gamma(j)}\,)^p_r = \eta^{k_p}_{j_r}$$

we find

$$(\,\xi^{\alpha(i)}_{\beta(k)}\eta^{\beta(k)}_{\gamma(j)}\,)^q_r = \sum_{p=1}^{s} \xi^{i_q}_{k_p}\eta^{k_p}_{j_r}\,.$$

Phrased differently, we have that the r-th column vector of the matrix is given by

$$\begin{aligned}
\left(\xi^{\alpha(i)}_{\beta(k)}\eta^{\beta(k)}_{\gamma(j)}\right)_r &= \begin{pmatrix} (\,\xi^{\alpha(i)}_{\beta(k)}\eta^{\beta(k)}_{\gamma(j)}\,)^1_r \\ \vdots \\ (\,\xi^{\alpha(i)}_{\beta(k)}\eta^{\beta(k)}_{\gamma(j)}\,)^s_r \end{pmatrix} = \begin{pmatrix} \sum_{p=1}^s \xi^{i_1}_{k_p}\eta^{k_p}_{j_r} \\ \vdots \\ \sum_{p=1}^s \xi^{i_s}_{k_p}\eta^{k_p}_{j_r} \end{pmatrix} \\
&= \sum_{p=1}^{s} \eta^{k_p}_{j_r} \begin{pmatrix} \xi^{i_1}_{k_p} \\ \vdots \\ \xi^{i_s}_{k_p} \end{pmatrix} = \sum_{p=1}^{s} \eta^{k_p}_{j_r}\,(\,\xi^{\alpha(i)}\,)_{k_p}\,.
\end{aligned}$$

We thus deduce that

$$\begin{aligned}
& (\operatorname{adj}_s \xi \operatorname{adj}_s \eta)_j^i \\
= \ & (-1)^{i+j} \sum_{k=1}^{\binom{n}{s}} \det\left(\left(\xi_{\beta(k)}^{\alpha(i)} \eta_{\gamma(j)}^{\beta(k)} \right)_1, \cdots, \left(\xi_{\beta(k)}^{\alpha(i)} \eta_{\gamma(j)}^{\beta(k)} \right)_s \right) \\
= \ & (-1)^{i+j} \sum_{k=1}^{\binom{n}{s}} \det\left(\sum_{p=1}^{s} \eta_{j_1}^{k_p} \left(\xi^{\alpha(i)} \right)_{k_p}, \cdots, \sum_{p=1}^{s} \eta_{j_s}^{k_p} \left(\xi^{\alpha(i)} \right)_{k_p} \right) \\
= \ & (-1)^{i+j} \sum_{k=1}^{\binom{n}{s}} \det\left(\sum_{p_1=1}^{s} \eta_{j_1}^{k_{p_1}} \left(\xi^{\alpha(i)} \right)_{k_{p_1}}, \cdots, \sum_{p_s=1}^{s} \eta_{j_s}^{k_{p_s}} \left(\xi^{\alpha(i)} \right)_{k_{p_s}} \right) \\
= \ & (-1)^{i+j} \sum_{k=1}^{\binom{n}{s}} \sum_{p_1, \cdots, p_s=1}^{s} \eta_{j_1}^{k_{p_1}} \cdots \eta_{j_s}^{k_{p_s}} \det\left(\left(\xi^{\alpha(i)} \right)_{k_{p_1}}, \cdots, \left(\xi^{\alpha(i)} \right)_{k_{p_s}} \right).
\end{aligned}$$

If $(p_1, \cdots, p_s) \in \{1, \cdots, s\}^s$ is not a permutation of $(1, \cdots, s)$, then

$$\det\left(\left(\xi^{\alpha(i)} \right)_{k_{p_1}}, \cdots, \left(\xi^{\alpha(i)} \right)_{k_{p_s}} \right) = 0.$$

Letting

$$\nu_r := k_{p_r}, \ r = 1, \cdots, s,$$

we note that, when $(p_1, \cdots, p_s) \in \{1, \cdots, s\}^s$ is a permutation of $(1, \cdots, s)$ and $k \in \left\{1, \cdots, \binom{n}{s}\right\}$, then $(\nu_1, \cdots, \nu_s) \in F_{n,s}$, the set of s-tuples $(\nu_1, \cdots, \nu_s)$ in $\{1, \cdots, n\}^s$ such that the ν_p are pairwise distinct. We therefore get that

$$\begin{aligned}
& (\operatorname{adj}_s \xi \operatorname{adj}_s \eta)_j^i \\
= \ & (-1)^{i+j} \sum_{(\nu_1, \cdots, \nu_s) \in F_{n,s}} \eta_{j_1}^{\nu_1} \cdots \eta_{j_s}^{\nu_s} \det\left(\left(\xi^{\alpha(i)} \right)_{\nu_1}, \cdots, \left(\xi^{\alpha(i)} \right)_{\nu_s} \right).
\end{aligned}$$

The above identity and (5.112) imply the result.

(ii) As above, let

$$\alpha := \varphi_s^N, \ \beta := \varphi_s^n.$$

We clearly have, for $1 \leq i \leq \binom{N}{s}$ and $1 \leq j \leq \binom{n}{s}$, that

$$\left(\xi^t\right)_{\beta(j)}^{\alpha(i)} = \left(\xi_{\alpha(i)}^{\beta(j)}\right)^t$$

since, for $\alpha(i) = (i_1, \cdots, i_s)$ and $\beta(j) = (j_1, \cdots, j_s)$, we can write

$$\begin{aligned}
\left(\xi^t\right)^{\alpha(i)}_{\beta(j)} &= \begin{pmatrix} (\xi^t)^{i_1}_{j_1} & \cdots & (\xi^t)^{i_1}_{j_s} \\ \vdots & \ddots & \vdots \\ (\xi^t)^{i_s}_{j_1} & \cdots & (\xi^t)^{i_s}_{j_s} \end{pmatrix} = \begin{pmatrix} \xi^{j_1}_{i_1} & \cdots & \xi^{j_s}_{i_1} \\ \vdots & \ddots & \vdots \\ \xi^{j_1}_{i_s} & \cdots & \xi^{j_s}_{i_s} \end{pmatrix} \\
&= \begin{pmatrix} \xi^{j_1}_{i_1} & \cdots & \xi^{j_1}_{i_s} \\ \vdots & \ddots & \vdots \\ \xi^{j_s}_{i_1} & \cdots & \xi^{j_s}_{i_s} \end{pmatrix}^t = \left(\xi^{\beta(j)}_{\alpha(i)}\right)^t.
\end{aligned}$$

We can therefore deduce that

$$\begin{aligned}
\left(\operatorname{adj}_s\left(\xi^t\right)\right)^i_j &= (-1)^{i+j}\det(\left(\xi^t\right)^{\alpha(i)}_{\beta(j)}) = (-1)^{i+j}\det((\xi^{\beta(j)}_{\alpha(i)})^t) \\
&= (-1)^{i+j}\det(\xi^{\beta(j)}_{\alpha(i)}) = (\operatorname{adj}_s \xi)^j_i
\end{aligned}$$

which is statement (ii).

(iii) From (i) and (ii) we immediately deduce the claim for $R \in O(n)$, since

$$\begin{aligned}
\operatorname{adj}_s R\,(\operatorname{adj}_s R)^t &= \operatorname{adj}_s R \operatorname{adj}_s R^t = \operatorname{adj}_s\left(RR^t\right) \\
&= \operatorname{adj}_s I_n = I_{\binom{n}{s}}
\end{aligned}$$

where for any integer m we have let I_m to be the identity matrix in $\mathbb{R}^{m\times m}$.

We now discuss the case where $R \in SO(n)$. We already know that

$$\operatorname{adj}_s R \in O\left(\tbinom{n}{s}\right).$$

It therefore remains to prove that

$$\det(\operatorname{adj}_s R) = 1.$$

We observe that $SO(n)$ is a connected manifold, meaning that, for every $R \in SO(n)$, there exists a continuous function

$$\theta : [0,1] \to SO(n),\ \theta(0) = I_n,\ \theta(1) = R.$$

We then define, for $t \in [0,1]$, the function

$$f(t) := \det(\operatorname{adj}_s \theta(t)).$$

We observe that since any $Q \in SO(n) \subset O(n)$ has

$$\det(\operatorname{adj}_s Q) \in \{\pm 1\},$$

then the function f takes only values in $\{\pm 1\}$. Since it is a continuous function, as a composition of three continuous functions, and since $f(0) = 1$, we deduce that $f(1) = 1$, which is the assertion.

(iv) This follows from (i) exactly as above. Indeed

$$\operatorname{adj}_s \xi \operatorname{adj}_s \left(\xi^{-1}\right) = \operatorname{adj}_s I_n = I_{\binom{n}{s}}.$$

(v) From (ii) of Proposition 5.65, we have, since $R \in SO(n)$,

$$R\left(\operatorname{adj}_{n-1} R\right)^t = I_n$$

and thus the claim. ■

We now want to write, for every $\xi, \eta \in \mathbb{R}^{n \times n}$, $\det(\xi + \eta)$. To this aim let us introduce the following notations.

- Let $\mathcal{N}_{\{1,\cdots,n\}}$ be the set of couples (I, J), each of them ordered, so that

$$I \cup J = \{1, \cdots, n\}, \ I \cap J = \emptyset.$$

- For all $(I, J) \in \mathcal{N}_{\{1,\cdots,n\}}$ and all matrices $\xi, \eta \in \mathbb{R}^{n \times n}$, we denote by

$$(\xi^I, \eta^J) \in \mathbb{R}^{n \times n}$$

the $n \times n$ matrix whose row of index k is ξ^k if $k \in I$ or η^k if $k \in J$. So, for example, if $n = 3$, $I = \{1, 3\}$, $J = \{2\}$, then

$$(\xi^I, \eta^J) = \begin{pmatrix} \xi^1 \\ \eta^2 \\ \xi^3 \end{pmatrix}.$$

Proposition 5.67 *Let $\xi, \eta \in \mathbb{R}^{n \times n}$, then*

$$\det(\xi + \eta) = \sum_{(I,J) \in \mathcal{N}_{\{1,\cdots,n\}}} \det(\xi^I, \eta^J).$$

Proof. Let us first examine the case $n = 2$, where we trivially have

$$\det(\xi + \eta) = \det(\xi^1, \xi^2) + \det(\xi^1, \eta^2) + \det(\eta^1, \xi^2) + \det(\eta^1, \eta^2).$$

The general case easily follows if we write the determinant as a multilinear form; namely, for $\xi \in \mathbb{R}^{n \times n}$, we write

$$\det \xi = \xi^1 \wedge \cdots \wedge \xi^n.$$

The claim follows by induction, since

$$\begin{aligned}\det(\xi+\eta) &= \xi^1 \wedge \left(\xi^2+\eta^2\right) \wedge \cdots \wedge (\xi^n+\eta^n) + \eta^1 \wedge \left(\xi^2+\eta^2\right) \wedge \cdots \wedge (\xi^n+\eta^n) \\ &= \sum_{(I,J)\in\mathcal{N}_{\{2,\cdots,n\}}} \det(\,\xi^1,\xi^I,\eta^J\,) + \sum_{(I,J)\in\mathcal{N}_{\{2,\cdots,n\}}} \det(\,\eta^1,\xi^I,\eta^J\,) \\ &= \sum_{(I,J)\in\mathcal{N}_{\{1,\cdots,n\}}} \det(\,\xi^I,\eta^J\,).\end{aligned}$$

This finishes the proof of the proposition. ■

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