2
Primes in Arithmetic Progressions

In 1837 Dirichlet proved by an ingenious analytic method that there are infinitely many primes in the arithmetic progression

$$a, \quad a+q, \quad a+2q, \quad a+3q, \quad \ldots$$

in which a and q have no common factor and q is prime. The general case, for arbitrary q, was completed only later by him, in 1840, when he had finished proving his celebrated class number formula. In fact, many are of the view that the subject of analytic number theory begins with these two papers. It is also accurate to say that character theory of finite abelian groups begins here.

In this chapter we will derive Dirichlet's theorem, not exactly following his approach, but at least initially tracing his inspiration.

2.1 Summation Techniques

A very useful result is the following.

Theorem 2.1.1 *Suppose* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of complex numbers and* $f(t)$ *is a continuously differentiable function on* $[1, x]$. *Set*

$$A(t) = \sum_{n \le t} a_n.$$

Then

$$\sum_{n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

Proof. First, suppose x is a natural number. We write the left-hand side as

$$\begin{aligned}\sum_{n\le x} a_n f(n) &= \sum_{n\le x}\{A(n)-A(n-1)\}f(n)\\ &= \sum_{n\le x} A(n)f(n) - \sum_{n\le x-1} A(n)f(n+1)\\ &= A(x)f(x) - \sum_{n\le x-1} A(n)\int_n^{n+1} f'(t)dt\\ &= A(x)f(x) - \sum_{n\le x-1}\int_n^{n+1} A(t)f'(t)dt,\end{aligned}$$

since $A(t)$ is a step function. Also,

$$\sum_{n\le x-1}\int_n^{n+1} A(t)f'(t)dt = \int_1^x A(t)f'(t)dt,$$

and we have proved the result if x is an integer. If x is not an integer, write $[x]$ for the greatest integer less than or equal to x, and observe that

$$A(x)\{f(x)-f([x])\} - \int_{[x]}^x A(t)f'(t)dt = 0,$$

which completes the proof.

Remark. Theorem 2.1.1 is often referred to as "partial summation."

Exercise 2.1.2 *Show that*

$$\sum_{n\le x}\log n = x\log x - x + O(\log x).$$

Exercise 2.1.3 *Show that*

$$\sum_{n\le x}\frac{1}{n} = \log x + O(1).$$

In fact, show that

$$\lim_{x\to\infty}\Big(\sum_{n\le x}\frac{1}{n}-\log x\Big)$$

exists. (The limit is denoted by γ and called **Euler's constant**.)

Exercise 2.1.4 *Let $d(n)$ denote the number of divisors of a natural number n. Show that*

$$\sum_{n\le x} d(n) = x\log x + O(x).$$

Exercise 2.1.5 *Suppose $A(x) = O(x^\delta)$. Show that for $s > \delta$,*

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s} = s\int_1^\infty \frac{A(t)}{t^{s+1}}dt.$$

Hence the Dirichlet series converges for $s > \delta$.

Exercise 2.1.6 *Show that for $s > 1$,*

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx,$$

where $\{x\} = x - [x]$. Deduce that $\lim_{s\to 1^+}(s-1)\zeta(s) = 1$.

Consider the sequence $\{b_r(x)\}_{r=0}^\infty$ of polynomials defined recursively as follows:

$$\begin{aligned} b_0(x) &= 1, \\ b_r'(x) &= rb_{r-1}(x) \quad (r\ge 1), \\ \int_0^1 b_r(x)dx &= 0 \quad (r \ge 1). \end{aligned}$$

Thus, from the penultimate equation, $b_r(x)$ is obtained by integrating $rb_{r-1}(x)$, and the constant of integration is determined from the last condition.

Exercise 2.1.7 *Prove that*

$$F(x,t) = \sum_{r=0}^\infty b_r(x)\frac{t^r}{r!} = \frac{te^{xt}}{e^t-1}.$$

It is easy to see that

$$b_0(x) = 1,$$

$$b_1(x) = x - \tfrac{1}{2},$$

$$b_2(x) = x^2 - x - \tfrac{1}{6},$$

$$b_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,$$

$$b_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30},$$

$$b_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x.$$

These are called the **Bernoulli polynomials**. One defines the rth **Bernoulli function** $B_r(x)$ as the periodic function that coincides with $b_r(x)$ on $[0, 1)$. The number $B_r := B_r(0)$ is called the rth **Bernoulli number**. Note that if we denote by $\{x\}$ the quantity $x - [x]$, $B_r(x) = b_r(\{x\})$.

Exercise 2.1.8 *Show that $B_{2r+1} = 0$ for $r \geq 1$.*

The Bernoulli polynomials are useful in deriving the **Euler - Maclaurin summation formula** (Theorem 2.1.9 below).

Let $a, b \in \mathbb{Z}$. We will use the Stieltjes integral with respect to the measure $d[t]$. Then

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)d[t].$$

Notice that the interval of summation is $a < n \leq b$, so that

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt - \int_a^b f(t)dB_1(t)$$

because $d[t] = dt - d\{t\}$ and $B_1(t) = \{t\} - \frac{1}{2}$, by the theory of the Stieltjes integral. We can evaluate the last integral by parts:

$$\int_a^b f(t)dB_1(t) = (f(b) - f(a))B_1 - \int_a^b B_1(t)f'(t)dt,$$

since $B_1(b) = B_1(a) = B_1(0)$. From $B_2'(t) = 2B_1(t)$, we can write

$$\int_a^b f(t)dB_1(t) = (f(b) - f(a))B_1 - \frac{1}{2!}\int_a^b f'(t)dB_2(t),$$

provided that f is differentiable on $[a, b]$. We can iterate this procedure to deduce he following theorem:

Theorem 2.1.9 (Euler-Maclaurin summation formula) *Let k be a nonnegative integer and f be $(k+1)$ times differentiable on $[a, b]$ with $a, b \in \mathbb{Z}$. Then*

$$\begin{aligned}\sum_{a<n\leq b} f(n) &= \int_a^b f(t)dt + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!}(f^{(r)}(b) - f^{(r)}(a))B_{r+1} \\ &\quad + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(t)f^{(k+1)}(t)dt.\end{aligned}$$

Example 2.1.10 *For integers $x \geq 1$,*

$$\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + \frac{1}{2x} + \frac{1}{12x^2} + O\left(\frac{1}{x^3}\right).$$

Solution. Put $f(t) = 1/t$ in Theorem 2.1.9, $a = 1$, $b = x$, and $k = 2$. Then

$$\sum_{2\leq n\leq x} \frac{1}{n} = \log x + \frac{1}{2}\Big(\frac{1}{x} - 1\Big) + \frac{1}{12}\Big(\frac{1}{x^2} - 1\Big) - \int_1^x \frac{B_3(t)}{t^4}dt,$$

so that

$$\sum_{n\leq x} \frac{1}{n} = \log x + \frac{1}{2} - \frac{1}{12} - \int_1^x \frac{B_3(t)}{t^4}dt + \frac{1}{2x} - \frac{1}{12x^2}.$$

Since

$$-\gamma = \lim_{x\to\infty}\left(\log x - \sum_{n\leq x}\frac{1}{n}\right),$$

we must have

$$\gamma = \frac{1}{2} - \frac{1}{12} - \int_1^\infty \frac{B_3(t)dt}{t^4}.$$

Also,

$$\int_x^\infty \frac{B_3(t)dt}{t^4} = O\Big(\frac{1}{x^3}\Big),$$

so that the result is now immediate.

Exercise 2.1.11 *Show that for some constant* B,

$$\sum_{n\le x}\frac{1}{\sqrt{n}} = 2\sqrt{x} + B + O\Big(\frac{1}{\sqrt{x}}\Big).$$

Exercise 2.1.12 *For* $z \in \mathbb{C}$, *and* $|\arg z| \le \pi - \delta$, *where* $\delta > 0$, *show that*

$$\begin{aligned}\sum_{j=0}^{n}\log(z+j) \quad &= \quad \Big(z+n+\frac{1}{2}\Big)\log(z+n) \\ &\qquad -n - \Big(z-\frac{1}{2}\Big)\log z + \int_0^n \frac{B_1(x)dx}{z+x}.\end{aligned}$$

2.2 Characters mod q

Consider the group $(\mathbb{Z}/q\mathbb{Z})^*$ of coprime residue classes mod q. A homomorphism

$$\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$$

into the multiplicative group of complex numbers is called a **character** (mod q). Since $(\mathbb{Z}/q\mathbb{Z})^*$ has order $\varphi(q)$, then by Euler's theorem we have

$$a^{\varphi(q)} \equiv 1 \text{ (mod } q),$$

and so we must have $\chi^{\varphi(q)}(a) = 1$ for all $a \in (\mathbb{Z}/q\mathbb{Z})^*$. Thus $\chi(a)$ must be a $\varphi(q)$th root of unity.

We extend the definition of χ to all natural numbers by setting

$$\chi(n) = \begin{cases} \chi(n\,(\operatorname{mod} q)) & \text{if } (n,q)=1, \\ 0 & \text{otherwise.}\end{cases}$$

Exercise 2.2.1 *Prove that* χ *is a completely multiplicative function.*

We now define the L-series,

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

Since $|\chi(n)| \le 1$, the series is absolutely convergent for $\operatorname{Re}(s) > 1$.

Exercise 2.2.2 *Prove that for* $\mathrm{Re}(s) > 1$,

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the product is over prime numbers p.

The character

$$\chi_0 : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$$

satisfying $\chi_0(a) = 1$ for all $(a, q) = 1$ is called the **trivial** character. Moreover, if χ and ψ are characters, so is $\chi\psi$, as well as $\overline{\chi}$ defined by

$$\overline{\chi}(a) = \overline{\chi(a)},$$

which is clearly a homomorphism of $(\mathbb{Z}/q\mathbb{Z})^*$. Thus, the set of characters forms a group. This is a finite group, as the value of $\chi(a)$ is a $\varphi(q)$th root of unity for $(a, q) = 1$.

But more can be said. If we write

$$q = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$$

as the unique factorization of q as a product of prime powers, then by the Chinese remainder theorem,

$$\mathbb{Z}/q\mathbb{Z} \simeq \oplus_i \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$$

is an isomorphism of rings. Thus,

$$(\mathbb{Z}/q\mathbb{Z})^* \simeq \oplus_i (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^*.$$

Exercise 2.2.3 *Show that* $(\mathbb{Z}/p\mathbb{Z})^*$ *is cyclic if* p *is a prime.*

An element g that generates $(\mathbb{Z}/p\mathbb{Z})^*$ is called a **primitive root** (mod p).

Exercise 2.2.4 *Let* p *be an odd prime. Show that* $(\mathbb{Z}/p^a\mathbb{Z})^*$ *is cyclic for any* $a \geq 1$.

In the previous exercise it is crucial that p is odd. For instance, $(\mathbb{Z}/8\mathbb{Z})^*$ is not cyclic but rather isomorphic to the Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. However, one can show that $(\mathbb{Z}/2^\alpha\mathbb{Z})^*$ is isomorphic to a direct product of a cyclic group and a group of order 2 for $\alpha \geq 3$.

Exercise 2.2.5 *Let $a \geq 3$. Show that $5 \pmod{2^a}$ has order 2^{a-2}.*

Exercise 2.2.6 *Show that $(\mathbb{Z}/2^a\mathbb{Z})^*$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2^{a-2}\mathbb{Z})$, for $a \geq 3$.*

Exercise 2.2.7 *Show that the group of characters $(\bmod\ q)$ has order $\varphi(q)$.*

Exercise 2.2.8 *If $\chi \neq \chi_0$, show that*

$$\sum_{a (\bmod\ q)} \chi(a) = 0.$$

Exercise 2.2.9 *Show that*

$$\sum_{\chi (\bmod\, q)} \chi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv 1 \pmod q, \\ 0 & \text{otherwise.} \end{cases}$$

2.3 Dirichlet's Theorem

The central idea of Dirichlet's argument is to show that

$$\lim_{s \to 1^+} \sum_{p \equiv a (\bmod\, q)} \frac{1}{p^s} = +\infty,$$

where the summation is over primes $p \equiv a \pmod q$.

If $q = 1$, this is clear, because

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

and

$$\begin{aligned} \log \zeta(s) &= -\sum_p \log\left(1 - \frac{1}{p^s}\right) \\ &= \sum_p \left(\sum_{n=1}^{\infty} \frac{1}{np^{ns}}\right) \end{aligned}$$

upon using the expression

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Observing that

$$\lim_{s\to 1^+} \zeta(s) = +\infty$$

by virtue of the divergence of the harmonic series, we get

$$\lim_{s\to 1^+} \log \zeta(s) = +\infty.$$

Consequently,

$$\lim_{s\to 1^+} \Big(\sum_p \frac{1}{p^s} + \sum_p \sum_{n\geq 2} \frac{1}{np^{ns}} \Big) = +\infty.$$

In view of the fact for $s \geq 1$,

$$\sum_p \sum_{n\geq 2} \frac{1}{np^{ns}} \leq \sum_p \sum_{n\geq 2} \frac{1}{np^{n}} \leq \sum_p \frac{1}{p(p-1)} < \infty,$$

we deduce

$$\lim_{s\to 1^+} \sum_p \frac{1}{p^s} = +\infty.$$

Exercise 2.3.1 *Let $\chi = \chi_0$ be the trivial character $(\bmod\, q)$. Show that*

$$\lim_{s\to 1^+} \log L(s, \chi_0) = +\infty.$$

Exercise 2.3.2 *Show that for $s > 1$,*

$$\sum_{\chi (\bmod\, q)} \log L(s, \chi) = \varphi(q) \sum_{n\geq 1} \sum_{p^n \equiv 1 (\bmod\, q)} \frac{1}{np^{ns}}.$$

Exercise 2.3.3 *Show that for $s > 1$ the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} := \prod_{\chi (\bmod\, q)} L(s, \chi)$$

has the property that $a_1 = 1$ and $a_n \geq 0$ for $n \geq 2$.

Exercise 2.3.4 *For $\chi \neq \chi_0$, a Dirichlet character $(\bmod\, q)$, show that $|\sum_{n\leq x} \chi(n)| \leq q$. Deduce that*

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

converges for $s > 0$.

Exercise 2.3.5 *If $L(1,\chi) \neq 0$, show that $L(1,\overline{\chi}) \neq 0$, for any character $\chi \neq \chi_0 \bmod q$.*

Exercise 2.3.6 *Show that*

$$\lim_{s\to 1^+}(s-1)L(s,\chi_0) = \varphi(q)/q.$$

Exercise 2.3.7 *If $L(1,\chi) \neq 0$ for every $\chi \neq \chi_0$, deduce that*

$$\lim_{s\to 1^+}(s-1)\prod_{\chi (\bmod q)} L(s,\chi) \neq 0$$

and hence

$$\sum_{p\equiv 1 (\bmod q)} \frac{1}{p} = +\infty.$$

Conclude that there are infinitely many primes $p \equiv 1 (\bmod q)$.

This exercise shows that the essential step in establishing the infinitude of primes congruent to 1 $(\bmod q)$ is the nonvanishing of $L(1,\chi)$. The exercise below establishes the same for other progressions (mod q).

Exercise 2.3.8 *Fix $(a,q)=1$. Show that*

$$\sum_{\chi (\bmod q)} \overline{\chi(a)}\chi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv a \,(\bmod q), \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2.3.9 *Fix $(a,q)=1$. If $L(1,\chi)\neq 0$, show that*

$$\lim_{s\to 1^+}(s-1)\prod_{\chi (\bmod q)} L(s,\chi)^{\overline{\chi(a)}} \neq 0.$$

Deduce that

$$\sum_{p\equiv a (\bmod q)} \frac{1}{p} = +\infty.$$

The essential thing now is to show that $L(1,\chi) \neq 0$ for $\chi \neq \chi_0$. Historically, this was a difficult step to surmount. Now, there are many ways to establish this. We will take the most expedient route. We will exploit the fact that

$$F(s) := \prod_{\chi (\bmod q)} L(s,\chi)$$

is a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ with $a_1 = 1$ and $a_n \geq 0$. If for some χ_1, $L(1, \chi_1) = 0$, we want to establish a contradiction.

Exercise 2.3.10 *Suppose $\chi_1 \neq \overline{\chi_1}$ (that is, χ_1 is not real-valued). Show that $L(1, \chi_1) \neq 0$ by considering $F(s)$.*

It remains to show that $L(1, \chi) \neq 0$ when χ is real and not equal to χ_0.

We will establish this in the next section by developing an interesting technique discovered by Dirichlet that was first developed by him not to tackle this question, but rather another problem, namely the Dirichlet divisor problem.

2.4 Dirichlet's Hyperbola Method

Suppose we have an arithmetical function $f = g * h$. That is,

$$f(n) = \sum_{d|n} g(d)h(n/d)$$

for two arithmetical functions g and h. Define

$$G(x) = \sum_{n \leq x} g(n),$$
$$H(x) = \sum_{n \leq x} h(n).$$

Theorem 2.4.1 *For any $y > 0$,*

$$\sum_{n \leq x} f(n) = \sum_{d \leq y} g(d) H\Big(\frac{x}{d}\Big) + \sum_{d \leq \frac{x}{y}} h(d) G\Big(\frac{x}{d}\Big) - G(y) H\Big(\frac{x}{y}\Big).$$

Proof. We have

$$\begin{aligned}
\sum_{n \leq x} f(n) &= \sum_{de \leq x} g(d)h(e) \\
&= \sum_{\substack{de \leq x \\ d \leq y}} g(d)h(e) + \sum_{\substack{de \leq x \\ d > y}} g(d)h(e) \\
&= \sum_{d \leq y} g(d) H\Big(\frac{x}{d}\Big) + \sum_{e \leq \frac{x}{y}} h(e) \Big\{ G\Big(\frac{x}{e}\Big) - G(y) \Big\} \\
&= \sum_{d \leq y} g(d) H\Big(\frac{x}{d}\Big) + \sum_{e \leq \frac{x}{y}} h(e) G\Big(\frac{x}{e}\Big) - G(y) H\Big(\frac{x}{y}\Big). \qquad \square
\end{aligned}$$

The method derives its name from the fact that the inequality $de \leq x$ is the area underneath a hyperbola. Historically, this method was first applied to the problem of estimating the error term $E(x)$ defined as

$$E(x) = \sum_{n \leq x} \sigma_0(n) - \{x(\log x) + (2\gamma - 1)x\},$$

where $\sigma_0(n)$ is the number of divisors of n and γ is Euler's constant.

Exercise 2.4.2 *Prove that*

$$\sum_{n \leq x} \sigma_0(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Exercise 2.4.3 *Let χ be a real character* $(\bmod\ q)$. *Define*

$$f(n) = \sum_{d|n} \chi(d).$$

Show that $f(1) = 1$ and $f(n) \geq 0$. In addition, show that $f(n) \geq 1$ whenever n is a perfect square.

Exercise 2.4.4 *Using Dirichlet's hyperbola method, show that*

$$\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2L(1, \chi)\sqrt{x} + O(1),$$

where $f(n) = \sum_{d|n} \chi(d)$ and $\chi \neq \chi_0$.

Exercise 2.4.5 *If $\chi \neq \chi_0$ is a real character, deduce from the previous exercise that $L(1, \chi) \neq 0$.*

Exercise 2.4.6 *Prove that*

$$\sum_{n > x} \frac{\chi(n)}{n} = O\left(\frac{1}{x}\right)$$

whenever χ is a nontrivial character $(\bmod\, q)$.

Exercise 2.4.7 *Let*

$$a_n = \sum_{d|n} \chi(d)$$

where χ is a nonprincipal character $(\bmod\, q)$. *Show that*

$$\sum_{n \le x} a_n = xL(1, \chi) + O(\sqrt{x}).$$

Exercise 2.4.8 *Deduce from the previous exercise that $L(1, \chi) \neq 0$ for χ real.*

Thus, we have proved the following Theorem:

Theorem 2.4.9 (Dirichlet) *For any natural number q, and a coprime residue class a* $(\bmod\, q)$, *there are infinitely many primes $p \equiv a$* $(\bmod\, q)$.

2.5 Supplementary Problems

Exercise 2.5.1 *Let $d_k(n)$ be the number of ways of writing n as a product of k numbers. Show that*

$$\sum_{n \le x} d_k(n) = \frac{x(\log x)^{k-1}}{(k-1)!} + O(x(\log x)^{k-2})$$

for every natural number $k \geq 2$.

Exercise 2.5.2 *Show that*

$$\sum_{n \le x} \log \frac{x}{n} = x + O(\log x).$$

Exercise 2.5.3 *Let $A(x) = \sum_{n \le x} a_n$. Show that for x a positive integer,*

$$\sum_{n \le x} a_n \log \frac{x}{n} = \int_1^x \frac{A(t)dt}{t}.$$

Exercise 2.5.4 *Let $\{x\}$ denote the fractional part of x. Show that*

$$\sum_{n \le x} \left\{ \frac{x}{n} \right\} = (1 - \gamma)x + O(x^{1/2}),$$

where γ is Euler's constant.

Exercise 2.5.5 *Prove that*

$$\sum_{n \le x} \log^k \frac{x}{n} = O(x)$$

for any $k > 0$.

Exercise 2.5.6 *Show that for* $x \ge 3$,

$$\sum_{3 \le n \le x} \frac{1}{n \log n} = \log\log x + B + O\Big(\frac{1}{x \log x}\Big).$$

Exercise 2.5.7 *Let* χ *be a nonprincipal character* $(\operatorname{mod} q)$. *Show that*

$$\sum_{n \ge x} \frac{\chi(n)}{\sqrt{n}} = O\Big(\frac{1}{\sqrt{x}}\Big).$$

Exercise 2.5.8 *For any integer* $k \ge 0$, *show that*

$$\sum_{n \le x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} + O(1).$$

Exercise 2.5.9 *Let* $d(n)$ *be the number of divisors of* n. *Show that for some constant* c,

$$\sum_{n \le x} \frac{d(n)}{n} = \frac{1}{2} \log^2 x + 2\gamma \log x + c + O\Big(\frac{1}{\sqrt{x}}\Big)$$

for $x \ge 1$.

Exercise 2.5.10 *Let* $\alpha \ge 0$ *and suppose* $a_n = O(n^\alpha)$ *and*

$$A(x) := \sum_{n \le x} a_n = O(x^\delta)$$

for some fixed $\delta < 1$. *Define*

$$b_n = \sum_{d|n} a_d.$$

Prove that

$$\sum_{n \le x} b_n = cx + O\left(x^{(1-\delta)(1+\alpha)/(2-\delta)}\right),$$

for some constant c.

Exercise 2.5.11 *Let χ be a nontrivial character $(\bmod q)$ and set*

$$f(n) = \sum_{d|n} \chi(d).$$

Show that

$$\sum_{n \le x} f(n) = xL(1, \chi) + O(q\sqrt{x}),$$

where the constant implied is independent of q.

Exercise 2.5.12 *Suppose that $a_n \ge 0$ and that for some $\delta > 0$, we have*

$$\sum_{n \le x} a_n \ll \frac{x}{(\log x)^\delta}.$$

Let b_n be defined by the formal Dirichlet series

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \Big(\sum_{n=1}^{\infty} \frac{a_n}{n^s} \Big)^2.$$

Show that

$$\sum_{n \le x} b_n \ll x(\log x)^{1-2\delta}.$$

Exercise 2.5.13 *Let $\{a_n\}$ be a sequence of nonnegative numbers. Show that there exists $\sigma_0 \in \mathbb{R}$ (possibly infinite) such that*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s) > \sigma_0$ and diverges for $\mathrm{Re}(s) < \sigma_0$. Moreover, show that the series converges uniformly in $\mathrm{Re}(s) \ge \sigma_0 + \delta$ for any $\delta > 0$ and that

$$f^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \frac{a_n (\log n)^k}{n^s}$$

for $\mathrm{Re}(s) > \sigma_0$ (σ_0 is called the **abscissa of convergence of the Dirichlet series** *$\sum_{n=1}^{\infty} a_n/n^s$).*

Exercise 2.5.14 (Landau's theorem) *Let $a_n \geq 0$ be a sequence of non-negative numbers. Let σ_0 be the abscissa of convergence of*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Show that $s = \sigma_0$ is a singular point of $f(s)$ (that is, $f(s)$ cannot be extended to define an analytic function at $s = s_0$).

Exercise 2.5.15 *Let χ be a nontrivial character $\pmod q$ and define*

$$\sigma_{a,\chi} = \sum_{d|n} \chi(d) d^a.$$

If χ_1, χ_2 are two characters $(\bmod\, q)$, prove that for $a, b \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} \sigma_{a,\chi_1}(n) \sigma_{b,\chi_2}(n) n^{-s}$$

$$= \frac{\zeta(s) L(s-a, \chi_1) L(s-b, \chi_2) L(s-a-b, \chi_1\chi_2)}{L(2s-a-b, \chi_1\chi_2)}.$$

as formal Dirichlet series.

Exercise 2.5.16 *Let χ be a nontrivial character $(\bmod\, q)$. Set $a = \overline{b}$, $\chi_1 = \chi$ and $\chi_2 = \overline{\chi}$ in the previous exercise to deduce that*

$$\sum_{n=1}^{\infty} |\sigma_{a,\chi}(n)|^2 n^{-s} = \frac{\zeta(s) L(s-a, \chi) L(s-\overline{a}, \overline{\chi}) L(s-a-\overline{a}, \chi_0)}{L(2s-a-\overline{a}, \chi_0)}$$

Exercise 2.5.17 *Using Landau's theorem and the previous exercise, show that $L(1, \chi) \neq 0$ for any non-trivial real character $\pmod q$.*

Exercise 2.5.18 *Show that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$.*

Exercise 2.5.19 (Landau's theorem for integrals) *Let $A(x)$ be right continuous for $x \geq 1$ and of bounded finite variation on each finite interval. Suppose that*

$$f(s) = \int_1^{\infty} \frac{A(x)}{x^{s+1}} dx,$$

with $A(x) \geq 0$. Let σ_0 be the infimum of all real s for which the integral converges. Show that $f(s)$ has a singularity at $s = \sigma_0$.

Exercise 2.5.20 *Let λ denote Liouville's function and set*

$$S(x) = \sum_{n \leq x} \lambda(n).$$

Show that if $S(x)$ is of constant sign for all x sufficiently large, then $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > \frac{1}{2}$. (The hypothesis is an old conjecture of Pólya. It was shown by Haselgrove in 1958 that $S(x)$ changes sign infinitely often.)

Exercise 2.5.21 *Prove that*

$$b_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k,$$

where $b_n(x)$ is the nth Bernoulli polynomial and B_n denotes the nth Bernoulli number.

Exercise 2.5.22 *Prove that*

$$b_n(1-x) = (-1)^n b_n(x),$$

where $b_n(x)$ denotes the nth Bernoulli polynomial.

Exercise 2.5.23 *Let*

$$s_k(n) = 1^k + 2^k + 3^k + \cdots + (n-1)^k.$$

Prove that for $k \geq 1$,

$$(k+1)s_k(n) = \sum_{i=0}^{k} \binom{k+1}{i} B_i n^{k+i-i}.$$

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