Preface

The theory of braid groups is one of the most fascinating chapters of low-dimensional topology. Its beauty stems from the attractive geometric nature of braids and from their close relations to other remarkable geometric objects such as knots, links, homeomorphisms of surfaces, and configuration spaces. On a deeper level, the interest of mathematicians in this subject is due to the important role played by braids in diverse areas of mathematics and theoretical physics. In particular, the study of braids naturally leads to various interesting algebras and their linear representations.

Braid groups first appeared, albeit in a disguised form, in an article by Adolf Hurwitz published in 1891 and devoted to ramified coverings of surfaces. The notion of a braid was explicitly introduced by Emil Artin in the 1920s to formalize topological objects that model the intertwining of several strings in Euclidean 3-space. Artin pointed out that braids with a fixed number n of strings form a group, called the nth braid group and denoted by B_n. Since then, the braids and the braid groups have been extensively studied by topologists and algebraists. This has led to a rich theory with numerous ramifications.

In 1983, Vaughan Jones, while working on operator algebras, discovered new representations of the braid groups, from which he derived his celebrated polynomial of knots and links. Jones's discovery resulted in a strong increase of interest in the braid groups. Among more recent important results in this field are the orderability of the braid group B_n, proved by Patrick Dehornoy in 1991, and the linearity of B_n, established by Daan Krammer and Stephen Bigelow in 2001–2002.

The principal objective of this book is to give a comprehensive introduction to the theory of braid groups and to exhibit the diversity of their facets. The book is intended for graduate and postdoctoral students, as well as for all mathematicians and physicists interested in braids. Assuming only a basic knowledge of topology and algebra, we provide a detailed exposition of the more advanced topics. This includes background material in topology and algebra that often goes beyond traditional presentations of the theory of braids.

In particular, we present the basic properties of the symmetric groups, the theory of semisimple algebras, and the language of partitions and Young tableaux.

We now detail the contents of the book. Chapter 1 is concerned with the foundations of the theory of braids and braid groups. In particular, we describe the connections with configuration spaces, with automorphisms of free groups, and with mapping class groups of punctured disks.

In Chapter 2 we study the relation between braids and links in Euclidean 3-space. The central result of this chapter is the Alexander–Markov description of oriented links in terms of Markov equivalence classes of braids.

Chapter 3 is devoted to two remarkable representations of the braid group B_n: the Burau representation, introduced by Werner Burau in 1936, and the Lawrence–Krammer–Bigelow representation, introduced by Ruth Lawrence in 1990. We use the technique of Dehn twists to show that the Burau representation is nonfaithful for large n, as was first established by John Moody in 1991. We employ the theory of noodles on punctured disks introduced by Stephen Bigelow to prove the Bigelow–Krammer theorem on the faithfulness of the Lawrence–Krammer–Bigelow representation. In this chapter we also construct the one-variable Alexander–Conway polynomial of links.

Chapter 4 is concerned with the symmetric groups and the Iwahori–Hecke algebras, both closely related to the braid groups. As an application, we construct the two-variable Jones–Conway polynomial of links, also known as the HOMFLY or HOMFLY-PT polynomial, which extends two fundamental one-variable link polynomials, namely the aforementioned Alexander–Conway polynomial and the Jones polynomial.

Chapter 5 is devoted to a classification of the finite-dimensional representations of the generic Iwahori–Hecke algebras in terms of Young diagrams. As an application, we show that the (reduced) Burau representation of B_n is irreducible. We also discuss the Temperley–Lieb algebras and classify their finite-dimensional representations.

Chapter 6 presents the Garside solution of the conjugacy problem in the braid groups. Following Patrick Dehornoy and Luis Paris, we introduce the concept of a Garside monoid, which is a monoid with appropriate divisibility properties. We show that the braid group B_n is the group of fractions of a Garside monoid of positive braids on n strings. We also describe similar results for the generalized braid groups associated with Coxeter matrices.

Chapter 7 is devoted to the orderability of the braid groups. Following Dehornoy, we prove that the braid group B_n is orderable for every n.

The book ends with four short appendices: Appendix A on the modular group $\mathrm{PSL}_2(\mathbf{Z})$, Appendix B on fibrations, Appendix C on the Birman–Murakami–Wenzl algebras, and Appendix D on self-distributive sets.

The chapters of the book are to a great degree independent. The reader may start with the first section of Chapter 1 and then freely explore the rest of the book.

The theory of braids is certainly too vast to be covered in a single volume. One important area entirely skipped in this book concerns the connections with mathematical physics, quantum groups, Hopf algebras, and braided monoidal categories. On these subjects we refer the reader to the monographs [Lus93], [CP94], [Tur94], [Kas95], [Maj95], [KRT97], [ES98].

Other areas not presented here include the homology and cohomology of the braid groups [Arn70], [Vai78], [Sal94], [CS96], automatic structures on the braid groups [ECHLPT92], [Mos95], and applications to cryptography [SCY93], [AAG99], [KLCHKP00].

For further aspects of the theory of braids, we refer the reader to the following monographs and survey articles: [Bir74], [BZ85], [Han89], [Kaw96], [Mur96], [MK99], [Ver99], [Iva02], [BB05].

This book grew out of the lectures [Kas02], [Tur02] given by the authors at the Bourbaki Seminar in 1999 and 2000 and from graduate courses given by the first-named author at Université Louis Pasteur, Strasbourg, in 2002–2003 and by the second-named author at Indiana University, Bloomington, in 2006.

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