2

Friezes and Mosaics

This chapter discusses the classification of friezes and several concepts related to mosaics. The first section introduces the concept of operations that leave a frieze unchanged, using basic geometry and intuition. It also describes what will be the main steps of the classification theorem. Section 2.2 defines affine transformations and their matrix representation, and isometries. The highlight of this chapter is the classification theorem shown in Section 2.3. In less detail, the last section discusses mosaics. There is no advanced section to this chapter, the proof of the classification theorem being the most difficult element. Sections 2.1 and 2.4 can be covered in three hours of class. The tools are then purely geometric and the possibility of classification is made clear. If the classification theorem is the goal, four hours should be devoted to the first three sections. In all cases, the lecturer should bring copies of Figure 2.2 on transparencies to the classroom. Their use on a projector helps students to understand quickly the concept of symmetry. Only a basic knowledge of linear algebra and Euclidean geometry is required to understand this chapter. The proof of the classification theorem requires a familiarity with abstract reasoning.

This subject offers several interesting directions for further study: aperiodic tilings (end of Section 2.4) is one such direction, while Exercises 13, 14, 15, and 16 present several others.

Friezes and mosaics have been used in decoration for several millennia. The ancient world's Sumerian, Egyptian, and Mayan civilizations all used them to great effect. It would be a lie, however, to pretend that ancient mathematics developed the "technology" behind the art. The formal mathematical study of tilings is relatively recent, having started no more than two centuries ago. The memoir of Bravais [1], a French physicist, is among the first scientific studies of the subject.

Mathematics is able to provide a way to systematically classify the friezes and mosaics commonly seen in architecture and art. These classifications have allowed mathematicians to better understand the rules behind them and to create truly new patterns by breaking some of these rules.

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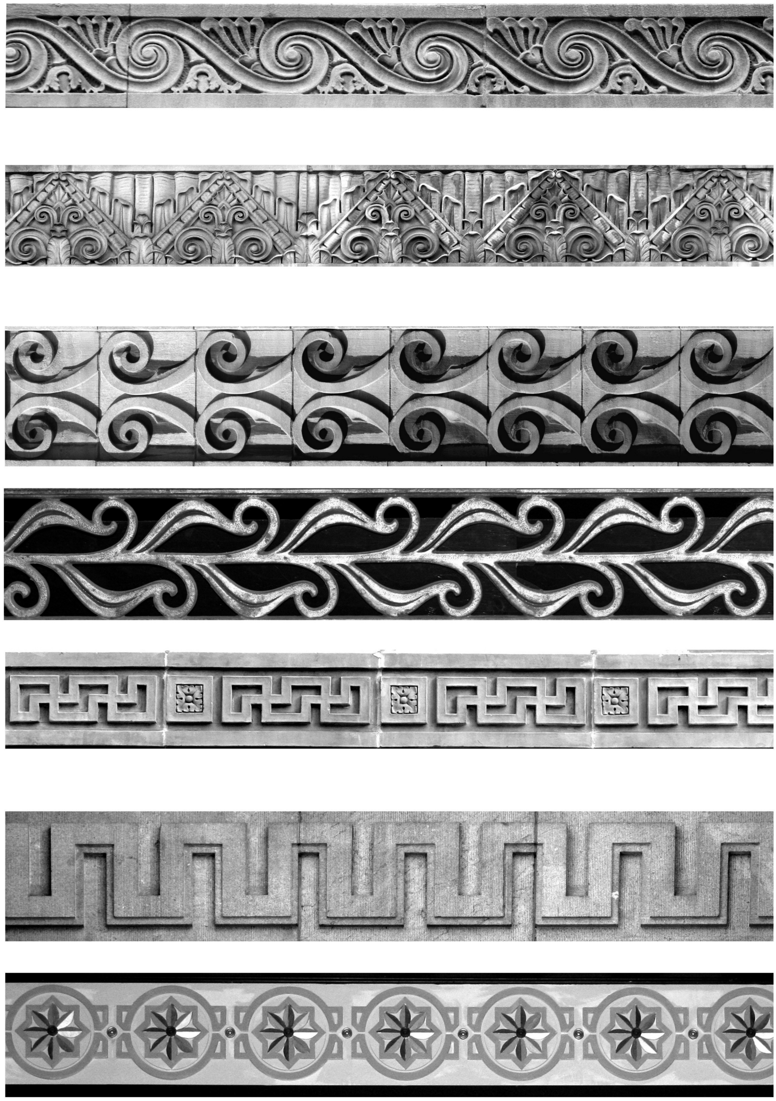

Fig. 2.1. Seven friezes. (Each of the above friezes has its pattern displayed in simplified form in Figure 2.2.)

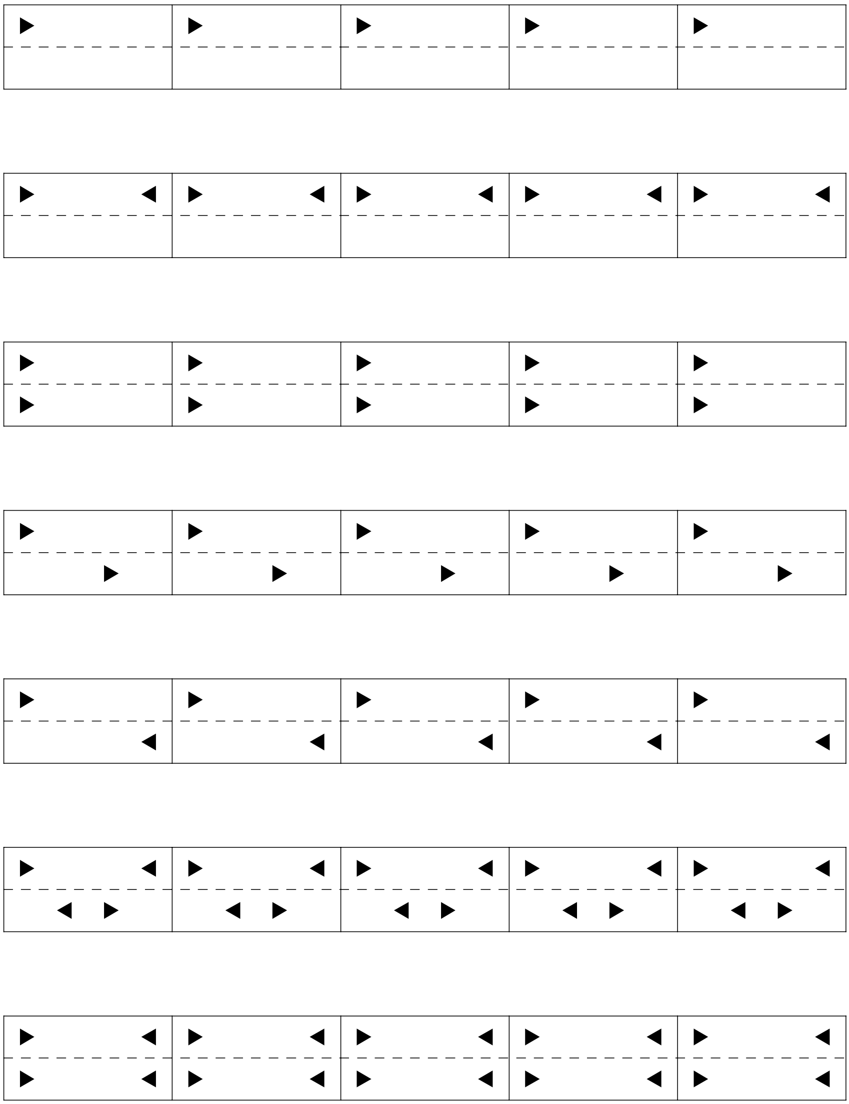

Fig. 2.2. Seven simplified friezes. (Each of the above friezes is a simplified form of the corresponding frieze in Figure 2.1.)

Classification of objects is a fairly common mathematical activity. The reader who has followed a course on multivariable calculus will remember the classification of extrema of a function of two variables using the second partial derivative test. If the matrix of second derivatives (the Hessian matrix) is nonsingular, the extremum can be classified as either a local minimum or a local maximum or a saddle point. The reader might also have encountered the classification of conics, either in an advanced linear algebra class or in Euclidean geometry. And for those having read Chapter 6 on error-correcting codes, Theorems 6.17 and 6.18 classify finite fields. These are examples of classifications of abstract objects. It may be surprizing to learn that mathematics can classify objects as concrete as architectural patterns. Here is how it is done.

2.1 Friezes and Symmetries

The *Oxford English Dictionary* defines *frieze* as *a band of painted or sculptured decoration.* It is also defined as *that member in the entablature of an order which comes between the architrave and cornice*, referring to the architectural location where such patterns are commonly used. Figure 2.1 shows seven friezes taken from architecture. To discuss these objects from a mathematical point of view, we will modify the definition to include the following elements: *(i)* a frieze has a constant and finite width (the height of the friezes in Figure 2.1) and is infinitely long in the perpendicular direction (the horizontal one in our examples); and, *(ii)* it is periodic, meaning that there exists some minimal distance $L > 0$ such that a translation of the frieze by a distance L along the direction in which it is infinite will leave the frieze unchanged. The length L is called the *period* of the frieze. This definition does not fit perfectly with real-world friezes (specifically those in Figure 2.1) because they are not infinitely long. However, we can easily imagine extending them infinitely in both directions by simply continuing the pattern.

Figure 2.2 presents seven more friezes. They are much less detailed but much simpler to study. Each of these seven friezes has the same period L, equal to the distance between two neighboring vertical bars. In the remaining discussion we will imagine that these vertical bars *do not appear* in the frieze pattern, since they have been drawn simply to make the period explicit. Some of these friezes are invariant under various geometric transformations other than translations. For example, the third and seventh friezes remain the same even if we flip them so as to exchange their top and bottom. In this case we say that they are invariant under *reflection by a horizontal mirror.* The second, sixth, and seventh friezes remain unchanged if flipped from left to right; we say that they are invariant under *reflection by a vertical mirror.* These distinctions between various friezes raise a natural question: *is it possible to classify all friezes by considering the set of operations under which they are invariant?* For example, the set of operations leaving the first frieze unchanged includes neither the horizontal nor the vertical reflection just discussed. This set of operations is distinct from that characterizing the third frieze, which may be reflected horizontally. Note that the friezes in Figures 2.1 and 2.2 have been ordered such that they each display the same respective symmetries. Thus, corresponding pairs will be left unchanged by the same operations. For example, the third frieze in both figures is invariant under translations and horizontal reflection.

When a geometric transformation preserving lengths (such as a translation or a reflection) leaves a frieze unchanged, it is said to be a *symmetry operation of the frieze* or, simply, a *symmetry*. The complete list of symmetries of a frieze is infinite. Indeed, we would like to distinguish in this list the translation by a distance of one period L from the translations by $2L$, $3L$, etc., and these already account for an infinite number of symmetry operations. Moreover, the list should also contain the inverse of each symmetry operation. The *inverse of a symmetry operation* is the usual inverse of a function: the composition of a function and its inverse is the identity in the plane (or on the subset defined by the frieze as in the present case). The inverse of a translation to the right by a distance L is a translation to the left by the same distance. (Exercise: what is the inverse of a reflection with respect to a given mirror? and that of a rotation by an angle θ?) If translations to the right (respectively to the left) are associated to positive distances (respectively negative distances), then the list of symmetries of a frieze of period L should contain all translations by a distance nL with $n \in \mathbb{Z}$. Instead of listing all symmetries of a frieze, it is common to give only a subset of elements whose compositions and inverses give the whole list. Such a subset is called a *set of generators*. This is what we are going to use from now on. (Mathematicians usually take this subset as small as possible. They call it minimal whenever the subset, after removal of one of its elements, fails to generate the whole set of symmetries.)

The goal for the remainder of this section is to build geometric intuition of key ideas leading to the classification theorem, Theorem 2.12. This theorem gives all possible lists of symmetry generators for friezes of a given period. The reader is urged to make a copy of Figure 2.2 on a transparency and cut it into seven strips, one for each frieze, before reading on. Experimentation is an ideal way to develop intuition!

The three generators t_L, r_h, and r_v. We have already introduced some possible symmetry operations: translations (by any integer multiple of the period), reflections by horizontal and vertical mirrors. We will use the symbol r_h and r_v for the latter. The set of translations of a frieze is generated by the unique translation t_L by a period L. (The inverse of t_L is t_{-L}. Composition of n operations t_L gives $t_L \circ t_L \circ \cdots \circ t_L = t_{nL}$.)

A subtlety should be cleared up right away. For the reflection r_h to leave a frieze unchanged, the horizontal mirror should be located along the middle line of the frieze (the dashed lines in Figure 2.2). Its position is therefore completely determined by the requirement of being a symmetry. This is not the case for reflections through a vertical mirror. Positions of vertical mirrors must be chosen according to the pattern. The frieze **2** (the second from the top in Figure 2.2) has an infinite set of vertical mirrors. All small vertical bars define a position for a vertical mirror. But these are not the only ones. A mirror located halfway between two adjacent vertical bars also defines a symmetry of this frieze. Exercise 7 shows that if a frieze of period L is unchanged under a given vertical mirror, it is also invariant under an infinite number of mirrors, any of those being at a distance $n\frac{L}{2}$, for $n \in \mathbb{Z}$, from the first. The notation r_v underlies therefore a choice for the position of one mirror and all other vertical mirrors at a distance equal to an integer multiple of $\frac{L}{2}$ from the first one. (Exercise: which other friezes of the figure have a symmetry r_v?)

Notation. Composition of symmetry operations will be used often in the following, and we shall drop the symbol "$\circ$". For example, $r_h \circ r_v$ will be simply noted $r_h r_v$. Soon will also appear the necessity of distinguishing the order of operations. It is important to note that operations are listed from right to left. The composition $r_h r_v$ stands for the operation r_v followed by r_h.

The rotation $r_h r_v$. The frieze **5** introduces a new generator. This frieze has neither r_h nor r_v as a symmetry, but if r_v and then r_h are both performed on it, the frieze remains unchanged. (The vertical mirror is along one of the vertical bars.) (Exercise: check this claim!) It can then happen that neither r_h nor r_v is a symmetry but their composition $r_h r_v$ is. The final result $r_h r_v$ of these two reflections is a rotation by an angle 180°. To see this, note that $r_h r_v$ exchanges the top and bottom, the left and the right, without altering the distances. This is exactly the action of rotation by 180°. (In terms of a coordinate system whose origin is on a vertical bar, a point (x, y) within the frieze is mapped into $(-x, -y)$ under this transformation. This is why this operation is also called *the symmetry through the origin.*) Exercise 8 proposes a geometrical proof of this property.

The following properties of the three generators r_h, r_v, and $r_h r_v$ are easily verified, geometrically or with the use of the copy on transparency that you have made of the figure. They could also be proved using the matrix representation that will be introduced in Section 2.2. (See Exercise 6.)

Proposition 2.1 *1. The operations r_h and r_v commute, that is, the two compositions $r_h r_v$ and $r_v r_h$ are equal.*
2. The inverse of r_h is r_h, that of r_v is r_v, and that of $r_h r_v$ is $r_h r_v$.
3. The composition of r_h and $r_h r_v$ gives r_v. That of r_v and $r_h r_v$ gives r_h. (This allows us to conclude that a frieze that would have any two of the three operations $r_h, r_v, r_h r_v$ as symmetries would automatically have the third also.)

With these properties, it should be easy to determine which of r_h, r_v, and $r_h r_v$ are symmetries of a given frieze of Figure 2.2. (Exercise: do it for all of them!)

The glide reflection symmetry $s_g = t_{L/2} r_h$. After the last proposition, the list of possible generators reads t_L, r_h, r_v, and $r_h r_v$. Any of r_h, r_v, and $r_h r_v$ is a symmetry of at least one frieze in Figure 2.2 and not a symmetry of at least one other frieze. But the frieze **4** shows that this list is not yet complete. None of $r_h, r_v, r_h r_v$ is a symmetry of this frieze. But a reflection r_h followed by a translation by a half-period $\frac{L}{2}$ leaves it unchanged. (See Figure 2.3. Recall that vertical bars *are not* part of the pattern.) We shall refer to this operation as the *glide reflection* and denote it by s_g. Using the composition we can write it as $s_g = t_{L/2} r_h$. (Exercise: only one other frieze among the seven of Figure 2.2 has s_g among its symmetries. Which one?)

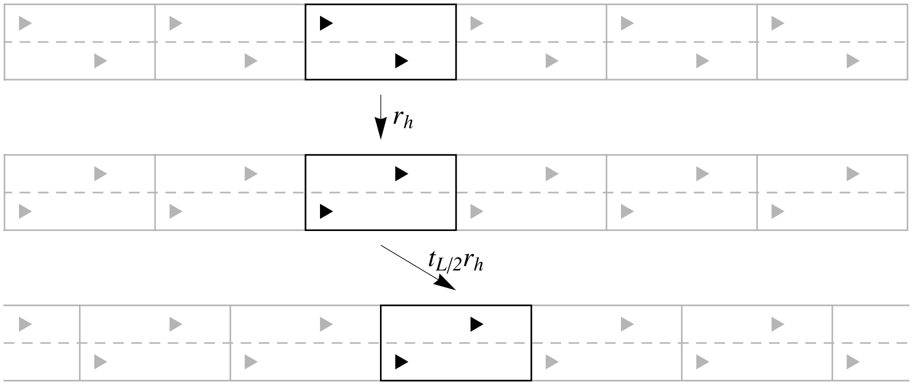

Fig. 2.3. A glide reflection. The frieze **4** as it appears in Figure 2.2 (top line), the same after the operation r_h (middle line), and after a translation by a half-period (bottom line).

Toward the classification theorem. The list of possible generators now contains five operations $(t_L, r_h, r_v, r_h r_v, s_g)$. It was obtained by studying Figure 2.2. To obtain the complete list of symmetry sets of friezes, we need all possible symmetry operations of friezes. What tells us that the list of five operations above is complete? Could there be another frieze that has a symmetry that cannot be obtained from these five? These will be the first questions to answer in order to prove the classification theorem.

Suppose for the time being that this list is complete. We can then enumerate potential sets of symmetries for friezes of period L. As stated above, we shall do this by identifying a set of generators. By definition, all sets will include the translation t_L by a distance L and no shorter ones. Any set may contain either zero or one or two of the three generators $r_h, r_v, r_h r_v$. (If the list contains two, it automatically contains the third one.) These observations lead to the following list.

1. $\langle t_L \rangle$
2. $\langle t_L, r_v \rangle$
3. $\langle t_L, r_h \rangle$
4. $\langle t_L, s_g \rangle$
5. $\langle t_L, r_h r_v \rangle$
6. $\langle t_L, s_g, r_h r_v \rangle$
7. $\langle t_L, r_h, r_v \rangle$
8. $\langle t_L, s_g, r_h \rangle$
9. $\langle t_L, s_g, r_v \rangle$
10. $\langle t_L, s_g, r_h, r_v \rangle$

All of the sets contain t_L. Sets **1** and **4** contain none of $r_h, r_v, r_h r_v$. Set **4** contains s_g, set **1** does not. Sets **2, 3, 5, 6, 8** and **9** contain one and only one of $r_h, r_v, r_h r_v$; **6, 8, 9** add the glide reflection s_g, but **2, 3, 5** do not. Sets **7** and **10** contain two of $r_h, r_v, r_h r_v$ (and therefore all three). Set **10** has moreover s_g.

The classification theorem will have to resolve two more questions. The first is whether this list contains repetitions. Since we are listing only generators, two in the list above could generate the same list of symmetries. The second question is whether some of the sets do not generate symmetries of friezes of period L. This question might be somewhat surprising. But one can easily see that set **8** needs to be crossed out of the list, since it does not generate symmetries of a frieze of period L.

To see this, it is crucial to remember that the glide reflection s_g is the composition of r_h and $t_{L/2}$. But it can be seen that the set of generators of a frieze of period L cannot contain both s_g and r_h. Why? We have noted that the inverse of r_h is r_h itself. Then the composition of r_h and s_g is $s_g r_h = t_{L/2} r_h r_h = t_{L/2}(\mathrm{Id}) = t_{L/2}$. Because compositions of symmetries are symmetries, the translation $t_{L/2}$ should also be a symmetry of the frieze. But the period of the frieze was assumed to be L, and by definition, this period should be the smallest translation leaving the frieze invariant. The translation $t_{L/2}$ cannot appear, and hence s_g and r_h cannot simultaneously be generators of the same frieze. Set **8** must be rejected. (Note that this set does generate a set of symmetries for a frieze. But that frieze is of period $\frac{L}{2}$ and it is then set **3**, that is, $\langle t_{L/2}, r_h \rangle$.) (Exercise: the classification theorem will end up keeping only seven of the ten lists above. The argument for rejecting **8** was given. Can you guess which other two must be discarded?)

We shall complete the proof of the Classification theorem after having discussed a powerful algebraic tool to study these geometric operations: the matrix representation of affine transformations.[1]

2.2 Symmetry Group and Affine Transformations

We will use affine transformations as the mathematical foundation for describing invariant operations on friezes. (If you have read Chapter 3 or 11, you will have already encountered them.)

Definition 2.2 *An* affine transformation *in the plane is a transformation* $\mathbb{R}^2 \to \mathbb{R}^2$ *of the form* $(x, y) \mapsto (x', y')$, *where*

$$\begin{aligned} x' &= ax + by + p, \\ y' &= cx + dy + q. \end{aligned}$$

An affine transformation is called proper *if it is bijective.*

Such a transformation can be described in matrix form as

[1] It is possible to give a purely geometric proof of this theorem. See, for example, [2] and [5].

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}. \tag{2.1}$$

The matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a *linear transformation*, while p and q represent a *translation* in the plane. For the rest of this chapter we will be considering only proper (or *regular*) affine transformations, that is, affine transformations that are one-to-one. As we shall see soon, this additional condition is equivalent to the invertibility of the linear transformation matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Observe that the following equation describes the same affine transformation:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{2.2}$$

In this modified form, a one-to-one correspondence is made between elements (x, y) of the plane $\mathbb{R}^2$ and elements $(x, y, 1)^t$ in the plane at $z = 1$ of $\mathbb{R}^3$. The mapping between affine transformations of the form (2.1) and the 3×3 matrices whose last line is (0 0 1),

$$\begin{pmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{pmatrix},$$

is also one-to-one.

If we compose two affine transformations $(x, y) \to (x', y')$ and $(x', y') \to (x'', y'')$ given by

$$\begin{aligned} x' &= a_1 x + b_1 y + p_1, \\ y' &= c_1 x + d_1 y + q_1, \end{aligned}$$

and

$$\begin{aligned} x'' &= a_2 x' + b_2 y' + p_2, \\ y'' &= c_2 x' + d_2 y' + q_2, \end{aligned}$$

the resulting (x'', y'') can be obtained as

$$\begin{aligned} x'' &= a_2 x' + b_2 y' + p_2 \\ &= a_2(a_1 x + b_1 y + p_1) + b_2(c_1 x + d_1 y + q_1) + p_2 \\ &= (a_2 a_1 + b_2 c_1)x + (a_2 b_1 + b_2 d_1)y + (a_2 p_1 + b_2 q_1 + p_2) \end{aligned}$$

and

$$\begin{aligned} y'' &= c_2 x' + d_2 y' + q_2 \\ &= c_2(a_1 x + b_1 y + p_1) + d_2(c_1 x + d_1 y + q_1) + q_2 \\ &= (c_2 a_1 + d_2 c_1)x + (c_2 b_1 + d_2 d_1)y + (c_2 p_1 + d_2 q_1 + q_2). \end{aligned}$$

Note that this compound transformation can itself be described in a 3×3 matrix form:

$$\begin{pmatrix} x'' \\ y'' \\ 1 \end{pmatrix} = \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 & a_2p_1 + b_2q_1 + p_2 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 & c_2p_1 + d_2q_1 + q_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

This last example demonstrates the utility of the 3×3 matrix notation, since composed transformations can themselves be expressed as the product of the matrices underlying the individual transformations:

$$\begin{pmatrix} a_2 & b_2 & p_2 \\ c_2 & d_2 & q_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & p_1 \\ c_1 & d_1 & q_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 & a_2p_1 + b_2q_1 + p_2 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 & c_2p_1 + d_2q_1 + q_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

This property allows us to study affine transformations and their *compositions* using this 3×3 representation and simple matrix multiplication. The geometric problem is thus reduced to a linear algebra problem. Because of this correspondence, we shall often use the matrix representation to describe an affine transformation. It should be stressed that an affine transformation can be defined without using a coordinate system, but its matrix representation exists only if one has been chosen.

To show the power of this notation we will now compute the inverse of a proper affine transformation. The inverse is the transform that associates $(x', y') \to (x, y)$, where $x' = ax + by + p$ and $y' = cx + dy + q$. Since the composition of affine transformations is represented by matrix multiplication, it must be that the matrix describing the inverse is the inverse of the matrix describing the original transform. This is easily calculated as

$$\begin{pmatrix} d/D & -b/D & (-dp + bq)/D \\ -c/D & a/D & (cp - aq)/D \\ 0 & 0 & 1 \end{pmatrix},$$

where $D = \det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = ad - bc$. This is also a matrix describing a proper affine transformation. (Exercise: what must you do to ensure that it actually describes a proper transform? Do it. This exercise confirms the claim that an affine transformation is proper if and only if the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible.) If we write the matrix describing the original transform in the form

$$B = \begin{pmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{t} = \begin{pmatrix} p \\ q \end{pmatrix},$$

then its inverse may be written as

$$B^{-1} = \begin{pmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}.$$

Note that B^{-1} is of the same form as B: its third row is $(0\ 0\ 1)$. Furthermore, note that the linear transformation A^{-1} is also invertible.

The set of all proper affine transformations forms a group.

Definition 2.3 *A set E equipped with a multiplication operation $E \times E \to E$ is a group if it satisfies the following properties:*

1. *associativity:* $(ab)c = a(bc), \forall a, b, c \in E$;
2. *existence of an identity element: there exists an element $e \in E$ such that $ea = ae = a, \forall a \in E$;*
3. *existence of inverses: $\forall a \in E$, $\exists b \in E$ such that $ab = ba = e$.*

The inverse of an element a is usually denoted by a^{-1}.

Groups play an important role in several other chapters. See, for example, Section 1.4 and Section 7.4.

It is easy to verify that the set of matrices representing proper affine transformations forms a group. Thus, the set of affine transformations itself forms a group. This is what we check now.

Proposition 2.4 *The set of matrices representing proper affine transformations forms a group under matrix multiplication. The set of proper affine transformations also forms a group under composition. The latter is called the* affine group.

Proof : Consider the matrix

$$B = \begin{pmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}$$

representing a proper affine transformation. Since the affine transformation is proper, A is an invertible 2×2 matrix and therefore the matrix B is itself invertible. Being of the same form as B, the matrix B^{-1} also represents a proper affine transformation, and condition 3 holds. Property 1 holds because matrix multiplication is itself associative, and property 2 holds using the 3×3 identity matrix, which represents the affine transformation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \longleftrightarrow \qquad \begin{cases} x' = x, \\ y' = y. \end{cases}$$

Therefore the set of matrices representing proper affine transformations forms a group. We have seen that there is a one-to-one correspondence between matrices (with last line $(0\ 0\ 1)$) and affine transformations. Moreover, the composition of affine transformations is represented by matrix multiplication through this correspondence. The verification above automatically holds for the proper affine transformations themselves. $\square$

Earlier, we introduced reflections with respect to horizontal and vertical mirrors. As examples, we now give their matrix representation. To obtain these, we need to fix the origin. We shall place it at equal distance between the top and bottom of the frieze.

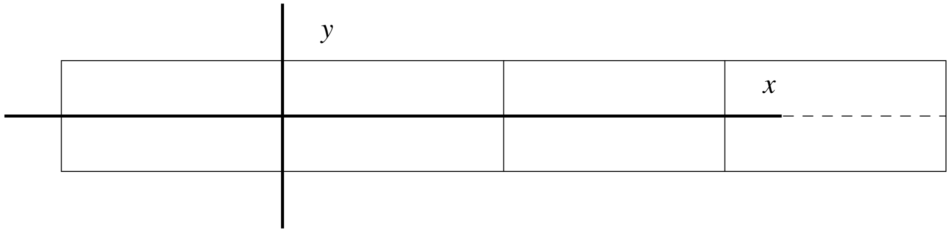

Fig. 2.4. The coordinate system.

(See Figure 2.4.) This still leaves some freedom, since any point on the horizontal axis in the middle of the frieze is a possible choice. (We have already underlined this freedom when discussing the position of vertical mirrors. We shall also use this freedom in the proof of Lemma 2.10.) For a given choice along the horizontal axis, the reflection r_h that exchanges top and bottom (that is, that exchanges the positive vertical axis with the negative one) is represented by the matrix

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{r_h}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{where } r_h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the reflection r_v that exchanges left and right is

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{r_v}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{where } r_v = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

if the origin is on the mirror. (Exercise: check these claims.) Note that

$$r_h r_v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We observe again that the rotation by an angle of 180° (or π) can be obtained by a reflection in a vertical mirror followed by a reflection in a horizontal one. (Exercise: determine the 3×3 matrices that represent the translation t_L and the glide reflection s_g.)

The definition of an affine transformation makes it a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. The requirement that these functions leave a frieze invariant restricts the set of affine transformations that we need to consider. But a second restriction is made that limits the affine transformations even more.

Definition 2.5 *An isometry of the plane (or of a region of the plane) is a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ (or $T : F \subset \mathbb{R}^2 \to \mathbb{R}^2$) that preserves lengths. Hence, if (x_1, y_1) and (x_2, y_2) are two points, then the distance between them is equal to the distance between their images $T(x_1, y_1)$ and $T(x_2, y_2)$.*

Definition 2.6 *A symmetry of a frieze is an isometry that maps the frieze onto the frieze.*

Exercise 9 will show that an isometry is an affine transformation. Lemma 2.7 shows that this restriction to isometric affine transformations limits significantly the possible linear transformations A that can play a role.

Lemma 2.7 *Let the isometry represented by the matrix*

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$$

be a symmetry of a frieze. Then the 2×2 *block is one of the four matrices*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad r_v = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \textit{and} \;\; r_h r_v = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

PROOF: A linear transformation is completely determined by its action on a basis. We shall use the basis $\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are horizontal and vertical vectors of length equal to half the width of the frieze. With this choice any point of the frieze is of the form $(x, y) = \alpha\mathbf{u} + \beta\mathbf{v}$ with $\alpha \in \mathbb{R}$ and $\beta \in [-1, 1]$. (The constraint $\beta \in [-1, 1]$ ensures that the point (x, y) is within the frieze.) The two basis vectors are perpendicular ($\mathbf{u} \perp \mathbf{v}$) or, equivalently, their inner product vanishes: $(\mathbf{u}, \mathbf{v}) = 0$.

To check whether $\left(\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{smallmatrix}\right)$ represents an isometry, it is sufficient to check that

$$|A\mathbf{u}| = |\mathbf{u}|, \qquad |A\mathbf{v}| = |\mathbf{v}|, \qquad \text{and} \qquad A\mathbf{u} \perp A\mathbf{v}. \tag{2.4}$$

Indeed, if P and Q are two points in the frieze and $Q - P = \alpha\mathbf{u} + \beta\mathbf{v}$ is the vector between them, then the image of $Q - P$ is $A(\alpha\mathbf{u} + \beta\mathbf{v})$ and the square of its length is given by

$$\begin{aligned} |A(\alpha\mathbf{u} + \beta\mathbf{v})|^2 &= (\alpha A\mathbf{u} + \beta A\mathbf{v}, \alpha A\mathbf{u} + \beta A\mathbf{v}) \\ &= \alpha^2 |A\mathbf{u}|^2 + 2\alpha\beta(A\mathbf{u}, A\mathbf{v}) + \beta^2 |A\mathbf{v}|^2 \\ &= \alpha^2 |\mathbf{u}|^2 + \beta^2 |\mathbf{v}|^2 \\ &= (\alpha\mathbf{u} + \beta\mathbf{v}, \alpha\mathbf{u} + \beta\mathbf{v}) \\ &= |\alpha\mathbf{u} + \beta\mathbf{v}|^2, \end{aligned}$$

where we have used, to obtain the third equality, the three relations of (2.4) and, for the fourth, the fact that the basis vectors are perpendicular. Then the distance between any pair of points P and Q is preserved by A if the relations (2.4) are satisfied. (Exercise: show that these relations are also necessary.)

Let $A\mathbf{u} = \gamma\mathbf{u} + \delta\mathbf{v}$ be the image of $\mathbf{u}$ by A. Since the transformation is linear, $A(\beta\mathbf{u}) = \beta(\gamma\mathbf{u} + \delta\mathbf{v})$. If δ is nonzero, then it is possible to choose $\beta \in \mathbb{R}$ sufficiently large that $|\beta\delta| > 1$. This means that the point $A(\beta\mathbf{u})$ is outside the frieze. Since this must be ruled out, δ has to be set to zero. (In other words, a transformation A such that δ is nonzero is a linear transformation that tilts the frieze out of the horizontal.) Thus $A\mathbf{u} = \gamma\mathbf{u}$, and if $|A\mathbf{u}| = |\mathbf{u}|$, we must have $\gamma = \pm 1$.

Now let $A\mathbf{v} = \rho\mathbf{u} + \sigma\mathbf{v}$ be the image of $\mathbf{v}$ under A. Since $A\mathbf{u}$ must be perpendicular to $A\mathbf{v}$, we must have

$$0 = (A\mathbf{u}, A\mathbf{v}) = (\gamma\mathbf{u}, \rho\mathbf{u} + \sigma\mathbf{v}) = \gamma\rho|\mathbf{u}|^2.$$

Since neither γ nor $|\mathbf{u}|$ is zero, ρ must be set to 0. And again the last condition $|A\mathbf{v}| = |\mathbf{v}|$ fixes σ to be ± 1. The matrix A representing the transformation in the basis $\{\mathbf{u}, \mathbf{v}\}$ is then $\left(\begin{smallmatrix} \gamma & 0 \\ 0 & \sigma \end{smallmatrix}\right)$. There are two choices for each γ and σ and thus four for the matrix A, precisely those appearing in the statement. $\square$

The composition of two isometries and the inverse of an isometry are themselves isometries. Thus the subset of isometric transformations of the affine group itself forms a group, called the *group of isometries.* Finally, the composition of two isometries leaving a frieze unchanged itself leaves the frieze unchanged. The subset of the group of isometries that leaves the frieze invariant is therefore a group. We are led to the following definition.

Definition 2.8 *The group of symmetry of a frieze is the group of all isometries that leave the frieze invariant.*

2.3 The Classification Theorem

Having a formal theory of isometries and affine transformations allows us to create a list of such transformations that could leave a frieze unchanged. This section will first establish a complete list of possible symmetry generators. The second part of this section uses this list of transformations to enumerate and classify all possible types of groups of frieze symmetries.

There are many affine transformations that simply cannot appear in the symmetry group of a frieze. Lemma 2.7 has already rejected the linear transformations that tilt the frieze out of its domain (the constraint $\delta = 0$ excludes these transformations). The following lemmas characterize the transformations that can appear in frieze symmetry groups. The first describes translations along the infinite axis of the frieze.

Lemma 2.9 *The symmetry group of any frieze of period L contains the translations*

$$\begin{pmatrix} 1 & 0 & nL \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n \in \mathbb{Z}.$$

These are the only translations that appear in the symmetry group.

PROOF: The translation

$$t_L = \begin{pmatrix} 1 & 0 & L \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

leaves any frieze with period L unchanged. Observe that the inverse of this translation is

$$t_{-L} = \begin{pmatrix} 1 & 0 & -L \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and that its composition n times yields

$$t_{nL} = \begin{pmatrix} 1 & 0 & nL \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Exercise!) The translation t_{nL} must therefore be in the symmetry group for all $n \in \mathbb{Z}$. No translation of the form

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with $b \neq 0$ can leave a frieze unchanged, since the vertical portion of the translation will map certain points of the frieze outside of its original vertical extent. We are left with possible translations of the form

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where a is not an integer multiple of L. After performing such a translation by $\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)$, one can repeatedly perform a translation by $\left(\begin{smallmatrix} L \\ 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -L \\ 0 \end{smallmatrix}\right)$ until the resulting translation is by $\left(\begin{smallmatrix} a' \\ 0 \end{smallmatrix}\right)$, where a' satisfies $0 \leq a' < L$. If $0 < a' < L$, it is a translation by a constant a' smaller than the period L, contradicting the definition of the period. And if $a' = 0$, then the original a was an integer multiple of the period L. The only translations left are therefore $t_{nL}, n \in \mathbb{Z}$. $\square$

Are there any other transformations of the form

$$\begin{pmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}$$

where A is not the identity matrix and $\mathbf{t}$ is nonzero? The next lemma answers this question.

Lemma 2.10 *Consider isometries of the form* $\left(\begin{smallmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{smallmatrix}\right)$, *where* $\mathbf{t}$ *is nonzero. By redefining the origin it is possible to reduce any such transformation to one of the form*

$$(i)\ \begin{pmatrix} \begin{matrix} & A & \end{matrix} & \begin{matrix} nL \\ 0 \end{matrix} \\ 0 \quad 0 & 1 \end{pmatrix}, \quad (ii)\ \begin{pmatrix} 1 & 0 & L/2+nL \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad (iii)\ \begin{pmatrix} -1 & 0 & L/2+nL \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $n \in \mathbb{Z}$ *and* A *is one of the four allowed by Lemma 2.7. Form (iii) may occur only if the rotation* $r_h r_v$ *is also a symmetry.*

PROOF: By definition of an isometry, lengths must be preserved. Since the distance between two points is the same as the distance between any translation of the same two points, the matrix A must be one of the four given in (2.3). Moreover, if $t_y \neq 0$ in

$$\begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix},$$

then $y' = cx + dy + t_y$ will be outside of the frieze for certain values of x and y. In fact, for the four possible matrices A, the image of the square $[-1, 1] \times [-1, 1]$ is the square itself. Every translation that has $t_y \neq 0$ moves the square vertically and takes some points of this square out of the frieze. Thus, t_y must be zero.

Since the symmetry group of a frieze contains all horizontal translations by integer multiples of L, the presence of

$$\begin{pmatrix} a & 0 & t_x \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the group implies the presence of

$$\begin{pmatrix} 1 & 0 & nL \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & t_x \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & t_x + nL \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for all $n \in \mathbb{Z}$. Out of the set of all such transformations there will be one such that $0 \leq t'_x = t_x + nL < L$.

We now consider the four possibilities for A. If A is the identity matrix, then Lemma 2.9 forces t'_x to be zero, and the resulting matrix is of the form *(i)*.

Let $A = r_h$. Then the square of

$$\begin{pmatrix} \begin{matrix} & r_h & \end{matrix} & \begin{matrix} t'_x \\ 0 \end{matrix} \\ 0 \quad 0 & 1 \end{pmatrix}$$

must also be in the symmetry group of the frieze. However,

$$\begin{pmatrix} 1 & 0 & t'_x \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 2t'_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a translation. Thus there exists $m \in \mathbb{Z}$ such that $2t'_x = mL$. Since $0 \le t'_x < L$, we have that $0 \le 2t'_x < 2L$. If $t'_x = 0$, the translation is trivial. Otherwise, we must have that $t'_x = L/2$, and the affine transformation becomes

$$\begin{pmatrix} 1 & 0 & L/2 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

It remains to consider the two cases $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Here we will use our freedom in choosing the origin. (See the remarks after the proof of Proposition 2.9.) Consider translating the origin along the x axis by a distance a. The matrix describing the coordinate change is given by

$$S = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If T is the matrix representing an affine transformation and S the matrix changing the coordinate system (x, y) to a new one (x', y'), the same affine transformation will be represented by the matrix STS^{-1} in the new system. To see this, we read as usual from right to left. This expression first transforms the coordinates (x', y') of a point into its coordinates (x, y) in the old system using S^{-1}, applies the affine transformation represented in these old coordinates by the matrix T, and transforms the result back with S into the new coordinate system. The affine transformation represented by

$$\begin{pmatrix} -1 & 0 & t'_x \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.6}$$

will therefore be represented by the matrix

$$\begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & t'_x \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 & t'_x - a \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & t'_x - 2a \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the new system. (Exercise: It is crucial to check that this coordinate change does not spoil the form of other symmetry operations. Show that transformations represented by $\left(\begin{smallmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{smallmatrix}\right)$ with A equal to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or r_h keep the same matrix representation after a horizontal translation of the origin.) Thus the affine transformation represented by (2.6) is now represented by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

if we displace the origin by precisely $a = t'_x/2$.

Note that if the symmetry group contains two transformations of the form (2.6) with distinct $t'_{x1}, t'_{x2} \in [0, L)$, then moving the origin assures us that the transformation with t'_{x1} can be written in the form (2.7). The second remains of the form (2.6) with t'_{x2} replaced by $t_{x2} = t'_{x2} - t'_{x1}$. If both transformations have the same A, then their composition will be a translation by t_{x_2}, forcing t_{x2} to be nL for some integer n. In this case both transformations are cast into form *(i)* by the change of origin. If, however, the two transformations have different A's, we may suppose that the first has $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and then it is a rotation $r_h r_v$ by 180°. The composition of the two is then

$$\begin{pmatrix} 1 & 0 & t_{x2} \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and by previous arguments, t_{x2} must be either nL or $nL + \frac{L}{2}$ for some integer n. The second transformation is then of the form *(i)* if t_{x2} is an integer multiple of L or of the form *(iii)* if not. □

The first two forms of isometries allowed by Lemma 2.10 are then *(i)* the composition of one of the linear transformations of Lemma 2.7 and a translation t_{nL} by an integer multiple of the period L and *(ii)* the composition of the glide reflection s_g and a translation t_{nL}. The third form *(iii)* may appear only if $r_h r_v$ is also present, and in this case, one can use $r_h r_v$ and the isometry of the form *(ii)* (with $n = 0$) as generators. Hence the three lemmas together show that the symmetry group of a frieze can be generated by a subset of $\{t_L, r_h, r_v, r_h r_v, s_g\}$. This answers the question of the list of possible generators, a question left open at the end of Section 2.1.

The lemmas will now allow us to finish our classification of the symmetry groups of various friezes, which will provide us with an affirmative answer to our earlier question: *is it possible to classify friezes based on the set of geometric operations under which they are invariant?* When describing the various possible symmetry groups we will simply reference the generators of each group. We recall formally the definition of such a list of generators.

Definition 2.11 *Let $\{a, b, \ldots, c\}$ be a subset of a group G. This set is a set of generators for G, and then we write $G = \langle a, b, \ldots, c \rangle$ if the set of all compositions of a finite number of elements of $\{a, b, \ldots, c\}$ and of their inverses is G.*

Theorem 2.12 (Classification of frieze groups) *The symmetry group of any frieze is one of the following seven groups:*

1. $\langle t_L \rangle$
2. $\langle t_L, r_v \rangle$
3. $\langle t_L, r_h \rangle$
4. $\langle t_L, t_{L/2} r_h \rangle$
5. $\langle t_L, r_h r_v \rangle$
6. $\langle t_L, t_{L/2} r_h, r_h r_v \rangle$
7. $\langle t_L, r_h, r_v \rangle$

Each of these groups is described by a set of generators, and they are presented in the same order as those in Figures 2.1 and 2.2.

PROOF: Let t_L represent translation by a distance L along the horizontal axis. All of the groups contain translations by integer multiples of L, the period of the frieze, and the list of generators must contain t_L. Through an appropriate choice for the origin, the only other generators of the symmetry groups will be the linear transformations denoted by $A = r_h, r_v$ or $r_h r_v$ and the glide reflection s_g allowed by Lemma 2.10. Note that if a symmetry group contains any two of r_h, r_v, and $r_h r_v$ then it must automatically contain all three. The list of all possible combinations of generators therefore consists of the seven given in the statement of the theorem as well as

8. $\langle t_L, t_{L/2} r_h, r_h \rangle$
9. $\langle t_L, t_{L/2} r_h, r_v \rangle$
10. $\langle t_L, t_{L/2} r_h, r_h, r_v \rangle$

(See the discussion at the end of Section 2.1, where this list was first constructed.) We repeat here the argument that forces us to reject the case **8**. The presence of $s_g = t_{L/2} r_h$ and r_h implies that the group must also contain their product $(t_{L/2} r_h) r_h = t_{L/2}(r_h^2) = t_{L/2}$, which is a translation by $L/2$ (since $r_h^2 = \mathrm{Id}$). This contradicts the fact that the frieze is periodic with a minimum period of L, and therefore this set must be rejected.

For case **9**, note that the product of s_g and r_v is of the form $t_{L/2} r_h r_v$ discussed in Lemma 2.10. Through a translation of the origin (by $a = \frac{L}{4}$), this product can be written in the form of (2.7) with $A = r_h r_v$. A simple calculation shows that the generators t_L and s_g are unchanged by this translation but that r_v becomes $s_g = t_{L/2} r_v$. Thus subgroup **9** is equally described by the generators $\langle t_L, t_{L/2} r_h, t_{L/2} r_v, r_h r_v \rangle$. Three of these generators belong to **6**, while the fourth $(t_{L/2} r_v)$ is simply the product of $t_{L/2} r_h$ and $r_h r_v$. Case **9** is in fact identical to case **6** and it may be omitted.

Finally, case **10** contains the generators of case **8** and can be eliminated for the same reason.

Thus the symmetry group of any frieze must be one of the seven listed groups. Is there any redundancy in this list? No, and with the help of Figure 2.2 we can easily convince ourselves of this fact. The full argument is rather tedious, and thus we will restrict ourselves to frieze **4**, whose symmetry group was determined to be $\langle t_L, s_g \rangle$. We first observe that the two generators t_L and s_g are both symmetries of this frieze. The group they generate must therefore be a subgroup of the actual symmetry group of the frieze. Can we add any other generators to these two? A quick inspection shows that no such addition (from among the remaining possibilities $r_h, r_v, r_h r_v$) is possible. Thus $\langle t_L, s_g \rangle$ is indeed the entire symmetry group of the frieze **4**. Finally, since group **1** is distinct from **4** and the remaining five groups each contain at least one of r_h, r_v, and $r_h r_v$ which group **4** does not have, then group **4** is in fact distinct from the other six. Repeating an argument of this type for each of the remaining friezes and symmetry groups shows that the list is exhaustive and does not contain any redundancy. □

2.4 Mosaics

In architecture, mosaics are as popular, if not more popular, than friezes. For us, a mosaic will be a pattern that can be repeated to fill the plane and that is periodic along two linearly independent directions. Thus, a mosaic has two linearly independent vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ along which it may be translated without change.

As with friezes, mosaics may be studied in terms of the symmetry operations that leave them unchanged. And as with friezes, they may also be classified by their symmetry groups. Due to their importance in the physics and chemistry of crystals, they are referred to as the *crystallographic groups*. There are 17 crystallographic groups. We will not derive this classification. We will limit ourselves to enumerating the rotations that may appear in the symmetry groups of mosaics, and to understanding the description of the classification.

Lemma 2.13 *Any rotation that leaves a mosaic unchanged must have one of the following angles:* $\pi, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}$.

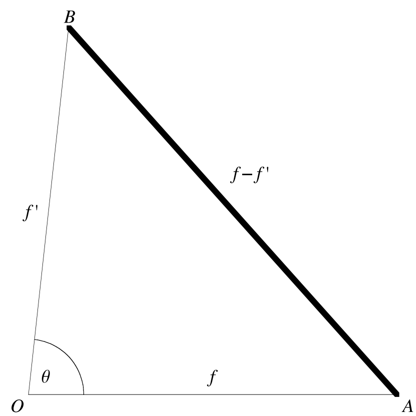

Fig. 2.5. The point $\mathcal{O}$ and two of its images $\mathcal{A}, \mathcal{B}$ under translation.

Proof: Let $\mathcal{O}$ be the center of a rotation leaving the mosaic unchanged. Let $\theta = \frac{2\pi}{n}$ be the smallest angle describing the rotation about this point. Since the mosaic is periodic in two linearly independent directions, there exists an infinity of such points. Let $\mathbf{f}$ be a vector joining $\mathcal{O}$ to a nearby image $\mathcal{A}$ chosen among the closest images of $\mathcal{O}$ obtained by translations. Then translation along the vector $\mathbf{f}$ belongs to the symmetry group of the mosaic.

By rotating the mosaic about $\mathcal{O}$ by an angle θ, the point $\mathcal{A}$ is mapped to $\mathcal{B}$. The vector $\mathbf{f}'$ joining $\mathcal{O}$ to $\mathcal{B}$ also describes a translation under which the mosaic is invariant (see Figure 2.5). The distance between $\mathcal{A}$ and $\mathcal{B}$ is the length of the vector $\mathbf{f}' - \mathbf{f}$, and since $\mathbf{f}' - \mathbf{f}$ is also a translation leaving the mosaic unchanged, this distance must be greater than or equal to the length of $\mathbf{f}$ by hypothesis. ($\mathcal{A}$ was one of the nearest images of $\mathcal{O}$.) Since $\mathbf{f}$ and $\mathbf{f}'$ are of the same length, it must be that the angle $\theta = \frac{2\pi}{n}$ is greater than or equal to $\frac{2\pi}{6} = \frac{\pi}{3}$ (which is 60°). In fact, $\frac{\pi}{3}$ is the precise angle such that $\mathbf{f}$, $\mathbf{f}'$, and $\mathbf{f}' - \mathbf{f}$ are all the same length. This first argument restricts the possibilities to $\frac{2\pi}{2} = \pi$, $\frac{2\pi}{3}$, $\frac{2\pi}{4} = \frac{\pi}{2}$, $\frac{2\pi}{5}$, and $\frac{2\pi}{6} = \frac{\pi}{3}$.

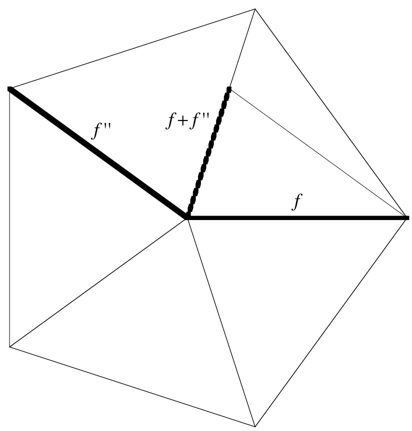

Fig. 2.6. The case of rotation by an angle $\frac{2\pi}{5}$.

However, no mosaic can be left unchanged after rotation by an angle of $\frac{2\pi}{5}$. Figure 2.6 shows $\mathbf{f}$ and its image $\mathbf{f}''$ after a rotation of $\frac{4\pi}{5}$. Translation along $\mathbf{f}+\mathbf{f}''$ must also be an invariant operation, but its length is shorter than that of $\mathbf{f}$, a contradiction. Thus, we can safely reject this angle. □

The elements of the crystallographic groups are similar to those found in the frieze symmetry groups: translations, reflections, reflections followed by translations (that is, glide reflections as for friezes), and rotations. Rather than exhaustively listing the generators for each of the 17 crystallographic groups, we will instead show an example of

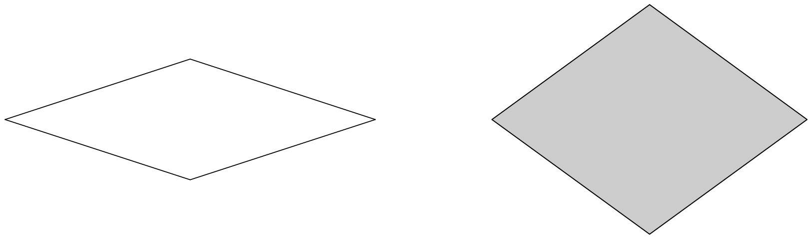

Fig. 2.7. Penrose tiles.

each type and highlight its symmetries (see Figures 2.17 through 2.22, starting on page 77). For each class we illustrate the basic shape of the mosaic at the left, overlaid with a shaded parallelogram whose sides indicate the two linearly independent directions in which the mosaic may be translated. These vectors have been chosen such that the parallelogram encloses the smallest possible area necessary to cover the plane by translations along them. There is usually more than one choice for this parallelogram. On the right, the same mosaic has been drawn again with axes of reflection or glide reflection and points of rotation overlaid. Finally, the legend of each graph identifies the *international symbols* commonly used to designate each crystallographic group [5]. Solid lines indicate that a simple reflection across the axis is a symmetry. Dashed lines indicate glide reflections; the required translations are not explicitly shown but are easily seen nonetheless. Various symbols are used to indicate points about which the mosaic may be rotated. If the center of rotation does not fall on an axis of reflection, the following are used:

$\diamond$ for rotations of angle π,
$\triangle$ for rotations of angle $\frac{2\pi}{3}$,
$\square$ for rotations of angle $\frac{\pi}{2}$,
and hexagons for rotations of angle $\frac{\pi}{3}$.

When the point of rotation lies along an axis of reflection, solid versions of the same symbols ($\blacktriangle$, $\blacksquare$, etc.) are employed.

The ancient city of Alhambra, seat of the Moorish government of Granada in the south of modern-day Spain, houses many mosaics that are as stunning in number as they are in complexity. For a long time it was debated whether all 17 crystallographic groups were represented by the Alhambra mosaics. Grünbaum, Grünbaum, and Shephard [4] claim that this is not the case, with only 13 groups being employed. Even with this negative response, it is still natural to ask whether the Moorish artists of the time were aware of such a system of classification.

The precise mathematical formalization of friezes and mosaics allowed mathematicians to study new generalized structures by relaxing certain rules in the definition. Aperiodic tilings are one such structure. All mosaics must fill the plane, meaning that repeating the pattern in all directions covers all points of $\mathbb{R}^2$ without leaving any gaps.

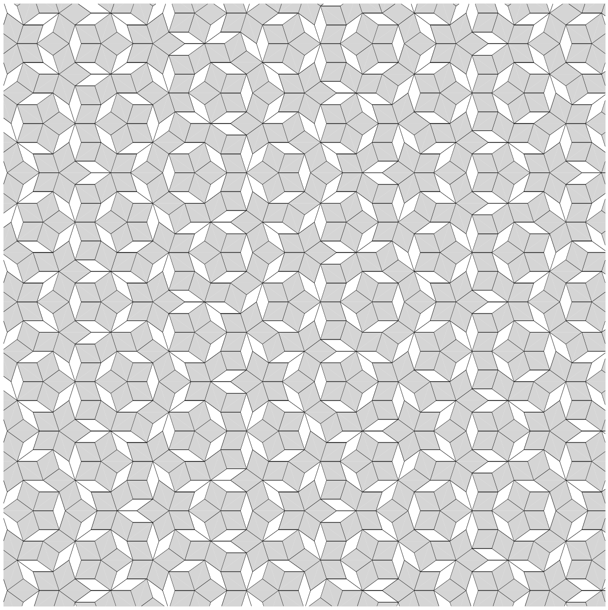

Fig. 2.8. An aperiodic Penrose tiling.

This condition is also satisfied by aperiodic tilings. For example, it is possible to tile the plane $\mathbb{R}^2$ with the two Penrose tiles (referred to as the Penrose rhombs) shown in Figure 2.7 [5]. Even if it is possible to tile the plane in a periodic manner with these tiles, it is also possible to arrange them in such a way that no translational symmetry is present; in other words, they may be used to tile the plane in an *aperiodic* manner. Figure 2.8 shows a fragment of an aperiodic tiling. Maybe these new generalized structures will find their way into architecture... (There are other sets of tiles, constructed by Penrose and others, that may be tiled *only* aperiodically!)

2.5 Exercises

1. We say that two operations $a, b \in E$ commute if $ab = ba$.

(a) Do translation operations commute?

(b) Do r_h, r_v, and $r_h r_v$ all commute with each other?

(c) Do the reflections r_h, r_v, and $r_h r_v$ commute with translations?

2. Find the conditions under which a linear transformation

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and a translation

$$\begin{pmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}$$

will commute with each other.

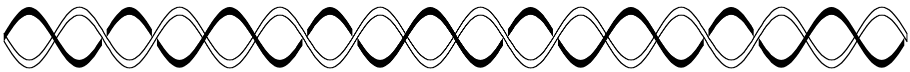

Fig. 2.9. The frieze of Exercise 3.

3. **(a)** Determine the period L of the frieze in Figure 2.9. Indicate it directly on the figure or a copy of it.
(b) Under which of the transformations $t_L, r_h, s_g, r_v, r_h r_v$ is the frieze invariant?
(c) Which of the seven symmetry groups does the frieze belong to?
(d) By drawing a single point per period on the frieze, reduce its symmetry group to $\langle t_L \rangle$ without changing the length of its period.

4. **(a)** Friezes are often used in architecture, with [3] giving several remarkable examples. Select a few such examples, and determine to which of the symmetry groups they belong.
(b) The artist M. C. Escher created several remarkable mosaics, with a large number of them being presented in [6]. Select a few of Escher's mosaics and determine to which of the 17 crystallographic groups they belong.

5. **(a)** Identify the symmetry group of the frieze shown in Figure 2.10.

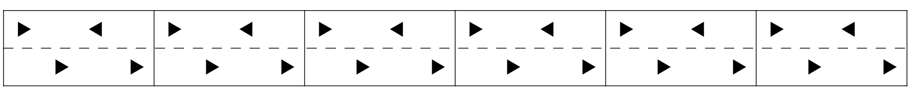

Fig. 2.10. Frieze for Exercise 5.

(b) By removing two triangles from each period of this frieze, construct a frieze belonging to the symmetry group **5**.

6. Prove the three statements of Proposition 2.1. Suggestion: these properties can be proved using only Euclidean geometry or using the matrix representation of affine transformations. Explore both approaches.

7. **(a)** Let m_1 and m_2 be parallel lines at a distance d and let r_{m_1} and r_{m_2} be the reflections through these lines. Show that the composition $r_{m_2}r_{m_1}$ is a translation by a distance $2d$ along a direction perpendicular to the lines (mirrors) m_1 and m_2. Hint: show this using only Euclidean geometry, that is, without use of a coordinate system. You may use the concept of distance or length of a segment.
(b) Let a frieze of period L be invariant under the reflection r_v. Show that it is invariant under reflection through a vertical mirror at distance $\frac{L}{2}$ from the first. Hint: study the composition of r_v and the translation t_L.

8. Let m_1 and m_2 be two lines intersecting at P and let r_{m_1} and r_{m_2} be the reflections through these lines. Show that the composition $r_{m_2}r_{m_1}$ is a rotation of center P by twice the angle between the two lines (mirrors) m_1 and m_2. Hint: show first that the images $r_{m_1}Q$ and $Q' = r_{m_2}r_{m_1}Q$ lie on a circle of center P and of radius $|PQ|$. Then study the angles made by the segments PQ and PQ' with a given line, say m_1.

9. The goal of this exercise is to show that an isometry is the composition of a linear transformation and a translation and therefore is an affine transformation. (Either the linear transformation or the translation could be the identity.) Recall that a linear transformation of the plane is a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies the following two conditions: *(i)* $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and *(ii)* $T(c\mathbf{u}) = cT(\mathbf{u})$ for all points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and constant $c \in \mathbb{R}$.
(a) Show that an isometry $T : \mathbb{R}^2 \to \mathbb{R}^2$ preserves angles. Hint: choose three (non-collinear) points P, Q, R. If P', Q', R' are their images under T, show that the triangles PQR are $P'Q'R'$ are congruent.
(b) Show that a translation is an isometry.
(c) Suppose that an isometry S has no fixed-point and that $S(P) = Q$. Show that the composition TS, where T is the translation that maps Q to P, has at least one fixed-point.
(d) Let S be an isometry that has (at least) one fixed-point O. Let P, Q, R be chosen such that $OPQR$ is a parallelogram. Let P', Q', R' be their image under S. Show that the sum of the vectors OP' and OR' is OQ'. (This amounts to $S(OP + OR) = S(OP) + S(OR)$.)
(e) Let S be an isometry that has (at least) one fixed-point O and let P and Q be two points, distinct and distinct from O, such that O, P, Q are collinear. Show that

$$S(OP) = \frac{|OP|}{|OQ|} S(OQ).$$

(f) Conclude that an isometry is a linear transformation followed by a translation and is therefore an affine transformation. (Either of the two operations could be the identity.)

10. **(a)** The pattern of Figure 2.11 consists of a series of ellipses centered along the x axis at the points $(2^i, 0)$ with principal axes $r_x = 2^{i-2}, r_y = 1$. Thus, this pattern exists over the infinite half-strip $(0, \infty) \times [-\frac{1}{2}, \frac{1}{2}]$. This pattern is not a frieze because it is not periodic. Replace the periodicity condition with another invariance condition such that this pattern is a "frieze."
(b) Describe the transformation that maps one ellipse to the first one on its left. Is it linear? Does the set of such transformations form a group?

Fig. 2.11. A pattern that is not periodic. (For Exercise 10.)

11. Let $r > 1$ be a real number and let

$$A_r = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{1}{r} \leq \sqrt{x^2 + y^2} \leq r \right\}$$

be the ring with center at the origin of the plane and delimited by the circles with radii r and $\frac{1}{r}$.
(a) Show that the set A_r is invariant under rotations of the form

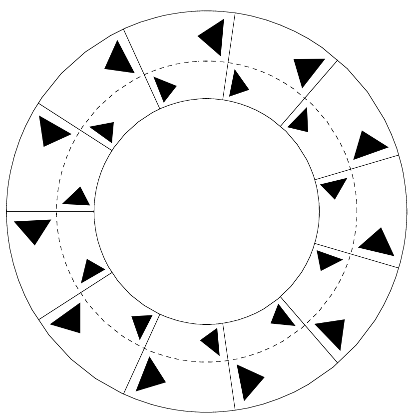

Fig. 2.12. A circular frieze. (See Exercise 11.)

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for all $\theta \in [0, 2\pi)$. (The invariance of A_r means that the transformation is invertible and that the image of A_r is A_r itself.)

(b) Consider the transformation $\mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$ defined by

$$\begin{aligned} x' &= \frac{x}{x^2+y^2}, \\ y' &= \frac{y}{x^2+y^2}. \end{aligned}$$

This transformation is called an *inversion*. Show that A_r is invariant under this transformation. Show that A_r^2 is the identity transformation. Is this transformation linear?

(c) Figure 2.12 represents a circular frieze drawn on a ring A_r. The dashed line represents the circle of radius 1. Unlike the band friezes discussed earlier, circular friezes are bounded. It is easy to construct a correspondence between the symmetries of a band frieze presented in Section 2.2 and those of a circular frieze. Translations become rotations, and reflection r_h across the horizontal axis becomes inversion as introduced in (b). Define the transformation that corresponds to reflection r_v across a vertical axis. We will call this last transformation *reflection*. Is reflection a linear transformation? (As before, this transformation can be defined only after a suitable origin has been chosen. You will have to carefully choose a particular point of A_r through which the "mirror" will pass.)

(d) Starting from the three operations of rotation, inversion, and reflection, construct a set of generators for the symmetry group of the circular frieze shown in Figure 2.12.

12. **(a)** This exercise continues the previous one. Let n be the largest integer for which a circular frieze is invariant under a rotation of $\frac{2\pi}{n}$. We will suppose that $n \geq 2$. Classify the symmetry groups of a circular frieze for a given n. Does the classification depend on n in any way?

(b) The *order* of a group is the number of elements in the group. The orders of the symmetry groups of regular friezes are infinite, but those of circular friezes are finite. Calculate the orders of the groups you constructed in (a).

13. For each Archimedean tiling shown in Figure 2.13, determine to which of the 17 crystallographic groups it belongs (certain tilings must belong to the same group). An *Archimedean tiling* is a tiling of the plane consisting of regular polygons such that each vertex is of the same *type*. For two vertices to be of the same type, they must be coincident with similar polygons, and the polygons must appear in the same order as we turn about the point in a given direction (clockwise, for example). It is possible that the mirror image of such a tiling is impossible to achieve through rotation and translation alone. If we assume that such tilings are unique up to their mirror image (when such an image is different from the original tiling), there are exactly 11 families of Archimedean

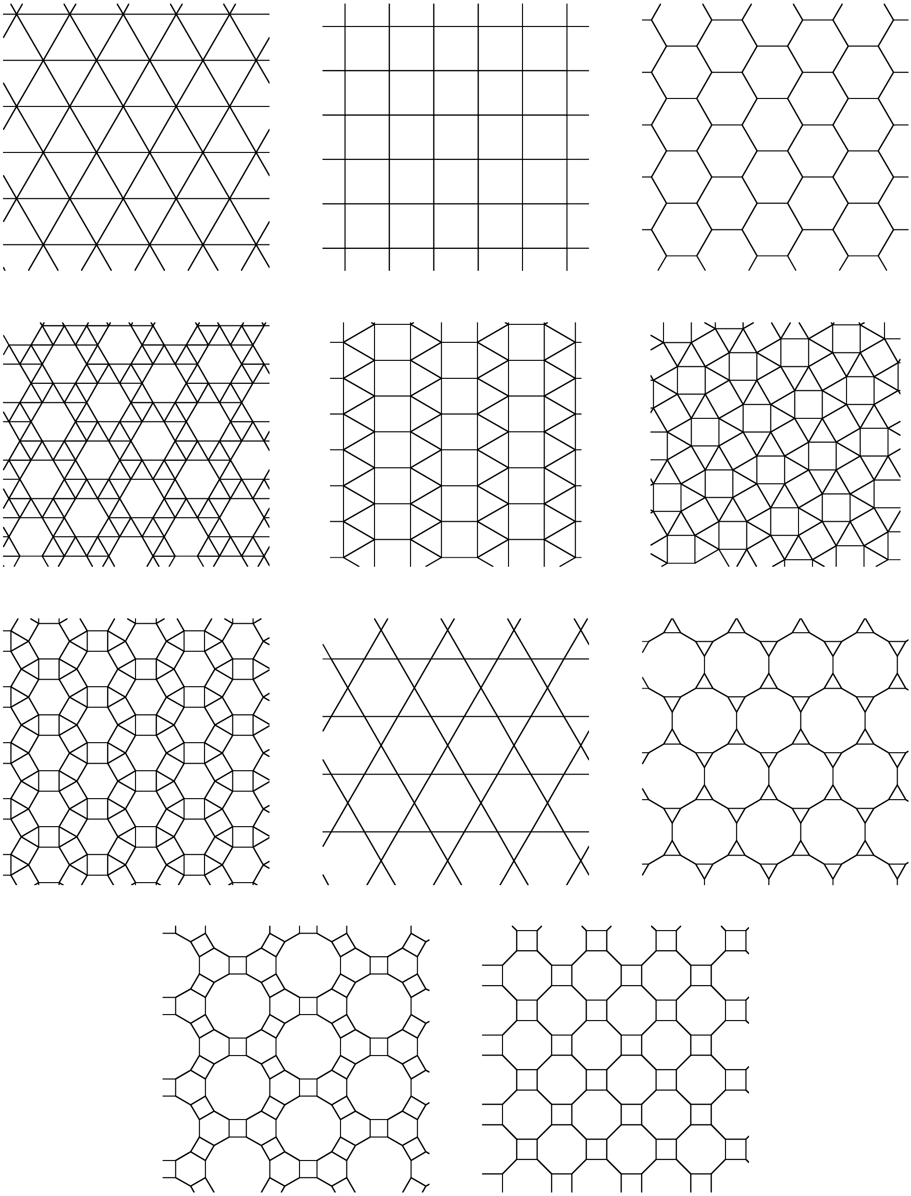

Fig. 2.13. Archimedean tilings. (See Exercise 13.)

tilings. The mirror image is distinct from the original tiling for exactly one of these tilings. Identify it.

14. A small challenge: classify the Archimedean tilings (see Exercise 13).
(a) Denote by n the regular polygon with n sides. Its internal angles are all equal to $\frac{(n-2)\pi}{n}$. (Prove this!) Consider an Archimidean tiling and let $(n_1, n_2, \ldots, n_m)$ be the list of the m polygons that meet at the vertices of this tiling. The sum of the angles at a given vertex must be 2π, and therefore

$$2\pi = \frac{(n_1-2)\pi}{n_1} + \frac{(n_2-2)\pi}{n_2} + \cdots + \frac{(n_m-2)\pi}{n_m}.$$

For example, for the Archimedean tiling of Figure 2.14, the polygons that meet at a vertex are enumerated by the list $(4, 3, 3, 4, 3)$, and as required, they satisfy

$$\frac{(4-2)\pi}{4} + \frac{(3-2)\pi}{3} + \frac{(3-2)\pi}{3} + \frac{(4-2)\pi}{4} + \frac{(3-2)\pi}{3} = 2\pi.$$

Enumerate all possible lists $(n_1, n_2, \ldots, n_m)$ of polygons that may meet at a vertex. Hint: there are 17 such lists if we distinguish between them using only their size, not the order of the n_i's.
(b) Why does the list $(5, 5, 10)$ not correspond to an Archimedean tiling of the plane?
(c) For each of the lists determined in (a), verify whether the set of polygons $(n_1, n_2, \ldots, n_m)$ meeting at a vertex actually describes a tiling of the plane. Caution: the order of the elements in the list $(n_1, n_2, \ldots, n_m)$ is important!

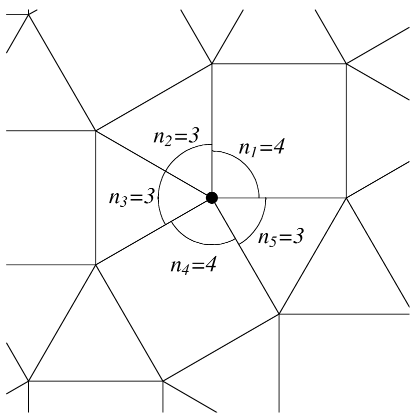

Fig. 2.14. A closer look at an Archimedean tiling (see Exercise 14). The list of polygons meeting at a vertex is denoted by $(4, 3, 3, 4, 3)$.

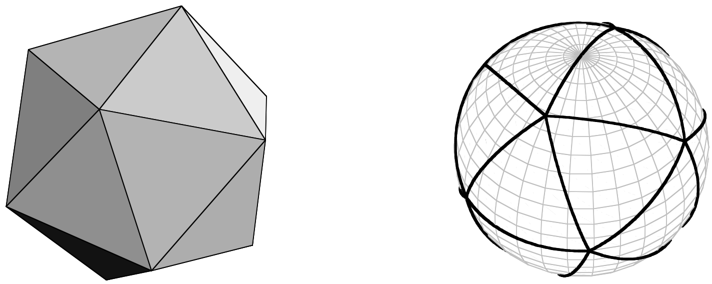

Fig. 2.15. An icosahedron and the corresponding tiling of the sphere (see Exercise 15).

15. A challenge: classify the Archimedean tilings of the sphere. In Section 15.8, we see that each regular polyhedron (the tetrahedron, the cube, the octahedron, the icosahedron, and the dodecahedron) corresponds to a regular tiling of the sphere. This correspondence is constructed as follows:

- the polyhedron is centered at the origin. The distance between the origin and each of the vertices is therefore the same, and we circumscribe a sphere with this radius that passes through all of the vertices;
- for every edge of the polyhedron, we join the vertices by an arc from the great circle between them.

The end result is the desired tiling of the sphere. Figure 2.15 shows such a construction for an icosahedron. The construction can be repeated for any polyhedron whose vertices all lie along the surface of a sphere. This is the case with Archimedean polyhedra: all of their faces are regular polygons with the same side length and all of their corners are incident to the same polygons. Even though regular polyhedra (also called Platonic polyhedra) meet these requirements, we reserve the adjective "Archimedean" for polyhedra whose faces consist of at least two different types of polygons. An example of an Archimedean polyhedron is the familiar shape of a soccer ball, formally called a *truncated icosahedron* (see Figure 2.16). Each vertex is shared by two hexagons and a pentagon. We denote it by the list $(5, 6, 6)$. Archimedean tilings of the sphere are classified as follows: prisms, antiprisms, and the 13 exceptional tilings. (Certain mathematicians prefer to exclude the prisms and antiprisms from the Archimedean tilings, and use the term to refer only to the 13 remaining tilings.)

(a) The list $(n_1, n_2, \ldots, n_m)$ of polygons meeting at a vertex must satisfy two simple conditions. In order for each vertex to be convex (and not planar), the sum of the internal angles meeting at the vertex must be less than 2π:

$$\pi \sum_{i=0}^{m} \frac{n_i - 2}{n_i} < 2\pi.$$

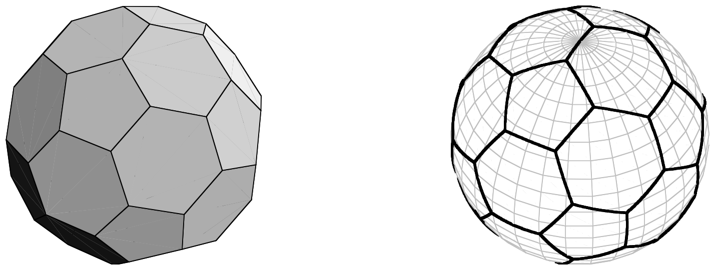

Fig. 2.16. A truncated icosahedron and the corresponding tiling of the sphere (see Exercise 15).

This is the first test. The second condition is based on Descartes's theorem. Each vertex of the polyhedron has associated with it an *angle deficiency* defined as $\Delta = 2\pi - \pi\sum_i (n_i - 2)/n_i$. Descartes's theorem states that the sum of the deficiencies across all vertices of a polyhedron must be equal to 4π. Since all vertices of an Archimedean solid are identical, we must therefore have that $4\pi/\Delta$ is an integer, equal to the number of vertices. This is the second test. Verify that the soccer ball satisfies both of these conditions. (We will see in (d) that these two tests alone are not sufficient to characterize the Archimedean solids.)

(b) A prism is a polyhedron consisting of two identical polygonal faces that are parallel. Each edge of these two faces is then connected by a square. They form an infinite family of solids denoted by $(4, 4, n)$, for $n \geq 3$. Convince yourself that all of the vertices of such a solid are identical and accurately described by the list $(4, 4, n)$. Draw an example of such a prism, for example $(4, 4, 5)$. Verify that the list $(4, 4, n)$ passes both of the tests described in (a) regardless of n. (When n is sufficiently large, these solids begin to resemble stout cylinders.)

(c) An antiprism also consists of two parallel identical polygons with n faces ($n \geq 4$). However, one of the faces is rotated with respect to the other by an angle of $\frac{\pi}{n}$ and the corners joined by equilateral triangles. The antiprisms form an infinite family of solids and are denoted by the list $(3, 3, 3, n)$ for $n \geq 4$. Answer the same questions as for prisms.

(d) Show that the list $(3, 4, 12)$ passes both of the tests described in (a). However, it is impossible to construct a regular polyhedron based on this list. Why? Hint: start by assembling a triangle, a square, and a polygon with twelve sides (a dodecagon) around a single vertex. Consider the other vertices of these three faces. Is it possible for these vertices to have the same configuration described by the list $(3, 4, 12)$? (This is the hardest part of this question!)

(e) Show that there exist 13 Archimedean tilings of the sphere (or, equivalently, 13 Archimedean polyhedra) that are neither prisms nor antiprisms. (The soccer ball is one of these 13 solids.)

16. A difficult challenge: derive the crystallographic groups (shown in Figures 2.17–2.22).

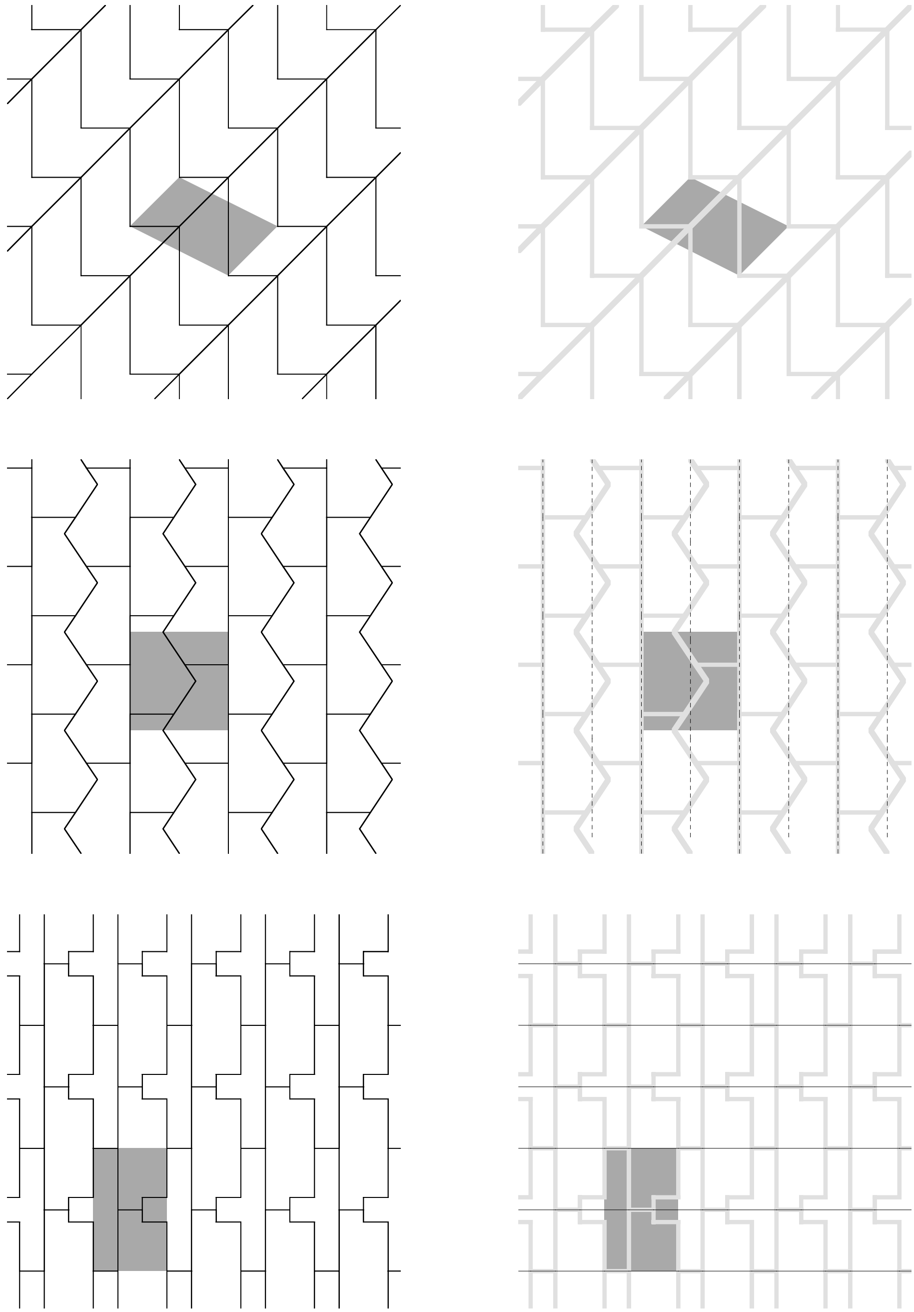

Fig. 2.17. The 17 crystallographic groups. From top to bottom: the groups *p1, pg, pm*.

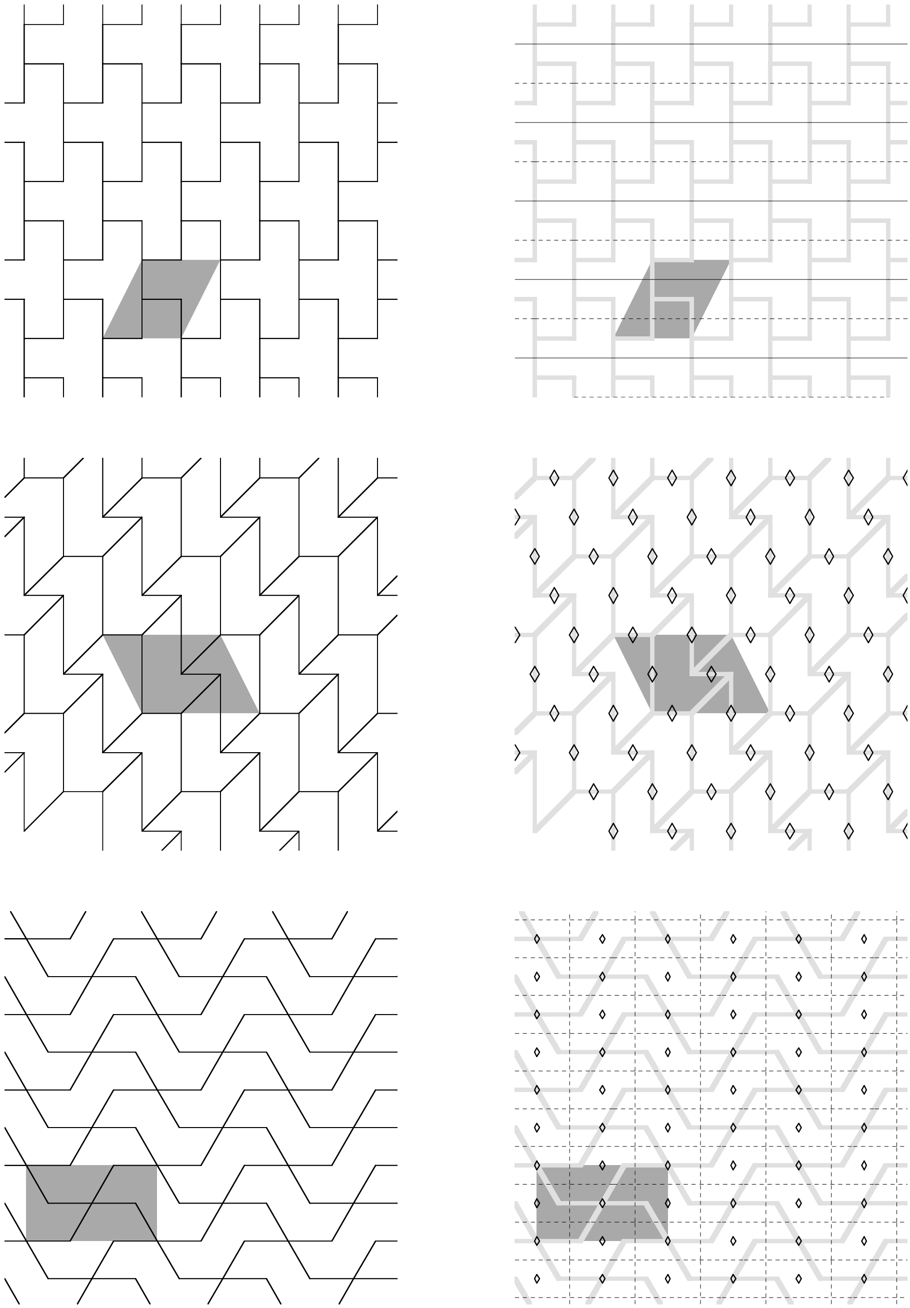

Fig. 2.18. The 17 crystallographic groups (continued). From top to bottom: the groups *cm*, *p2*, *pgg*.

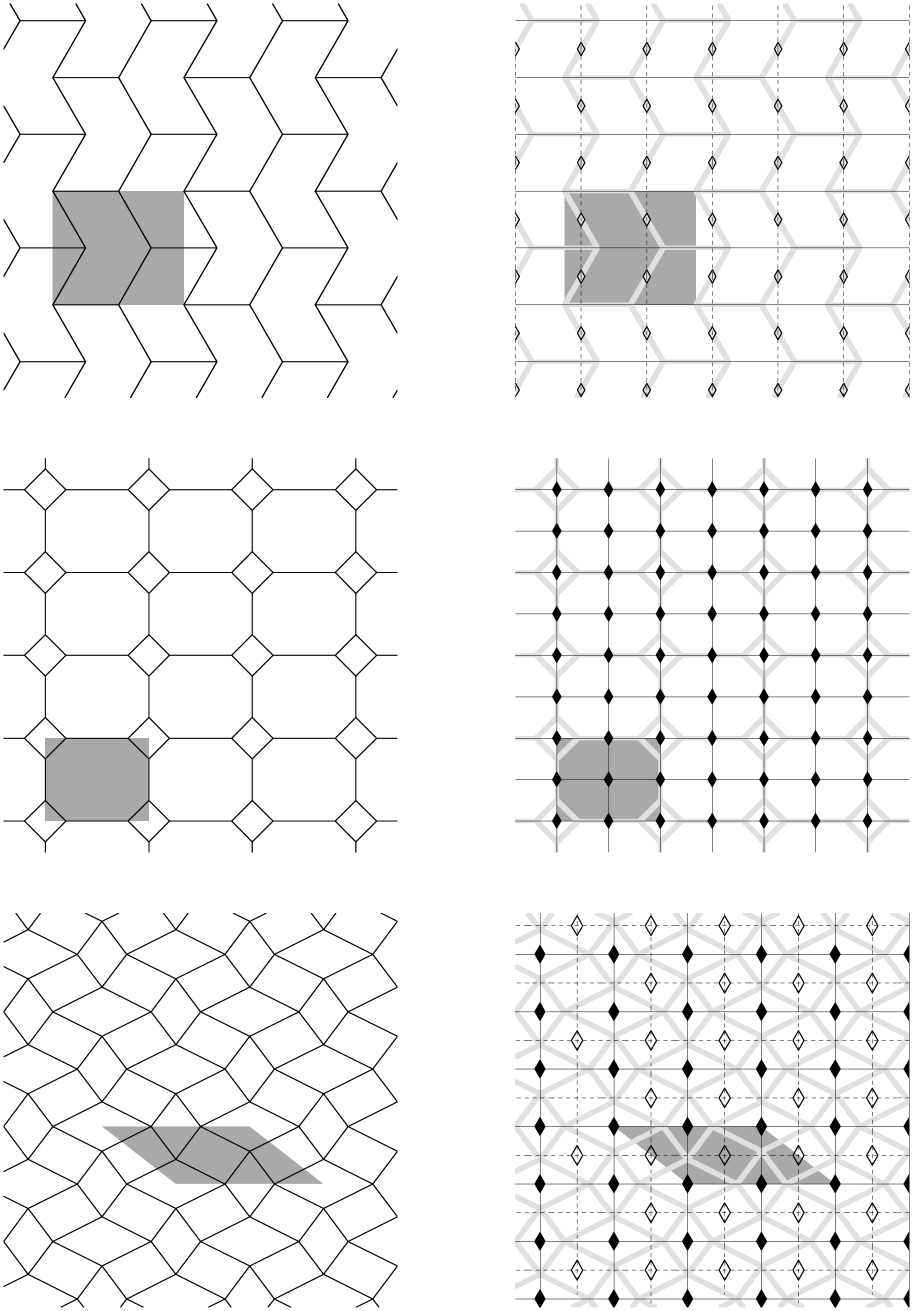

Fig. 2.19. The 17 crystallographic groups (continued). From top to bottom: the groups *pmg*, *pmm*, *cmm*.

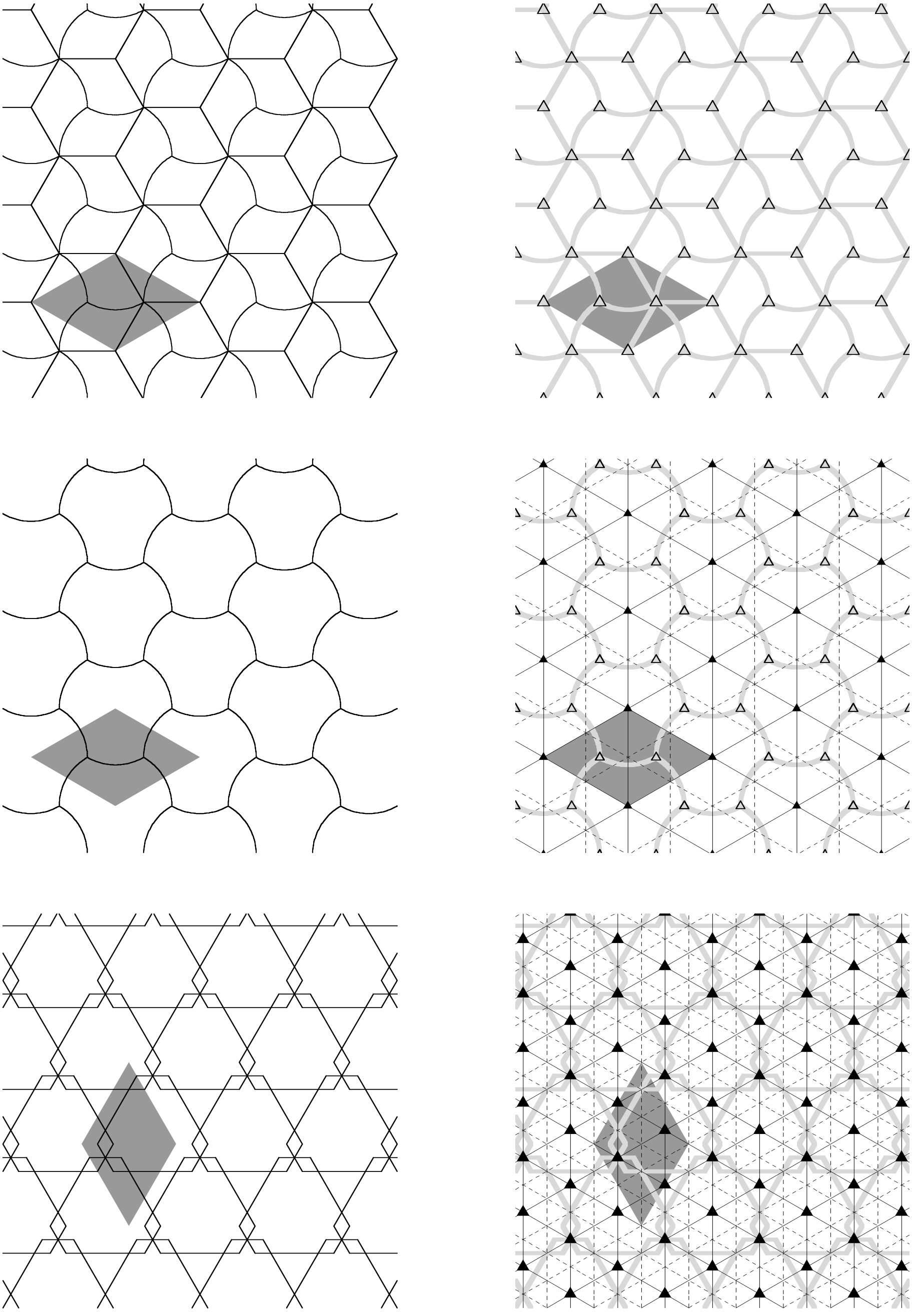

Fig. 2.20. The 17 crystallographic groups (continued). From top to bottom: the groups *p3*, *p31m*, *p3m1*.

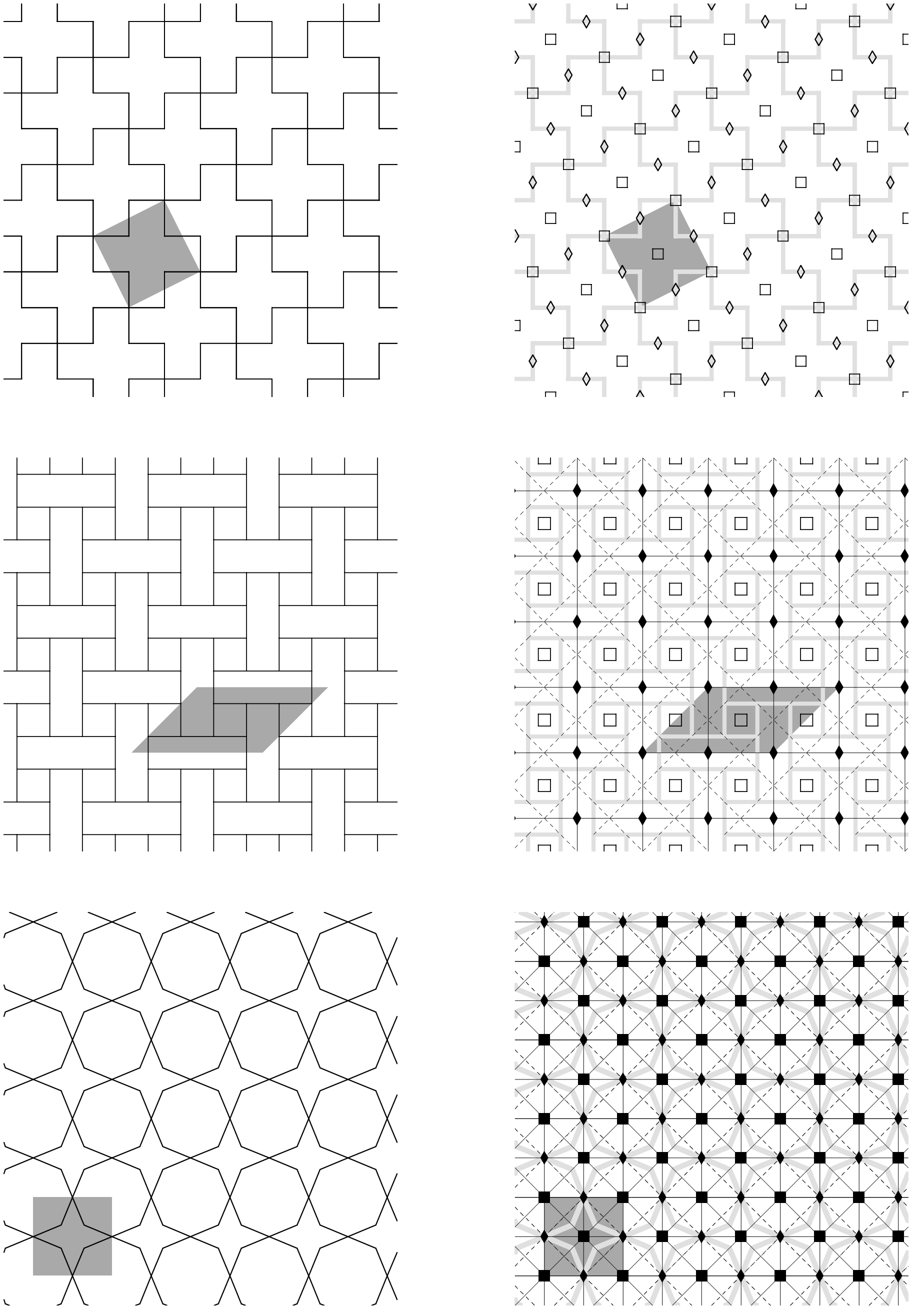

Fig. 2.21. The 17 crystallographic groups (continued). From top to bottom: the groups $p4$, $p4g$, $p4m$.

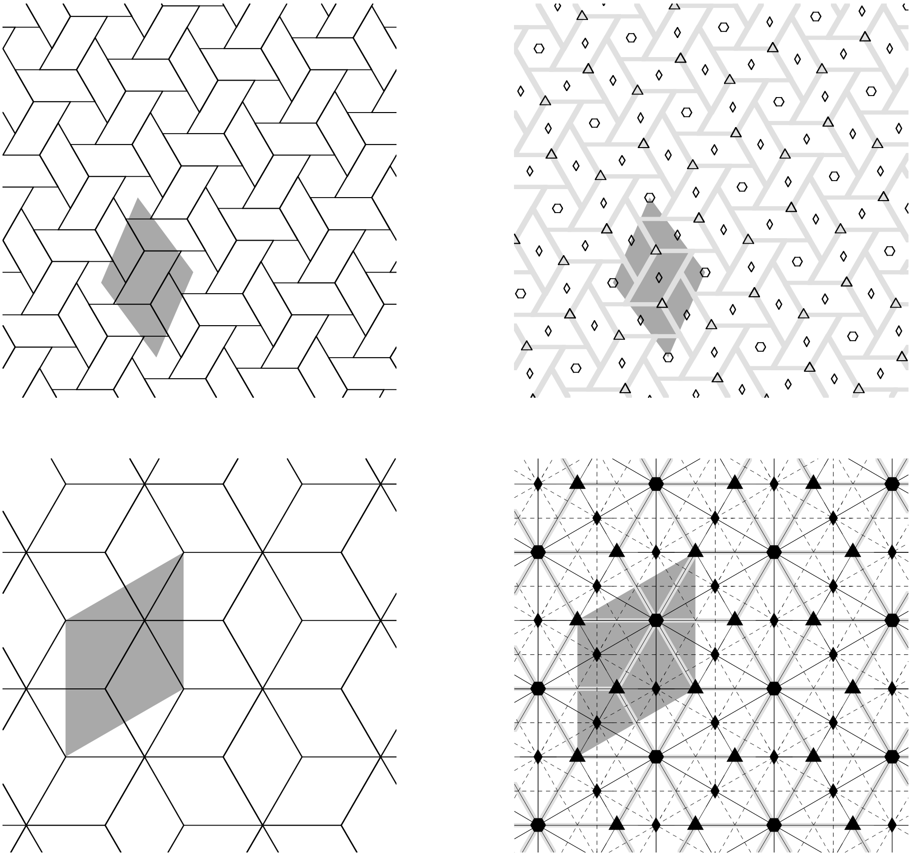

Fig. 2.22. The 17 crystallographic groups (continued). From top to bottom: the groups *p6*, *p6m*.

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