Computability and Numberings

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Introduction

The theory of computable numberings is one of the main parts of the theory of numberings. The papers of H. Rogers [36] and R. Friedberg [21] are the starting points in the systematical investigation of computable numberings. The general notion of a computable numbering was proposed in 1954 by A.N. Kolmogorov and V.A. Uspensky (see [40, p. 398]), and the monograph of Uspensky [41] was the first textbook that contained several basic results of the theory of computable numberings. The theory was developed further by many authors, and the most important contribution to it and its applications was made by A.I. Malt'sev, Yu.L. Ershov, A. Lachlan, S.S. Goncharov, S.A. Badaev, A.B. Khutoretskii, V.L. Selivanov, M. Kummer, M.B. Pouer-El, I.A. Lavrov, S.D. Denisov, and many other authors.

S.S. Goncharov and A.T. Nurtazin found applications of the theory of computable numberings to the theory of computable models, more precisely, to the problem of decidability of prime and saturated models [30]. Later S.S. Goncharov applied computable numberings of families of partial computable functions to the problem of characterizing autostability on the base of Scott's families [22] and established the existence of models with finite algorithmic dimension (on the base of the duality, founded by him, between the problem of the possible number of computable Friedberg numberings for families of c.e. sets and the problem of the existence of models with finite algorithmic dimension) by constructing families of c.e. sets with any finite number of Friedberg numberings up to equivalence [23, 24]. The problems arising in

- estimate of the complexity of isomorphisms between different representations of computable models,
- description of the autostable models,
- classification of the definable relations in computable models, etc.

led to investigation of computable numberings not only for families of partial computable functions and c.e. sets but also for families of constructive objects of a more general nature. In addition, in computability theory, one meets uniform computations for families of a special kind of relations and functions having high algorithmic complexity, and in the theory of computable models, very often we have to deal with computable classes of computable models.

All this was a strong motivation for S.S. Goncharov and A. Sorbi to propose in 1997 a new approach to the notion of computable numbering for general families of objects, which admit a constructive description in formal languages with a Gödel numbering for formulas [31].

Since then a lot of problems have been considered in the study of computable numberings for families of sets in the arithmetical hierarchy (see [7]–[15], and [33]–[35]) and in the hierarchy of Ershov (see [6], [16], [29], and [39]). Applications of generalized computable numberings were also pursued. The paper of S.S. Goncharov and J. Knight [27] offered an approach to the classification problem based on computable Friedberg numberings. And, in [26], and [28], computable numberings in all levels of the hyperarithmetical hierarchy, including the infinite ones, already have been applied to study problems in the theory of computable models.

In this paper, we study some problems relative to computable numberings in the sense of Goncharov–Sorbi for families of sets in the hyperarithmetical hierarchy. In section 1, we introduce a notion of computable numbering of a family of hyperarithmetical sets. In section 2, we continue to go along the line of research devoted to the problem of the isomorphism types of Rogers semilattices for families of arithmetical sets, which was initiated in the papers [7], [11]–[14].

We refer to the handbooks [37], [38], and [5] for the notions and standard notations on computability theory and computable infinite formulas. Undefined notions of the theory of numberings can be found in [18], and [19]. For more background on generalized computable numberings, see the articles [7], and [9].

1 Computable numberings in the hyperarithmetical hierarchy

A surjective mapping ν of the set ω of natural numbers onto a nonempty set $\mathcal{A}$ is called a *numbering* of $\mathcal{A}$. Suppose that $\mathcal{A}$ is a family of objects that admit constructive descriptions. By this we mean that one can define a language $\mathcal{L}$ (henceforth identified with a corresponding set of "well-formed formulas") and an interpretation of (fragments of) this language via an onto partial mapping $i\colon\ \mathcal{L} \longrightarrow \mathcal{A}$. For any object $a \in \mathcal{A}$, each formula Φ of $\mathcal{L}$ such that $i(\Phi) = a$ is interpreted as a "description" of a. Suppose further that $G\colon\ \omega \longrightarrow \mathcal{L}$ is a Gödel numbering.

Following [31], we propose:

Definition 1.1. A numbering ν of $\mathcal{A}$ is called *computable in* $\mathcal{L}$ *with respect to an interpretation* i if there exists a computable mapping f such that $\nu = i \circ G \circ f$.

It is immediate to see that Definition 1.1 does not depend on the choice of the Gödel numbering G. Hence, via identification of $\mathcal{L}$ with ω through some fixed Gödel numbering, the above definition states that ν is computable if there is some computable function f from ω to $\mathcal{L}$ such that $\nu = i \circ f$.

Definition 1.1 has a wide scope of applications, based on suitable choices of $\mathcal{L}$ and i. Let's, at first, consider the families of arithmetical sets. As language $\mathcal{L}$ we take in this case the collection of arithmetical first-order formulas in the signature $\langle +, \cdot, 0, s, \leqslant \rangle$, and i will be a mapping associating each formula with the corresponding set defined by that formula in the standard model $\mathfrak{N}$ of Peano arithmetic. For $v \in N$, denote by $\mathbf{v}$ an arithmetic term defining v, that is, the term $s(s(\ldots s(0)\ldots))$, in which the symbol s occurs v times. Then $A \in \Sigma^0_{n+1}$ if and only if there exists an arithmetic Σ_{n+1} formula $\Phi(v)$ such that

$$v \in A \Leftrightarrow \mathfrak{N} \models \Phi(\mathbf{v}).$$

If $\mathcal{A}$ is a family of Σ^0_{n+1} sets, then, by [31, Proposition 2.1], a numbering $\nu\colon N \to \mathcal{A}$ is computable with respect to interpretation i if and only if

$$\{\langle m, v\rangle \ \mid v \in \nu(m)\} \in \Sigma^0_{n+1}.$$

Thus, despite the strong hierarchy theorem, [37, §14.5], a computable numbering ν of a family $\mathcal{A} \subseteq \Sigma^0_{n+1}$ may be thought of as an enumeration procedure for the sequence $\nu(0), \nu(1), \ldots$ of Σ^0_{n+1} sets, which is uniformly computable with respect to the oracle $\emptyset^{(n)}$.

It seems promising to generalize in a straightforward manner this description of a computable numbering ν in terms of relative computability of the relation $\{\langle m, v\rangle \ \mid v \in \nu(m)\}$ for families of sets from any level of the arithmetical hierarchy, to families from any arbitrary level of the hyperarithmetical hierarchy.

Classes of the hyperarithmetical hierarchy

We need some notions from the textbook [37] to make our paper self-contained. First we remember Kleene's system of notations for computable ordinals. This system consists of a set $\mathcal{O}$ of notations, together with a partial ordering $<_{\mathcal{O}}$.

The ordinal 0 gets notation 1.

If a is a notation for α, then 2^a is a notation for $\alpha + 1$. Then $a <_{\mathcal{O}} 2^a$, and also, if $b <_{\mathcal{O}} a$, then $b <_{\mathcal{O}} 2^a$.

Suppose α is a limit ordinal. If φ_e is a total function, giving notations for an increasing sequence of ordinals with limit α, then $3 \cdot 5^e$ is a notation for α. For all n, $\varphi_e(n) <_{\mathcal{O}} 3 \cdot 5^e$, and if $b <_{\mathcal{O}} \varphi_e(n)$, then $b <_{\mathcal{O}} 3 \cdot 5^e$.

The sequence of oracles $\{\emptyset^{(n)}\}_{n\in\omega}$ is extended with the family of sets $H(a), a \in \mathcal{O}$, by transfinite induction on the ordinals $|a|_{\mathcal{O}}$ as follows.

(1) $H(1) = \emptyset$,

(2) $H(2^a) = H(a)'$,

(3) $H(3 \cdot 5^e) = \{\langle u, v\rangle \mid u <_{\mathcal{O}} 3 \cdot 5^e \ \&\ v \in H(u)\}$.

Now, following Kleene, we define the classes $\Sigma^0_\alpha, \Pi^0_\alpha, \Delta^0_\alpha$ of the hyperarithmetical hierarchy for all computable ordinals $\alpha \geq \omega$. For infinite α, a relation is said to be $\Sigma^0_\alpha, \Pi^0_\alpha$, or Δ^0_α if it is, respectively, c.e., co-c.e., or computable relative to $H(a)$, *for some* $a \in \mathcal{O}$ with $|a|_{\mathcal{O}} = \alpha$. By a theorem of Spector, such a relation will be c.e., co-c.e., or computable relative to $H(a)$ *for every* $a \in \mathcal{O}$ with $|a|_{\mathcal{O}} = \alpha$. It is important for us to recall the well-known lack of uniformity in the definition of the classes $\Sigma^0_\alpha, \Pi^0_\alpha, \Delta^0_\alpha$ when we pass from finite to infinite computable ordinals. For finite α, say $\alpha = n$, the Σ^0_n relations are the ones that are c.e. relative to $H(a)$ where $|a|$ is $n-1$. There is the same lack of uniformity for Π^0_α and Δ^0_α relations.

If α is a computable limit ordinal and $a \in \mathcal{O}$ is a notation for α, then for every $n \in \omega$, the classes of the hyperarithmetical hierarchy may be also defined by $H(a)$–forms.

- $\Sigma^0_{\alpha+n} = \Sigma^{H(a)}_{n+1}$,
- $\Pi^0_{\alpha+n} = \Pi^{H(a)}_{n+1}$,
- $\Delta^0_{\alpha+n} = \Delta^{H(a)}_{n+1}$.

We often use relativized forms of the sets $H(a)$. Let X be any set of natural numbers. Then

(1) $H^X(1) = X$,

(2) $H(2^a) = H^X(a)'$,

(3) $H(3 \cdot 5^e) = \{\langle u, v\rangle \mid u <_{\mathcal{O}} 3 \cdot 5^e \ \&\ v \in H^X(u)\}$.

We will need some details of Kleene's notion of partial recursive function relative to an oracle X, as is done in [37, § 9.2.]:

$$\varphi^X_z = \{\langle x, y\rangle \mid \exists u \exists v \left(\langle x, y, u, v\rangle \in W_{\rho(z)} \ \&\ D_u \subseteq X \ \&\ D_v \subseteq \overline{X}\right)\}; \qquad (1)$$

here $\rho(z)$ is a computable function with some special properties.

Ash–Knight's classification of the infinitary computable formulas

Let α be any constructive ordinal, and let $\mathcal{A}$ be a family of Σ^0_α sets. Our aim is to show that a numbering ν: $N \to \mathcal{A}$ is computable in the sense of Goncharov–Sorbi if and only if

$$\{\langle m, v\rangle \ \mid v \in \nu(m)\} \in \Sigma^0_\alpha.$$

But we still have not defined the notion of a computable numbering for a family $\mathcal{A} \subseteq \Sigma^0_\alpha$, for constructive ordinals $\alpha \geq \omega$. To do this, we need to specify a language suitable for descriptions of Σ^0_α sets, as well as an interpretation of these descriptions.

Evidently, we cannot restrict our descriptions to finite first-order formulas of arithmetics. We need a language with more expressive opportunities, namely, the language $L_{\omega_1\omega}$ with countable disjunctions and conjunctions. Indeed, to keep valid very productive tools like the compactness theorem, one is forced to consider some admissible fragments of the language $L_{\omega_1\omega}$. We will give an inductive definition of a language of computable infinitary formulas that has been used to characterize isomorphism types of computable models in terms of so-called Scott's rank (see [5]).

We follow [5] to give a classification of the family of computable infinitary formulas for any computable signature. Let $\{x_i : i \in \omega\}$ be a countable set of variables of a language L. Computable infinitary formulas are classified as *computable* Σ_α, or *computable* Π_α, for various computable ordinals α. Roughly speaking, they are infinitary formulas in which the disjunctions and conjunctions are over c.e. sets. In predicate formulas, only finitely many free variables are allowed, and for both predicate and propositional languages, only formulas in normal form are considered.

To each formula Φ, we associate a tuple of variables $\overline{x}$, including the free variables of Φ. We define the class of computable infinitary formulas by induction on the complexity, which is a computable ordinal. The computable Σ_0– and Π_0–formulas are the finitary open formulas.

For a computable ordinal $\alpha > 0$, a computable Σ_α formula $\Phi(\overline{x})$ is the disjunction of a c.e. set of formulas of the form $(\exists \overline{y})\psi$, where ψ is a computable Π_β formula for some $\beta < \alpha$ and $\overline{y}$ includes the variables of ψ that are not in $\overline{x}$ ($\overline{y}$ may also include some variables from $\overline{x}$).

Similarly, a computable Π_α formula $\Phi(\overline{x})$ is the conjunction of a c.e. set of formulas of the form $(\forall \overline{y})\psi$, where ψ is a computable Σ_β formula for some $\beta < \alpha$ and $\overline{y}$ includes the variables of ψ not in $\overline{x}$.

The informal notions given above are sufficient for us, but, for $\alpha \geq 2$, they are not precise. We refer to the textbook of C. Ash and J. Knight [5] for formal definitions as well as for their original Gödel numbering of the family of computable infinitary formulas.

Infinitary computable formulas have remarkable properties and have many applications in computable model theory (see [1]–[5], and [26]). For instance, we can illustrate this by the following two statements.

Proposition 1.2 ([5],[26]). *If $\mathcal{A}$, $\mathcal{B}$ are computable structures satisfying the same computable infinitary sentences, then $\mathcal{A} \cong \mathcal{B}$.*

Proposition 1.3 ([5],[26]). *Suppose $\overline{a}$, $\overline{b}$ are tuples satisfying the same computable infinitary formulas in a computable structure $\mathcal{A}$. Then there is an automorphism of $\mathcal{A}$ taking $\overline{a}$ to $\overline{b}$.*

To study Scott's ranks and the problems of auto-stability and algorithmic dimension as well as definability problems for computable structures, one has to extend the notion of a computable numbering for families of sets from finite levels of the arithmetical hierarchy to a notion of computable numbering for families of sets from infinite levels of the hyperarithmetical hierarchy [28].

Relations definable by computable infinitary formulas

Theorem 1.4 ([5, Theorem 7.5 (a)]). *For any computable structure $\mathcal{A}$, if Φ is a computable Σ_α formula, then $\Phi^{\mathcal{A}}$ is in the class Σ^0_α, and if Φ is a computable Π_α formula, then $\Phi^{\mathcal{A}}$ is in the class Π^0_α of the hyperarithmetical hierarchy. Moreover, uniformity holds.*

Therefore, every relation definable in the standard model $\mathfrak{N}$ of arithmetic by an infinitary computable Σ_α or a Π_α formula is, respectively, a Σ^0_α or a Π^0_α set. We will show that there are no new definable relations in $\mathfrak{N}$.

Lemma 1.5. *For every computable ordinal α, if $a \in \mathcal{O}$ is a notation for α, then $H(2^a)$ is definable in $\mathfrak{N}$ by a Σ_α formula, which is computable uniformly in a.*

Proof. We prove this lemma by transfinite induction on α. The statement of the lemma holds for any finite ordinal α (see [37]).

Let $a = 3 \cdot 5^e$ for some e. Then

$$x \in H(2^a) \Leftrightarrow \exists y \exists u \exists v \left(\langle x, y, u, v \rangle \in W_{\rho(x)} \;\&\; D_u \subseteq H(a) \;\&\; D_v \subseteq \overline{H(a)} \right).$$

Since $H(a) = \{\langle b, m \rangle \mid b <_{\mathcal{O}} a \;\&\; m \in H(b)\}$, it follows that

$$x \in H(a) \Leftrightarrow (\exists b <_{\mathcal{O}} a) \exists m \, (x = \langle b, m \rangle \;\&\; m \in H(b)) .$$

The relation $x = \langle b, m \rangle \;\&\; m \in H(b)$ is definable in a $\mathfrak{N}$ by Σ_β formula $\psi_b(x, m)$ for some ordinal $\beta = |b|_{\mathcal{O}}$, which is less than the limit ordinal α. We can consider

$\psi_b(x,m)$ as a $\Pi_{\beta+1}$ formula, which is computable uniformly in b. Therefore the relation $x \in H(a)$ is definable in $\mathfrak{N}$ by an infinite disjunction of formulas $\exists m\psi_b(x,m)$ over the c.e. set $\{b \mid b <_{\mathcal{O}} a\}$. So the relation $D_u \subseteq H(a)$ is definable by a Σ_α formula computable uniformly in a and u.

It is also true that

$$x \notin H(a) \Leftrightarrow (\forall b <_{\mathcal{O}} a)\forall m(x \neq \langle b,m\rangle)\vee$$
$$(\exists b <_{\mathcal{O}} a)\exists m\,(x = \langle b,m\rangle \;\&\; m \notin H(b))\,.$$

The relation $x = \langle b,m\rangle$ & $m \notin H(b)$ is definable in $\mathfrak{N}$ by a Π_β formula $\theta_b(x,m)$ for some ordinal $\beta = |b|_{\mathcal{O}}$ that is less than the ordinal α. By induction, $\theta_b(x,m)$ is computable uniformly in b and therefore the relation $x \notin H(a)$ is definable in $\mathfrak{N}$ by an infinite disjunction of formulas $\exists m\theta_b(x,m)$ over the c.e. set $\{b \mid b <_{\mathcal{O}} a\}$ and one Π_1 formula. So the relation $D_v \subseteq \overline{H(a)}$ is definable by a Σ_α formula computable uniformly in a and v.

Gathering all the facts proved above we obtain that the relation $x \in H(2^a)$ is definable in $\mathfrak{N}$ by a Σ_α formula, which is computable uniformly on a.

Finally suppose that α is infinite and $a = 2^b$ for some b. By induction we have that the relation $x \in H(a)$ is definable in $\mathfrak{N}$ by a Σ_β formula where $\beta = |b|_{\mathcal{O}}$. As in the previous case we have

$$x \in H(2^a) \Leftrightarrow \exists y\exists u\exists v\left(\langle x,y,u,v\rangle \in W_{\rho(x)} \;\&\; D_u \subseteq H(a) \;\&\; D_v \subseteq \overline{H(a)}\right).$$

By the definitions we obtain that the relation $D_u \subseteq H(a)$ is definable in $\mathfrak{N}$ by a finite conjunction of Σ_β formulas, and therefore, it is definable by a Σ_α formula computable uniformly in u and a. Similarly the relation $D_v \subseteq \overline{H(a)}$ is presented by a finite conjunction of Π_β formulas, and therefore, it is definable in $\mathfrak{N}$ by a Σ_α formula that is computable uniformly in v and a. Taking infinite disjunction over all u, v, and y such that $\langle x,y,u,v\rangle \in W_{\rho(x)}$, we conclude that in this case the relation $x \in H(2^a)$ is also definable in $\mathfrak{N}$ by a Σ_α formula that is computable uniformly on a. □

Theorem 1.6. *For every computable ordinal α and every set X from the class Σ^0_α of the hyperarithmetical hierarchy, X is definable in $\mathfrak{N}$ by an infinitary computable Σ_α formula.*

Proof. The claim of the theorem is true for any finite ordinal. Suppose that α is a computable infinite ordinal, and let a be a notation for α. Let $X \in \Sigma^0_\alpha$. Then X is c.e. relative to $H(a)$, and hence, X is 1-reducible to $(H(a))' = H(2^a)$. If f is a computable function that does the reduction of X to $H(a)$, then

$$\forall x(x \in X \Leftrightarrow \exists y(y = f(x) \;\&\; y \in H(2^a)).$$

Now Lemma 1.5 implies that X is definable in $\mathfrak{N}$ by a computable an infinitary Σ_α formula. □

Corollary 1.7. *For every computable ordinal α and every set $X \in \omega$, $X \in \Sigma^0_\alpha$ if and only if X is definable in $\mathfrak{N}$ by an infinitary computable Σ_α formula.*

Hyperarithmetical numberings

Let $\mathcal{L}$ be the family of infinitary computable formulas. We will denote by Φ the Gödel numbering of $\mathcal{L}$ given by C. Ash and J. Knight in [5].

Definition 1.8. Let α be a computable ordinal. A numbering ν of a family $\mathcal{A} \subseteq \Sigma^0_\alpha$ is called Σ^0_α*– computable* if there exists a computable function f such that $\{\Phi_{f(i)} | i \in \omega\}$ is a set of Σ_α formulas of Peano arithmetic and

$$\nu(m) = \{x \in \omega \mid \mathfrak{N} \models \Phi_{f(m)}(\mathbf{x})\};$$

here $\mathbf{x}$ stands for the numeral for x. The set of Σ^0_α– computable numberings of $\mathcal{A}$ will be denoted by $\mathrm{Com}^0_\alpha(\mathcal{A})$.

In other words, a Σ^0_α– computable numbering is just a computable numbering in the sense of Definition 1.1.

Theorem 1.9. *A numbering ν of a family $\mathcal{A} \subseteq \Sigma^0_\alpha$ is Σ^0_α– computable if and only if $\{\langle m, x\rangle \mid x \in \nu(m)\}$ is Σ^0_α.*

Proof. $\Rightarrow$. Let f be a computable function such that $\{\Phi_{f(i)} | i \in \omega\}$ is a set of Σ_α formulas and for all x, m

$$x \in \nu(m) \Leftrightarrow \mathfrak{N} \models \Phi_{f(m)}(\mathbf{x}).$$

Let $\theta(m, x)$ be the infinite disjunction

$$\bigvee_{n \in \omega} (m = f(n) \;\&\; \Phi_{f(n)}(x)).$$

Every $\Phi_{f(n)}$ is a Σ_α formula; i.e., $\Phi_{f(n)}$ is a disjunction of formulas of form $\exists \overline{y} \Psi_{g(n,i)}$ over some c.e. set $W_{h(n)}$:

$$\Phi_{f(n)} = \bigvee_{i \in W_{h(n)}} \exists \overline{y} \Psi_{g(n,i)}.$$

The functions g, h are computable (see [5]), and for every n and every i, $\Psi_{g(n,i)}$ is a Π_β formula with $\beta < \alpha$. Therefore, $\theta(m, x)$ is the disjunction of the formulas $\exists \overline{y}(m = f(n) \;\&\; \Psi_{g(n,i)})$ over a c.e. set, and hence, θ is a Σ_α formula. By Theorem 1.4, the set $\{\langle m, x\rangle \mid x \in \nu(m)\}$ is Σ^0_α.

$\Leftarrow$. Let $\{\langle m, x\rangle \mid x \in \nu(m)\}$ be Σ^0_α. By Theorem 1.6, there exists a Σ_α formula $\eta(m, x)$ such that

$$x \in \nu(m) \Leftrightarrow \mathfrak{N} \models \eta(\mathbf{m}, \mathbf{x}).$$

It is easy to check by transfinite induction on the ordinal notations that, for every $m \in \omega$, the formula $\eta(\mathbf{m}, x)$, of one free variable x, is Σ_α. Obviously, an index of this formula can be effectively found, uniformly from m. $\square$

Corollary 1.10. *A numbering ν of a family $\mathcal{A} \subseteq \Sigma^0_\alpha$ is Σ^0_α– computable if and only if $\{\langle m, x\rangle \mid x \in \nu(m)\}$ is definable in $\mathfrak{N}$ by some Σ_α formula.*

Numbering $\nu : \omega \mapsto \mathcal{A}$ of a family $\mathcal{A}$ of hyperarithmetical sets is also called *hyperarithmetical* if $\{\langle m, x\rangle \mid x \in \nu(m)\}$ is definable in $\mathfrak{N}$ by some computable (infinitary) formula of Peano arithmetic. A family $\mathcal{A}$ for which $\mathrm{Com}^0_\alpha(\mathcal{A}) \neq \emptyset$ is called *Σ^0_α– computable*. If the ordinal α is finite, then we usually use use the term *arithmetical numbering*.

The function f in Definition 1.8 can be chosen Σ^0_α computable because in the definition of Σ_α formulas, one can replace disjunctions over c.e. sets with disjunctions over hyperarithmetical sets (see [5, Proposition 7.11]).

We now revise some of the basic definitions of the theory of numberings. Two numberings ν, μ of $\mathcal{A}$ can be compared by defining $\nu \leq \mu$ (ν is *reducible to* μ) if there is a computable function f such that $\nu = \mu \circ f$. Two numberings ν and μ are *equivalent* (written $\nu \equiv \mu$) if $\nu \leq \mu$ and $\mu \leq \nu$.

The equivalence $\equiv$ partitions the set $\mathrm{Com}^0_\alpha(\mathcal{A})$ into the equivalence classes $\deg(\mu)$ of all Σ^0_α– computable numberings μ of $\mathcal{A}$, thus originating a quotient structure, denoted by $\mathcal{R}^0_\alpha(\mathcal{A})$. The latter forms an upper semilattice under the partial ordering induced by $\leq$, where the join of two numberings ν and μ is defined by $(\nu \oplus \mu)(2n) = \nu(n)$ and $(\nu \oplus \mu)(2n+1) = \mu(n)$ induces the least upper bound of $\deg(\mu)$ and $\deg(\nu)$. $\mathcal{R}^0_\alpha(\mathcal{A})$ is called the *Rogers semilattice* of $\mathcal{A}$.

2 Isomorphism types of Rogers semilattices

One of the global aims of the theory of computable numberings is to investigate the isomorphism types of the Rogers semilattices. Furthermore, we will consider nontrivial families of sets only, i.e., families that contain at least two sets. As the first stage of this research we study the differences in the isomorphism types of Rogers semilattices of computable numberings for the families of sets lying in different levels of the arithmetical hierarchy (see [11]–[14]). The strongest result that has been obtained in this direction is as follows. For every two nontrivial families of sets taken from two different finite levels of the arithmetical hierarchy, if the gap in the levels is not less than 3, then the corresponding Rogers semilattices of computable numberings are not isomorphic [14]. Roughly speaking, we try to extend this statement to the families of sets taken from infinite levels of the hyperarithmetical hierarchy. We should note that this is mainly a straightforward relativization of the proofs from [14].

Theorem 2.1. *For any computable ordinals $\alpha > 0$ and β and for every Σ^0_α– computable family $\mathcal{A}$ and every nontrivial Σ^0_β– computable family $\mathcal{B}$, if $\alpha + 3 \le \beta$, then the Rogers semilattices $\mathcal{R}^0_\alpha(\mathcal{A})$ and $\mathcal{R}^0_\beta(\mathcal{B})$ are not isomorphic.*

Proof. Let $\alpha > 0$ and β be any computable ordinals such that $\beta \ge \alpha + 3$. Let $a, b \in \mathcal{O}$ stand for some notations of ordinals α and $\alpha + 3$, respectively. Let $\mathcal{A}$ be any Σ^0_α– computable family, and let $\mathcal{B}$ be a Σ^0_β– computable family, which contains at least two sets.

We will construct in the Rogers semilattice $\mathcal{R}^0_\beta(\mathcal{B})$ an interval that forms a Boolean algebra not isomorphic to any interval in the Rogers semilattice $\mathcal{R}^0_\alpha(\mathcal{A})$.

For the ease of the reader we recall only necessary notions and statements that allow us to formulate the requirements for constructing the desired interval.

Definition 2.2. If ρ is a numbering of a family $\mathcal{A}$, and C is a nonempty c.e. set, with f a computable function such that $\mathrm{range}(f) = C$, then we define $\rho_C \leftrightharpoons \rho \circ f$.

The definition does not depend on f: If we define ρ_C starting from any other computable function g such that $\mathrm{range}(g) = C$, then we get a numbering that is equivalent to the one given by f. The assignment $C \mapsto \rho_C$ from c.e. sets to numberings (up to equivalence of numberings) is called *Lachlan operator*.

Lemma 2.3. *For every pair A, B of c.e. sets and for every pair of numberings τ, ρ, we have:*

(1) The following are equivalent:

(a) $\rho_A \leqslant \rho_B$;

(b) there is a partial computable function φ satisfying $\mathrm{dom}(\varphi) \supseteq A$, $\varphi[A] \subseteq B$ and for all $x \in A$, $\rho(x) = \rho(\varphi(x))$;

(2) if $A \subseteq B$, then $\rho_A \leqslant \rho_B$;

(3) if $\rho_A \leqslant \rho_B$, then $\rho_B \equiv \rho_{A\cup B}$;

(4) if $\tau \leqslant \rho$, then $\tau \equiv \rho_C$ for some c.e. set C;

(5) if $\tau \leqslant \rho$, and $\tau \equiv \rho_C$, for some c.e. set C, then for every γ such that $\tau \leqslant \gamma \leqslant \rho$ there exists a c.e. set D with $C \subseteq D$ and $\gamma \equiv \rho_D$;

(6) $\rho_{A\cup B} \equiv \rho_A \oplus \rho_B$.

Proof. See Lemma 2.2 in [10]. □

In what follows, the symbol $[\eta, \theta]$ denotes the following interval of degrees in $\mathcal{R}^0_\alpha(\mathcal{A})$:

$$[\eta, \theta] \leftrightharpoons \{\deg(\mu) \mid \eta \le \mu \le \theta\}.$$

Now we estimate the complexity of any interval of Rogers semilattice $\mathcal{R}^0_\alpha(\mathcal{A})$ if it is Boolean algebra. The notion of an $\mathbf{X}$– computable Boolean algebra plays a key role in establishing our claim. Recall (see [25]) that a Boolean algebra $\mathfrak{A}$ is called $\mathbf{X}$–*computable* if its universe, operations, and relations are $\mathbf{X}$– computable.

Lemma 2.4. *Let $\eta, \theta \in \mathrm{Com}^0_\alpha(\mathcal{A})$. If $[\eta, \theta]$ is a Boolean algebra, then it is $H(b)$–computable.*

Proof. Given η and θ as in the hypothesis of the lemma, we first observe that by (4) and (5) of Lemma 2.3 there exists a c.e. set C such $\eta \equiv \theta_C$ and

$$[\eta, \theta] = \{\deg(\theta_X) \mid X \text{ is c.e. and } X \supseteq C\}.$$

For every i, let $U_i \risingdotseq C \cup W_i$. This gives an effective listing of all c.e. supersets of C. By Lemma 2.3 (1b), for every i, j, we have $\theta_{U_i} \leqslant \theta_{U_j}$ if and only if

$$\exists p[\forall x(x \in U_i \Rightarrow \exists y(\varphi_p(x) = y \ \&\ y \in U_j))$$
$$\&\ \forall x \forall y(x \in U_i \ \&\ \varphi_p(x) = y \Rightarrow \theta(x) = \theta(y))].$$

Since $\theta \in \mathrm{Com}^0_\alpha(\mathcal{A})$, this implies, by Theorem 1.9, that the binary relation $z \in \theta(x)$ is c.e. relative to the oracle $H(a)$. Therefore, the binary relation $\theta(x) = \theta(y)$ is a $\forall\exists$–predicate relative to the oracle $H(a)$.

Simple calculations show now that $\theta_{U_i} \leqslant \theta_{U_j}$ is a $\Sigma^0_{\alpha+2}$– relation in i, j.

Let us consider the equivalence relation ε on ω defined by

$$(i, j) \in \varepsilon \Leftrightarrow \theta_{U_i} \leqslant \theta_{U_j} \ \&\ \theta_{U_j} \leqslant \theta_{U_i}.$$

Let $B \risingdotseq \{x \mid \forall y(y < x \Rightarrow (x, y) \notin \varepsilon)\}$. Define a bijection $\psi_1 : B \to [\eta, \theta]$, by letting $\psi_1(i) = \deg(\theta_{U_i})$, for all $i \in B$. It is evident that ψ_1 induces on B a partially ordered set $\mathfrak{B}$, which is a Boolean algebra isomorphic to $[\eta, \theta]$. The interval $\mathfrak{B}$ is an $H(b)$– computable partially ordered set since $(H(a))''' = H(b)$. It follows from [17] (see also [25, Theorem 3.3.4]) that $\mathfrak{B}$ with respect to the corresponding Boolean operations is $H(b)$-computable too. □

Lemma 2.5 (L. Feiner). *Let $\mathfrak{F}$ be a computable atomless Boolean algebra. Then for every $\mathbf{X}$ there is an ideal J such that J is $\mathbf{X}$– c.e. and the quotient $\mathfrak{F}/J$ is not isomorphic to any $\mathbf{X}$– computable Boolean algebra.*

Proof. See [20]. □

Below, we will use the following notations. For a given c.e. set H, $\{V_i \mid i \in \omega\}$ denotes an effective listing of all c.e. supersets of the set H defined, for instance, by $V_i \risingdotseq H \cup W_i$, for all i. We will assume for convenience that $V_0 = H$. Let ε_H stand for the distributive lattice of the c.e. supersets of H. For a given c.e. set $V \supseteq H$, let V^* denote the image of V under the canonical homomorphism of ε_H onto ε_H^* (i.e., ε_H modulo the finite sets), and let $\subseteq^*$ denote the partial ordering relation of ε_H^*. Obviously, if J is an ideal in ε_H, then $J^* \leftrightharpoons \{V^* \mid V \in J\}$ is an ideal in ε_H^*.

As is known (see, for instance, [25]), if $\mathfrak{A}$ is a Boolean algebra and J is an ideal of $\mathfrak{A}$, then the universe of the quotient Boolean algebra $\mathfrak{A}/J$ is given by the set of equivalence classes $\{[a]_J \mid a \in \mathfrak{A}\}$ under the equivalence relation $\equiv_J$ given by

$$a \equiv_J b \Leftrightarrow \exists c_1, c_2 \in J(a \vee c_1 = b \vee c_2),$$

and the partial ordering relation is given by

$$[a]_J \leq_J [b]_J \Leftrightarrow a - b \in J,$$

where $a - b$ stands for $a \wedge \neg b$.

Lemma 2.6. *Let $\mathcal{B}$ be a Σ^0_β– computable family, $\mu \in \mathrm{Com}^0_\beta(\mathcal{B})$, and let H be any c.e. set such that $\mu(H) = \mathcal{B}$ and ε_H^* is a Boolean algebra. Let $\psi_2 : \varepsilon_H \longrightarrow [\mu_H, \mu]$ be the mapping given by $\psi_2(V_i) = \deg(\mu_{V_i})$ for all i, and let I be any ideal of ε_H. Then ψ_2 induces an isomorphism of ε_H^*/I^* onto $[\mu_H, \mu]$ if and only if for every i, j*

(1) $V_i \in I \Rightarrow \mu_{V_i} \leqslant \mu_H$;

(2) $V_i - V_j \notin I \Rightarrow \mu_{V_i} \nleqslant \mu_{V_j}$ (where $V_i - V_j \leftrightharpoons (V_i \setminus V_j) \cup H$).

Proof. See Lemma 4 in [14]. □

By Lemma 2.4, all Boolean intervals of $\mathcal{R}^0_\alpha(\mathcal{A})$ are $H(b)$– computable Boolean algebras. Therefore, to show the theorem, it is sufficient:

(i) to consider a computable atomless Boolean algebra $\mathfrak{F}$ and an ideal J of $\mathfrak{F}$ as in Feiner's Lemma such that J is c.e. in $H(b)$ and $\mathfrak{F}/J$ is not isomorphic to any $H(b)$– computable Boolean algebra,

(ii) to find Σ^0_β– computable numberings ν and μ of $\mathcal{B}$ such that the interval $[\nu, \mu]$ of $\mathcal{R}^0_\beta(\mathcal{B})$ is a Boolean algebra isomorphic to $\mathfrak{F}/J$.

First, we consider item (i) above. Let $\mathfrak{F}$ be a computable atomless Boolean algebra. According to a famous result of Lachlan [32], there exists a hyperhypersimple set H such that ε_H^* is isomorphic to $\mathfrak{F}$. We fix such a set H.

We refer to the textbook of Soare [38] for the details of a suitable isomorphism χ of ε_H^* onto $\mathfrak{F}$. We only notice that starting from a computable listing $\{b_0, b_1, \ldots\}$ of the elements of $\mathfrak{F}$, one can find a Σ^0_3– computable Friedberg numbering $\{B_0, B_1, \ldots\}$ of a subfamily of the family ε_H such that $\varepsilon_H^* = \{B_0^*, B_1^*, \ldots\}$ and $\chi(B_i^*) = b_i$.

We will use the techniques for embedding posets into intervals of Rogers semilattices, which have been developed in [10]. Let J be any $H(b)$ –c.e. ideal of $\mathfrak{F}$ satisfying the conclusions of Lemma 2.5, and let $\hat{J} = \{j \in \omega \mid b_j \in J\}$. Then $\hat{J}$ is an

$H(b)$ –c.e. set, $I^* \leftrightharpoons \{B_j^* \mid j \in \hat{J}\}$ is an ideal of ε_H^*, and $\mathfrak{F}/J$ is isomorphic to ε_H^*/I^*. So, instead of the Boolean algebra $\mathfrak{F}/J$ in item (ii) above, we can consider ε_H^*/I^*.

Let $I \leftrightharpoons \{V \mid V \in \varepsilon_H \ \&\ V^* \in I^*\}$, and let $\hat{I} = \{i \in \omega \mid V_i^* \in I^*\}$. Obviously, I is an ideal of ε_H.

Lemma 2.7. *The relations "$V_i \in I$" (equivalently: "$i \in \hat{I}$"), in i, and "$V_i - V_j \in I$", in i, j, are both $H(b)$ –c.e.*

Proof. Straightforward relativization of the proof of Lemma 5 in [14]. □

Since $\beta \geq \alpha + 3$ we can use the oracle $H(b)$ and apply Lemma 2.6 to construct a suitable numbering μ of $\mathcal{B}$ and consider the corresponding mapping ψ_2 that will give us an isomorphism of ε_H^*/I^* onto the interval $[\mu_H, \mu]$.

The requirements

First of all, we need the numbering μ to satisfy the requirement:

$$\mathbf{B} : \mu[H] = \mathcal{B}$$

to guarantee that μ_H is a numbering of the whole family $\mathcal{B}$. Then in view of Lemma 2.6, we must satisfy, for every i, j, p, the requirements:

$$\mathbf{P}_i : V_i \in I \Rightarrow \mu_{V_i} \leqslant \mu_H,$$
$$\mathbf{R}_{i,j,p} : V_i - V_j \notin I \Rightarrow \mu_{V_i} \nleqslant \mu_{V_j} \text{ via } \varphi_p,$$

where by "$\mu_{V_i} \nleqslant \mu_{V_j}$ via φ_p" we mean that φ_p does not reduce μ_{V_i} to μ_{V_j} in the sense of Lemma 2.3(1b).

We take any numbering $\nu \in \mathrm{Com}_\beta^0(\mathcal{B})$ and try to construct a numbering μ that meets the above requirements $\mathbf{B}$, $\mathbf{P}_i$, $\mathbf{R}_{i,j,p}$. Evidently, to get $\mu \in \mathrm{Com}_\beta^0(\mathcal{B})$ starting from any uniform enumeration of the numbering ν, we have to avoid using oracles that are stronger than the oracles used in ν–computations. Since $\beta \geq \alpha + 3$, the oracles relative to which we can make uniform computations in the numbering ν have complexity at least $H(b)$. This lower complexity boundary for the oracles that we intend to use in our strategies to construct μ is essential. And this forces us to partition the rest of the proof into two cases.

CASE 1 $\beta > \alpha + 3$. In this case we can even use oracles of complexity equal to or greater than $H(2^b)$. And Lemma 2.7 implies that the conditions $V_i \in I$ or $V_i - V_j \notin I$ in the requirements $\mathbf{P}_i$ and $\mathbf{R}_{i,j,p}$ are effectively recognizable by any oracle of complexity equal or higher than $H(2^b)$. The strategies and the construction for building the numbering μ in this case are relativized versions of the corresponding strategies and construction from [14, Theorem 1].

CASE 2 $\beta = \alpha + 3$. In this case we use only oracle $H(b)$. Lemma 2.7 implies that a condition $V_i \in I$ is eventually recognizable by this oracle. As to the negative condition $V_i - V_j \notin I$ in the requirement $\mathbf{R}_{i,j,p}$, we constantly try to destroy the reducibility $\mu_{V_i} \nleq \mu_{V_j}$ via φ_p until (if ever) the condition $V_i - V_j \in I$ is eventually recognized by the oracle $H(b)$.

We refer to [14, Theorem 2] for the corresponding versions of the strategies and the construction in the setting of the arithmetical hierarchy. □

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