2

The Newton-Kantorovich (NK) Method

We study the problem of approximating locally unique solution of an equation in a Banach space. The Newton-Kantorovich method is undoubtedly the most popular method for solving such equations.

2.1 Linearization of equations

Let F be a Fréchet-differentiable operator mapping a convex subset D of a Banach space X into a Banach space Y. Consider the equation

$$F(x) = 0. \tag{2.1.1}$$

We will assume D is an open set, unless otherwise stated.

The principal method for constructing successive approximations x_n to a solution x^* (if it exists) of Equations (2.1.1) is based on successive linearization of the equation.

The interpretation of (2.1.1) is that we model F at the current iterate x_n with a linear function:

$$L_n(x) = F(x_n) + F'(x_n)(x - x_n). \tag{2.1.2}$$

L_n is called the local linear model. If $F'(x_n)^{-1} \in L(Y, X)$ the space of bounded linear operators from Y into X, then approximation x_{n+1}, which is the root of $L_n(x_{n+1}) = 0$, is given by

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n \geq 0). \tag{2.1.3}$$

The iterative procedure generated by (2.1.3) is the famous Newton-Kantorovich (NK) method [125]. The geometric interpretation of this method is well-known, if F is a real function. In such a case, x_{n+1} is the point where the tangential line $y - F(x_n) = F'(x_n)(x - x_n)$ of function F at the point $(x, F(x_n))$ intersects the x-axis.

The basic defect of method (2.1.3) is that each step involves the solution of a linear equation with a different linear operator $F'(x_n)$. For this reason, one often

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constructs successive approximations that employ linear equations other than (2.1.2), though similar to them.

The most frequently used substitute for (2.1.2) is the equation

$$F(x_n) + F'(x_0)(x - x_n), \tag{2.1.4}$$

where x_0 is the initial approximation. The successive approximations are then defined by the recurrence relation

$$x_{n+1} = x_n - F'(x_0)^{-1} F(x_n) \quad (n \geq 0). \tag{2.1.5}$$

We will call this method the modified Newton-Kantorovich method (MNK).

We are concerned about the following aspects:

(a) finding effectively verifiable conditions for its applicability;
(b) computing convergence rates and a priori error estimates;
(c) choosing an initial approximation x_0 for which the method converges; and
(d) the degree of "stability" of the method.

2.2 Semilocal convergence of the NK method

Define the operator P by

$$P(x) = x - F'(x)^{-1} F(x) \tag{2.2.1}$$

Then the NK method (2.1.3) may be regarded as the usual iterative method

$$x_{n+1} = P(x_n) \quad (n \geq 0), \tag{2.2.2}$$

for approximating solution x^* of the equation

$$x = P(x) \tag{2.2.3}$$

Suppose that

$$\lim_{n \to \infty} x_n = x^*. \tag{2.2.4}$$

We would like to know under what conditions on F and F' the point x^* is a solution of Equation (2.1.1).

Proposition 2.2.1. *If F' is continuous at $x = x^*$, then we have*

$$F\left(x^*\right) = 0. \tag{2.2.5}$$

Proof. The approximations x_n satisfy the equation

$$F'(x_n)(x_{n+1} - x_n) = -F(x_n). \tag{2.2.6}$$

Because the continuity of F at x^* follows from the continuity of F', taking the limit as $n \to \infty$ in (2.2.6) we obtain (2.2.5).

Proposition 2.2.2. *If*

$$\left\| F'(x) \right\| \leq b \tag{2.2.7}$$

in some closed ball that contains $\{x_n\}$, then x^* is a solution of $F(x) = 0$.

Proof. By (2.2.7) we get

$$\lim_{n \to \infty} F(x_n) = F\left(x^*\right), \tag{2.2.8}$$

and as

$$\|F(x_n)\| \leq b \|x_{n+1} - x_n\|, \tag{2.2.9}$$

(2.2.5) is obtained by taking the limit as $n \to \infty$ in (2.2.4).

Proposition 2.2.3. *If*

$$\left\| F''(x) \right\| \leq K \tag{2.2.10}$$

in some closed ball $\overline{U}(x_0, r) = \{x \in X \mid \|x - x_0\| \leq r\}$, $0 < r < \infty$, which contains $\{x_n\}$, then x^* is a solution of equation $F(x) = 0$.

Proof. By (2.2.10)

$$\left\| F'(x) - F'(x_0) \right\| \leq K \|x - x_0\| \leq Kr \tag{2.2.11}$$

for all $x \in \overline{U}(x_0, r)$. Moreover we can write

$$\left\| F'(x) \right\| \leq \left\| F'(x_0) \right\| + \left\| F'(x) - F'(x_0) \right\|, \tag{2.2.12}$$

so the conditions of Proposition 2.2.2 hold with

$$b = \left\| F'(x_0) \right\| + Kr \tag{2.2.13}$$

Let us assume that the operator F is Fréchet-differentiable on D, where x_0 is an initial approximation for the NK method (2.1.3) and that the operator $F'(x)$ satisfies a Lipschitz condition

$$\left\| F'(x) - F'(y) \right\| \leq \ell \|x - y\|, \quad \text{for all } x, y \in D. \tag{2.2.14}$$

Throughout the sequel, we shall assume that the operator $\Gamma_0 = F'(x_0)^{-1}$ exists.

We shall now state and prove the famous Newton-Kantorovich theorem for approximating solutions of equation (2.1.1) [125]:

Theorem 2.2.4. *Assume that*

$$\Gamma_0 \leq b_0, \tag{2.2.15}$$

$$\|\Gamma_0 F(x_0)\| \leq \eta_0 = \eta \tag{2.2.16}$$

$$h_0 = b_0 \ell \eta_0 \leq \tfrac{1}{2} \tag{2.2.17}$$

$$r_0 = \tfrac{1 - \sqrt{1 - 2h_0}}{h_0} \eta_0, \tag{2.2.18}$$

and

$$\bar{U}(x_0, r) \subseteq D$$

then the NK method (2.1.3) converges to a solution x^* of equation (2.1.1) in the ball $U(x_0, r)$.

There are several proofs of this theorem; we present one due to Kantorovich [125].

Proof. Define a number sequence

$$b_{n+1} = \tfrac{b_n}{1-h_n}, \quad \eta_{n+1} = \tfrac{h_n}{2(1-h_n)}\eta_n, \quad h_{n+1} = b_{n+1}\ell\eta_{n+1}, \tag{2.2.19}$$

$$r_{n+1} = \tfrac{1-\sqrt{1-2h_{n+1}}}{h_{n+1}}\eta_{n+1}. \tag{2.2.20}$$

We claim that under the assumptions (2.2.15), (2.2.18) the successive approximations (2.1.3) exists; moreover

$$\|\Gamma(x_n)\| \le b_n, \quad \|\Gamma(x_n)F(x_n)\| \le \eta_n, \quad h_n \le \tfrac{1}{2} \tag{2.2.21}$$

and

$$U(x_n, r_n) \subset U(x_{n-1}, r_{n-1}). \tag{2.2.22}$$

The proof is by induction. Assume that (2.2.21) and (2.2.22) hold for $n = m$. Because $\|x_{m+1} - x_m\| = \|\Gamma(x_m)F(x_m)\| \le \eta_m$, it follows from the definition of r_m that $x_{m+1} \in U(x_m, r_m)$; a fortiori, $x_{m+1} \in D$. The derivative $F'(x_{m+1})$ therefore exists. By (2.2.14)

$$\left\|\Gamma(x_m)\left(F'(x_{m+1}) - F'(x_m)\right)\right\| \le b_m\ell\,\|x_{m+1} - x_m\| \le h_m \le \tfrac{1}{2};$$

the operator $\Gamma(x_{m+1}) = F'(x_{m+1})^{-1}$ therefore exists, and has the representation

$$\begin{aligned}\Gamma(x_{m+1}) &= \left\{I + \Gamma(x_m)\left[F'(x_{m+1}) - F'(x_m)\right]\right\}^{-1}\Gamma(x_m)\\ &= \sum_{i=0}^{\infty}(-1)^i\left\{\Gamma(x_m)\left[F'(x_{m+1}) - F'(x_m)\right]\right\}^i\Gamma(x_m)\end{aligned} \tag{2.2.23}$$

Hence

$$\|\Gamma(x_{m+1})\| \le \sum_{i=0}^{\infty}\left\|\Gamma(x_m)\left[F'(x_{m+1}) - F'(x_m)\right]\right\|^i b_m \le \tfrac{b_m}{1-h_m} = b_{m+1}. \tag{2.2.24}$$

Now consider the second inequality of (2.2.21) (for $n = m + 1$). It follows from the identity

$$F(x_{m+1}) = F(x_{m+1}) - F(x_m) - F'(x_m)(x_{m+1} - x_m) \tag{2.2.25}$$

that

$$\|F(x_{m+1})\| \le \tfrac{\ell}{2}\|x_{m+1} - x_m\|^2 \le \tfrac{\ell}{2}\eta_m^2, \tag{2.2.26}$$

and, by (2.2.24),

$$\|\Gamma(x_{m+1})F(x_{m+1})\| \le \tfrac{b_m\ell\eta_m^2}{2(1-h_m)} = \tfrac{h_m}{2(1-h_m)}\eta_m = \eta_{m+1}.$$

The third inequality of (2.2.21) is easily proved; by definition,

$$h_{m+1} = b_{m+1}\ell\eta_{m+1} = \frac{b_m}{1-h_m}\ell\frac{h_m}{2(1-h_m)}\eta_m = \frac{h_m^2}{2(1-h_m)^2} \le \frac{1}{2}. \tag{2.2.27}$$

To prove the inclusion (2.2.22) it suffices to note that if $\|x - x_{k+1}\| \le r_{k+1}$ then

$$\|x - x_k\| \le \|x - x_{k+1}\| + \|x_{k+1} - x_k\| \le r_{k+1} + \eta_k, \tag{2.2.28}$$

as the right-hand side is identically equal to r_k (the simple computation is left to the reader).

Thus the successive approximations (2.1.3) are well defined.

The third inequality of (2.2.21) implies that $\eta_{m+1} \le \frac{1}{2}\eta_m$; therefore $r_n \to 0$ as $n \to \infty$. Thus the successive approximations converge to some point $x^* \in U(x_0, r_0)$. To complete the proof, it suffices to leave $m \to \infty$ in (2.2.26).

Remark 2.2.5. (a) It is clear from the proof that, under the assumptions of Theorem 2.2.4,

$$\left\|x_n - x^*\right\| \le r_n \quad (n = 1, 2, ...).$$

(b) Under the assumptions of Theorem 2.2.4, one can easily prove that

$$\left\|x_n - x^*\right\| \le \frac{1}{2^n}\left(2h_0\right)^{2^n-1}\eta_0.$$

(c) In Exercise 2.16.9 we have provided a list of error bounds and the relationship between them.

It is natural to call a solution x^* of equation (2.1.1) a simple zero of the operator F if the operator $\Gamma(x^*)$ exists and is continuous.

Theorem 2.2.6. *If, under the assumptions of Theorem 2.2.4, $h_0 < \frac{1}{2}$, then the zero x^* of F to which the successive approximations (2.1.3) converge is simple.*

Proof. It suffices to note that $r_0 < (\ell b_0)^{-1}$ for $h_0 < \frac{1}{2}$, and that $\|x - x_0\| < (\ell b_0)^{-1}$ implies $\left\|\Gamma_0 F'(x) - I\right\| < 1$. Thus both operators $\Gamma_0 F'(x)$ and $F'(x)$ are invertible.

Note that when $h_0 = \frac{1}{2}$ the successive approximations may converge to a "multiple" zero. An example is the scalar equation $x^2 = 0$ for any $x_0 \neq 0$.

We now examine the convergence of the MNK method (2.1.5).

The method (2.1.5) coincides with the usual iterative method

$$x_{n+1} = Ax_n \quad (n = 0, 1, 2, ...) \tag{2.2.29}$$

for approximate solution of the equation

$$x = Ax \tag{2.2.30}$$

where

$$Ax = x - \Gamma_0 Fx. \tag{2.2.31}$$

Theorem 2.2.7. *Under the hypotheses of Theorem 2.2.4 with (2.2.15) holding as strict inequality the successive approximations (2.1.5) converge to a solution $x^* \in U(x_0, r_0)$ of equation (2.1.1).*

Proof. First note that equation (2.1.1) indeed has a solution in the ball $U(x_0, r_0)$—this follows from Theorem 2.2.4. Below we shall prove that the operator (2.2.31) satisfies the assumptions of the contractive mapping principle Theorem 1.3.4 in the ball $U(x_0, r_0)$. This will imply that the solution x^* in the ball $U(x_0, r_0)$ is unique, and that the approximations (2.1.5) converge.

Obviously, for any $x, y \in U(x_0, r)$ $(r \leq R)$,

$$\begin{aligned} Ax - Ay &= x - y - \Gamma_0(Fx - Fy) \\ &= \Gamma_0 \int_0^1 \left[F'(x_0) - F'(y + t(x-y))\right](x-y)\, dt. \end{aligned} \tag{2.2.32}$$

This identity, together with (2.2.14) implies the estimate

$$\|Ax - Ay\| \leq b_0 L r \|x - y\|. \tag{2.2.33}$$

Consequently, A is a contraction operator in the ball $U(x_0, r_0)$. To complete the proof, it remains to show that

$$AU(x_0, r_0) \subseteq U(x_0, r_0).$$

Let $x_0 \in D$. Then, by (2.2.32)

$$\begin{aligned} \|Ax - x_0\| &\leq \|Ax - Ax_0\| + \|Ax_0 - x_0\| \\ &\leq \left\| \Gamma_0 \int_0^1 \left[F'(x_0) - F'(x_0 + t(x - x_0))\right](x - x_0)\, dt \right\| + \eta_0. \end{aligned}$$

Therefore, when $\|x - x_0\| \leq r_0$,

$$\|Ax - x_0\| \leq \frac{b_0 L r_0^2}{2} + \eta_0 = r_0.$$

Note that, by (2.2.33), the operator A satisfies a Lipschitz condition with constant $q = 1 - \sqrt{1 - 2h_0}$ (see also (1.3.3)).

The above analysis of the modified Newton-Kantorovich method relates the simplest case. More subtle arguments (see, e.g., Kantorovich and Akilov [67]) show that Theorem 2.2.7 remains valid if the sign $<$ in (2.2.17) is replaced by $\leq$.

If $D = U(x_0, R)$, consider the operator A defined by (2.2.31). Assume that the conditions of Theorem 2.2.6 hold, and set

$$\alpha(r) = \sup_{\|x - x_0\| \leq r} \|Ax - x_0\|. \tag{2.2.34}$$

The function $\alpha(r)$ is obviously continuous and nondecreasing. It was shown in the proof of Theorem 2.2.6 that

$$\|Ax - x_0\| \leq \frac{b_0 L \|x - x_0\|^2}{2} + \eta_0 \quad (\|x - x_0\| \leq R). \tag{2.2.35}$$

Hence it follows:

Lemma 2.2.8. *The function $\alpha(r)$ satisfies the inequality*

$$\alpha(r) \leq \frac{b_0 L r^2}{2} + \eta_0 \quad (r_0 \leq r \leq R).$$

Theorem 2.2.9. *If, under the assumptions of Theorem 2.2.4,*

$$r_0 = \frac{1-\sqrt{1-2h_0}}{h_0}\eta_0 \leq R < \frac{1+\sqrt{1-2h_0}}{h_0}\eta_0,$$

as the quadratic trinomial $\frac{1}{2}b_0 L r^2 - r + \eta_0$ is negative in the interval

$$\left(r_0, \frac{1+\sqrt{1-2h_0}}{h_0}\eta_0\right).$$

Remark 2.2.10. If one repeats the proofs of Theorem 2.2.4 and 2.2.7 using $F'(x_0)^{-1} F(x)$ instead of the operator $F(x)$, condition

$$\left\| F'(x_0)^{-1} \left(F'(x) - F'(y)\right) \right\| \leq \ell \|x-y\| \tag{2.2.36}$$

for all $x, y \in U(x_0, R)$ instead of (2.2.14), condition

$$h = \frac{1}{2}\ell\eta \tag{2.2.37}$$

instead of (2.2.17), and finally

$$s^* = \frac{1-\sqrt{1-2h}}{h}\eta, \quad U\left(x_0, s^*\right) \subseteq D \tag{2.2.38}$$

instead of (2.2.38) then the results hold in an affine invariant setting. The advantages of such an approach have elegantly been explained in [78] and also in [43]. From now on we shall be referring to (2.2.37) as the famous for its simplicity and clarity Newton-Kantorovich hypothesis.

Note that we are using for simplicity the same symbol ℓ to denote the Lipschitz constant in both conditions (2.2.14) and (2.2.36).

It also turns out from the proof of Theorem 2.2.4 that the scalar sequence $\{s_n\}$ $(n \geq 0)$ given by

$$s_0 = 0, \ s_1 = \eta, \ s_{n+2} = s_{n+1} + \frac{\ell\left(s_{n+1} - s_n\right)^2}{2\left(1 - \ell s_n\right)} \tag{2.2.39}$$

is a majorizing sequence for $\{x_n\}$ such that

$$0 \leq s_0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq s^* \tag{2.2.40}$$

and

$$\lim_{n\to\infty} s_n = s^*$$

[43], [67], [96].

Moreover the following error estimates hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n \tag{2.2.41}$$

and

$$\left\|x_n - x^*\right\| \leq s^* - s_n. \tag{2.2.42}$$

In the rest of the book motivated by optimization considerations using the same information (F, x_0, ℓ, η) we attempt to weaken crucial condition (2.2.37) and also provide a finer majorizing sequence than $\{s_n\}$. We also investigate in applications how is this effecting results by others based on (2.2.37).

To achieve all the above we introduce the center-Lipschitz condition

$$\left\|F'(x_0)^{-1}\left(F'(x) - F'(x_0)\right)\right\| \leq \ell_0 \|x - x_0\| \tag{2.2.43}$$

for all $x \in D$, where D is an open convex subset of X.

We also define scalar sequence $\{t_n\}$ by

$$t_0 = 0,\ t_1 = \eta,\ t_{n+2} = t_{n+1} + \frac{\ell\,(t_{n+1} - t_n)^2}{2\,(1 - \ell_0 t_{n+1})}\quad (n \geq 0). \tag{2.2.44}$$

In [24] we showed:

Theorem 2.2.11. *Let $F\colon D \subseteq X \to Y$ be a Fréchet-differentiable operator and for $x_0 \in D$, assume*

$$F'(x_0)^{-1} \in L(Y, X); \tag{2.2.45}$$

conditions (2.2.36), (2.2.43), and

$$\bar{U}(x_0, t^*) \subseteq D \tag{2.2.46}$$

hold, where

$$t^* = \lim_{n\to\infty} t_n. \tag{2.2.47}$$

Moreover, assume that the following conditions hold:

$$h_\delta = (\delta \ell_0 + \ell)\,\eta \leq \delta,\ \textit{for}\ \delta \in [0, 1] \tag{2.2.48}$$

or

$$h_\delta \leq \delta,\ \frac{2\ell_0\eta}{2-\delta} \leq 1,\ \frac{\ell_0\delta^2}{2-\delta} \leq \ell,\ \textit{for}\ \delta \in [0, 2) \tag{2.2.49}$$

or

$$h_\delta \leq \delta,\ \ell_0\eta \leq 1 - \frac{1}{2}\delta,\ \textit{for}\ \delta \in [\delta_0, 2)\,, \tag{2.2.50}$$

where

$$\delta_0 = \frac{-\frac{\ell}{\ell_0} + \sqrt{\left(\frac{\ell}{\ell_0}\right)^2 + 8\frac{\ell}{\ell_0}}}{2}\quad (\ell_0 \neq 0) \tag{2.2.51}$$

Then sequence $\{x_n\}$ *generated by the NK method (2.1.3) is well defined, remains in* $\bar{U}(x_0, t^*)$ *for all* $n \geq 0$ *and converges to a solution* $x^* \in \bar{U}(x_0, t^*)$ *of equation* $F(x) = 0$.

Moreover the following error estimates hold for all $n \geq 0$*:*

$$\|x_{n+2} - x_{n+1}\| \leq \frac{\ell \|x_{n+1} - x_n\|^2}{2\left[1 - \ell_0 \|x_{n+1} - x_0\|\right]} \leq t_{n+2} - t_{n+1} \tag{2.2.52}$$

and

$$\left\|x_n - x^*\right\| \leq t^* - t_n, \tag{2.2.53}$$

where t_n, t^* *are given by (2.2.44) and (2.2.47) respectively.*

Furthermore, if there exists $t^{**} \geq t^*$ *such that*

$$U\left(x_0, t^{**}\right) \in D \tag{2.2.54}$$

and

$$\ell_0\left(t^* + t^{**}\right) \leq 2, \tag{2.2.55}$$

the solution x^* *is unique in* $U(x_0, t^{**})$.

Note that optimum condition is given by (2.2.50) for $\delta = \delta_0$. However, we will be mostly using condition (2.2.48) for $\delta = 1$, which is the simplest, in the rest of this book.

We now compare our results with the ones obtained in Theorem 2.2.4 for the NK method (2.1.3).

Remark 2.2.12. Let us set $\delta = 1$ in condition (2.2.48). That is, consider

$$h_1 = (\ell + \ell_0) \leq 1. \tag{2.2.56}$$

Although (2.2.56) is not the weakest condition among (2.2.48)–(2.2.50) we will only compare this one with condition (2.2.37), since it seems to be the simplest.

(a) Note that

$$\ell_0 \leq \ell \tag{2.2.57}$$

holds in general and $\frac{\ell}{\ell_0}$ can be arbitrarily large as the following example indicates:

Example 2.2.13. Let $X = Y = D = \mathbf{R}$, $x_0 = 0$ and define function F on D by

$$F(x) = c_0 + c_1 x + c_2 \sin e^{c_3 x} \tag{2.2.58}$$

where c_i, $i = 0, 1, 2, 3$ are given parameters. Then it can easily be seen that for c_3 large and c_2 sufficiently small, $\frac{\ell}{\ell_0}$ can be arbitrarily large.

(b) We have

$$h \leq \frac{1}{2} \Rightarrow h_1 \leq 1 \tag{2.2.59}$$

but not vice versa unless if $\ell = \ell_0$. Indeed, for $\ell = \ell_0$, the NK theorem 2.2.4 is a special case of our Theorem 2.2.11. Otherwise our Theorem 2.2.11 can double the applicability of the NK theorem 2.2.4 as $\ell_0 \in [0, \ell]$:

Example 2.2.14. Let $X = Y = \mathbf{R}$, $D = [a, 2-a]$, $a \in \left[0, \frac{1}{2}\right)$, $x_0 = 1$, and define function F on D by

$$F(x) = x^3 - a. \tag{2.2.60}$$

Using (2.2.16), (2.2.36), and (2.2.43), we obtain

$$\eta = \frac{1}{3}(1-a), \;\; \ell = 2(2-a) > \ell_0 = 3 - a. \tag{2.2.61}$$

The Newton-Kantorovich hypothesis (2.2.37) cannot hold since

$$h = \frac{2}{3}(1-a)(2-a) > \frac{1}{2}, \text{ for all } a \in \left[0, \frac{1}{2}\right). \tag{2.2.62}$$

That is, there is no guarantee that NK method (2.1.3) converges to a solution of equation $F(x) = 0$.

However our condition (2.2.56), which becomes

$$h_1 = \frac{1}{3}(1-a)[3 - a + 2(2-a)] \le 1, \tag{2.2.63}$$

holds for all $a \in \left[\frac{5-\sqrt{13}}{3}, \frac{1}{2}\right)$.

In fact we can do better if we use (2.2.50) for

$$.4505 < \frac{5-\sqrt{13}}{3} = .464816242...,$$

as we get $\eta = .183166...$, $\ell_0 = 2.5495$, $\ell = 3.099$, and $\delta_0 = 1.0656867$.

Choose $\delta = \delta_0$. Then we get that the interval $\left[\frac{5-\sqrt{13}}{3}, \frac{1}{2}\right)$ can be extended to $\left[.450339002, \frac{1}{2}\right)$.

(c) Using simple induction (see [24]) we showed:

$$t_n \le s_n \tag{2.2.64}$$

$$t_{n+1} - t_n \le s_{n+1} - s_n \tag{2.2.65}$$

$$t^* \le s^* = \lim_{n\to\infty} s_n \tag{2.2.66}$$

and

$$t^* - t_n \le s^* - s_n. \tag{2.2.67}$$

Note also that strict inequality holds in (2.2.64) and (2.2.65) if $\ell_0 < \ell$.

That is, in this case our error estimates are finer and the information on the location of the solution at least as precise.

Note that all the above advantages are obtained using the same information and with the same computational cost since in practice the evaluation of ℓ requires the evaluation of ℓ_0.

We now compare our results with the ones obtained in Theorem 2.2.7 for the MNK method (2.1.5).

Remark 2.2.15. (a) Conditions (2.2.36), (2.2.37), and (2.2.38) can be replaced by

$$h^0 = \ell_0 \eta \leq \frac{1}{2}, \tag{2.2.68}$$

$$s_0^* = \frac{1 - \sqrt{1 - 2h^0}}{h^0} \eta, \tag{2.2.69}$$

and (2.2.43), respectively.

Indeed the proof of Theorem 2.2.7 can simply be rewritten with the above changes as condition (2.2.36) is never used full strength. This observation is important in computational mathematics for the following reasons: condition (2.2.68) is weaker than (2.2.37) if $\ell_0 < \ell$. That increases the applicability of Theorem 2.2.7 (see also Example 2.2.13); the error estimates are finer since the ratio q becomes smaller for $\ell_0 < \ell$; it is easier to compute ℓ_0 than computing ℓ (see also the three examples that follow). Finally by comparing (2.2.38) with (2.2.69) for $\ell_0 < \ell$ we obtain

$$s_0^* < s^*. \tag{2.2.70}$$

That is, we also obtain a more precise information on the location of the solution s^*.

Example 2.2.16. Returning back to Example 2.2.14 we see that Theorem 2.2.7 cannot be applied as condition (2.2.37) is violated. However, our condition (2.2.68) which becomes

$$h^0 = \frac{1}{3}(1 - a)(3 - a) \leq \frac{1}{2} \tag{2.2.71}$$

holds for $a \in \left[\frac{4 - \sqrt{10}}{2}, \frac{1}{2}\right)$.

Our motivation for introducing condition (2.2.39) instead of (2.2.36) can also be seen in the following example:

Example 2.2.17. Let $X = Y = \mathbf{R}$, $D = [0, \infty)$, $x_0 = 1$ and define function F on D by

$$F(x) = \frac{x^{1 + \frac{1}{i}}}{1 + \frac{1}{i}} + c_1 x + c_2, \tag{2.2.72}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{\frac{1}{i}} + c_1$ is not Lipschitz on D. However, center-Lipschitz condition (2.2.43) holds for $\ell_0 = (1 + c_1)^{-1}$ $(c_1 \neq -1)$.

Indeed, we have

$$\begin{aligned} \left\| F'(x_0)^{-1}\left[F'(x) - F'(x_0)\right]\right\| &= (1 + c_1)^{-1}\left|x^{\frac{1}{i}} - x_0^{\frac{1}{i}}\right| \\ &= \frac{(1 + c_1)^{-1}|x - x_0|}{x_0^{\frac{i-1}{i}} + \dots + x^{\frac{i-1}{i}}} \\ &\leq \ell_0 |x - x_0|. \end{aligned} \tag{2.2.73}$$

Example 2.2.18. We consider the integral equation

$$u(s) = f(s) + \lambda \int_a^b G(s,t)\, u(t)^{1+\frac{1}{n}}\, dt, \quad n \in \mathbf{N}. \tag{2.2.74}$$

Here, f is a given continuous function satisfying $f(s) > 0$, $s \in [a,b]$, λ is a real number, and the kernel G is continuous and positive in $[a,b] \times [a,b]$.

For example, when $G(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$\begin{aligned} u'' &= \lambda u^{1+\frac{1}{n}}, \\ u(a) &= f(a), \; u(b) = f(b). \end{aligned}$$

These type of problems have been considered in [71].

Equations of the form (2.2.74) generalize equations of the form

$$u(s) = \int_a^b G(s,t)\, u(t)^n\, dt \tag{2.2.75}$$

studied in [45].

Instead of (2.2.74), we can try to solve the equation $F(u) = 0$ where

$$F \colon \Omega \subseteq C[a,b] \to C[a,b], \quad \Omega = \{u \in C[a,b] : u(s) \geq 0, s \in [a,b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s,t)\, u(t)^{1+\frac{1}{n}}\, dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)\, v(s) = v(s) - \lambda \left(1 + \frac{1}{n}\right) \int_a^b G(s,t)\, u(t)^{\frac{1}{n}}\, v(t)\, dt, \; v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1]$, $G(s,t) = 1$ and $y(t) = 0$. Then $F'(y)\, v(s) = v(s)$ and

$$\left\| F'(x) - F'(y) \right\| = |\lambda| \left(1 + \frac{1}{n}\right) \int_0^1 x(t)^{\frac{1}{n}}\, dt.$$

If F' were a Lipschitz function, then

$$\left\| F'(x) - F'(y) \right\| \leq L_1 \|x - y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{\frac{1}{n}}\, dt \leq L_2 \max_{x \in [0,1]} x(s), \tag{2.2.76}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \quad j \geq 1, \quad t \in [0,1].$$

If these are substituted into (2.2.76)

$$\frac{1}{j^{1/n}\left(1+\frac{1}{n}\right)} \leq \frac{L_2}{j} \Longleftrightarrow j^{1-1/n} \leq L_2\left(1+\frac{1}{n}\right), \quad \forall j \geq 1.$$

This inequality is not true when $j \to \infty$.

Therefore, condition (2.2.36) fails in this case. However, condition (2.2.43) holds. To show this, let $x_0(t) = f(t)$ and $\alpha = \min_{s\in[a,b]} f(s)$, $\alpha > 0$. Then, for $v \in \Omega$,

$$\begin{aligned}
&\left\|\left[F'(x) - F'(x_0)\right]v\right\| \\
&\quad = |\lambda|\left(1+\frac{1}{n}\right)\max_{s\in[a,b]}\left|\int_a^b G(s,t)\left(x(t)^{\frac{1}{n}} - f(t)^{\frac{1}{n}}\right)v(t)\,dt\right| \\
&\quad \leq |\lambda|\left(1+\frac{1}{n}\right) \\
&\qquad \cdot \int_a^b \max_{s\in[a,b]} \frac{G(s,t)\,|x(t) - f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n} f(t)^{1/n} + \ldots + f(t)^{(n-1)/n}}\,dt\,\|v\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|F'(x) - F'(x_0)\right\| &\leq \frac{|\lambda|\left(1+\frac{1}{n}\right)}{\alpha^{(n-1)/n}}\max_{s\in[a,b]}\int_a^b G(s,t)\,dt\,\|x - x_0\| \\
&\leq K\|x - x_0\|,
\end{aligned}$$

where $K = \frac{|\lambda|\left(1+\frac{1}{n}\right)}{\alpha^{(n-1)/n}}N$ and $N = \max_{s\in[a,b]}\int_a^b G(s,t)\,dt$.

Set $\ell_0 = \left\|F'(x_0)^{-1}K\right\|$. Then condition (2.2.68) holds for sufficiently small λ.

Remark 2.2.19. (a) We showed above that although the convergence of NK method (2.1.5) is quadratic (for $h < \frac{1}{2}$) there are cases when MNK method is preferred over the NK method.

(b) Although $t^* \in [\eta, 2\eta]$ say if condition (2.2.56) holds, we do not have an explicit form for it like, e.g., (2.2.38).

In practice though we can handle this problem in several ways. It follows from (2.2.56) that condition (2.2.68) also holds. Therefore we know that the solution x^* is unique in $U\left(x_0, s_0^*\right)$. That is, there exists a finite $n_0 \geq 1$ such that if $n \geq n_0$ the sequence $\{x_n\}$ will enter the ball $U\left(x_0, s_0^*\right)$ and enjoy quadratic convergence according to Theorem 2.2.11. Note that if $t^* \leq s_0^*$ then we can take $n_0 = 1$. Moreover, if (2.2.37) also holds, then $t^* \in \left[\eta, s^*\right]$, with $s^* \leq 2\eta$.

2.3 New sufficient conditions for the secant method

It turns out that the ideas introduced in Section 2.2 for Newton's method can be extended to the method of chord or the secant method.

In this section, we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{2.3.1}$$

where F is a nonlinear operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

We consider the secant method in the form

$$x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1} F(x_n) \quad (n \geq 0), \tag{2.3.2}$$

where $\delta F(x, y) \in L(X, Y)$ $(x, y \in D)$ is a consistent approximation of the Fréchet derivative of F, Dennis [74], Potra [162], Argyros [12], [43], Hernandez [116], [117], and others have provided sufficient convergence conditions for the secant method based on "Lipschitz-type" conditions on δF (see also Section 1.2). Here using "Lipschitz-type" and center-"Lipschitz-type" conditions, we provide a semilocal convergence analysis for (2.3.2). It turns out that our error bounds are more precise and our convergence conditions hold in cases where the corresponding hypotheses mentioned in earlier references mentioned above are violated.

We need the following result on majorizing sequences.

Lemma 2.3.1. *Assume there exist nonnegative parameters ℓ, ℓ_0, η, c, and $a \in [0, 1]$,*

$$\delta \in \begin{cases} \left[0, \dfrac{-1 + \sqrt{1 + 4a}}{2a}\right], & a \neq 0 \\ [0, 1), & a = 0 \end{cases} \tag{2.3.3}$$

such that:

$$(\ell + \delta\ell_0)(c + \eta) \leq \delta, \tag{2.3.4}$$

$$\eta \leq \delta c, \tag{2.3.5}$$

$$\ell_0 \leq a\ell. \tag{2.3.6}$$

Then,

(a) iteration $\{t_n\}$ $(n \geq -1)$ given by

$$t_{-1} = 0, \quad t_0 = c, \quad t_1 = c + \eta,$$
$$t_{n+2} = t_{n+1} + \frac{\ell(t_{n+1} - t_{n-1})}{1 - \ell_0 \left[t_{n+1} - t_0 + t_n\right]}(t_{n+1} - t_n) \quad (n \geq 0) \tag{2.3.7}$$

is nondecreasing, bounded above by

$$t^{**} = \frac{\eta}{1 - \delta} + c \tag{2.3.8}$$

and converges to some t^* *such that*

$$0 \le t^* \le t^{**}. \tag{2.3.9}$$

Moreover, the following estimates hold for all $n \ge 0$

$$0 \le t_{n+2} - t_{n+1} \le \delta(t_{n+1} - t_n) \le \delta^{n+1}\eta. \tag{2.3.10}$$

(b) Iteration $\{s_n\}$ $(n \ge 0)$ *given by*

$$\begin{aligned} &s_{-1} - s_0 = c, \quad s_0 - s_1 = \eta, \\ &s_{n+1} - s_{n+2} = \frac{\ell(s_{n-1} - s_{n+1})}{1 - \ell_0\left[(s_0 + s_{-1}) - (s_n + s_{n+1})\right]}(s_n - s_{n+1}) \quad (n \ge 0) \end{aligned} \tag{2.3.11}$$

for s_{-1}, s_0, $s_1 \ge 0$ *is nonincreasing, bounded below by*

$$s^{**} = s_0 - \frac{\eta}{1-\delta} \tag{2.3.12}$$

and converges to some s^* *such that*

$$0 \le s^{**} \le s^*. \tag{2.3.13}$$

Moreover, the following estimates hold for all $n \ge 0$

$$0 \le s_{n+1} - s_{n+2} \le \delta(s_n - s_{n+1}) \le \delta^{n+1}\eta. \tag{2.3.14}$$

Proof. (a) The result clearly holds if $\delta = 0$ or $\ell = 0$ or $c = 0$. Let us assume $\delta \ne 0$, $\ell \ne 0$ and $c \ne 0$. We must show for all $k \ge 0$:

$$\ell(t_{k+1} - t_{k-1}) + \delta\ell_0\left[(t_{k+1} - t_0) + t_k\right] \le \delta, \;\; 1 - \ell_0\left[(t_{k+1} - t_0) + t_k\right] > 0. \tag{2.3.15}$$

Inequalities (2.3.15) hold for $k = 0$ by the initial conditions. But then (2.3.7) gives

$$0 \le t_2 - t_1 \le \delta(t_1 - t_0).$$

Let us assume (2.3.10) and (2.3.15) hold for all $k \le n+1$. By the induction hypotheses we can have in turn:

$$\begin{aligned} &\ell(t_{k+2} - t_k) + \delta\ell_0\left[(t_{k+2} - t_0) + t_{k+1}\right] \\ &\quad \le \ell\left[(t_{k+2} - t_{k+1}) + (t_{k+1} - t_k)\right] + \delta\ell_0\left[\tfrac{1-\delta^{k+2}}{1-\delta} + \tfrac{1-\delta^{k+1}}{1-\delta}\right]\eta + \delta\ell_0 c \\ &\quad \le \ell(\delta^{k+1} + \delta^k)\eta + \tfrac{\delta\ell_0}{1-\delta}(2 - \delta^{k+1} - \delta^{k+2})\eta + \delta\ell_0 c. \end{aligned} \tag{2.3.16}$$

We must show that δ is the upper bound in (2.3.16). Instead by (2.3.5) we can show

$$\ell\delta^k(1+\delta)\eta + \tfrac{\delta\ell_0}{1-\delta}(2 - \delta^{k+2} - \delta^{k+1})\eta + \delta\ell_0 c \le (\ell + \delta\ell_0)(c + \eta)$$

or

$$\delta\ell_0\left[\frac{2-\delta^{k+2}-\delta^{k+1}}{1-\delta}-1\right]\eta \le \ell\left[c+\eta-\delta^k(1+\delta)\eta\right]$$

or

$$a\delta\ell\frac{1+\delta-\delta^{k+1}(1+\delta)}{1-\delta}\eta \le \ell\left[\frac{\eta}{\delta}+\eta-\delta^k(1+\delta)\eta\right]$$

or

$$a\delta^2(1+\delta)(1-\delta^{k+1}) \le (1-\delta)(1+\delta)(1-\delta^{k+1})$$

or

$$a\delta^2+\delta-1\le 0,$$

which is true by the choice of δ. Moreover, by (2.3.5) and (2.3.10)

$$\begin{aligned}\delta\ell_0\left[(t_{k+2}-t_0)+t_{k+1}\right] &\le \frac{\delta\ell_0}{1-\delta}(2-\delta^{k+2}-\delta^{k+1})\eta+\delta\ell oc\\ &< (\ell+\delta\ell_0)(c+\eta)\le\delta, \end{aligned}\tag{2.3.17}$$

which shows the second inequality in (2.3.15). We must also show:

$$t_k \le t^{**} \quad (k\ge -1). \tag{2.3.18}$$

For $k=-1,0,1,2$ we have $t_{-1}=0\le t^{**}$, $t_0=\eta\le t^{**}$, $t_1=\eta+c\le t^{**}$ by (2.3.8), and $t_2=c+\eta+\delta\eta=c+(1+\delta)\eta\le t^{**}$ by the choice of δ. Assume (2.3.18) holds for all $k\le n+1$. It follows from (2.3.10)

$$\begin{aligned}t_{k+2} &\le t_{k+1}+\delta(t_{k+1}-t_k)\le t_k+\delta(t_k-t_{k-1})+\delta(t_{k+1}-t_k)\\ &\le\cdots\le t_1+\delta(t_1-t_0)+\cdots+\delta(t_{k+1}-t_k)\\ &\le c+\eta+\delta\eta+\cdots+\delta^{k+1}\eta=c+\frac{1-\delta^{k+2}}{1-\delta}\eta\\ &< \frac{\eta}{1-\delta}+c=t^{**}.\end{aligned}$$

That is $\{t_n\}$ $(n\ge -1)$ is bounded above by t^{**}. It also follows from (2.3.7) and (2.3.15) that it is also nondecreasing and as such it converges to some t^* satisfying (2.3.9).

(b) As in part (a) but we show $\{s_n\}$ $(n\ge -1)$ is nonincreasing and bounded below by s^{**}. Note that the inequality corresponding with (2.3.16) is

$$\ell(s_k-s_{k+2})\le\delta\left[1-\beta(s_0+s_{-1})+\beta(s_{k+1}+s_{k+2})\right]$$

or

$$\begin{aligned}&\ell\left[\delta^k(s_0-s_1)+\delta^{k+1}(s_0-s_1)\right]\\ &\quad\le\ \delta\left[1-\ell_0(s_0+s_{-1})+\ell_0\left(s_0-\frac{1-\delta^{k+1}}{1-\delta}(s_0-s_1)\right)+\ell\left(s_0-\frac{1-\delta^{k+2}}{1-\delta}(s_0-s_1)\right)\right]\end{aligned}$$

or

$$\ell\delta^k(1+\delta)\eta+\delta\ell_0\left[\frac{2-\delta^{k+1}-\delta^{k+2}}{1-\delta}\eta+c\right]$$

must be bounded above by δ which was shown in part (a).

Remark 2.3.2. It follows from (2.3.16) and (2.3.17) that the conclusions of Lemma 2.3.1 hold if (2.3.3), (2.3.5), (2.3.6) are replaced by the weaker conditions:

for all $n \geq 0$ there exists $\delta \in [0, 1)$ such that:

$$\ell\delta^n(1+\delta)\eta + \tfrac{\delta\ell_0}{1-\delta}(2-\delta^{n+2}-\delta^{n+1})\eta + \delta\ell_0 c \leq \delta,$$

and

$$\tfrac{\delta\ell_0}{1-\delta}(2-\delta^{n+2}-\delta^{n+1})\eta + \delta\ell oc < 1.$$

The above conditions hold in many cases for all $n \geq 0$. One such stronger case is

$$\ell(1+\delta)\eta + \tfrac{2\delta\ell_0\eta}{1-\delta} + \delta\ell_0 c \leq \delta,$$

and

$$\tfrac{2\delta\ell_0\eta}{1-\delta} + \delta\ell_0 c < 1.$$

We shall study the iterative procedure (2.3.2) for triplets (F, x_{-1}, x_0) belonging to the class $C(\ell, \ell_0, \eta, c)$ defined as follows:

Definition 2.3.3. *Let ℓ, ℓ_0, η, c be nonnegative parameters satisfying the hypotheses of Lemma 2.3.1 or Remark 2.3.2 (including (2.3.4)).*

We say that a triplet (F, x_{-1}, x_0) belongs to the class $C(\ell, \ell_0, \eta, c)$ if:

(c_1) F is a nonlinear operator defined on a convex subset D of a Banach space X with values in a Banach space Y;

(c_2) x_{-1} and x_0 are two points belonging to the interior D^0 of D and satisfying the inequality

$$\|x_0 - x_{-1}\| \leq c; \tag{2.3.19}$$

(c_3) F is Fréchet-differentiable on D^0 and there exists an operator $\delta F\colon D^0 \times D^0 \to L(X, Y)$ such that:

the linear operator $A = \delta F(x_{-1}, x_0)$ is invertible, its inverse A^{-1} is bounded and:

$$\|A^{-1}F(x_0)\| \leq \eta; \tag{2.3.20}$$

$$\|A\left[\delta F(x, y) - F'(z)\right]\| \leq \ell(\|x - z\| + \|y - z\|), \tag{2.3.21}$$

$$\|A\left[\delta F(x, y) - F'(x_0)\right]\| \leq \ell_0(\|x - x_0\| + \|y - x_0\|) \tag{2.3.22}$$

for all $x, y, z \in D$.

(c_4) the set $D_c = \{x \in D;\ F$ is continuous at $x\}$ contains the closed ball $\bar{U}(x_0, s^*)$ where s^* is given in Lemma 2.3.1.

We present the following semilocal convergence theorem for secant method (2.3.2).

Theorem 2.3.4. *If $(F, x_{-1}, x_0) \in C(\ell, \ell_0, \eta, c)$ then sequence $\{x_n\}$ $(n \geq -1)$ generated by secant method (2.3.2) is well defined, remains in $\bar{U}(x_0, s^*)$ for all $n \geq 0$ and converges to a solution $x^* \in \bar{U}(x_0, s^*)$ of equation $F(x) = 0$.*

Moreover the following estimates hold for all $n \geq 0$

$$\|x_{n+2} - x_{n+1}\| \leq s_{n+1} - s_{n+2}, \tag{2.3.23}$$

$$\|x_n - x^*\| \leq \alpha_n \tag{2.3.24}$$

and

$$\|x_n - x^*\| \geq \beta_n \tag{2.3.25}$$

where,

$$s_{-1} = \frac{1 + \ell_0 c}{2\ell_0}, \quad s_0 = \frac{1 - \ell_0 c}{2\ell_0} \text{ for } \ell_0 \neq 0, \tag{2.3.26}$$

sequence $\{s_n\}$ $(n \geq 0)$ given by (2.3.11), α_n, β_n are respectively the nonnegative solutions of equations

$$\ell_0 t^2 - 2\ell_0(s_0 - \|x_n - x_0\|)t - \ell(\|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\|)\|x_n - x_{n-1}\| = 0, \tag{2.3.27}$$

and

$$\begin{aligned} \ell t^2 &+ \left[\ell\|x_n - x_{n-1}\| + 1 - \ell_0(\|x_n - x_0\| + \|x_{n-1} - x_0\| + c)\right]t \\ &+ \left[\ell_0(\|x_n - x_0\| + \|x_{n-1} - x_0\| + c) - 1\right]\|x_{n+1} - x_n\| = 0. \end{aligned} \tag{2.3.28}$$

Proof. We first show operator $L = \delta F(u, v)$ is invertible for all $u, v \in D^0$ with

$$\|u - x_0\| + \|v - x_0\| < 2s_0. \tag{2.3.29}$$

It follows from (2.3.22) and (2.3.29)

$$\begin{aligned} \|I - A^{-1}L\| &= \|A^{-1}(L - A)\| \leq \|A^{-1}(L - F'(x_0))\| + \|A^{-1}(F'(x_0) - A)\| \\ &\leq \ell_0(\|u - x_0\| + \|v - x_0\| + \|x_0 - x_{-1}\|) < 1. \end{aligned} \tag{2.3.30}$$

According to the Banach Lemma on invertible operators and (2.3.30), L is invertible and

$$\|L^{-1}A\| \leq [1 - \ell_0(\|u - x_0\| + \|v - x_0\| + c)]^{-1}. \tag{2.3.31}$$

Condition (2.3.21) implies the Lipschitz condition for F'

$$\|A^{-1}(F'(u) - F'(v))\| \leq 2\ell\|u - v\|, \quad u, v \in D^0. \tag{2.3.32}$$

By the identity,

$$F(x) - F(y) = \int_0^1 F'(y + t(x - y))dt(x - y) \tag{2.3.33}$$

we get

$$\|A_0^{-1}\left[F(x) - F(y) - F'(u)(x - y)\right]\| \leq \ell(\|x - u\| + \|y - u\|)\|x - y\| \tag{2.3.34}$$

and

$$\|A_0^{-1}[F(x)-F(y)-\delta F(u,v)(x-y)]\| \le \ell(\|x-v\|+\|y-v\|+\|u-v\|)\|x-y\| \tag{2.3.35}$$

for all $x, y, u, v \in D^0$. By a continuity argument (2.3.33)–(2.3.35) remain valid if x and/or y belong to D_c.

We first show (2.3.23). If (2.3.23) hold for all $\eta \le k$ and if $\{x_n\}$ $(n \ge 0)$ is well defined for $n = 0, 1, 2, \ldots, k$ then

$$\|x_0 - x_n\| \le s_0 - s_n < s_0 - s^*, \quad n \le k. \tag{2.3.36}$$

Hence (2.3.29) holds for $u = x_i$ and $v = x_j$ $(i, j \le k)$. That is (2.3.2) is well defined for $n = k+1$. For $n = -1$ and $n = 0$, (2.3.23) reduces to $\|x_{-1} - x_0\| \le c$ and $\|x_0 - x_1\| \le \eta$. Suppose (2.3.23) holds for $n = -1, 0, 1, \ldots, k$ $(k \ge 0)$. Using (2.3.31), (2.3.35) and

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - \delta F(x_{k-1}, x_k)(x_{k+1} - x_k) \tag{2.3.37}$$

we obtain in turn

$$\begin{aligned}\|x_{k+2} - x_{k+1}\| &= \|\delta F(x_k, x_{k+1})^{-1} F(x_{k+1})\| \\ &\le \|\delta F(x_k, x_{k+1})^{-1} A\| \, \|A^{-1} F(x_{k+1})\| \\ &\le \tfrac{\ell(\|x_{k+1}-x_k\|+\|x_k-x_{k-1}\|)}{1-\ell_0[\|x_{k+1}-x_0\|+\|x_k-x_0\|+c]} \|x_{k+1} - x_k\| \\ &\le \tfrac{\ell(s_k-s_{k+1}+s_{k-1}-s_k)}{1-\ell_0[s_0-s_{k+1}+s_0-s_k+s_{-1}-s_0]} (s_k - s_{k+1}) = s_{k+1} - s_{k+2}. \end{aligned} \tag{2.3.38}$$

The induction for (2.3.23) is now complete. It follows from (2.3.23) and Lemma 2.3.1 that sequence $\{x_n\}$ $(n \ge -1)$ is Cauchy in a Banach space X and as such it converges to some $x^* \in \bar{U}(x_0, s^*)$ (as $\bar{U}(x_0, s^*)$ is a closed set) so that

$$\|x_n - x^*\| \le s_n - s^*. \tag{2.3.39}$$

By letting $k \to \infty$ in (2.3.38), we obtain $F(x^*) = 0$.

Set $x = x_n$ and $y = x^*$ in (2.3.33), $M = \int_0^1 F'(x^* + t(x_n - x^*))dt$. Using (2.3.23) and (2.3.39) we get in turn

$$\begin{aligned}\|x_n - x_0\| + \|x^* - x_0\| + \|x_0 - x_{-1}\| &\le 2\|x_n - x_0\| + \|x_n - x^*\| + c \\ &< 2(\|x_n - x_0\| + \|x_n - x^*\|) \\ &\le 2(s_0 - s_n + s_n - s^*) + c \le 2s_0 + c = \frac{1}{\ell_0}. \end{aligned} \tag{2.3.40}$$

By (2.3.40) and the Banach Lemma on invertible operators we get

$$\|M^{-1}A\| \le \left[1 - \ell_0(2\|x_n - x_0\| + \|x_n - x^*\| + c)\right]^{-1}. \tag{2.3.41}$$

It follows from (2.3.2) and (2.3.41)

$$\begin{aligned}\|x_n - x^*\| &\le \|M^{-1}A\| \cdot \|A^{-1}F(x_n)\| \\ &\le \tfrac{\ell[\|x_n-x_{n-1}\|+\|x_{n-1}-x_{n-2}\|]}{1-\ell_0[2\|x_n-x_0\|+\|x_n-x^*\|+c]} \|x_n - x_{n-1}\|, \end{aligned} \tag{2.3.42}$$

which shows (2.3.24).

Using the approximation

$$x_{n+1} - x^* = x^* - x_n + [A\delta F(x_{n-1}, x_n)]^{-1} \\ \cdot A\,[F(x^*) - F(x_n) - \delta F(x_{n-1}, x_n)(x^* - x_n)] \quad (2.3.43)$$

and estimates (2.3.30), (2.3.35) we get

$$\|x_{n+1} - x_n\| \leq \frac{\ell[\|x^* - x_n\| + \|x_n - x_{n-1}\|]}{1 - \ell_0[\|x_n - x_0\| + \|x_{n-1} + x_0\| + c]}\|x_n - x^*\| + \|x_n - x^*\|, \quad (2.3.44)$$

which shows (2.3.25).

In the next result we examine the uniqueness of the solution x^*.

Theorem 2.3.5. *If $(F, x_{-1}, x_0) \in C(\ell, \ell_0, \eta, c)$ equation (2.3.1) has a solution $x^* \in \bar{U}(x_0, s^*)$. This solution is unique in the set $U_1 = \{x \in D_c \mid \|x - x_0\| < s_0 + \gamma\}$ if $\gamma > 0$ or in the set $U_2 = \{x \in D_c \mid \|x - x_0\| \leq s_0\}$ if $\gamma = 0$.*

Proof. *Case 1:* $\gamma > 0$. Let $x^* \in \bar{U}(x_0, s^*)$ and $y^* \in U_1$ be solutions of equation $F(x) = 0$. Set $P = \int_0^1 F'(y + t(x - y))dt$. Using (2.3.22) we get

$$\|I - A^{-1}P\| = \|A^{-1}(A - P)\| \leq \ell_0(\|y^* - x_0\| + \|x^* - x_0\| + \|x_0 - x_{-1}\|) \\ < \ell_0(s_0 + \gamma + s_0 - \gamma + c) = 1.$$

Hence, P is invertible and from (2.3.33) we get $x^* = y^*$.

Case 2: $\gamma = 0$. Consider the modified secant method

$$s_{n+1} = s_n - A^{-1}F(y_n) \quad (n \geq 0). \quad (2.3.45)$$

By Theorem 2.3.4 sequence $\{y_n\}$ $(n \geq 0)$ converges to x^* and

$$\|x_n - x_{n+1}\| \leq \bar{s}_n - \bar{s}_{n+1} \quad (2.3.46)$$

where,

$$\bar{s}_0 = \sqrt{\tfrac{n}{\ell}}, \quad \bar{s}_{n+1} = \bar{s}_n - \ell s_n^2 \quad (n \geq 0), \quad \text{for } \ell > 0. \quad (2.3.47)$$

Using induction on $n \geq 0$ we get

$$\bar{s}_n \geq \frac{\sqrt{\frac{\eta}{\ell}}}{n+1} \quad (n \geq 0). \quad (2.3.48)$$

Let y^* be a solution of $F(x) = 0$. Set $P_n = \int_0^1 F'(y^* + t(x_n - y^*))dt$. It follows from (2.3.22), (2.3.33), (2.3.45), and (2.3.48)

$$\|x_{n+1} - y^*\| = \|A^{-1}(A - P_n)(x_n - y^*)\| \\ \leq \ell(\|y^* - x_0\| + \|x_n - x_0\| + \|x_0 - x_{-1}\|)\|x_n - y^*\| \\ \leq (1 - \ell\bar{s}_n)\|x_n - y^*\| \leq \cdots \leq \prod_{i=1}^{n}(1 - \ell\bar{s}_i)\|x_1 - y^*\|. \quad (2.3.49)$$

By (2.3.49), we get $\lim\limits_{n\to\infty} \prod_{i=1}^{n}(1 - \ell\bar{s}_i) = 0$. Hence, we deduce $x^* = y^*$.

That completes the proof of the theorem.

Remark 2.3.6. The parameter s^* can be computed as the limit of sequence $\{s_n\}$ $(n \geq -1)$ using (2.3.11). Simply set

$$s^* = \lim_{n \to \infty} s_n. \tag{2.3.50}$$

Remark 2.3.7. A similar convergence analysis can be provided if sequence $\{s_n\}$ is replaced by $\{t_n\}$. Indeed under the hypotheses of Theorem 2.3.4 we have for all $n \geq 0$

$$\|x_{n+2} - x_{n+1}\| \leq t_{n+2} - t_{n+1} \tag{2.3.51}$$

and

$$\|x^* - x_n\| \leq t^* - t_n. \tag{2.3.52}$$

In order for us to compare with earlier results we first need the definition:

Definition 2.3.8. *Let ℓ, η, c be three nonnegative numbers satisfying the inequality*

$$\ell c + 2\sqrt{\ell\eta} \leq 1. \tag{2.3.53}$$

We say that a triplet $(F, x_{-1}, x_0) \in C_1(\ell, \eta, c)$ $(\ell > 0$ if conditions (c_1)–(c_4) hold (excluding (2.3.22)). Define iteration $\{p_n\}$ $(n \geq -1)$ by

$$p_{-1} = \tfrac{1+\ell c}{2\ell}, \quad p_0 = \tfrac{1-\ell c}{2\ell}, \quad p_{n+1} = p_n - \tfrac{p_n^2 - p^2}{p_n + p_{n-1}}, \tag{2.3.54}$$

where,

$$p = \tfrac{1}{2\ell}\sqrt{(1-\ell c)^2 - 4\ell\eta}\,. \tag{2.3.55}$$

The proof of the following semilocal convergence theorem can be found in [164].

Theorem 2.3.9. *If $(F, x_{-1}, x_0) \in C_1(\ell, \eta, c)$ sequence $\{x_n\}$ $(n \geq -1)$ generated by secant method (2.3.2) is well defined, remains in $\bar{U}(x_0, p)$ for all $n \geq 0$ and converges to a unique solution $x^* \in \bar{U}(x_0, p)$ of equation $F(x) = 0$.*

Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq p_n - p_{n+1} \tag{2.3.56}$$

and

$$\|x_n - x^*\| \leq p_n - p. \tag{2.3.57}$$

Using induction on n we can easily show the following favorable comparison of error bounds between Theorems 2.3.4 and 2.3.9.

Proposition 2.3.10. *Under the hypotheses of Theorems 2.3.4 and 2.3.9 the following estimates hold for all $n \geq 0$*

$$p_n \leq s_n \tag{2.3.58}$$

$$s_n - s_{n+1} \leq p_n - p_{n+1} \tag{2.3.59}$$

and

$$s_n - s^* \leq p_n - p. \tag{2.3.60}$$

Remark 2.3.11. We cannot compare conditions (2.3.4) and (2.3.53) in general because of ℓ_0. However in the special case $\ell = \ell_0 \neq 0$, we can set $a = 1$ to obtain $\delta = \frac{\sqrt{5}-1}{2}$. Condition (2.3.4) can be written

$$\ell c + \ell\eta \leq \beta = \frac{\delta}{1+\delta} = .381966011.$$

It can then easily be seen that if

$$0 < \ell c < 2\sqrt{\beta} - 1 = .236067977,$$

condition (2.3.4) holds but (2.3.53) is violated. That is, even in the special case of $\ell = \ell_0$, our Theorem 2.3.4 can be applied in cases not covered by Theorem 2.3.9.

2.4 Concerning the "terra incognita" between convergence regions of two Newton methods

There is an unknown area, between the convergence regions ("terra incognita") of the NK method, and the corresponding MNK method, when F' is an λ-Hölder continuous operator, $\lambda \in [0, 1)$. Note that according to Kantorovich theorems 2.2.4 and 2.2.7, these regions coincide when $\lambda = 1$. However, we already showed (see (2.2.70)) that this is not the case unless if $\ell_0 = \ell$. Here, we show how to investigate this region and improve on earlier attempts in this direction for $\lambda \in [0, 1)$ [32], [35], [64].

To make the study as self-contained as possible, we briefly reintroduce some results (until Remark 2.4.3) that can originally be found in [32], [64].

Let $x_0 \in D$ be such that $F'(x_0)^{-1} \in L(Y, X)$. Assume F' satisfies a center-Hölder condition

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq \ell_0 \|x - x_0\|^\lambda, \tag{2.4.1}$$

and a Hölder condition

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq \ell \|x - y\|^\lambda \tag{2.4.2}$$

for all $x, y \in U(x_0, R) \subseteq D$.

The results in [64] were given in non-affine invariant form. Here we reproduce them in affine invariant form. The advantages of such an approach have been well explained in [43], [78].

Define:

$$h_0 = \ell_0 \eta^\lambda, \tag{2.4.3}$$

$$h = \ell \eta^\lambda \tag{2.4.4}$$

and function

$$\psi(r) = \tfrac{\ell}{1+\lambda} r^{1+\lambda} - r + \eta, \tag{2.4.5}$$

where η is given by (2.2.16).

The first semilocal convergence result for methods NK and MNK under Hölder conditions were given in [135]:

Theorem 2.4.1. *Assume:*

$$h \leq \left(\frac{\lambda}{1+\lambda}\right)^{\lambda} \tag{2.4.6}$$

and

$$r^* \leq R, \tag{2.4.7}$$

where r^* *is the smallest positive zero of function* ψ*. Then sequence* $\{x_n\}$ $(n \geq 0)$ *generated by MNK method is well defined, remains in* $U(x_0, r^*)$ *for all* $n \geq 0$ *and converges to a unique solution* x^* *of equation (2.1.1) in* $U(x_0, r^*)$*. If* r^* *is the unique zero of* ψ *on* $[0, R]$ *and* $\psi(R) \leq 0$*, then* x^* *is unique in* $U(x_0, R)$*.*

Moreover, if

$$h \leq h_\nu, \tag{2.4.8}$$

where h_ν *is the unique solution in* $(0, 1)$ *of equation*

$$\left(\frac{t}{1+\lambda}\right)^{\lambda} = (1-t)^{1+\lambda} \tag{2.4.9}$$

method NK converges as well.

Theorem 2.4.1 holds [135] if condition (2.4.6) is replaced by the weaker

$$h \leq 2^{\lambda-1}\left(\frac{\lambda}{1+\lambda}\right)^{\lambda}. \tag{2.4.10}$$

Later in [64], (2.4.10) was replaced by an even weaker condition

$$h \leq \frac{1}{g(\lambda)}\left(\frac{\lambda}{1+\lambda}\right)^{\lambda}, \tag{2.4.11}$$

where,

$$g(\lambda) = \max_{t \geq 0} f(t), \tag{2.4.12}$$

$$f(t) = \frac{t^{1+\lambda}+(1+\lambda)t}{(1+t)^{1+\lambda}-1} \tag{2.4.13}$$

with

$$g(\lambda) < 2^{1-\lambda} \text{ for all } \lambda \in (0, 1). \tag{2.4.14}$$

Recently in [64], (2.4.11) was replaced by

$$h \leq \frac{1}{a(\lambda)}\left(\frac{\lambda}{1+\lambda}\right)^{\lambda}, \tag{2.4.15}$$

where,

$$a(\lambda) = \min\left\{b \geq 1\colon \max_{0 \leq t \leq t(b)} f(t) \leq b\right\}, \tag{2.4.16}$$

$$t(b) = \frac{b\lambda^{\lambda}}{(1+\lambda)\left[b(1+\lambda)^{\lambda}-\lambda^{\lambda}\right]}. \tag{2.4.17}$$

The idea is to optimize b in the equation

$$\psi_b(r) = 0, \tag{2.4.18}$$

where,

$$\psi_b(r) = \tfrac{b\ell}{1+\lambda} r^{1+\lambda} - r + \eta \tag{2.4.19}$$

assuming

$$h \leq \tfrac{1}{b} \left(\tfrac{\lambda}{1+\lambda}\right)^{\lambda}. \tag{2.4.20}$$

Note that condition (2.4.20) guarantees that equation (2.4.18) is solvable (see Proposition 1.1 in [64]).

With the above notation it was shown in [64] (Theorem 2.2, p. 719):

Theorem 2.4.2. *Assume (2.4.15) holds and that $r^* \leq R$, where r^* is the smallest solution of the scalar equation*

$$\psi_a(r) = \tfrac{a(\lambda)\ell}{1+\lambda} r^{1+\lambda} - r + \eta = 0. \tag{2.4.21}$$

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by NK method is well defined, remains in $U(x_0, r^*)$ for all $n \geq 0$, and converges to a unique solution x^* of equation $F(x) = 0$ in $U(x_0, r^*)$.

Moreover if sequence r_n is defined by

$$r_0 = 0, \quad r_n = r_{n-1} - \tfrac{\psi_a(r_{n-1})}{\psi'(r_{n-1})} \quad (n \geq 1) \tag{2.4.22}$$

then the following estimates hold for all $n \geq 1$:

$$\|x_n - x_{n-1}\| \leq r_n - r_{n-1} \tag{2.4.23}$$

and

$$\|x_n - x^*\| \leq r^* - r_n. \tag{2.4.24}$$

Remark 2.4.3. It was also shown in [64] (see Theorem 2.3) that

$$a(\lambda) < f(2) < g(\lambda) \ \text{ for all } \ \lambda \in (0, 1), \tag{2.4.25}$$

which shows that (2.4.15) is a real improvement over (2.4.10) and (2.4.11).

We can summarize as follows:

$$\begin{aligned} h_\nu &< 2^{\lambda-1} \left(\tfrac{\lambda}{1+\lambda}\right)^{\lambda} < \tfrac{1}{g(\lambda)} \left(\tfrac{\lambda}{1+\lambda}\right)^{\lambda} \\ &< \tfrac{1}{a(\lambda)} \left(\tfrac{\lambda}{1+\lambda}\right)^{\lambda} \leq \left(\tfrac{\lambda}{1+\lambda}\right)^{\lambda} = h_{exi}. \end{aligned} \tag{2.4.26}$$

Below we present our contributions/improvements in the exploration of "terra incognita."

First of all, we have observed that the Vertgeim result given in Theorem 2.4.1 holds under weaker conditions. Indeed:

Theorem 2.4.4. *Assume:*

$$h_0 \leq \left(\tfrac{\lambda}{1+\lambda}\right)^{\lambda} \tag{2.4.27}$$

replaces condition (2.4.6) in Theorem 2.4.1. Then under the rest of the hypotheses of Theorem 2.4.1, the conclusions for method (2.1.5) and equation (2.1.1) hold.

Proof. We note that (2.4.1) can be used instead of (2.4.2) in the proof of Theorem 1 given in [135].

Remark 2.4.5. Condition (2.4.27) is weaker than (2.4.6) because

$$h \leq \left(\tfrac{\lambda}{1+\lambda}\right)^{\lambda} \Rightarrow h_0 \leq \left(\tfrac{\lambda}{1+\lambda}\right)^{\lambda} \tag{2.4.28}$$

but not vice versa unless if $\ell = \ell_0$. Therefore our Theorem 2.4.4 improves the convergence region for MNK method under weaker conditions and cheaper computational cost.

It turns out that we can improve on the error bounds given in Theorem 2.4.2 under the same hypotheses and computational cost. Indeed:

Theorem 2.4.6. *Assume hypotheses of Theorem 2.4.1 and condition (2.4.1) hold.*

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by NK method is well defined, remains in $U(x_0, r^*)$ for all $n \geq 0$, and converges to a unique solution x^* of equation $F(x) = 0$ in $U(x_0, r^*)$. Moreover, if scalar sequence s_n is defined by

$$s_0 = 0, \quad s_n = s_{n-1} - \tfrac{\psi_a(s_{n-1})}{a(\lambda)\ell_0 s_{n-1}^{\lambda} - 1} \quad (n \geq 1), \tag{2.4.29}$$

then the following estimates hold for all $n \geq 1$

$$\|x_n - x_{n-1}\| \leq s_n - s_{n-1} \tag{2.4.30}$$

and

$$\|x_n - x^*\| \leq r^* - s_n. \tag{2.4.31}$$

Furthermore, if $\ell_0 < \ell$, then we have:

$$s_n < r_n \quad (n \geq 2), \tag{2.4.32}$$

$$s_n - s_{n-1} < r_n - r_{n-1} \quad (n \geq 2), \tag{2.4.33}$$

and

$$s^* - s_n \leq r^* - r_n \quad (n \geq 0). \tag{2.4.34}$$

Proof. We simply arrive at the more precise estimate

$$\|F'(x)^{-1}F'(x_0)\| \leq \left[1 - \ell_0\|x - x_0\|^{\lambda}\right]^{-1} \tag{2.4.35}$$

instead of

$$\|F'(x)^{-1}F'(x_0)\| \leq (1 - \ell\|x - x_0\|^{\lambda}) \tag{2.4.36}$$

used in the proof of Theorem 1.4.2 in [64, pp. 720], for all $x \in U(x_0, R)$. Moreover note that if $\ell_0 < \ell$, $\{s_n\}$ is a more precise majorizing sequence of $\{x_n\}$ than sequence $\{r_n\}$ otherwise $r_n = s_n$ $(n \geq 0)$). With the above changes, the proof of Theorem 2.4.2 can be utilized so we can reach until (2.4.31).

Using (2.4.22), (2.4.29), and simple induction on n, we immediately obtain (2.4.32) and (2.4.33), whereas (2.4.34) is obtained from (2.4.33) by using standard majorization techniques.

At this point we wonder if:

(a) condition (2.4.15) can be weakened, by using more precise majorizing sequences along the lines of the proof of Theorem 2.4.4;

(b) even more precise majorizing sequences than $\{s_n\}$ can be found.

We need the following result on majorizing sequences for the NK method.

Lemma 2.4.7. *Assume there exist parameters* $\ell \geq 0$, $\ell_0 \geq 0$, $\eta \geq 0$, $\lambda \in [0, 1]$, *and* $q \in [0, 1)$ *with* η *and* λ *not zero at the same time such that:*

(a)

$$\left[\ell + \frac{\delta \ell_0}{(1-q)^\lambda}\right] \eta^\lambda \leq \delta, \quad \textit{for} \quad \delta = (1+\lambda) q \quad \lambda \in [0, 1)\,, \tag{2.4.37}$$

or

(b)

$$(\ell + \overline{\delta} \ell_0) \eta \leq \overline{\delta}, \quad \textit{for} \quad \lambda = 1, \quad \ell_0 \leq \ell, \quad \textit{and} \quad \overline{\delta} \in [0, 1]\,. \tag{2.4.38}$$

Then, iteration $\{t_n\}$ $(n \geq 0)$ *given by*

$$\begin{aligned} t_0 &= 0, \quad t_1 = \eta, \\ t_{n+2} &= t_{n+1} + \frac{\ell}{(1+\lambda)\left[1 - \ell_0 t_{n+1}^\lambda\right]} (t_{n+1} - t_n)^{1+\lambda} \end{aligned} \tag{2.4.39}$$

is nondecreasing, bounded above by

$$\text{(a)} \quad t^{**} = \frac{\eta}{1-q}, \quad \textit{or} \quad \text{(b)} \quad t^{**} = \frac{2\eta}{2-\overline{\delta}}, \quad \overline{\delta} \in [0, 1] \tag{2.4.40}$$

and converges to some t^* *such that*

$$0 \leq t^* \leq t^{**}. \tag{2.4.41}$$

Moreover, the following estimates hold for all $n \geq 0$*:*

$$\text{(a)} \qquad 0 \leq t_{n+2} - t_{n+1} \leq q(t_{n+1} - t_n) \leq q^{n+1} \eta, \tag{2.4.42}$$

or

$$\text{(b)} \qquad 0 \leq t_{n+2} - t_{n+1} \leq \frac{\overline{\delta}}{2}(t_{n+1} - t_n) \leq \left(\frac{\overline{\delta}}{2}\right)^{n+1} \eta,$$

respectively.

Proof. (a) The result clearly holds if q or ℓ or n or $\ell_0 = 0$. Let us assume q, ℓ, η, $\ell_0 \neq 0$. We must show:

$$\ell(t_{k+1}-t_k)^\lambda+\delta\ell_0 t_{k+1}^\lambda \leq \delta,\;\; t_{k+1}-t_k \geq 0,\;\; 1-\ell_0 t_{k+1}^\lambda > 0 \text{ for all } \quad k \geq 0. \tag{2.4.43}$$

Estimate (2.4.42) can then follow immediately from (2.4.39) and (2.4.43). Using induction on the integer k, we have for $k = 0$, $\ell\eta^\lambda + \delta\ell_0\eta^\lambda = (\ell + \delta\ell_0)\eta^\lambda \leq \delta$ (by (2.4.37)) and $1 - \ell_0\eta^p > 0$. But then (2.4.43) gives

$$0 \leq t_2 - t_1 \leq q(t_1 - t_0).$$

Assume (2.4.43) holds for all $k \leq n + 1$. We can have in turn

$$\begin{aligned}
&\ell(t_{k+2} - t_{k+1})^\lambda + \delta\ell_0^\lambda t_{k+2} \\
&\quad\leq\; \ell\eta^\lambda q^{k+1} + \delta\ell_0 \left[t_1 + q(t_1 - t_0) + q^2(t_1 - t_0) + \cdots + q^{k+1}(t_1 - t_0)\right]^\lambda \\
&\quad\leq\; \ell\eta^\lambda q^{(k+1)\lambda} + \delta\ell_0\eta^\lambda \left[\tfrac{1-q^{k+2}}{1-q}\right]^\lambda \\
&\quad=\left[\ell q^{(k+1)\lambda} + \tfrac{\delta\ell_0}{(1-q)^\lambda}(1 - q^{k+2})^\lambda\right]\eta^\lambda \leq \left[\ell + \tfrac{\delta\ell_0}{(1-q)^\lambda}\right]\eta^\lambda
\end{aligned} \tag{2.4.44}$$

which is smaller or equal to δ by (2.4.37). Hence, the first estimate in (2.4.43) holds for all $k \geq 0$. We must also show:

$$t_k \leq t^{**} \quad (k \geq 0). \tag{2.4.45}$$

For $k = 0, 1, 2$ we have

$$t_0 = \eta \leq t^{**},\;\; t_1 = \eta \leq t^{**} \quad \text{and} \quad t_2 \leq \eta + q\eta = (1 + q)\eta \leq t^{**}.$$

Assume (2.4.45) holds for all $k \leq n + 1$. We also can get

$$\begin{aligned}
t_{k+2} &\leq t_{k+1} + q(t_{k+1} - t_k) \leq t_k + q(t_k - t_{k-1}) + q(t_{k+1} - t_k) \\
&\leq \cdots \leq t_1 + q(t_1 - t_0) + \cdots + q(t_k - t_{k-1}) + q(t_{k+1} - t_k) \\
&\leq \eta + q\eta + q^2\eta + \cdots + q^{k+1}\eta = \tfrac{1-q^{k+2}}{1-q}\eta < \tfrac{\eta}{1-q} = t^{**}.
\end{aligned} \tag{2.4.46}$$

Moreover the second inequality in (2.4.43) holds since

$$\ell_0 t_{k+2}^\lambda \leq \ell_0 \left(\tfrac{\eta}{1-q}\right)^\lambda < 1 \quad \text{by (2.4.37).}$$

Furthermore the third inequality in (2.4.43) holds by (2.4.39), (2.4.44), and (2.4.46). Hence (2.4.43) holds for all $k \geq 0$. Iteration $\{t_n\}$ is nondecreasing and bounded above by t^{**} and as such it converges to some t^* satisfying (2.4.41).

(b) See [39] and the proof of part (a).

We can show the main semilocal convergence theorem for the NK method:

Theorem 2.4.8. *Let $F: D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume: there exist a point $x_0 \in D$ and parameters $\eta \geq 0$, $\ell_0 \geq 0$, $\ell \geq 0$, $\lambda \in [0,1]$, $q \in [0,1)$, $\overline{\delta} \in [0,1]$, $R \geq 0$ such that: conditions (2.4.1), (2.4.2), and hypotheses of Lemma 2.4.7 hold, and*

$$\overline{U}(x_0, t^*) \subseteq U(x_0, R). \tag{2.4.47}$$

Then, $\{x_n\}$ $(n \geq 0)$ generated by NK method is well defined, remains in $\overline{U}(x_0, t^*)$ for all $n \geq 0$ and converges to a unique solution $x^* \in \overline{U}(x_0, t^*)$ of equation $F(x) = 0$.

Moreover the following estimates hold for all $n \geq 0$:

$$\|x_{n+2} - x_{n+1}\| \leq \frac{\ell\|x_{n+1}-x_n\|^{1+\lambda}}{(1+\lambda)\left[1-\ell_0\|x_{n+1}-x_0\|^{\lambda}\right]} \leq t_{n+2} - t_{n+1} \tag{2.4.48}$$

and

$$\|x_n - x^*\| \leq t^* - t_n, \tag{2.4.49}$$

where iteration $\{t_n\}$ $(n \geq 0)$ and point t^* are given in Lemma 2.4.7.

Furthermore, if there exists $R > t^*$ such that

$$R_0 \leq R \tag{2.4.50}$$

and

$$\ell_0 \int_0^1 \left[\theta t^* + (1-\theta)R\right]^{\lambda} d\theta \leq 1, \tag{2.4.51}$$

the solution x^* is unique in $U(x_0, R_0)$.

Proof. We shall prove:

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \tag{2.4.52}$$

and

$$\overline{U}(x_{k+1}, t^* - t_{k+1}) \subseteq \overline{U}(x_k, t^* - t_k) \tag{2.4.53}$$

hold for all $n \geq 0$.

For every $z \in \overline{U}(x_1, t^* - t_1)$

$$\|z - x_0\| \leq \|z - x_1\| + \|x_1 - x_0\| \leq t^* - t_1 + t_1 = t^* - t_0$$

implies $z \in \overline{U}(x_0, t^* - t_0)$. Because also

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = t_1$$

(2.4.52) and (2.4.53) hold for $n = 0$. Given they hold for $n = 0, 1, \ldots, k$, then

$$\|x_{k+1} - x_0\| \leq \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 = t_{k+1} \tag{2.4.54}$$

and

$$\|x_k + \theta(x_{k+1} - x_k) - x_0\| \leq t_k + \theta(t_{k+1} - t_k) < t^*, \quad \theta \in [0,1]. \tag{2.4.55}$$

Using NK we obtain the approximation

$$\begin{aligned}F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k)\\ &= \int_0^1 \left[F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)\right](x_{k+1} - x_k)d\theta,\end{aligned} \tag{2.4.56}$$

and by (2.4.2)

$$\begin{aligned}&\|F'(x_0)^{-1}F(x_{k+1})\| \leq\\ &\leq \int_0^1 \|F'(x_0)^{-1}\left[F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)\right]\|d\theta\|x_{k+1} - x_k\|\\ &\leq \tfrac{\ell}{1+\lambda}\|x_{k+1} - x_k\|^{1+\lambda}.\end{aligned} \tag{2.4.57}$$

By (2.4.1), the estimate

$$\|F'(x_0)^{-1}\left[F'(x_{k+1}) - F'(x_0)\right]\| \leq \ell_0\|x_{k+1} - x_0\|^\lambda \leq \ell_0 t_{k+1}^\lambda < 1$$

and the Banach Lemma on invertible operators $F'(x_{k+1})^{-1}$ exists and

$$\|F'(x_0)F'(x_{k+1})^{-1}\| \leq \tfrac{1}{1-\ell_0\|x_{k+1}-x_0\|^\lambda} \leq \tfrac{1}{1-\ell_0 t_{k+1}^\lambda}. \tag{2.4.58}$$

Therefore, by NK, (2.4.39), (2.4.57), and (2.4.58) we obtain in turn

$$\begin{aligned}\|x_{k+2} - x_{k+1}\| &= \|F'(x_{k+1})^{-1}F(x_{k+1})\|\\ &\leq \|F'(x_{k+1})^{-1}F'(x_0)\| \cdot \|F'(x_0)^{-1}F(x_{k+1})\|\\ &\leq \tfrac{\ell\|x_{k+1}-x_k\|^{1+\lambda}}{(1+\lambda)\left[1-\ell_0\|x_{k+1}-x_0\|^\lambda\right]}\\ &\leq \tfrac{\ell(t_{k+1}-t_k)^{1+\lambda}}{(1+\lambda)\left[1-\ell_0 t_{k+1}^\lambda\right]} = t_{k+2} - t_{k+1}.\end{aligned} \tag{2.4.59}$$

Thus for every $z \in \overline{U}(x_{k+2}, t^* - t_{k+2})$, we have

$$\|z - x_{k+1}\| \leq \|z - x_{k+2}\| + \|x_{k+2} - x_{k+2}\| \leq t^* - t_{k+2} + t_{k+2} - t_{k+2} = t^* - t_{k+1}.$$

That is

$$z \in \overline{U}(x_{k+1}, t^* - t_{k+1}). \tag{2.4.60}$$

Estimates (2.4.59) and (2.4.60) imply that (2.4.52) and (2.4.53) hold for $n = k + 1$. By induction the proof of (2.4.52) and (2.4.53) is completed.

Lemma 2.4.7 implies that $\{t_n\}$ $(n \geq 0)$ is a Cauchy sequence. From (2.4.52) and (2.4.53) $\{x_n\}$ $(n \geq 0)$ becomes a Cauchy sequence, too, and as such it converges to some $x^* \in \overline{U}(x_0, t^*)$ so that (2.4.49) holds.

The combination of (2.4.59) and (2.4.60) yields $F(x^*) = 0$. Finally to show uniqueness let y^* be a solution of equation $F(x) = 0$ in $U(x_0, R)$. It follows from (2.4.1), the estimate

$$\begin{aligned}\left\| F'(x_0)^{-1} \int_0^1 \left[F'(y^* + \theta(x^* - y^*)) - F'(x_0)\right] d\theta \right\| &\le \\ \le \ \ell_0 \int_0^1 \|y^* + \theta(x^* - y^*) - x_0\|^\lambda d\theta & \\ \le \ \ell_0 \int_0^1 \left[\theta \|x^* - x_0\| + (1-\theta)\|y^* - x_0\|\right]^\lambda d\theta & \\ < \ \ell_0 \int_0^1 \left[\theta t^* + (1-\theta) R_0\right]^\lambda d\theta \le 1 \quad \text{(by (2.4.51))} & \end{aligned} \tag{2.4.61}$$

and the Banach Lemma on invertible operators that linear operator

$$L = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta \tag{2.4.62}$$

is invertible.

Using the identity

$$0 = F(y^*) - F(x^*) = L(x^* - y^*) \tag{2.4.63}$$

we deduce $x^* = y^*$. To show uniqueness in $\overline{U}(x_0, t^*)$ as in (2.4.61), we get:

$$\|F'(x_0)^{-1}(L - F'(x_0))\| \le \tfrac{\ell_0}{1+\lambda}(t^*)^{1+\lambda} < 1 \ \text{ (by Lemma 2.4.7)}$$

which implies again $x^* = y^*$.

Remark 2.4.9. In the result that follows we show that our error bounds on the distances involved are finer and the location of the solution x^* at least as precise.

Proposition 2.4.10. *Under hypotheses of Theorems 2.4.6 and 2.4.8 with $\ell_0 < \ell$ the following estimates hold:*

$$r_0 = t_0 = s_0 = 0, \quad r_1 = t_1 = s_1 = \eta,$$

$$t_{n+1} < s_{n+1} < r_{n+1} \quad (n \ge 1), \tag{2.4.64}$$

$$t_{n+1} - t_n < s_{n+1} - s_n < r_{n+1} - r_n \quad (n \ge 1), \tag{2.4.65}$$

$$t^* - t_n \le s^* - s_n \le r^* - r_n \quad (n \ge 0), \tag{2.4.66}$$

and

$$t^* \le s^* \le r^*. \tag{2.4.67}$$

Proof. We use induction on the integer k to show the left-hand sides of (2.4.64) and (2.4.65) first. By (2.4.29) and (2.4.39), we obtain

$$t_2 - t_1 = \tfrac{\ell \eta^{1+\lambda}}{(1+\lambda)[1-\ell_0\eta^\lambda]} < \tfrac{\psi_a(s_1)}{(1+\lambda)[1-\ell_0\eta^\lambda]} = s_2 - s_1,$$

and

$$t_2 < s_2.$$

Assume:

$$t_{k+1} < s_{k+1}, \quad t_{k+1} - t_k < s_{k+1} - s_k \quad (k \leq n). \tag{2.4.68}$$

Using (2.4.29), and (2.4.39), we get

$$t_{k+2} - t_{k+1} = \frac{\ell(t_{k+1} - t_k)^{1+\lambda}}{(1+\lambda)\left[1 - \ell_0 t_{k+1}^{\lambda}\right]} < \frac{\ell(s_{k+1} - s_k)^{1+\lambda}}{(1+\lambda)\left[1 - \ell t_{k+1}^{\lambda}\right]} \leq s_{k+2} - s_{k+1},$$

(by the proof of Theorem 2.2 in [64], end of page 720 and first half of page 721) and

$$t_{k+2} < s_{k+2}.$$

Let $m \geq 0$, we can obtain

$$\begin{aligned} t_{k+m} - t_k &< (t_{k+m} - t_{k+m-1}) + (t_{k+m-1} - t_{k+m-2}) + \cdots + (t_{k+1} - t_k) \\ &< (s_{k+m} - s_{k+m-1}) + (s_{k+m-1} - s_{k+m-2}) + \cdots + (s_{k+1} - s_k) \\ &= s_{k+m} - s_k. \end{aligned} \tag{2.4.69}$$

By letting $m \to \infty$ in (2.4.69) we obtain (2.4.66). For $n = 1$ in (2.4.66) we get (2.4.67).

That completes the proof of Proposition 2.4.10, as the right-hand side estimates in (2.4.65)–(2.4.67) were shown in Theorem 2.4.6.

In the next remark, we also show that our sufficient convergence conditions are weaker in general than the earlier ones (i.e., the Lipschitz case):

Remark 2.4.11. Case $\lambda = 1$. (see Section 2.2 of Chapter 2)

Case $\lambda = 0$. It was examined here but not in [64], [78], [135].

Case $\lambda \in (0, 1)$. We can compare condition (2.4.37) with (2.4.15) (or (2.4.11) or (2.4.10) or (2.4.6)). For example set

$$q = \frac{1}{\lambda + 1}. \tag{2.4.70}$$

Then for

$$\ell_0 = \ell d, \quad d \in [0, 1], \tag{2.4.71}$$

and

$$c(\lambda, d) = d + \left(\frac{\lambda}{1+\lambda}\right)^{\lambda}, \tag{2.4.72}$$

condition (2.4.37) becomes:

$$h \leq \tfrac{1}{c(\lambda,d)} \left(\frac{\lambda}{1+\lambda}\right)^{\lambda}. \tag{2.4.73}$$

(a) Choose $d = \frac{1}{2}$, then using Mathematica we compare the magnitude of $a(\lambda)$ with $c\left(\lambda, \frac{1}{2}\right)$ to obtain the following favorable for our approach table:

Comparison table

λ	.1	.2	.3	.4	.5	.6	.7	.8	.9
$a(\lambda)$	1.842	1.695	1.562	1.445	1.341	1.252	1.174	1.108	1.050
$c\left(\lambda, \frac{1}{2}\right)$	1.287	1.200	1.444	1.106	1.080	1.055	1.037	1.023	1.010

See also the corresponding table in [64, pp. 722].

(b) If $d = 1$ (i.e., $\ell = \ell_0$), say for $\lambda = .1$ we found

$$c(.1, 1) = 1.787 < a(.1) = 1.842.$$

(c) Because $\frac{\ell}{\ell_0}$ can be arbitrarily large (see Example 2.2.13) for

$$d = 1 - \left(\frac{\lambda}{1+\lambda}\right)^{\lambda} = p, \tag{2.4.74}$$

condition (2.4.73) reduces to (2.4.6), whereas for

$$0 \le d < p \tag{2.4.75}$$

(2.4.73) improves (2.4.6), which is the weakest of all conditions given before (see [64]).

Other favorable comparisons can also be made when q is not necessarily given by (2.4.70). However we leave the details to the motivated reader.

We state the following local convergence result for the NK method.

Theorem 2.4.12. *Let $F\colon D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume:*
(a) there exist a simple zero $x^* \in D$ of equation $F(x) = 0$, parameters $\overline{\ell}_0 \ge 0$, $\overline{\ell} \ge 0$, $\mu \in [0, 1]$ not all zero at the same time such that:

$$\|F'(x^*)^{-1}\left[F'(x) - F'(y)\right]\| \le \overline{\ell}\|x - y\|^{\mu}, \tag{2.4.76}$$

$$\|F'(x^*)^{-1}\left[F'(x) - F'(x^*)\right]\| \le \overline{\ell}_0\|x - x^*\|^{\mu} \tag{2.4.77}$$

for all $x, y \in \overline{U}(x_0, R) \subseteq D$ $(R \ge 0)$;
(b) Define:

$$q = \begin{cases} \left[\dfrac{1+\mu}{\overline{\ell} + (1+\mu)\overline{\ell}_0}\right]^{1/\mu} & \mu \ne 0 \\ R \quad \text{and} \quad \overline{\ell} + \overline{\ell}_0 \le 1 & \text{for } \mu = 0 \end{cases} \tag{2.4.78}$$

and

$$q \le R. \tag{2.4.79}$$

Then, sequence $\{x_n\}$ $(n \ge 0)$ generated by NK is well defined, remains in $U(x^*, q)$ for all $n \ge 0$ and converges to x^*, provided that $x_0 \in U(x^*, q)$. Moreover the following estimates hold for all $n \ge 0$:

$$\|x_{n+1} - x^*\| \le \frac{\overline{\ell}\|x_n - x^*\|^{1+\mu}}{(1+\mu)\left[1 - \overline{\ell}_0\|x_n - x^*\|^{\mu}\right]}. \tag{2.4.80}$$

Proof. Inequality (2.4.80) follows from the approximation

$$\begin{aligned} x_{n+1} - x^* &= \\ &= x_n - x^* - F'(x_n)^{-1} F(x_n) \\ &= -\left[F'(x_n)^{-1} F'(x^*)\right] \times \\ &\times \left\{ F'(x^*)^{-1} \int_0^1 \left[F'(x^* + t(x_n - x^*)) - F'(x_n)\right](x_n - x^*)dt \right\}, \end{aligned} \tag{2.4.81}$$

and estimates

$$\|F'(x_n)^{-1} F'(x^*)\| \leq \left[1 - \overline{\ell}_0 \|x_n - x^*\|^{\mu}\right]^{-1} \quad \text{(see (2.4.58))} \tag{2.4.82}$$

$$\left\| F'(x^*)^{-1} \int_0^1 \left[F'(x^* + t(x_n - x^*)) - F'(x_n)\right](x_n - x^*)dt \right\| \leq$$

$$\leq \tfrac{\overline{\ell}}{1+\mu} \|x_n - x^*\|^{1+\mu}, \quad \text{(see (2.4.57))} \tag{2.4.83}$$

The rest follows using induction on the integer n, (2.4.81)–(2.4.83), and along the lines of the proof of Theorem 2.4.8.

The corresponding local result for the MNK method is:

Remark 2.4.13. Using only condition (2.4.76) and the approximation

$$y_{n+1} - x^* = F'(y_0)^{-1} \int_0^1 \left[F'(x^* + t(y_n - x^*)) - F'(y_0)\right](y_n - x^*)dt, \tag{2.4.84}$$

as in the proof of Theorem 2.4.12 we obtain the convergence radius

$$\bar{q}_0 = \begin{cases} \left[\dfrac{1+\mu}{(2^{1+\mu} - 1)\overline{\ell}}\right]^{1/\mu}, & \overline{\ell} \neq 0, \ \mu \neq 0 \\ R, & \mu = 0, \end{cases} \tag{2.4.85}$$

and the corresponding estimates

$$\begin{aligned} \|y_{n+1} - x^*\| &\leq \overline{\ell} \int_0^1 \left[\|x^* - y_0\| + t\|y_n - x^*\|\right]^{\mu} dt \|y_n - x^*\| \\ &\leq \frac{\overline{\ell}(2^{1+\mu} - 1)}{1+\mu} \bar{q}_0^{\mu} \|y_n - x^*\| \quad (n \geq 0). \end{aligned} \tag{2.4.86}$$

Remark 2.4.14. As noted in [43] and [216], the local results obtained here can be used for projection methods such as Arnoldi’s, the Generalized Minimum Residual method (GMRES), the generalized conjugate residual method (GCR), for combined Newton/finite-difference projection methods, and in connection with the mesh independence principle in order to develop the cheapest mesh refinement strategies.

Remark 2.4.15. The local results obtained here can also be used to solve equations of the form $F(x) = 0$, where F' satisfies the autonomous differential equation [71]:

$$F'(x) = T(F(x)), \tag{2.4.87}$$

where $T: Y \to X$ is a known continuous operator. Because $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results obtained here without actually knowing the solution x^* of the equation $F(x) = 0$.

We complete this section with a numerical example to show that through Theorem 2.4.6 we can obtain a wider choice of initial guesses x_0 than before.

Example 2.4.16. Let $X = Y = \mathbf{R}$, $D = U(0, 1)$ and define function F on D by

$$F(x) = e^x - 1. \tag{2.4.88}$$

Then it can easily be seen that we can set $T(x) = x + 1$ in [35]. Because $F'(x^*) = 1$, we get $\|F'(x) - F'(y)\| \le e\|x - y\|$. Hence we set $\overline{\ell} = e$, $\mu = 1$. Moreover, because $x^* = 0$, we obtain in turn

$$\begin{aligned} F'(x) - F'(x^*) &= e^x - 1 = x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\ &= \left(1 + \frac{x}{2!} + \cdots + \frac{x^{n-1}}{n!} + \cdots\right)(x - x^*) \end{aligned}$$

and for $x \in U(0, 1)$,

$$\|F'(x) - F'(x^*)\| \le (e - 1)\|x - x^*\|.$$

That is, $\overline{\ell}_0 = e - 1$. Using (2.4.85) we obtain

$$r^* = .254028662\,.$$

Rheinboldt's radius [175] is given by

$$p = \frac{2}{3\overline{\ell}}\,.$$

Note that

$$p < r^* \qquad (\text{as} \;\; \overline{\ell}_0 < \overline{\ell}\,).$$

In particular, in this case we obtain

$$p = .245252961\,.$$

Note also that our error estimates are finer as $\overline{\ell}_0 < \overline{\ell}$. That is our convergence radius r^* is larger than the corresponding one p due to Rheinboldt [175]. This observation is very important in computational mathematics (see Remark 2.4.15). Note also that local results were not given in [64].

The case $\mu \in [0, 1)$ was not covered in [64]. The "terra incognita" can be examined along the lines of the semilocal case studied above. However, we leave the details to the motivated reader.

2.5 Enlarging the convergence domain of the NK method under regular smoothness conditions

Sufficient convergence conditions such as the famous Newton-Kantorovich hypothesis (see Chapter 2, Sections 2.2 and 2.4) have been given under the hypotheses for all $x, y \in D$

$$\left\| F'(x) - F'(y) \right\| \leq L \left\| x - y \right\|^{\lambda} \quad \lambda \in [0, 1], \tag{2.5.1}$$

or the w-smoothness

$$\left\| F'(x) - F'(y) \right\| \leq w\left(\| x - y \| \right) \tag{2.5.2}$$

for some increasing conditions function $w\colon [0, \infty) \to [0, \infty)$ with $w(0) = 0$ [6], [35], [43], [58], [98], [99], [146].

Under (2.5.1) the error bound

$$\left\| F(y) - F(x) - F'(x)(y - x) \right\| \leq \tfrac{L}{1+\lambda} \left\| x - y \right\|^{1+\lambda} \tag{2.5.3}$$

crucial in any convergence analysis of method NK has been improved only if $\lambda \in (0, 1)$ under an even more flexible condition than (2.5.2) called w-regular smoothness (to be precised later).

Here motivated by the elegant works in [98], [99] but using more precise majorizing sequences and under the same computational cost, we provide a semilocal convergence analysis for NK method under w-regular smoothness conditions on F' with the following advantages:

(a) finer estimates on the distances

$$\| x_{n+1} - x_n \|, \quad \left\| x_n - x^* \right\| \quad (n \geq 0);$$

(b) an at least as precise information on the location of the solution;

and

(c) a larger convergence domain.

Expressions $r \to cr^{\lambda}$, $\lambda \in (0, 1]$ are typical representations of the class C of nondecreasing functions $w\colon (0, \infty] \to (0, \infty]$ that are concave and vanishing at zero. By w^{-1} we denote the function whose closed epigraph $cl\{(s, t),\ s \geq 0, \text{ and } t \geq w^{-1}(s)\}$ is symmetrical to closure of the subgraph of w with respect to the axis $t = s$ [98], [99]. Consider $T \in L(X, Y)$ and denote $\underline{h}(T)$ the inf $\| T(x) \|$. Given an $w_0 \in C$ and $x_0 \in D$, we say that T is w_0-regularly continuous on D with respect to $x = x_0 \in D$ or, equivalently, that w_0 is a regular continuity modulus of T on D relative to x_0, if there exists $\underline{h} = \underline{h}(x_0) \in \left[0, \underline{h}(T)\right]$ such that for all $x \in D$:

$$w_0^{-1}\left(h_T(x_0, x) + \| T(x) - T(x_0) \| \right) - w_0^{-1}\left(h_T(x_0, x) \right) \leq \| x - x_0 \|, \tag{2.5.4}$$

where

$$h_{0T}(x_0, x) = \| T(x) \| - \underline{h}_0.$$

Given $w \in C$, we say T is w-regularly continuous on D if there exists $\underline{h} \in \left[0, \underline{h}(T)\right]$ such that for all $x, y \in D$

$$w^{-1}\left(h_T(x,y)+\|T(y)-T(x)\|\right)-w^{-1}\left(h_T(x,y)\right)\le\|y-x\|, \tag{2.5.5}$$

where

$$h_T(y,x)=\min\{\|T(x)\|,\|T(y)\|\}-\underline{h}.$$

The operator F is w_0-regularly smooth on D with respect to a given point $x_0\in D$, if its Fréchet derivative F' is w_0-regularly continuous with respect to x_0. Operator F is w-regularly smooth on D, if its Fréchet derivative F' is w-regularly continuous there [98], [99].

Note that in general

$$w_0(r)\le w(r)\quad\text{for all}\quad r\in[0,\infty) \tag{2.5.6}$$

holds.

Given $w\in C$, set $Q_0(t)=\int_0^t w(\theta)\,d\theta$ and define function Q by

$$Q(u,t)=\begin{cases} tw(u)-Q_0(u)+Q_0(u-t) & \text{for } t\in[0,u],\ u\ge 0\\ uw(u)-2Q_0(u)+Q_0(t) & \text{for } t\ge u,\ u\ge 0.\end{cases} \tag{2.5.7}$$

Denote by the superscript$^+$ the nonnegative part of a real number

$$a^+=\max\{a,0\}. \tag{2.5.8}$$

Given $x_0\in D$, if the operator $F_0=F'(x_0)^{-1}F$ is w_0-regularly smooth with respect to x_0 and w-regularly smooth on D, define the sequence $\overline{u}_n=\left(\overline{t}_n,\overline{\alpha}_n,\overline{\overline{\alpha}}_n,\overline{\varepsilon}_n\right)$ by

$$\begin{aligned}&\overline{t}_n=\|x_n-x_0\|,\ \overline{\alpha}_n=w^{-1}\left(\left\|F_0'(x_n)\right\|-\underline{h}\right),\overline{\overline{\alpha}}_n=w_0^{-1}\left(\left\|F_0'(x_n)\right\|-\underline{h}_0\right),\\ &\quad\left(\text{or },\overline{\overline{\alpha}}_n=w_0^{-1}\left(\left\|F_0'(x_n)\right\|-\underline{h}\right)\right)\\ &\overline{\varepsilon}_n=\left\|F_0'(x_n)^{-1}F_0(x_n)\right\|\quad(n\ge 0).\end{aligned} \tag{2.5.9}$$

As in Theorem 4.3 in [98, pp. 831] but using w_0 (i.e., (2.5.4)) instead of w (i.e., (2.5.5)) for the computation of the upper bounds of the inverses $F_0'(x_n)^{-1}$ we show:

$$\overline{t}_n\le\overline{t}_n+\overline{\varepsilon}_n,\ \overline{\alpha}_{n+1}\ge(\overline{\alpha}_n-\overline{\varepsilon}_n)^+,\ \overline{\overline{\alpha}}_{n+1}\ge\left(\overline{\overline{\alpha}}_n-\overline{\varepsilon}_n\right)^+, \tag{2.5.10}$$

$$\overline{\varepsilon}_{n+1}\le\frac{Q(\overline{\alpha}_n,\overline{\varepsilon}_n)}{1-w_0\left(\overline{\overline{\alpha}}_{n+1}+\overline{t}_{n+1}\right)+w_0\left(\overline{\overline{\alpha}}_{n+1}\right)} \tag{2.5.11}$$

where function Q is given by (2.5.7).

Consider the sequence $u_n=\left(t_n,\alpha_n,\alpha_n^0,\varepsilon_n\right)$ given by

$$t_{n+1}=t_n+\varepsilon_n,\ \alpha_{n+1}=(\alpha_n-\varepsilon_n)^+,\ \alpha_{n+1}^0=\left(\alpha_{n+1}^0-\varepsilon_n\right), \tag{2.5.12}$$

$$(\text{or } \alpha_{n+1}^0=\alpha_{n+1}\quad n\ge 0)$$

$$\varepsilon_{n+1} = \frac{Q\left(\alpha_n, \varepsilon_n\right)}{1 - w_0\left(\alpha_{n+1}^0 + t_{n+1}\right) + w_0\left(\alpha_{n+1}^0\right)} \tag{2.5.13}$$

for

$$t_0 = 0,\ \alpha_0 = w^{-1}\left(1 - \underline{h}\right),\ \alpha_0^0 = w_0^{-1}\left(1 - \underline{h}_0\right) \text{ and } \varepsilon_0 \geq \overline{\varepsilon}_0 \text{ [98], [99].} \tag{2.5.14}$$

The sequence $\{u_n\}$ is well defined and converges if for all $n \geq 0$

$$w_0\left(\alpha_{n+1}^0 + t_{n+1}\right) + w_0\left(\alpha_{n+1}^0\right) < 1, \tag{2.5.15}$$

or, equivalently

$$t_n < w_0^{-1}(1) \tag{2.5.16}$$

(as sequence u_n will then be increasing and bounded above by the number $w_0^{-1}(1)$).

Denote by s_n the sequence given by (2.5.12), (2.5.13) when $w_0 = w$. If strict inequality holds in (2.5.6) we get by induction on $n \geq 0$

$$t_n < s_n \tag{2.5.17}$$

$$t_{n+1} - t_n < s_{n+1} - s_n \tag{2.5.18}$$

and

$$t^* = \lim_{n\to\infty} \leq \lim_{n\to\infty} s_n = s^* \tag{2.5.19}$$

We can now show the following semilocal convergence result for Newton's method under regular smoothness:

Theorem 2.5.1. *Assume:*
Operator F_0 is w_0-regularly smooth with respect to $x_0 \in D$, and w-regularly smooth on D;
condition (2.5.16) holds;
and for $t^* = \lim_{n\to\infty} t_n$

$$\overline{U}\left(x_0, t^*\right) \subseteq D. \tag{2.5.20}$$

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by NK is well defined, remains in $U(x_0, t^*)$ for all $n \geq 0$, and converges to a solution $x^* \in \overline{U}(x_0, t^*)$ of equation $F(x) = 0$. Moreover the following bounds hold for all $n \geq 0$:

$$\left\| F_0'(x_n)^{-1} \right\| \leq \gamma_n^{-1} = \left[1 - w_0\left(\alpha_n^0 + t_n\right) + w_0\left(\alpha_n^0\right)\right]^{-1}, \tag{2.5.21}$$

$$(or\ [1 - w_0(\alpha_n + t_n) + w_0(\alpha_n)]^{-1})$$

$$\|x_{n+1} - x_n\| \leq \overline{\varepsilon}_{n+1} \leq \varepsilon_{n+1}, \tag{2.5.22}$$

and

$$\left\|x_n - x^*\right\| \leq t^* - t_n. \tag{2.5.23}$$

Furthermore if ε_0 is such that

$$t^* \leq \alpha_0^0, \tag{2.5.24}$$

then the solution x^* is unique in $\overline{U}\left(x_0, \mathcal{P}_{\underline{h},2}^{-1}(0)\right)$, where function $\mathcal{P}_{\underline{h},2}^{-1}$ is the inverse of the restriction of $\mathcal{P}_{\underline{h}}$ to $\left[w_0^{-1}(1), \infty\right]$, and function $\mathcal{P}_{\underline{h}}$ was defined in [98, pp. 830].

Proof. We state that the derivation of (2.5.21) requires only (2.5.4) and not the stronger (2.5.5) used in [98, pp. 831] (see also (2.5.6)). The rest follows exactly in the proof of Theorem 4.3 in [98, pp. 831].

That completes the proof of the theorem.

Remark 2.5.2. (a) If equality holds in (2.5.6) then our Theorem 2.5.1 reduces to Theorem 4.3 in [98]. However if strict inequality holds in (2.5.6) then our error bounds on the distances $\|x_{n+1} - x_n\|$ (see (2.5.18) and (2.5.22)) are finer (smaller) than the corresponding ones in [98], [99]. Moreover condition (2.5.16) is weaker than the corresponding one in [98] (see 4.4 there) given by

$$s_n < w^{-1}(1)\ (n \geq 0)\,. \tag{2.5.25}$$

Furthermore the information on the location of the solution x^* is at least as precise, as our majorizing sequence is finer (smaller) (see (2.5.19)).

All the above advantages hold even if we choose

$$\overline{\alpha}_n = \overline{\overline{\alpha}}_n \text{ and we set } \alpha_n = \alpha_n^0\ \ (n \geq 0)\,. \tag{2.5.26}$$

Note also that the above results are obtained under the same computational cost since computing function w requires the computation of w_0.

Definition 2.5.3. *Given a continuous operator $f\colon R^m \to U \subseteq D$, the set*

$$U(p) = \left\{u_0 \in U \,\middle|\, f^n(u_0) \to p\right\} \tag{2.5.27}$$

is called the attraction basin of p [81].

This set is not empty if and only of p is a fixed point f as it can be seen from the equality

$$u_{n+1} = f(u_n)\,. \tag{2.5.28}$$

It follows that the convergence domain

$$\begin{aligned} U_c &= \{u_0 \in U \,|\text{sequence } \{u_n\} \text{ converges}\} \\ &= \bigcup_{p \in U} U(p) = \bigcup_{a \in U_f} U(p) \end{aligned} \tag{2.5.29}$$

where

$$U_f = \{p \,|\, f(p) = p\}\,. \tag{2.5.30}$$

Hence, the convergence domain of f can be constructed as the union of the attraction basins of its fixed points.

Example 2.5.4. Iteration (2.5.12), (2.5.13) can be rewritten as

$$t_+ = t + \varepsilon,\ \varepsilon_+ = \frac{Q\left((\alpha_0 - t)^+ , \varepsilon\right)}{1 - w_0\left(\alpha_0^0 + t_+\right) + w_0\left(\alpha_0^0\right)} \tag{2.5.31}$$

(see also [99, p. 789]).

Its fixed points constitute the segment $\left[0, w_0^{-1}(1)\right)$ of the t-axis. When F_0 is Lipschitz smooth at x_0 with $w_0(t) = L_0 t$ ($L_0 \geq 0$) and Lipschitz smooth on D with $w(t) = Lt$ ($L \geq 0$), (2.5.31) reduces to:

$$t_+ = t + \varepsilon,\ \varepsilon_+ = \frac{.5L\varepsilon^2}{1 - L_0 t_+}, \tag{2.5.32}$$

and in the special case $L_0 = L$

$$s_+ = t + \varepsilon,\ \varepsilon_+ = \frac{.5L\varepsilon^2}{1 - Ls_+}. \tag{2.5.33}$$

It was shown in [99, p. 789] (i.e., using (2.5.33)) that

$$U_G = U_c^{L=L_0} = \left\{(t,\varepsilon)\,|0 \leq \varepsilon < .5\left(L^{-1} - t\right)\right\}. \tag{2.5.34}$$

Denote by U_A the convergence domain if

$$L_0 = L \tag{2.5.35}$$

or

$$L_0 < L. \tag{2.5.36}$$

Then we can show that our convergence domain U_A contains U_w:

Proposition 2.5.5. *Under hypotheses of Theorem 4.3 in [98] (i.e., (2.5.25))*

$$U_G \subset U_A \tag{2.5.37}$$

where $\subset$ denotes strict inequality if (2.5.36) holds.

Proof. Condition (2.5.16) follows from (2.5.25). Hence the conclusions of Theorem 2.5.1 also hold. The rest follows from (2.5.32), (2.5.33), (2.5.35) and the definitions of sets U_G and U_A.

Remark 2.5.6. It was shown in Section 2.2 of this chapter that for $\delta \in [0, 2)$ the set

$$U_\delta(L_0, L) = \tag{2.5.38}$$
$$= \left\{(t,\varepsilon)\,\middle|\,K_\delta = L(t+\varepsilon) + \delta L_0 \varepsilon \leq \delta,\ L_0\left(t + \tfrac{2\delta\varepsilon}{2-\delta}\right) \leq 1,\ \tfrac{L_0\delta^2}{2-\delta} \leq L\right\}$$

contains pairs (t,ε) such that method (2.5.32) converges.

Clearly we have:

$$U_\delta(L_0, L) \subset U_A. \tag{2.5.39}$$

Moreover, we have

$$U_G \subseteq U_\delta(L_0, L), \tag{2.5.40}$$

as

$$U_1(L_0, L) \subseteq U_\delta(L_0, L). \tag{2.5.41}$$

Furthermore if $t = 0$ we get from (2.5.34), and (2.5.38) (for say $\delta = 1$)

$$K = 2Ln \leq 1, \tag{2.5.42}$$

and

$$K_1 = (L_0 + L)\, n \leq 1, \tag{2.5.43}$$

respectively.

Remark 2.5.7. It follows from the above that we also managed to enlarge the convergence domain U_G found in [99], and under the same computational cost.

Note also that condition (2.5.37) holds obviously for all possible choices of functions w_0 and w satisfying (2.5.6) (not only the ones given in Example 2.5.4). Claims (a)–(c) made in the introduction have now been justified.

Finally note that our technique used here only for NK method has been also illustrative and can be used easily on other methods appearing in [99] or elsewhere [43].

2.6 Convergence of NK method and operators with values in a cone

In this section, we are concerned with the solution of problems of the form

$$\text{Find } x^* \text{ such that } F\left(x^*\right) \in C, \tag{2.6.1}$$

where C is a nonempty closed convex cone in a Banach space Y, and F is a reflexive and continuously Fréchet-differentiable operator from a subset D_0 of a Banach space X into Y.

We used an extension of Newton's method to solve (2.6.1). The usual Newton's method corresponds with the special case when C is the degenerate cone $\{0\} \subseteq Y$. We provide a semilocal convergence analysis for Newton's method that generalizes the Newton-Kantorovich theorem. It turns out that our sufficient convergence conditions are weaker than the ones given by Robinson in [178], and under the same computational cost.

Let $p \in D_0$ be fixed. Define set-valued operator $G\,(p)$ from X into Y and its inverse by

$$G\,(p)\,x = F'\,(p)\,x - C, \quad x \in X, \tag{2.6.2}$$

$$G^{-1}\,(p)\,y = \left\{z \in F'\,(p)\,z \in y + C\right\}, \quad y \in Y, \tag{2.6.3}$$

where $F'\,(x)$ denotes the Fréchet-decreative of F evaluated at x. It is well-known that operator $G\,(p)$ as well as its inverse are convex [178]. Assume there exists an initial guess $x_0 \in D_0$ such that

$$G^{-1}\,(x_0)\,[-F\,(x_0)] \neq \varnothing. \tag{2.6.4}$$

Introduce algorithm so that given x_n, we choose x_{n+1} to be any solution of

$$\underset{n \geq 0}{\text{minimize}} \left\{ \|x - x_n\| \mid F(x_n) + F'(x_n)(x - x_n) \in C \right\}. \tag{2.6.5}$$

The similarity with the usual NK method is now clear. We expect that problem (2.6.2) will be easier to solve than problem (2.6.1) [178].

We need the following lemma on majorizing sequences. The proof can essentially be found in Lemma 2.4.7 (see also Section 2.3):

Lemma 2.6.1. *Assume there exist parameters $b > 0$, $\ell \geq 0$, $\ell_0 \geq 0$, with $\ell_0 \leq \ell$, $\eta \geq 0$, and*

$$h_\delta = b(\delta\ell_0 + \ell) n \leq \delta, \ \ \delta \in [0, 1] \tag{2.6.6}$$

or

$$h_\delta \leq \delta, \quad \frac{2b\ell_0 n}{2\delta} \leq 1, \quad \frac{\ell_0 \delta^2}{2 - \delta} \leq \ell, \quad \delta \in [0, 2) \tag{2.6.7}$$

or

$$\ell_0 n \leq 1 - \tfrac{1}{2}\delta, \quad \delta \in [\delta_0, 2) \tag{2.6.8}$$

where

$$\delta_0 = \frac{-b_0 + \sqrt{b_0^2 + 8b_0}}{2}, \quad b_0 = \frac{\ell}{\ell_0} \quad \text{for } \ell_0 \neq 0. \tag{2.6.9}$$

Then, iteration $\{t_n\}$ $(n \geq 0)$ given by

$$t_0 = 0, \ t_1 = \eta, \ t_{n+2} = t_{n+1} + \frac{b\ell(t_{n+1} - t_n)^2}{2(1 - b\ell_0 t_{n+1})} \quad (n \geq 0) \tag{2.6.10}$$

is nondecreasing, bounded by $t^{**} = \frac{2\eta}{2-\delta}$, and converges to some t^* such that

$$0 \leq t^* \leq t^{**}. \tag{2.6.11}$$

Moreover, the following error bounds hold for all $n \geq 0$:

$$0 \leq t_{n+2} - t_{n+1} \leq \frac{\delta}{2}(t_{n+1} - t_n) \leq \left(\frac{\delta}{2}\right)^{n+1} \eta. \tag{2.6.12}$$

We can show the following generalization of the Newton-Kantorovich theorem:

Theorem 2.6.2. *Let D_0, X, Y, C, F, and G be as above.*
Assume: there exists a point $x_0 \in D_0$ and nonnegative numbers b, ℓ_0, ℓ, η, δ such that (2.6.6) or (2.6.7) or (2.6.8) hold;

$$\left\| G^{-1}(x_0) \right\| \leq b, \tag{2.6.13}$$

$$\left\| F'(x) - F'(x_0) \right\| \leq \ell_0 \|x - x_0\|, \tag{2.6.14}$$

$$\left\| F'(x) - F'(y) \right\| \leq \ell \|x - y\|, \tag{2.6.15}$$

$$U(x_0, t^*) \subseteq D_0, \tag{2.6.16}$$

and

$$\|x_1 - x_0\| \leq \eta, \tag{2.6.17}$$

where x_1 is any point obtained from (2.6.5) (given x_0 satisfying (2.6.4) and t^* is given in Lemma 2.6.1).

Then, algorithm (2.6.4)–(2.6.5) generates at least one Newton-iteration $\{x_n\}$ $(n \geq 0)$, which is well defined, remains in $U(x_0, t^*)$ for all $n \geq 0$ and converges to some $x^* \in U(x_0, t^*)$ such that $F(x^*) \in C$. Moreover the following estimates hold for all $n \geq 0$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{2.6.18}$$

and

$$\left\|x_n - x^*\right\| \leq t^* - t_n. \tag{2.6.19}$$

Proof. We first show sequence $\{x_n\}$ $(n \geq 0)$ exists, $x_n \in \overline{U}(x_0, t^*)$ and

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \quad (n \geq 0). \tag{2.6.20}$$

Point x_1 exists as $G(x_0)$ is an onto operator, which solves (2.6.5) for $n = 0$, and (2.6.20) holds for $k = 0$ (by (2.6.17)). Moreover we get

$$x_1 \in \overline{U}\left(x_0, t^*\right).$$

If (2.6.5) is feasible it must be solvable. Indeed, because $F'(x_k)$ is continuous and C is closed and convex the feasible set of (2.6.5) is also closed and convex. The existence of a feasible point q implies that any solution of (2.6.5) lie in the intersection of the feasible set of (2.6.5) and $\overline{U}(|x_k, 1\ \|q - x_k\|)$. Moreover this intersection is a closed, convex and bounded set. Furthermore because X is reflexive and function $\|x - x_k\|$ is weakly lower semicontinuous a solution of (2.6.5) exists [178]. Finally, because (2.6.5) is a convex minimization problem, any solution will be a global solution.

Now assume $x_1, x_2, ..., x_{n+1}$ exists satisfying (2.6.20).

Then, we get

$$\begin{aligned} \|x_{k+1} - x_0\| &\leq \|x_{k+1} - x_k\| + \|x_n - x_{k-1}\| + \cdots + \|x_1 - x_0\| \\ &\leq (t_{k+1} - t_k) + (t_k - t_{k-1}) + \cdots + (t_1 - t_0) = t_{k+1} \leq t^*. \end{aligned} \tag{2.6.21}$$

Hence, $x_{k+1} \in \overline{U}(x_0, t^*)$.

By (2.6.14) we have:

$$\begin{aligned} \left\|G^{-1}(x_0)\right\| \left\|F'(x_{k+1}) - F'(x_0)\right\| &\leq b\ell_0 \|x_{k+1} - x_0\| \\ &\leq b\ell_0 t_{k+1} < 1. \end{aligned} \tag{2.6.22}$$

Therefore the convexity of $G(x_{k+1})$ carries to

$$F'(x_{k+1})\,x - C = \left\{F'(x_0) + \left[F(x_{k+1}) - F'(x_0)\right]\right\} x - C, \tag{2.6.23}$$

and by the Banach Lemma

$$\left\|G^{-1}(x_{k+1})\right\| \leq \frac{\left\|G^{-1}(x_0)\right\|}{1-\left\|G^{-1}(x_0)\right\|\left\|F'(x_{k+1})-F'(x_0)\right\|} \leq \frac{b}{1-b\ell_0\left\|x_{k+1}-x_0\right\|}. \tag{2.6.24}$$

It follows that (2.6.5) is feasible, and hence solvable for $n=k=1$, so that x_{k+1} exists. We need to solve for x:

$$F(x_{k+1})+F'(x_{k+1})(x-x_{k+1}) \in F(x_k)+F'(x_k)(x_{k+1}-x_k)+C. \tag{2.6.25}$$

But x_{k+1} solves (2.6.5) with $n=k$, so the right-hand side of (2.6.25) is contained in C. Hence any x satisfying (2.6.25) also satisfies (2.6.5) for $n=k+1$. We can rewrite (2.6.25) as

$$x-x_{k+1} \in G^{-1}(x_{k+1})\left[-F(x_{k+1})+F(x_k)+F'(x_k)(x_{k+1}-x_k)\right]. \tag{2.6.26}$$

Using (2.6.15) we get

$$\left\|-F(x_{n+1})+F(x_k)+F'(x_k)(x_{k+1}-x_k)\right\| \leq \tfrac{1}{2}\ell\left\|x_{k+1}-x_k\right\|^2. \tag{2.6.27}$$

Because the right-hand side of (2.6.26) contains an element of least norms, there exists some q satisfying (2.6.26) and consequently (2.6.25) so that

$$\begin{aligned}\left\|q-x_{k+1}\right\| &\leq \left\|G^{-1}(x_{k+1})\right\|\left\|-F(x_{k+1})+F'(x_k)(x_{k+1}-x_k)\right\| \\ &\leq \frac{\frac{1}{2}b\ell\left\|x_{k+1}-x_k\right\|^2}{1-b\ell_0\left\|x_{k+1}-x_0\right\|} \leq \frac{\frac{1}{2}b\ell\left(t_{k+1}-t_k\right)^2}{1-b\ell_0 t_{k+1}}\end{aligned} \tag{2.6.28}$$

That is q is also feasible for (b) with $n=k$, we have

$$\left\|x_{k+2}-x_{k+1}\right\| \leq \left\|q-x_{k+1}\right\| \leq t_{k+2}-t_{k+1} \tag{2.6.29}$$

and $x_{k+2} \in \overline{U}(x_0, t^*)$ which completes the induction. Hence sequence $\{x_n\}$ is Cauchy in X and as such it converges to some $x^* \in U(x_0, t^*)$. Then for any k

$$\left[F(x_{k+1})-F\left(x^*\right)\right]-\left[F(x_{k+1})-F(x_k)-F'(x_k)(x_{k+1}-x_k)\right] \in C-F\left(x^*\right). \tag{2.6.30}$$

The left-hand side of (2.6.30) approaches zero by the continuity assumptions, and as $C-F(x^*)$ is closed we get $F(x^*) \in C$.

Finally (2.6.19) follows from (2.6.18) by standard majorization techniques.

Remark 2.6.3. Our Theorem 2.6.2 reduces to Theorem 2 in [178, pp. 343] if $\ell_0=\ell$.

The advantages of this approach have already been explained in Section 2.2 of Chapter 2.

2.7 Convergence theorems involving center-Lipschitz conditions

In this section, we are concerned with the problem of approximating a locally unique solution x^* of equation (2.1.1).

Most authors have used a Lipschitz-type hypotheses of the form

$$\left\| F'(x_0)^{-1}\left[F'(x) - F'(y)\right]\right\| \leq \alpha(r) \tag{2.7.1}$$

for all $x, y, \in \overline{U}(x_0, r) \subseteq \overline{U}(x_0, R) \subseteq D$ for some $R > 0$ and a continuous nonnegative function α in connection with the NK method. The computation of function α is very difficult or impossible in general (see Example 2.2.18). That is why we use instead hypotheses (2.7.4) in which the corresponding function w_0 is easier to compute.

Based on this idea, we produce local and semilocal convergence theorems for the NK. Our results can be weaker than the corresponding ones using (2.7.1). In the local case, we show that a larger convergence radius can be obtained.

We provide the following semilocal convergence results involving center-Lipschitz conditions:

Theorem 2.7.1. *Let $F: D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume:*

there exist a point $x_0 \in D$, $\eta \geq 0$, $R > 0$, and nonnegative continuous functions w_0, w such that:

$$F'(x_0)^{-1} \in L(X, Y), \tag{2.7.2}$$

$$\left\| F'(x_0)^{-1} F(x_0)\right\| \leq \eta \tag{2.7.3}$$

$$\left\| F'(x_0)^{-1}\left[F'(x) - F'(x_0)\right]\right\| \leq w_0(\|x - x_0\|), \tag{2.7.4}$$

$$\left\| F'(x)^{-1} F'(x_0)\right\| \leq w(\|x - x_0\|) \tag{2.7.5}$$

for all $x \in \overline{D}(x_0, r) \subseteq \overline{U}(x_0, r)$;

equation

$$w(r)\left\{\left[\int_0^1 w_0(tr)\, dt + w_0(r)\right] r + \eta\right\} = r \tag{2.7.6}$$

has solutions on $(0, R]$. Denote by r_0 the smallest positive solution of equation (2.7.6):

$$q = 2w_0(r_0)\, w(r_0) < 1; \tag{2.7.7}$$

and

$$\overline{U}(x_0, R) \subseteq D. \tag{2.7.8}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by NK method is well defined and remains in $\overline{U}(x_0, r_0)$ for all $n \geq 0$ and converges to a unique solution x^* of equation $F(x) = 0$ in $\overline{U}(x_0, r_0)$. Moreover the following estimates hold for all $n \geq 0$:

$$\|x_{n+2} - x_{n+1}\| \leq q\, \|x_{n+1} - x_n\| \tag{2.7.9}$$

and

$$\left\| x_n - x^* \right\| \leq \frac{\eta}{1-q} q^n. \tag{2.7.10}$$

Moreover if there exists $r_1 \in (r_0, R]$ *such that*

$$w(r_0) \left[\int_0^1 w_0 [(1-t) r_0 + t r_1] \, dt + w_0 (r_0) \right] \leq 1 \tag{2.7.11}$$

the solution x^* *is in* $U(x_0, r_1)$.

Proof. By (2.7.1), (2.7.3), (2.7.6) and the definition of r_0 $\|x_1 - x_0\| \leq \eta \leq r_0$. Hence $x_1 \in \overline{U}(x_0, r_0)$. Assume $x_k \in \overline{U}(x_0, r_0)$, $k = 0, 1, ..., n$. Using (2.7.1) we obtain the approximation

$$\begin{aligned}
&x_{k+1} - x_0 = \\
&= x_k - F'(x_k)^{-1} F(x_k) - x_0 \\
&= -\left[F'(x_k)^{-1} F'(x_0) \right] F'(x_0)^{-1} \left\{ \int_0^1 \left[F'(x_0 + t(x_k - x_0)) - F'(x_0) \right] \right. \\
&\left. \cdot (x_k - x_0) \, dt + \left(F'(x_0) - F'(x_k) \right) (x_k - x_0) + F(x_0) \right\}.
\end{aligned} \tag{2.7.12}$$

By (2.7.3)–(2.7.6) and (2.7.12) we get in turn

$$\begin{aligned}
\|x_{k+1} - x_0\| \leq & \left\| F'(x_k)^{-1} F'(x_0) \right\| \\
& \cdot \left\{ \left\| F'(x_0)^{-1} \left[F'(x_0 + t(x_k - x_0)) - F'(x_0) \right] \right\| \|x_k - x_0\| \, dt \right. \\
& + \left\| F'(x_0)^{-1} \left[F'(x_0) - F'(x_k) \right] \right\| \\
& \left. \cdot \|x_k - x_0\| + \left\| F'(x_0)^{-1} F(x_0) \right\| \right\} \\
& \leq w(\|x_k - x_0\|) \left\{ \left[\int_0^1 w_0 (t \|x_k - x_0\|) \, dt \right. \right. \\
& \left. \left. + w_0 (\|x_k - x_0\|) \right] \|x_k - x_0\| + \eta \right\} \\
& \leq w(r) \left\{ \left[\int_0^1 w_0 (tr) \, dt + w_0 (r) \right] r + \eta \right\} = r.
\end{aligned} \tag{2.7.13}$$

That is $x_{k+1} \in \overline{U}(x_0, r_0)$. Moreover by (2.7.1) we obtain the approximation

$$\begin{aligned}
F'(x_0)^{-1} F(x_{k+1}) &= F'(x_0)^{-1} \left[F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k) \right] \\
&= F'(x_0)^{-1} \left\{ \int_0^1 \left[F'(x_k + t(x_{k+1} - x_k)) - F'(x_0) \right] dt \right. \\
&\left. + F'(x_0)^{-1} \left[F'(x_0) - F'(x_k) \right] \right\} (x_{k+1} - x_k).
\end{aligned} \tag{2.7.14}$$

By (2.7.4) and (2.7.14) we get in turn

$$\begin{aligned}
&\left\| F'(x_0)^{-1} F(x_{k+1}) \right\| \\
&\leq \left\{ \left\| F'(x_0)^{-1} \int_0^{1-1} \left[F'(x_k + t(x_{k+1} - x_k)) - F'(x_0) \right] \right\| dt \right. \\
&\left. + \left\| F'(x_0)^{-1} \left[F'(x_0) - F'(x_k) \right] \right\| \right\} \| x_{k+1} - x_k \| \\
&\leq \left\{ \int_0^1 w_0 \left[(1-t) \| x_k - x_0 \| + t \| x_{k+1} - x_0 \| \right] dt \right. \\
&\left. + w_0 (\| x_k - x_0 \|) \right\} \| x_{k+1} - x_k \| \\
&\leq \left[\int_0^1 w_0 \left[(1-t) r_0 + t r_0 \right] dt + w_0(r_0) \right] \| x_{k+1} - x_k \| \\
&= 2 w_0(r_0) \| x_{k+1} - x_k \| .
\end{aligned} \tag{2.7.15}$$

Hence by (2.7.1), (2.7.5) and (2.7.15) we obtain

$$\begin{aligned}
\| x_{k+2} - x_{k+1} \| &= \left\| \left[F'(x_{k+1})^{-1} F'(x_0) \right] \left[F'(x_0)^{-1} F(x_{k+1}) \right] \right\| \\
&\leq \left\| F'(x_{k+1})^{-1} F'(x_0) \right\| \cdot \left\| F'(x_0)^{-1} F(x_{k+1}) \right\| \\
&\leq w(r_0) \, 2w(r_0) \| x_{k+1} - x_k \| \\
&= q \| x_{k+1} - x_k \| \leq q^{k+1} \eta,
\end{aligned} \tag{2.7.16}$$

which shows (2.7.9) for all $n \geq 0$.

Let $m > 1$, then we get using (2.7.9),

$$\begin{aligned}
x_{n+m} - x_{n+1} &= (x_{n+m} - x_{n+m-1}) + (x_{n+m-1} - x_{n+m-2}) \\
&\quad + \cdots + (x_{n+2} - x_{n+1}),
\end{aligned} \tag{2.7.17}$$

and

$$\| x_{n+m} - x_{n+1} \| \leq q \| x_{n+m-1} - x_{n+m-2} \| \leq \cdots \leq q^{m-1} \| x_{n+1} - x_n \| \tag{2.7.18}$$

that

$$\begin{aligned}
\| x_{n+m} - x_{n+1} \| &\leq \left(q + \cdots + q^{m-2} + q^{m-1} \right) \| x_{n+1} - x_n \| \\
&\leq q \frac{1 - q^{m-1}}{1 - q} q^n \eta.
\end{aligned} \tag{2.7.19}$$

It follows from (2.7.19) and (2.7.7) that sequence $\{x_n\}$ $(n \geq 0)$ is Cauchy in a Banach space X and as such it converges to some $x^* \in \overline{U}(x_0, r_0)$. By letting $k \to \infty$, $m \to \infty$ in (2.7.16) and (2.7.19) we get (2.7.10) and (2.7.19) we get (2.7.10) and $F(x^*) = 0$ respectively.

To show uniqueness in $\overline{U}(x_0, r_0)$ let x_1^* be a solution in $\overline{U}(x_0, r_0)$. Using the approximation

$$
\begin{aligned}
x_{n+1} - x_1^* &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.7.20)\\
&= x_n - F'(x_n)^{-1} F(x_n) - x_1^* \\
&= F'(x_n)^{-1} \left[F\left(x^*\right) - F(x_n) - F'(x_n)\left(x_1^* - x_n\right)\right] \\
&= \left[F'(x_n)^{-1} F'(x_0)\right] F'(x_0)^{-1} \left\{ \int_0^1 \left[F'\left(x_n + t\left(x_1^* - x_n\right)\right) - F'(x_0)\right] dt \right. \\
&\qquad \left. + \left[F'(x_0) - F'(x_n)\right] \right\} \left(x_1^* - x_n\right)
\end{aligned}
$$

as in (2.7.13) we get

$$
\begin{aligned}
\left\|x_{n+1} - x_1^*\right\| &\le w(\|x_n - x_0\|) \left[\int_0^1 w_0 \left[(1-t)\|x_n - x_0\| \right.\right. \qquad (2.7.21)\\
&\qquad \left.\left. + t\left\|x_1^* - x_0\right\|\right] + w_0(\|x_n - x_0\|) \right] \left\|x_n - x_1^*\right\| \\
&\le q\left\|x_n - x_1^*\right\|. \qquad (2.7.22)
\end{aligned}
$$

By (2.7.7) and (2.7.22) $\lim_{n\to\infty} x_n = x_1^*$. But we already showed $\lim_{n\to\infty} x_n = x^*$. Hence, we conclude

$$x^* = x_1^*.$$

Finally to show uniqueness in $U(x_0, r_1)$, let x_1^* be a solution of $F(x) = 0$ in $U(x_0, r_1)$. As in (2.7.21) we get

$$
\begin{aligned}
\left\|x_{n+1} - x_1^*\right\| &< w(r_0) \left[\int_0^1 w_0 \left[(1-t) r_0 + t r_1\right] dt + w_0(r_0) \right] \left\|x_n - x_1^*\right\| \\
&\le \left\|x_n - x_1^*\right\|. \qquad (2.7.23)
\end{aligned}
$$

By (2.7.23) we get

$$\lim_{n\to\infty} x_n = x_1^*.$$

Hence, again we deduce:

$$x^* = x_1^*.$$

Remark 2.7.2. In order for us to compare our results with earlier ones, consider the Lipschitz condition (2.2.36) and the Newton-Kantorovich hypothesis (2.2.37). Define

$$w_0(r) = \ell_0 r^\lambda, \tag{2.7.24}$$

$$w(r) = \ell_1 \tag{2.7.25}$$

for some $\lambda \ge 0$, $\ell_0 \ge 0$, $\ell_1 > 0$ and all $r \in [0, R]$. Assuming conditions (2.7.4), (2.7.5) and (2.7.7) hold with the above choices then (2.7.6) and (2.7.7) reduce to

$$\frac{(\lambda+2)}{\lambda+1}\ell_0 r^{\lambda+1} - \frac{r}{\ell_1} + \eta = 0, \tag{2.7.26}$$

and

$$2\ell_0\ell_1 r^{\lambda} < 1. \tag{2.7.27}$$

Set $\lambda = 1$ then (2.7.26) and (2.7.27) are satisfied if

$$h_0 = 6\ell_0\ell_1^2\eta \leq 1 \tag{2.7.28}$$

with r_0 being the small solution of equation (2.7.26). By comparing (2.2.37) and (2.7.28) we see that (2.7.28) is weaker if (2.7.28) holds, and

$$3\ell_1^2 < \frac{\ell}{\ell_0} \quad (\ell_0 \neq 0). \tag{2.7.29}$$

This can happen in practice as $\frac{\ell}{\ell_0}$ can be arbitrarily large and hence larger than $3\ell_1^2$ (see Section 2.2).

This comparison can become even more favorable if $\lambda > 1$. Such a case is provided in Example 2.7.4.

Assume there exist a zero x^* of F, $R > 0$, and nonnegative continuous functions v_0, v such that

$$F'\left(x^*\right)^{-1} \in L(Y, X), \tag{2.7.30}$$

$$\left\| F'\left(x^*\right)^{-1}\left[F'(x) - F'\left(x^*\right)\right]\right\| \leq v_0\left(\left\|x - x^*\right\|\right), \tag{2.7.31}$$

$$\left\| F'(x)^{-1} F'\left(x^*\right)\right\| \leq v\left(\left\|x - x^*\right\|\right) \tag{2.7.32}$$

for all $x \in \overline{U}(x^*, r) \subseteq \overline{U}(x^*, R)$; equation

$$v(r)\left[\int_0^1 v_0[(1-t)r]\,dt + v_0(r)\right] = 1 \tag{2.7.33}$$

has solutions in $[0, R]$. Denote by r^* the smallest;

$$\overline{U}\left(x^*, R\right) \subseteq D. \tag{2.7.34}$$

Then the following local convergence result holds for NK method.

Theorem 2.7.3. *Under the above stated hypotheses sequence $\{x_n\}$ $(n \geq 0)$ generated by NK method is well defined, remains in $\overline{U}(x^*, r^*)$ for all $n \geq 0$, and converges to x^* provided that $x_0 \in U(x^*, r^*)$.*

Moreover the following estimates hold for all $n \geq 0$

$$\left\|x_{n+1} - x^*\right\| \leq a_n \left\|x_n - x^*\right\| \leq a \left\|x_n - x^*\right\|, \tag{2.7.35}$$

where

$$a_n = w\left(\left\|x_n - x^*\right\|\right)\left[\int_0^1 w_0\left((1-t)\left\|x_n - x^*\right\|\right)dt + w_0\left(\left\|x_n - x^*\right\|\right)\right] \quad (2.7.36)$$

and

$$a = w\left(r^*\right)\left[\int_0^1 w_0\left((1-t)\,r^*\right)dt + w_0\left(r^*\right)\right]. \quad (2.7.37)$$

Proof. It follows as in Theorem 2.7.1 by using (2.7.30)–(2.7.34), induction on n and the approximation

$$\begin{aligned} x_{n+1} - x^* &= \quad (2.7.38) \\ &= \left[F'(x_n)^{-1} F'\left(x^*\right)\right]\left\{\int_0^1 \left[F'\left(x_n + t\left(x^* - x_n\right)\right)\right.\right. \\ &\qquad \left.\left. - F'\left(x^*\right)\right]dt + \left[F'\left(x^*\right) - F'(x_n)\right]\right\}\left(x^* - x_n\right). \end{aligned}$$

We complete this section with a numerical example to show that we can obtain a larger convergence radius than in earlier results.

Example 2.7.4. Let $X = Y = \mathbf{R}$, $D = U(0, 1)$, and define function F on D by

$$F(x) = \tfrac{1}{5}e^{x^5} - x - \tfrac{1}{5}. \quad (2.7.39)$$

Choose $v_0(r) = \ell_0 r^\mu$, $v(r) = b$. Then it can easily be seen from (2.7.30)–(2.7.32), (2.7.39) that $\ell_0 = e$, $\mu = 4$, and

$$b = 1.581976707 = \left\|F'(-1)^{-1} F'(0)\right\|.$$

Equation (2.7.33) becomes

$$\left[\int_0^1 e\left[(1-t)\,r\right]^4 dt + er^4\right]b = 1 \quad (2.7.40)$$

or

$$r^* = \left[\frac{5}{6eb}\right]^{\frac{1}{4}} = .663484905. \quad (2.7.41)$$

We saw earlier in Example 2.4.16 that Rheinboldt radius [175] is given by

$$r_1^* = .245252961. \quad (2.7.42)$$

Hence, we conclude:

$$r_1^* < r^*. \quad (2.7.43)$$

Example 2.7.5. We refer the reader to Example 2.2.18.

2.8 The radius of convergence for the NK method

Let $F\colon D \subseteq X \to Y$ be an m-times continuously Fréchet-differentiable operator ($m \geq 2$ an integer) defined on an open convex subset D of a Banach space X with values in a Banach space Y. Suppose there exists $x^* \in D$ that is a solution of the equation

$$F(x) = 0. \tag{2.8.1}$$

The most popular method for approximating such a point x^* is Newton's method

$$x_{n+1} = G(x_n) \quad (n \geq 0), \quad (x_0 \in D), \tag{2.8.2}$$

where

$$G(x) \equiv x - F'(x)^{-1} F(x) \quad (x \in D). \tag{2.8.3}$$

In the elegant paper by Ypma [216], affine invariant results have been given concerning the radius of convergence of Newton's method. Ypma used Lipschitz conditions on the first Fréchet derivative as the basis for his analysis. In this study, we use Lipschitz-like conditions on the mth Fréchet derivative $F^{(m)}(x) \in L\left(X_1^m, Y_2\right)$ $(x \in D)$ $(m \geq 2)$ an integer. This way we manage to enlarge the radius of convergence for Newton's method (2.8.2). Finally we provide numerical examples to show that our results guarantee convergence, where earlier ones do not [216]. This is important in numerical computations [43], [216].

We give an affine invariant form of the Banach lemma on invertible operators.

Lemma 2.8.1. *Let $m \geq 2$ be an integer, $\alpha_i \geq 2m$ $(2 \leq i \leq m)$, $\eta \geq 0$, X, Y Banach spaces, D a convex subset of X and $F\colon D \to Y$ an m-times Fréchet-differentiable operator. Assume there exist $z \in D$ so that $F'(z)^{-1}$ exists, and some convex neighborhood $N(z) \subseteq D$*

$$\left\| F'(z)^{-1} F^{(i)}(z) \right\| \leq \alpha_i, \; i = 2, ..., m \tag{2.8.4}$$

and

$$\left\| F'(z)^{-1} \left[F^{(m)}(x) - F^{(m)}(z) \right] \right\| \leq \varepsilon_0 \;\; \textit{for all} \;\; x \in N(z), \varepsilon_0 > 0. \tag{2.8.5}$$

If $x \in N(z) \cap U(z, \delta)$, where δ is the positive zero of the equation $f'(t) = 0$, where

$$f(t) = \frac{\alpha_m + \varepsilon_0}{m!} t^m + \cdots + \frac{\alpha^2}{2!} t^2 - t + d \tag{2.8.6}$$

then $F'(x)^{-1}$ exists and for $\|x - z\| < t \leq \delta$

$$\left\| F'(z)^{-1} F''(x) \right\| s < f''(t) \tag{2.8.7}$$

and

$$\left\| F''(x)^{-1} F'(z) \right\| \leq -f'(t)^{-1}. \tag{2.8.8}$$

Proof. It is convenient to define $\varepsilon, b_1, b_i, i = 2, ..., m$ by

$$\begin{aligned} \varepsilon &= x - z_0, \\ b_1 &= z + \theta\varepsilon, \\ b_i &= z + \theta_i \left(b_{i-1} - z\right), \qquad \theta \in [0, 1]. \end{aligned}$$

We can have in turn

$$\begin{aligned} F''(x) & \\ &= F''(z) + \left[F''(x) - F''(z)\right] \\ &= F''(z) + \int_0^1 F'''\left[z + \theta_1 (x - z)\right](x - z)\, d\theta_1 \\ &= F''(z) + \int_0^1 \left[F'''(z + \theta_1 (x - z)) - F'''(z)\right](x - z)\, d\theta_1 \\ &\quad + \int_0^1 F'''(z)(x - z)\, d\theta_1 \\ &= F''(z) + \int_0^1 F'''(z)(x - z)\, d\theta_1 + \int_0^1 \int_0^1 F^{(4)}\{z + \theta_2 \\ &\qquad \cdot [z + \theta_1 (x - z) - z]\}[z + \theta_1 (x - z) z](x - z)\, d\theta_2 d\theta_1 \\ &= F''(z) + \int_0^1 F'''(z)\, \varepsilon d\theta_1 + \int_0^1 \int_0^1 F^{(4)}(b_2)(b_1 - z_0)\, \varepsilon d\theta_2 d\theta_1 \\ &= \cdots \\ &= F''(z) + \int_0^1 F'''(z)\, \varepsilon d\theta_1 + \cdots + \int_0^1 \cdots \int_0^1 F^{(m)}(b_{m-2})(b_{m-3} - z) \\ &\qquad \cdots (b_1 - z)\, d\theta_{m-2} \cdots d\theta_1 \\ &= F''(z) + \int_0^1 F'''(z)\, \varepsilon d\theta_1 + \cdots + \int_0^1 \cdots \int_0^1 F^{(m)}(z)(b_{m-3} - z) \\ &\qquad \cdots (b_1 - z)\, \varepsilon d\theta_{m-2} \cdots d\theta_1 \\ &\quad + \int_0^1 \cdots \int_0^1 \left[F^{(m)}(b_{m-2}) - F^{(m)}(z)\right](b_{m-3} - z) \\ &\qquad \cdots (b_1 - z)\, \varepsilon d\theta_{m-2} \cdots d\theta_1. \end{aligned} \tag{2.8.9}$$

Using the triangle inequality, (2.8.4), (2.8.5), (2.8.6) in (2.8.9) after composing by $F'(z)^{-1}$, we obtain (2.8.7).

We also get

$$
\begin{aligned}
&-F'(z)^{-1}\left[F'(z)-F'(x)\right]\\
&\quad = F'(z)^{-1}\left[F'(x)-F'(z)-F''(z)(x-z)+F''(z)(x-z)\right]\\
&\quad = \int_0^1 F'(z)^{-1}\left\{F''[z+\theta_1\varepsilon]-F''(z)\right\}d\theta_1\varepsilon + F'(z)^{-1}\int_0^1 F''(z)\,\varepsilon d\theta_1\\
&\quad = \int_0^1\int_0^1 F'(z)\,F'''(b_2)(b_1-z)\,\varepsilon d\theta_2 d\theta_1 + F'(z)^{-1}\int_0^1 F''(z)\,\varepsilon d\theta_1\\
&\quad = \cdots\\
&\quad = \int_0^1\cdots\int_0^1 F^{(m)}(b_{m-1})(b_{m-2}-z)\\
&\qquad\qquad\qquad \cdots(b_1-z)\,\varepsilon d\theta_{m-1}\varepsilon d\theta_{m-2}\cdots d\theta_2 d\theta_1\\
&\qquad + \int_0^1\cdots\int_0^1 F^{(m-1)}(b_{m-2})(b_{m-3}-z)\cdots(b_1-z)\,\varepsilon d\theta_{m-2}\cdots d\theta_2 d\theta_1\\
&\qquad + \cdots + \int_0^1 F'(z)^{-1}F''(z)\,\varepsilon d\theta_1\\
&\quad = \int_0^1\int_0^1 F'(z)^{-1}\left[F^{(m)}(b_{m-1})-F^{(m)}(z)\right](b_{m-2}-z)\\
&\qquad\qquad\qquad \cdots(b_1-z)\,\varepsilon d\theta_{m-2}\cdots d\theta_1\\
&\qquad + \int_0^1\cdots\int_0^1 F'(z)^{-1}F^{(m)}(z)(b_{m-2}-z)\cdots(b_1-z)\,\varepsilon d\theta_{m-1}\cdots d\theta_1\\
&\qquad + \int_0^1\cdots\int_0^1 F'(z)^{-1}F^{(m-1)}(z)(b_{m-3}-z)\cdots(b_1-z)\,\varepsilon d\theta_{m-2}\cdots d\theta_1\\
&\qquad + \cdots + \int_0^1 F'(z)^{-1}F''(z)\,\varepsilon d\theta_1.
\end{aligned}
\tag{2.8.10}
$$

Because $f'(t) < 0$ on $[0, \delta]$, using (2.8.4), (2.8.5), (2.8.6) in (2.8.10) we obtain for $\|x - z\| < t$

$$\left\|-F'(z)^{-1}\left[F'(z)-F'(x)\right]\right\| \le 1 + f'(\|x-z\|) < 1 + f'(t) < 1. \tag{2.8.11}$$

It follows from the Banach Lemma on invertible operators (2.8.11) $F'(x)^{-1}$ exists, and

$$\left\|F'(x)^{-1}F'(z)\right\| \le \left[1 - \left\|F'(z)^{-1}\left[F'(z)-F'(x)\right]\right\|\right]^{-1} \le -f'(t)^{-1}.$$

which shows (2.8.8).

We need the following affine invariant form of the mean value theorem for m-Fréchet-differentiable operators.

Lemma 2.8.2. *Let $m \geq 2$ be an integer, $\alpha_i \geq 0$ $(2 \leq i \leq m)$, X, Y Banach spaces, D a convex subset of X and $F; D \to Y$ an m-times Fréchet-differentiable operator. Assume there exist $z \in D$ so that $F'(z)^{-1}$ exists, and some convex neighborhood $N(z)$ of z such that $N(z) \subseteq D$,*

$$\left\| F'(z)^{-1} F^{(i)}(z) \right\| \leq \alpha_i, \quad i = 2, ..., m,$$

and

$$\left\| F'(z)^{-1} \left[F^{(m)}(x) - F^{(m)}(z) \right] \right\| \leq \varepsilon_0 \ \ \text{for all} \ \ x \in N(z), \ \varepsilon_0 > 0.$$

Then for all $x \in N(z)$

$$\begin{aligned} &\left\| F'(z)^{-1} \left[F(z) - F(x)(z-x) \right] \right\| \\ &\leq \frac{\alpha_m + \varepsilon}{m!} \|x-z\|^m + \frac{\alpha_{m-1}}{(m-1)!} \|x-z\|^{m-1} + \cdots + \frac{\alpha_2}{2!} \|x-z\|^2 . \end{aligned} \tag{2.8.12}$$

Proof. We can write in turn:

$$\begin{aligned} &F(z) - F(x) - F'(x)(z-x) \\ &= \int_0^1 \left[F'(x + \theta_1(z-x)) - F'(x) \right] (z-x)\, d\theta_1 \\ &= \int_0^1 \left[F''(z + \theta_1(x-z)) - F''(z) \right] \theta_1 d\theta_1 (x-z)^2 + \int_0^1 \theta_1 F''(z)(x-z)^2 \theta_1 \\ &= \int_0^1 \int_0^1 \left[F'''(z + \theta_2\theta_1(x-z)) - F'''(z) \right] \theta_1 (x-z)\, d\theta_2 \theta_1 d\theta_1 (x-z)^2 \\ &\quad + \int_0^1 \int_0^1 F'''(z)\theta_1(x-z)\, d\theta_2\theta_1 d\theta_1 (x-z)^2 + \int_0^1 \theta_1 F''(z)(x-z)^2 d\theta_1 \\ &= \cdots \\ &= \int_0^1 \int_0^1 \cdots \int_0^1 \left[F^{(m)}(z + \theta_{m-1}\theta_{m-2}\cdots\theta_1(x-z)) - F^{(m)}(z) \right] \theta_{m-2}^1 \\ &\qquad \cdots \theta_3^{m-4}\theta_2^{m-3}\theta_1^{m-1}(x-z)^m d\theta_{m-1} d\theta_{m-2} \cdots d\theta_3 d\theta_2 d\theta_1 \\ &\quad + \cdots + \int_0^1 \int_0^1 F'''(z)\theta_1^2(x-z)^3 d\theta_2 d\theta_1 + \int_0^1 \theta_1 F''(z)(x-z)^2 d\theta_1 . \end{aligned} \tag{2.8.13}$$

Composing both sides by $F'(z)^{-1}$, using the triangle inequality, (2.8.5) and (2.8.6) we obtain (2.8.12).

Based on the above lemmas, we derive affine invariant convergence results for the class $T \equiv T(\{\alpha_i\}, 2 \leq i \leq m, \alpha)$ $(\alpha > 0, \alpha_i \geq 0, 2 \leq i \leq m)$ of operators F defined by $T \equiv \{F | F\colon D \subseteq X \to Y;$ D open and convex set, F m-times continuously Fréchet-differentiable on D; there exists $x^* \in D$ such that $F(x^*) = 0$; $F'(x)^{-1}$ exists; $U(x^*, \alpha) \subseteq D$; x^* is the only solution of equation $F(x) = 0$ in $U(x^*, \alpha)$; and for all $x \in U(x^*, \alpha)$,

$$\left\| F'\left(x^*\right)^{-1} \left[F^{(m)}\left(x^*\right) - F^{(m)}(x) \right] \right\| s < \varepsilon_0, \ \varepsilon_0 > 0, \tag{2.8.14}$$

and

$$\left\| F'\left(x^*\right)^{-1} F^{(i)}\left(x^*\right) \right\| \leq \alpha_i, \quad i = 2, ..., m. \tag{2.8.15}$$

Let $F \in T$ and $x \in U(x^*, b)$ where $b \leq \min\{\alpha, \delta\}$. By Lemma 2.8.1, $F'(x)^{-1}$ exists. Define

$$\mu(F, x) \equiv \sup \left\{ \left\| F'(x)^{-1} \left[F^{(m)}(y) - F^{(m)} \right] \right\| \mid y \in U\left(x^*, b\right) \right\}, \tag{2.8.16}$$

$$q_i = q_i(F, x) \equiv \left\| F'(x)^{-1} F^{(i)}\left(x^*\right) \right\|, \quad 2 \leq i \leq m, \ x \in U\left(x^*, b\right). \tag{2.8.17}$$

It follows from (2.8.14)–(2.8.17) that

$$\mu\left(F, x^*\right) \leq \varepsilon_0 = \varepsilon\left(x^*\right), \quad q_i\left(F, x^*\right) \leq \alpha_i, \quad 2s < i \leq m, \tag{2.8.18}$$

$F \in T(\{q_i\}, \ 2 \leq i \leq m, \ \mu(F, x^*), \alpha)$, and by Lemma 2.8.1

$$\mu(F, x) \leq \frac{\mu(F, x^*)}{1 - q_2 \|x - x^*\| - \cdots - \frac{\mu(F, x^*) + \varepsilon_0}{(m-1)!} \|x - x^*\|^{m-1}} \equiv \overline{\mu}(x). \tag{2.8.19}$$

We also have the estimates

$$\begin{aligned} \left\| F'(x)^{-1} F^{(i)}\left(x^*\right) \right\| &\leq \left\| F'(x)^{-1} F'\left(x^*\right) \right\| \left\| F'\left(x^*\right)^{-1} F^{(i)}\left(x^*\right) \right\| \\ &\leq q_i \left\| F'(x)^{-1} F'\left(x^*\right) \right\| \\ &\leq \frac{q_i}{1 - \alpha_2 \|x - x^*\| - \cdots - \frac{\alpha_m + \varepsilon_0}{(m-1)!} \|x - x^*\|^{m-1}} \equiv \overline{q}_i(x). \end{aligned} \tag{2.8.20}$$

The following lemma on fixed points is important.

Lemma 2.8.3. *Let F, x be as above. Then, the Newton operator G defined in (2.8.3) satisfies:*

$$\begin{aligned} \left\| G(x) - x^* \right\| \leq \frac{\mu(F, x) + q_m}{m!} \left\| x - x^* \right\| + \frac{q_{m-1}}{(m-1)!} \left\| x - x^* \right\|^{m-1} \\ + \cdots + \frac{q_2}{2!} \left\| x - x^* \right\|^2 \end{aligned} \tag{2.8.21}$$

and

$$\left\| G(x) - x^* \right\| \leq \frac{\frac{\alpha_m + \varepsilon_0}{m!} \|x - x^*\| + \frac{\alpha_{m-1}}{(m-1)!} \|x - x^*\|^{m-1} + \cdots + \frac{\alpha_2}{2!} \|x - x^*\|^2}{1 - \alpha_2 \|x - x^*\| - \cdots - \frac{(\alpha_m + \varepsilon_0)}{(m-1)!} \|x - x^*\|^{m-1}}. \tag{2.8.22}$$

Proof. Using (2.8.3), we can write

$$\begin{aligned}
&G(x) - x^* = \\
&\quad = x - F'(x)^{-1} F(x) - x^* = F'(x)^{-1} \left[F'(x) \left(x - x^* \right) - F(x) \right] \\
&\quad = F'(x)^{-1} \left[F\left(x^*\right) - F(x) - F'(x) \left(x^* - x \right) \right] \\
&\quad = \left[F'(x)^{-1} F'\left(x^*\right) \right] \left\{ F'\left(x^*\right)^{-1} \left[F\left(x^*\right) - F(x) - F'(x) \left(x^* - x \right) \right] \right\} \quad (2.8.23)
\end{aligned}$$

As in Lemma 2.8.1 by taking norms in (2.8.23) and using (2.8.14), (2.8.15) we obtain (2.8.21). Moreover using Lemma 2.8.2 and (2.8.12) we get (2.8.22).

Remark 2.8.4. Consider Newton method (2.8.2)–(2.8.3) for some $x_0 \in U(x^*, b)$. Define sequence $\{c_n\}$ $(n \geq 0)$ by

$$c_n \equiv \left\| x_n - x^* \right\| \quad (n \geq 0) \tag{2.8.24}$$

and function g on $[0, \delta)$ by

$$g(t) \equiv \frac{\frac{\alpha_m + \varepsilon_0}{m!} t^m + \frac{\alpha_{m-1}}{(m-1)!} t^{m-1} + \cdots + \frac{\alpha_2}{2!} t^2}{1 - \alpha_2 t - \cdots - \frac{\alpha_m + \varepsilon_0}{(m-1)!} t^{m-1}}. \tag{2.8.25}$$

Using (2.8.24) and (2.8.25), estimate (2.8.22) becomes

$$c_{n+1} \leq g(c_n) \quad (n \geq 0). \tag{2.8.26}$$

It is simple algebra to show that $g(t) < t$ iff $t < \delta_0$, where δ_0 is the positive zero of the equation

$$h(t) = 0, \tag{2.8.27}$$

where

$$h(t) = \frac{(\alpha_m + \varepsilon_0)(m+1)}{m!} t^{m-1} + \frac{m\alpha_{m-1}}{(m-1)!} t^{m-2} + \cdots + \frac{3}{2!} \alpha_2 t - 1. \tag{2.8.28}$$

Note that for $m = 2$, using (2.8.28) we obtain

$$\delta_0 = \frac{2}{3(\alpha_2 + \varepsilon_0)}. \tag{2.8.29}$$

Hence, we proved the following local convergence result for the NK method (2.8.2)–(2.8.3).

Theorem 2.8.5. *NK method $\{x_n\}$ $(n \geq 0)$ generated by (2.8.2)–(2.8.3) converges to the solution x^* of equation $F(x) = 0$, for all $F \in T$, iff the initial guess x_0 satisfies*

$$\left\| x_0 - x^* \right\| < \min\{\alpha, \delta_0\}. \tag{2.8.30}$$

We also have the following consequence of Theorem 2.8.5.

Theorem 2.8.6. *NK method $\{x_n\}$ $(n \geq 0)$ generated by (2.8.2)–(2.8.3) converges to the solution x^* of equation $F(x) = 0$, for all $F \in T$, if $F'(x_0)^{-1}$ exists at the initial guess x_0, and*

$$\left\| x_0 - x^* \right\| < \min\left\{\alpha, \overline{\delta}_0\right\}, \tag{2.8.31}$$

where $\overline{\delta}_0$ is the positive zero of the equation resulting from (2.8.28) by replacing α_{m+1} by $\mu(F, x_0)$ (defined by (2.8.10)) and α_i, $2 \leq i \leq m$ by $q_i(F, x_0)$ (defined by (2.8.17)).

Proof. By Lemma 2.8.1, because $F'(x_0)^{-1}$ exists and $\|x_0 - x^*\| < \overline{\delta}_0$, we get

$$\mu\left(F, x^*\right) \leq m_0 \equiv \frac{\mu(F,x_0)}{1 - q_2(F,x_0)\|x_0 - x^*\| - \cdots - \frac{\mu(F,x_0)+\varepsilon_0}{(m-1)!}\|x - x_0\|^{m-1}}. \tag{2.8.32}$$

Moreover, we have

$$\begin{aligned} q_i\left(F, x^*\right) &= \\ &= \left\| F'\left(x^*\right)^{-1} F^{(i)}\left(x^*\right) \right\| \leq \left\| F'\left(x^*\right)^{-1} F'(x_0) \right\| \left\| F'(x_0)^{-1} F^{(i)}\left(x^*\right) \right\| \\ &\leq q_0^i \equiv \frac{q_i(F,x_0)}{1 - q_2(F,x_0)\|x_0 - x^*\| - \cdots - \frac{\mu(F,x_0)+\varepsilon_0}{(m-1)!}\|x_0 - x^*\|^{m-1}}. \end{aligned} \tag{2.8.33}$$

Denote by $\overset{=}{\delta}_0$ the positive zero of the equation resulting from (2.8.28) by replacing ε_0 by $\mu(F, x^*)$ (defined by (2.8.16)) and α_i, $2 \leq i \leq m$ by $q_i(F, x^*)$. Furthermore denote by $\overset{\equiv}{\delta}_0$ the positive zero of the equation resulting from (2.8.28) by replacing ε_0 by m_0 and α_i, $2 \leq i \leq m$ by q_0^i.

Using the above definitions we get

$$\begin{aligned} \overset{=}{\delta}_0 \geq \overset{\equiv}{\delta}_0 \geq \frac{q_m(F, x_0) + \mu(F, x_0)}{m!} \left\| x_0 - x^* \right\|^m + \frac{q_2(F, x_0)}{(m-1)!} \left\| x_0 - x^* \right\|^{m-1} \\ + \cdots + \frac{q_2(F, x_0)}{2!} \left\| x_0 - x^* \right\| \geq \left\| G(x_0) - x^* \right\|. \end{aligned} \tag{2.8.34}$$

The result now follows from (2.8.34) and Theorem 2.8.5.

Remark 2.8.7. Let us assume equality in (2.8.26) and consider the iteration $c_{n+1} = g(c_n)$ $(n \geq 0)$. Denote the numerator of function g by g_1 and the denominator by g_2. By Ostrowski's theorem for convex functions [155] iteration $\{c_n\}$ $(n \geq 0)$ converges to 0 if $c_0 \in \left[0, \overset{=}{\delta}\right)$, $g'(c_0) < 1$. Define the real function h_0 by

$$h_0(t) = g_2(t)^2 - g_1'(t)\, g_2(t) + g_2'(t)\, g_1(t), \tag{2.8.35}$$

where $\varepsilon_0(x^*) = \mu(F, x^*)$ and $\overline{\alpha}_i = q_i(F, x^*)$, $2 \leq i \leq m$ replace α_{m+1} and α_i in the definition of g respectively. Note that h is a polynomial of degree $2(m-1)$ and can be written in the form

$$h_0(t) = \frac{m^2 - m + 2}{(m-1)!} \varepsilon_0^2\left(x^*\right) t^{2(m-1)} + (\text{other lower order terms}) + 1. \tag{2.8.36}$$

Because h_0 is continuous, and

$$h_0(0) = 1 > 0, \tag{2.8.37}$$

we deduce that there exists $t_0 > 0$ such that $h_0(t) > 0$ for all $t \in [0, t_0)$.

Set

$$\overline{c}_0 = \min\left\{t_0, \overline{\overline{\delta}}\right\}. \tag{2.8.38}$$

It is simple algebra to show that $g'(c_0) < 1$ iff $h_0(c_0) > 0$. Hence, NK method converges to x^* for all $F \in T$ if the initial guess x_0 satisfies

$$\left\|x_0 - x^*\right\| \le \min\{\alpha, \overline{c}_0\}. \tag{2.8.39}$$

Condition (2.8.39) is weaker than (2.8.31).

Although Theorem 2.8.5 gives an optimal domain of convergence for Newton's method, the rate of convergence may be slow for x_0 near the boundaries of that domain. However, it is known that if the conditions of the Newton-Kantorovich theorem are satisfied at x_0 then convergence is rapid. The proof of this theorem can be found in [27].

Theorem 2.8.8. *Let $m \ge 2$ be an integer, X, Y be Banach spaces, D an open convex subset of X, $F\colon D \to Y$, and an m-times Fréchet-differentiable operator. Let $x_0 \in D$ be such that $F'(x_0)^{-1}$ exists, and suppose the positive numbers δ^*, $d(F, x_0)$, $\alpha_i(F, x_0)$, $2 \le i \le m$ satisfy*

$$\left\|F'(x_0)^{-1} F(x_0)\right\| \le d(F, x_0), \tag{2.8.40}$$

$$\left\|F'(x_0)^{-1} F^{(i)}(x_0)\right\| \le \alpha_i(F, x_0), \quad i = 2, \ldots, m, \tag{2.8.41}$$

and

$$\left\|F'(x_0)^{-1}\left[F^{(m)}(x) - F^{(m)}(x_0)\right]\right\| \le \varepsilon_0, \quad \varepsilon_0 = \varepsilon_0(F, x_0) \tag{2.8.42}$$

for all $x \in U(x_0, \delta^*) \subseteq D$.

Denote by s the positive zero of the scalar equation

$$p'(t) = 0, \tag{2.8.43}$$

where

$$p(t) = \frac{\alpha_m(F, x_0) + \varepsilon_0}{m!} t^m + \frac{\alpha_{m-1}(F, x_0)}{(m-1)!} t^{m-1} + \cdots + \frac{\alpha_2(F, x_0)}{2!} t^2 - t + d(F, x_0). \tag{2.8.44}$$

If

$$p(s) \le 0, \tag{2.8.45}$$

and

$$\delta^* \geq r_1, \tag{2.8.46}$$

where r_1 is the smallest nonnegative zero of equation

$$p(t) = 0$$

guaranteed to exist by (2.8.45), then NK method (2.8.2)–(2.8.3) starting from x_0 generates a sequence that converges quadratically to an isolated solution x^* of equation $F(x) = 0$.

Remark 2.8.9. Using this theorem we obtain two further sufficiency conditions for the convergence of NK method.

It is convenient for us to set $\varepsilon_0 = \mu(F, x_0)$, and $\alpha_i(F, x_0) = q_i$ (q_i evaluated at x_0) $2 \leq i \leq m$. Condition can be written as

$$d(F, x_0) \leq s_0, \tag{2.8.47}$$

where

$$s_0 = s - \left[\tfrac{q_2}{2!}s^2 + \cdots + \tfrac{\varepsilon_0 + q_m}{m!}s^m\right] > 0 \tag{2.8.48}$$

by the definition of s. Define functions h_1, h_2 by

$$h_1(t) = \tfrac{q_m + \varepsilon_0}{m!}t^m + \tfrac{q_{m-1}}{(m-1)!}t^{m-1} + \cdots + \tfrac{q_2}{2!}t^2 + t - s_0, \tag{2.8.49}$$

and

$$h_2(t) = \tfrac{\overline{q}_m(x_0) + \varepsilon_0}{m!}t^m + \tfrac{\overline{q}_{m-1}(x_0)}{(m-1)!}t^{m-1} + \cdots + \tfrac{\overline{q}_2(x_0)}{2!}t^2 + t - s_0. \tag{2.8.50}$$

Because $h_1(0) = h_2(0) = -s_0 < 0$, we deduce that there exist minimum $t_1 > 0$, $t_2 > 0$ such that

$$h_1(t) \leq 0 \quad \text{for all} \quad t \in [0, t_1] \tag{2.8.51}$$

and

$$h_2(t) \leq 0 \quad \text{for all} \quad t \in [0, t_2]. \tag{2.8.52}$$

Theorem 2.8.10. *Let $F \in T$, and $x_0 \in U(x^*, \alpha)$. Then condition (2.8.45) holds, if either*
(a) $F'(x_0)^{-1}$ exists and $\|x_0 - x^*\| \leq \min\{\alpha, t_1\}$;
(b) $F'(x_0)^{-1}$ exists and $\|x_0 - x^*\| \leq \min\{\alpha, t_2\}$,
where t_1 and t_2 are defined in (2.8.51), and (2.8.52), respectively.

Proof. Choose $\delta^* > 0$ such that $U(x_0, \delta^*) \subseteq U(x^*, \alpha)$. By (2.8.3), and (2.8.21), we get (for $\varepsilon_0(G, x_0) = \mu(F, x_0)$, and $\alpha_i(F, x_0) = q_i$ (q_i evaluated at x_0) $g \leq i \leq m$):

$$\begin{aligned}\left\|F'(x_0)^{-1}F(x_0)\right\| &= \|G(x_0) - x_0\| \leq \left\|F(x_0) - x^*\right\| + \left\|x^* - x_0\right\| \\ &\leq \tfrac{q_m + \varepsilon_0}{m!}\left\|x_0 - x^*\right\|^m + \tfrac{q_{m-1}}{(m-1)!}\left\|x_0 - x^*\right\|^{m-1} \\ &\quad + \cdots + \tfrac{q_2}{2!}\left\|x_0 - x^*\right\|^2 + \left\|x_0 - x^*\right\|. \end{aligned} \tag{2.8.53}$$

Using (2.8.53) to replace $d(F, x_0)$ in (2.8.44), and setting $\|x_0 - x^*\| \leq t$, we deduce that (2.8.45) holds if $h_1(t) \leq 0$, which is true by the choice of t_1, and (a). Moreover, by replacing $\mu(G, x_0)$ and q_i, $2 \leq i \leq m$ using (2.8.19), and (2.8.20), respectively, condition (2.8.45) holds if $h_2(t) \leq 0$, which is true by the choice of t_2, and (b).

In order for us to cover the case $m = 1$, we start from the identity

$$
\begin{aligned}
&x_{n+1} - x^* = \\
&= x_n - F'(x_n)^{-1} F(x_n) \\
&= \left[F'(x_n)^{-1} F'\left(x^*\right)\right] F'\left(x^*\right)^{-1} 1\left[F\left(x^*\right) - F(x_n) - F'(x_n)\left(x^* - x\right)\right] \\
&= \left[F'(x_n)^{-1} F'\left(x^*\right)\right] F'\left(x^*\right)^{-1} \int_0^1 \left[F'\left(x_n + t\left(x^* - x_n\right)\right) - F'(x_n)\right]\left(x^* - x_n\right) dt \\
&= \left[F'(x_n)^{-1} F'\left(x^*\right)\right] F'\left(x^*\right)^{-1} \int_0^1 \left[F'\left(x_n + t\left(x^* - x_n\right)\right) - F'\left(x^*\right)\right]\left(x^* - x_n\right) dt \\
&\qquad + \left[F'(x_n)^{-1} F'\left(x^*\right)\right] F'\left(x^*\right)^{-1} \left[F'\left(x^*\right) - F'(x_n)\right]\left(x^* - x_n\right).
\end{aligned}
$$

to show as in Lemma 2.8.3.

Theorem 2.8.11. *Let $F\colon D \to Y$ be a Fréchet-differentiable operator. Assume there exists a simple zero x^* of $F(x) = 0$, and for $\varepsilon_1 > 0$ there exists $\ell > 0$ such that*

$$\left\| F'\left(x^*\right)^{-1}\left[F'(x) - F'\left(x^*\right)\right]\right\| < \varepsilon_1$$

for all $x \in U(x^*, \ell)$.

Then, NK method $\{x_n\}$ $(n \geq 0)$ generated by (2.8.2)–(2.8.3) is well defined, remains in $U(x^*, l)$, and converges to x^* with

$$\left\|x_{n+1} - x^*\right\| \leq \tfrac{2\varepsilon_1}{1-\varepsilon_1}\left\|x_n - x^*\right\| \quad (n \geq 0)$$

provided that

$$3\varepsilon_1 < 1$$

and

$$x_0 \in U\left(x^*, l_1\right).$$

Example 2.8.12. Returning back to Example 2.4.16, for $m = 3$, $\alpha_2 = \alpha_3 = 1$ and $\varepsilon_0 = e - 1$.
We get using (2.8.28)

$$\delta_0^3 = .43649019. \tag{2.8.54}$$

To compare our results with earlier ones, note that in Theorem 3.7 [216, p. 111] the condition is

$$\left\|x_0 - x^*\right\| < \min\left\{\sigma, \tfrac{2}{3\rho}\right\} = \rho_0, \tag{2.8.55}$$

where σ, ρ are such that $U(x^*, \sigma) \subseteq D$, and

$$\left\| F'\left(x^*\right)^{-1}\left(F'(x) - F'(y)\right)\right\| \leq \rho \left\| x - y \right\| \quad \text{for all} \quad x, y \in U\left(x^*, \sigma\right). \quad (2.8.56)$$

Letting $\sigma = \alpha = 1$, we get using (2.8.56) $\rho = e$, and condition (2.8.55) becomes

$$\left\| x_0 - x^* \right\| < \rho \equiv .245253. \quad (2.8.57)$$

Remark 2.8.13. For $m = 2$, (2.8.28) gives (2.8.29). In general $\alpha_2 < \rho$. Hence, there exists $\varepsilon_0 > 0$ such that $\alpha_2 + \varepsilon_0 < \rho$, which shows that

$$\rho_0 > \rho_0. \quad (2.8.58)$$

(See Example 2.8.15 for such a case.)

Remark 2.8.14. Our analysis can be simplified if instead of (2.8.22) we consider the following estimate: because $x \in U(x^*, \alpha)$, there exist γ_1, γ_2 such that

$$2\left[\frac{\alpha_m + \varepsilon_0}{m!}\left\| x_0 - x^* \right\|^{m-2} + \cdots + \frac{\alpha_2}{2!}\right] \leq \gamma_1, \quad (2.8.59)$$

and

$$\frac{\alpha_m + \varepsilon_0}{(m-1)!}\left\| x_0 - x^* \right\|^{m-2} + \cdots + \alpha_2 \leq \gamma_2. \quad (2.8.60)$$

Hence estimate (2.8.22) can be written

$$\left\| G(x) - x^* \right\| \leq \frac{\gamma_1}{2(1 - \gamma_2 \|x - x^*\|)}\left\| x - x^* \right\|^2, \quad (2.8.61)$$

and for $\gamma^* = \max\{\gamma_1, \gamma_2\}$

$$\left\| G(x) - x^* \right\| \leq \frac{\gamma^*}{2(1 - \gamma^* \|x - x^*\|)}\left\| x - x^* \right\|^2. \quad (2.8.62)$$

The convergence condition of Theorem 3.7 [216, p. 111] and (2.8.61), (2.8.62), becomes respectively

$$\left\| x_0 - x^* \right\| \leq \min\{\alpha, \gamma\}, \ \gamma = \frac{2}{\gamma_1 + 2\gamma_2}, \quad (2.8.63)$$

and

$$\left\| x_0 - x^* \right\| \leq \min\left\{\sigma, \frac{2}{3\gamma^*}\right\}. \quad (2.8.64)$$

In particular, estimate (2.8.64) is similar to (2.8.55), and if $\gamma < \rho$, then (2.8.63) allows a wider range for the initial guess x_0 than (2.8.55).

Furthermore, assuming (2.8.4), (2.8.5), and (2.8.55) hold, our analysis can be based on the following variations of (2.8.22):

$$\left\| G(x) - x^* \right\| \leq \frac{\frac{q_m + \varepsilon_0}{m!}\|x - x^*\|^m + \cdots + \frac{q_2}{2!}\|x - x^*\|^2}{1 - \rho\|x - x^*\|}, \quad (2.8.65)$$

and

$$\left\| G(x) - x^* \right\| \leq \frac{\rho}{2\left[1 - \alpha_2\|x - x^*\| - \cdots - \frac{\alpha_m + \varepsilon_0}{(m-1)!}\|x - x^*\|^{m-1}\right]}\left\| x - x^* \right\|^2. \quad (2.8.66)$$

Example 2.8.15. Let us consider the system of equations

$$F(x, y) = 0,$$

where

$$F\colon \mathbf{R}^2 \to \mathbf{R}^2,$$

and

$$F(x, y) = (xy - 1, xy + x - 2y).$$

Then, we get

$$F'(x, y) = \begin{bmatrix} y & x \\ y+1 & x-2 \end{bmatrix},$$

and

$$F'(x, y)^{-1} = \frac{1}{x+2y} \begin{bmatrix} 2-x & x \\ y+1 & -y \end{bmatrix},$$

provided that (x, y) does not belong on the straight line $x + 2y = 0$. The second derivative is a bilinear operator on $\mathbf{R}^2$ given by the following matrix

$$F''(x, y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ - & - \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We consider the max-norm in $\mathbf{R}^2$. Moreover in $L\left(\mathbf{R}^2, \mathbf{R}^2\right)$ we use for

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the norm

$$\|A\| = \max\left\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\right\}.$$

As in [7], we define the norm of a bilinear operator B on $\mathbf{R}^2$ by

$$\|B\| = \sup_{\|z\|=1} \max_i \sum_{j=1}^{2} \left| \sum_{k=1}^{2} b_i^{jk} z_k \right|,$$

where

$$z = (z_1, z_2) \quad \text{and} \quad B = \begin{bmatrix} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \\ - & - \\ b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \end{bmatrix}.$$

Using (2.8.4), (2.8.5), (2.8.29), (2.8.55), (2.8.56), for $m = 2$ and $(x^*, y^*) = (1, 1)$, we get $\rho = \frac{4}{3}$, $\rho_0 = .5$, $\alpha_2 = 1$. We can set $\varepsilon_0 = .001$ to obtain $\delta_0^2 = .666444519$. Because $\rho_0 < \delta_0^2$, a remark similar to the one at the end of Example 2.8.12 can now follow.

2.9 On a weak NK method

R. Tapia in [188] showed that the weak Newton method (to be precised later) converges in cases NK cannot under the famous Newton-Kantorovich hypothesis (see (2.9.9)). Using the technique we recently developed in Section 2.2, we show that (2.9.9) can always be replaced by the weaker (2.9.6), which is obtained under the same computational cost. This way we can cover cases [39] that cannot be handled by the work in [188].

We need the following definitions and Lemma whose proof can be found in [188, p. 540]:

Definition 2.9.1. *Let $D_0 \subseteq D$ be a closed subset of X, D_1 and open subset of D_0. For $x \in D_1$, $M(x)$ is a left inverse for $F'(x)$ relative to D_0 if:*

(a) $M(x) \in L(Y_x, X)$, where Y_x is a closed linear subspace of Y containing $F(D_1)$;

(b) $M(x) F(D_1) \subseteq D_0$;

and

(c) $M(x) F'(x) = I$

where I is the identity operator from D_0 into D_0.

Lemma 2.9.2. *Hypotheses (a) and (b) imply that for all $y \in D_1$;*

(d) $F'(y)(D_0) \subseteq Y_x$,

and

(e) $M(x) F'(y)(D_0) \subseteq D_0$.

Definition 2.9.3. *If $x_0 \in D_1$, then*

$$x_{n+1} = x_n - M(x_n) F(x_n) \quad (n \geq 0) \tag{2.9.1}$$

is called the weak Newton method.

The following result is a version of Theorem 2 in [39] (see also Section 2.3):

Theorem 2.9.4. *If there exist $M \in L(Y, X)$ and constants $\eta, \delta, \ell, \ell_0, t^*$ such that:*

$$M^{-1} \text{ exists;}$$

$$\|M F(x_0)\| \leq \eta; \tag{2.9.2}$$

$$\left\| I - M F'(x_0) \right\| \leq \delta < 1; \tag{2.9.3}$$

$$\left\| M \left(F'(x) - F'(x_0) \right) \right\| \leq \ell_0 \|x - x_0\|; \tag{2.9.4}$$

$$\left\| M \left(F'(x) - F'(y) \right) \right\| \leq \ell \|x - y\| \tag{2.9.5}$$

for all $x, y \in D$;

$$h_0 = \frac{\overline{\ell}\eta}{(1-\delta)^2} \leq \frac{1}{2}, \ \overline{\ell} = \frac{\ell_0 + \ell}{2} \tag{2.9.6}$$

and

$$\overline{U}\left(x_0, t^*\right) \subseteq \overline{U}(x_0, 2\eta) \subseteq D. \tag{2.9.7}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by NK is well defined, remains in $U(x_0, t^*)$ for all $n \geq 0$, and converges to a unique solution x^* of equation $F(x) = 0$ in $\overline{U}(x_0, t^*)$.

Proof. Simply use MF, D_0 instead of F, X respectively in the proof of Theorem 2 in [39] (see also Section 2.2).

Lemma 2.9.5. *[188] Newton sequences in Theorem 2.9.4 exists if and only if M is invertible.*
If M is not invertible, we can have the following Corollary of Theorem 2.9.4.

Corollary 2.9.6. *If there exists $M \in L(S, X)$, such that $MF(U(x_0, 2\eta)) \subseteq D_0$, where S is a closed linear subspace of Y containing $F(U(x_0, 2\eta))$, and (2.9.2)–(2.9.7) hold, then*
(a) sequence $\{x_n\}$ $(n \geq 0)$ generated by the weak Newton method (2.9.1) is well defined, remains in $U(x_0, t^*)$ for all $n \geq 0$, and converges to some point $x^* \in \overline{U}(x_0, t^*)$.
(b) If M is one-to-one, then $F(x^*) = 0$; or if $t^* < 2\eta$, and F has a solution in $\overline{U}(x_0, t^*)$, then again $F(x^*) = 0$.

Proof. It follows from Lemma 2.9.5 that for any $x \in D_1$, $F'(x): D_0 \to S$, and $MF'(x): D_1 \to D_0$ so that $\left[MF'(x)\right]^{-1} MF: D_1 \to D_0$ whenever it exists. The rest follows as in the proof of Theorem 2 in [39] with F, X replaced by MF, D_0 respectively.

Remark 2.9.7. If

$$\ell_0 = \ell, \tag{2.9.8}$$

then Theorem 2.9.4 and Corollary 2.9.6 reduce to Theorem 3.1 and Corollary 3.1 respectively in [188] (if F is twice Fréchet-differentiable on D). However $\ell_0 \leq \ell$, holds in general (see Section 2.2). It follows that the Newton-Kantorovich hypothesis

$$h = \frac{\ell\eta}{(1-\delta)^2} \leq \frac{1}{2} \tag{2.9.9}$$

used in the results in [188] mentioned above always implies (2.9.6) but not vice versa unless if (2.9.8) holds.

2.10 Bounds on manifolds

We recently showed the following weaker version of the Newton-Kantorovich theorem [39] (see Section 2.2):

Theorem 2.10.1. *Let $F: D = U(0, r) \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume:*

$$F'(0)^{-1} \in L(Y, X), \tag{2.10.1}$$

and there exist positive constants a, ℓ, ℓ_0, η such that:

$$\| F'(0)^{-1} \| \leq a^{-1}, \tag{2.10.2}$$

$$\| F'(0)^{-1} F(0) \| \leq \eta_0 \leq \eta \tag{2.10.3}$$

$$\| F'(x) - F'(y) \| \leq \ell \|x - y\|, \tag{2.10.4}$$

$$\| F'(x) - F'(0) \| \leq \ell_0 \|x\| \tag{2.10.5}$$

for all $x, y \in D$,

$$h_0 = La^{-1}\eta < \tfrac{1}{2}, \quad L = \tfrac{\ell_0+\ell}{2}, \tag{2.10.6}$$

$$M = \lim_{n\to\infty} t_n < r, \tag{2.10.7}$$

where,

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} - \tfrac{\ell(t_{n+1}-t_n)^2}{2(1-\ell_0 t_{n+1})}. \tag{2.10.8}$$

Then equation $F(x) = 0$ *has a solution* $x^* \in D$ *that is the unique zero of* F *in* $U(0, 2\eta)$.

Remark 2.10.2. The above result was also shown in affine invariant form and for any initial guess including 0. However, we want the result in the above form for simplicity, and in order to compare it with earlier ones [156].

Let us assume for X, Y being Hilbert spaces:
$A = F'(0) \in L(X, Y)$ is surjective, $A^+ \in L(Y, X)$ is a right inverse of A, and

$$\left\|A^+\right\| \le a^{-1}, \tag{2.10.9}$$

$$\left\|A^+ F(0)\right\| = \bar{\eta} \le \eta; \tag{2.10.10}$$

Conditions (2.10.3)–(2.10.6) hold (for $F'(0)^{-1}$ replaced by A^+).

It is convenient for us to introduce:

$$S = \{x \in D \mid F(x) = 0\}, \tag{2.10.11}$$

$$S_0 = \{x \in D \mid F(x) = F(0)\}, \tag{2.10.12}$$

$$N_1 = \operatorname{Ker}(A), \tag{2.10.13}$$

and

$$N_2 \text{ the orthogonal complement of } N_1.$$

In Theorem 2.10.3 we provide an analysis in the normal space N_2 at 0 of S_0, which leads to an upper bound of $d(0, S)$.

Newton-Kantorovich-type condition (2.10.6) effects S to be locally in a convex cone. Theorem 2.10.8 gives the distance of 0 to that cone as a lower bound of $d(0, S)$.

This technique leads to a manageable way of determining for example sharp error bounds for an approximate solution of an undetermined system.

Finally we show that our approach provides better bounds than the ones given before in [156] (and the references there), and under the same computational cost.

The following results can be shown by simply using (2.10.4), (2.10.5) instead of (2.10.4) in the proofs of Theorem 2.10.3, Lemmas 2.10.4–2.10.7, Theorem 2.10.8, and Corollary 2.10.9, respectively.

Theorem 2.10.3. *Operator* $F\colon D \subseteq X \to Y$ *has a zero* x^* *in* $U(0, M) \cap N_2$; x^* *is the unique zero of* F/N_2 *in* $U(0, 2\eta) \cap N_2$.

Lemma 2.10.4. *The following bounds hold:*

$$r^* \leq \left\| x^* \right\| = b, \tag{2.10.14}$$

$$b < M < 2\eta, \tag{2.10.15}$$

and

$$\ell_0 b < 2\ell_0 \eta < a, \tag{2.10.16}$$

where,

$$r^* = \tfrac{a}{\ell_0} \left(-1 + \sqrt{1 + 2h_1} \right), \tag{2.10.17}$$

and

$$h_1 = \ell_0 \eta_0 a^{-1}. \tag{2.10.18}$$

It is convenient for us to introduce the notion:

$$V = \{x \in X | \, \|x\| < b\}, \;\; P(x) = F'(x)/N_1, \tag{2.10.19}$$

$$Q(x) = F'(x)/N_2, \quad x \in V, \tag{2.10.20}$$

and

$$\alpha = \tfrac{\ell_0 b}{\lambda - \ell_0 b}. \tag{2.10.21}$$

Lemma 2.10.5. *The following hold*

$$Q(x) \text{ is regular for all } x \in V$$

and

$$\| Q(x)^{-1} P(x) \| \leq \alpha, \text{ for all } x \in V. \tag{2.10.22}$$

Let us define:

$$W = \{x = x_1 + x_2 \in X \mid \alpha \|x_1\| + \|x_2\| < b, \; x_i \in N_i, \; i = 1, 2, \} \tag{2.10.23}$$

and

$$K(w) = \{(1 - \theta) w_1 + x_2 \mid \theta \in [0, 1], \; x_2 \in N_2, \; \|x_2 - w_2\| \leq \alpha \|w_1\| \theta\} \tag{2.10.24}$$

where,

$$w = w_1 + w_2 \in X, \; w_i \in N_i, \; i = 1, 2.$$

Lemma 2.10.6. *If $w = w_1 + w_2 \in W \cap V$, then $K(w) \subseteq V \cap W$.*

Lemma 2.10.7. *Operator F has no zeros in $W \cap V$.*

Theorem 2.10.8. *Let us define real function g by*

$$g(t) = t\left(\frac{a}{\ell_0} - t\right)\left(t^2 + \left(\frac{a}{\ell_0} - t\right)^2\right)^{-\frac{1}{2}}, \text{ for } t \in \left(0, \frac{a}{\ell_0}\right). \tag{2.10.25}$$

Then, the following bounds hold:

$$d(0, S) \geq m = \begin{cases} g(M), & \text{if } \frac{1}{4}\sqrt{3} \leq h_0 < \frac{1}{2}, \\ & \quad \sqrt{1-2h_0} + \frac{1}{2}(1-2h_0) \leq h_1 \leq h_0 \\ g(r^*), & \text{if } 0 < h_0 < \frac{1}{2}, \\ & \quad h_1 \leq \min\left\{h_0, \sqrt{1-2h_0} + \frac{1}{2}(1-2h_0)\right\}, \end{cases} \tag{2.10.26}$$

where M and r^* are given by (2.10.7) and (2.10.14), respectively.

Corollary 2.10.9. *If $\eta_0 = \eta$ then the following bounds hold:*

$$m = d(0, S) \geq \begin{cases} g(M), \ \text{if } \frac{\sqrt{3}}{4} \leq h_0 < \frac{1}{2} \\ g(r^*), \ \text{if } h_0 < \frac{\sqrt{3}}{4}. \end{cases} \tag{2.10.27}$$

Remark 2.10.10. (a) Theorem 2.10.1 reduces to the corresponding one in [156] if

$$\ell_0 = \ell. \tag{2.10.28}$$

However, in general,

$$\ell_0 \leq \ell \tag{2.10.29}$$

holds. Let

$$h = \ell\eta < \frac{1}{2}. \tag{2.10.30}$$

Then note by (2.10.6) and (2.10.30)

$$h < \frac{1}{2} \Longrightarrow h_0 < \frac{1}{2} \tag{2.10.31}$$

but not necessarily vice versa unless if (2.10.28) holds.

(b) Our results reduce to the corresponding ones in [156] again if (2.10.28) holds. However, if strict inequality holds in (2.10.29), then our interval of bounds $[m, M]$ is always more precise than the corresponding one in [156] and are found under the same computational cost.

2.11 The radius of convergence and one-parameter operator embedding

In this section, we are concerned with the problem of approximating a locally unique solution of the nonlinear equation

$$F(x) = 0, \tag{2.11.1}$$

where F is a Fréchet-differentiable operator defined on closed convex subset of the mth Euclidean space X^m into X^m.

NK method when applied to (2.11.1) converges if the initial guess is "close" enough to the root of F. If no approximate roots are known, this and other iterative methods may be of little use. Here we are concerned with enlarging the region of convergence for Newton's method. Our technique applies to other iterative methods. That is, the use of Newton's method is only illustrative. The results obtained here extend immediately to hold when F is defined on a Banach spaces with values in a Banach space.

We assume operator F is differentiably embedded into a one-parameter family of operators $\{H(t, \cdot)\}$ such that $H(t_0, x_0) = 0$, and $H(t_1, x) = F(x)$ for some $x_0 \in D$ and two values t_0 and t_1 of the parameter.

We consider the commonly used embedding (homotopy)

$$H(t, x) = F(x) + (t-1) F(x_0) \tag{2.11.2}$$

The solution of $F(x) = 0$ is then found by continuing the solution curve $x(t)$ of $H(t, x) = 0$ from t_0 until t_1.

Homotopies have been employed to prove existence results for linear and nonlinear equations (see [139] and the references there).

In particular, the results obtained in [139] can be weakened, and the region of convergence for NK method can be enlarged if we simply replace the Lipschitz constant by the average between the Lipschitz constant and the center-Lipschitz constant. Moreover our results can be used in cases not covered in [139].

Motivated by advantages of the weaker version of the Newton-Kantorovich theorem that we provided in Section 2.2, we hope that the homotopy approach will be successful under the same hypotheses. In particular, we can show:

Theorem 2.11.1. *Let $F \colon D \subseteq X^m \to X^m$ be Fréchet-differentiable. Assume there exist $x_0 \in D$, $\ell_0 \geq 0$, $\ell \geq 0$ and $\eta \geq 0$ such that:*

$$F'(x_0)^{-1} \in L\left(X^m, X^m\right), \tag{2.11.3}$$

$$\| F'(x_0)^{-1} \left[F'(x) - F'(x_0)\right] \| \leq \ell_0 \|x - x_0\|, \tag{2.11.4}$$

$$\| F'(x_0)^{-1} \left[F'(x) - F'(y)\right] \| \leq \ell \|x - y\|, \tag{2.11.5}$$

$$\| F'(x_0)^{-1} F(x_0) \| \leq \eta, \tag{2.11.6}$$

$$h_0 = L_0 \eta \leq \tfrac{1}{2} \textit{ for } \dot{\ell} < \ell, \tag{2.11.7}$$

or

$$h_0 < \tfrac{1}{2} \textit{ for } \ell_0 = \ell, \tag{2.11.8}$$

where

$$L_0 = \tfrac{\ell_0 + \ell}{2}, \tag{2.11.9}$$

and

$$\overline{U}(x_0, 2\eta) \subseteq D. \tag{2.11.10}$$

Then the solution

$$x = -F'(x)^{-1} F(x_0), \quad x(0) = x_0 \tag{2.11.11}$$

exists, belongs in $\overline{U}(x_0, r_0)$ *for all* $t \in [0,1]$, *and* $F(x(1)) = 0$.

Proof. Using (2.11.4), (2.11.6) for $x \in \overline{U} \in (x_0, r_0)$, we obtain in turn

$$\begin{aligned}\|F'(x)^{-1} F(x_0)\| \le& \|F'(x_0)^{-1}\left[F'(x_0) - F'(x)\right] F'(x)^{-1} F(x_0)\| \\ &+ \|F'(x_0)^{-1} F(x_0)\| \\ \le& \ell_0 \|x - x_0\| \, \|F'(x)^{-1} F(x_0)\| + \eta\end{aligned}$$

or

$$\|F'(x)^{-1} F(x_0)\| \le \tfrac{\eta}{1-\ell_0\|x-x_0\|} \tag{2.11.12}$$

(as $\ell_0 \|x - x_0\| \le \ell_0 2\eta < 1$ by (2.11.7)).

Define function h by

$$h(t, r) = \tfrac{\eta}{1-\ell_0 r}. \tag{2.11.13}$$

Then it is simple calculus to see that

$$r'(t) = h(t, r) \tag{2.11.14}$$

has a unique solution $r \le r_0$, $r(0) = 0$ for $t \in [0,1]$. Moreover by (2.11.7) we get $h(t, r) < \infty$ for $t \in [0,1]$. Hence by Lemma 1.2 in [139], equation (2.11.2) has a solution $x(t)$ and $F(x(1)) = 0$.

Remark 2.11.2. (a) If F is twice Fréchet-differentiable, and

$$\ell_0 = \ell, \tag{2.11.15}$$

then Theorem 2.11.1 reduces to Corollary 2.1 in [139, p. 743]. However

$$\ell_0 \le \ell \tag{2.11.16}$$

holds in general. Meyer in [139] used the famous Newton-Kantorovich hypothesis

$$h = \ell\eta < \tfrac{1}{2} \tag{2.11.17}$$

to show Corollary 2.1 in [139]. Note that

$$h \le \tfrac{1}{2} \Longrightarrow h_0 \le \tfrac{1}{2}. \tag{2.11.18}$$

(b) The conclusion of the theorem holds if $r^* = 2\eta$ is replaced by

$$r_0 = \lim_{n\to\infty} t_n \le 2\eta, \tag{2.11.19}$$

where,

$$t_0 = 0, \ t_1 = \eta, \ t_{n+2} = t_{n+1} + \tfrac{\ell(t_{n+1}-t_n)^2}{2(1-\ell_0 t_{n+1})} \quad (n \ge 0) \quad \text{(see Section 2.2), [39].} \tag{2.11.20}$$

Note also that

$$r_0 \le \tfrac{\eta}{h}\left(1 - \sqrt{1-2h}\right) = r^* \tag{2.11.21}$$

in case (2.11.17) holds [39]. Note that r^* was used in Corollary 2.1 [139].

NK method corresponds with integrating

$$x'(\lambda) = -F'(x)^{-1} F(x), \quad x(0) = x_0, \quad \lambda \in [0, \infty), \tag{2.11.22}$$

with Euler's method of step size 1, or equivalently,

$$x'(t) = -F'(x)^{-1} F(x_0), \quad x(0) = x_0, \quad t \in [0, 1], \tag{2.11.23}$$

with Euler's method and variable step size

$$h_{k+1} = e^{-k}\left(1 - e^{-1}\right), \quad k \geq 0. \tag{2.11.24}$$

Hence the initial step is $h_1 \cong .63$, which is too large. This is a large step for approximating $x\left(1 - e^{-1}\right)$ unless if $F(x)$ is sufficiently controlled.

As in Meyer [139] we suggest an alternative: Integrate (2.11.23) with step size $h = \frac{1}{N}$. Choose the approximate solution

$$x(N_h) = x_N \tag{2.11.25}$$

as the initial guess for NK method.

This way we have the result:

Theorem 2.11.3. *Let $F\colon X^m \to X^m$ be Fréchet-differentiable and satisfying*

$$\|F'(x)^{-1}\| \leq a\|x\| + b \text{ for all } x \in X^m. \tag{2.11.26}$$

Let $x_0 \in X^m$ be arbitrary and define ball $U(x_0, \overline{r} + \delta)$ for $\delta > 0$ by

$$\overline{r} = \begin{cases} \left[\|x_0\| + \frac{b}{a}\right] \exp\left(a\|F(x_0)\|\right) - \left(\|x_0\| + \frac{b}{a}\right), & \text{if } a \neq 0, \\ b\|F(x_0)\|, & \text{if } a = 0. \end{cases} \tag{2.11.27}$$

Assume (2.11.23) is integrated from 0 to 1 with a numerical method of order h^p, denoted by

$$x_{k+1} = G(x_k, h) \tag{2.11.28}$$

and satisfying

$$\|x(1) - x_N\| \leq ch^p, \tag{2.11.29}$$

where c does not depend on h.

Moreover assume there exist constants d, ℓ_0, ℓ such that:

$$\|F'(x)^{-1}\| \leq d, \tag{2.11.30}$$

$$\left\|F'(x) - F'(x_0)\right\| \leq \ell_0\|x - x_0\| \tag{2.11.31}$$

and

$$\left\|F'(x) - F'(y)\right\| \leq \ell\|x - y\| \tag{2.11.32}$$

for all $x, y \in \overline{U}(x_0, \overline{r} + \delta)$.

Then iteration

$$x_{k+1} = G(x_k, h) \quad k = 1, ..., N - 1 \tag{2.11.33}$$

$$x_{k+1} = x_n - F'(x_k)^{-1} F(x_k) \quad k = N, ... , \tag{2.11.34}$$

converges to the unique solution of equation $F(x) = 0$ *in* $U(x_0, \overline{r} + \delta)$ *provided that*

$$h = \frac{1}{N} \leq \left(\frac{\sqrt{2}-1}{dL_0 c}\right)^{\frac{1}{p}}, \tag{2.11.35}$$

and

$$ch^{\mathbf{p}} < S. \tag{2.11.36}$$

Proof. Simply replace ℓ (Meyer denotes ℓ by L in [139]) by L_0 in the proof of Theorem 4.1 in [139, p. 750].

Remark 2.11.4. If $\ell_0 = \ell$ and F is twice Fréchet-differentiable then Theorem 2.11.3 reduces to Theorem 4.1 in [139]. However if strict inequality holds in (2.11.16), because the corresponding estimate (2.11.35) in [139] is given by

$$h_M = \frac{1}{N} \leq \left(\frac{\sqrt{2}-1}{d\ell c}\right)^{\frac{1}{p}} \tag{2.11.37}$$

we get

$$h_M < h. \tag{2.11.38}$$

Hence our technique allows a under step size h, and under the same computational cost, as the computations of ℓ require in practice the computation of ℓ_0.

2.12 NK method and Riemannian manifolds

In this section, we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{2.12.1}$$

where F is C^1 and defined on an open convex subset S of R^m (m a natural number) with values in R^m.

Newton-like methods are the most efficient iterative procedures for solving (2.12.1) when F is sufficiently many times continuously differentiable. In particular, Newton's method is given by

$$y_n = -F'(x_n)^{-1} F(x_n) \quad (x_0 \in S) \tag{2.12.2}$$

$$x_{n+1} = x_n + y_n \quad (n \geq 0). \tag{2.12.3}$$

We can extend this method to approximate a singularity of a vectorial field G defined on a Riemannian manifold M:

$$G(z) = 0, \quad z \in M. \tag{2.12.4}$$

Operator $F'(x_n)$ is replaced by the covariant derivative of G at z_n:

$$\begin{aligned} \nabla G_{z_n} \colon T_{z_n}(M) &\to T_{z_n}(M) \\ y &\to \nabla_y G \end{aligned} \tag{2.12.5}$$

(we use $\nabla G_z y = \nabla_y G$). Therefore approximation (2.12.2) becomes:

$$y_n = -\nabla G_{z_n}^{-1} G(z_n), \tag{2.12.6}$$

and $y_n \in T_{z_n}(M)$ (if (2.12.6) is well defined for all $n \geq 0$).

In R^m, x_{n+1} is obtained from x_n using the secant line which passes through x_n with direction y_n, and at n distance $\|y_n\|$. In a Riemannian manifold, geodesics replace straight lines. Hence Newton's method in a Riemannian manifold becomes:

$$z_{n+1} = \exp_{z_n}(y_n) \quad (n \geq 0), \tag{2.12.7}$$

where y_n is given by (2.12.6) for all $n \geq 0$.

Ferreira and Svaiter in [94] extended the Newton-Kantorovich theorem to Riemannian manifolds. This elegant semilocal convergence theorem for Newton's method is based on the Newton-Kantorovich hypothesis (see (2.12.19)). Recently [39] they developed a new technique that on the one hand weakens (2.12.19) under the same computational cost, and on the other hand applies to cases not covered by the Newton-Kantorovich theorem (i.e., (2.12.19) is violated whereas (2.12.12) holds); fine error bounds on the distances involved are obtained and an at least as precise information on the location of the solution (if center-Lipschitz constant L_0 is smaller than Lipschitz constant L).

Here we extend our result from Banach spaces to Riemannian manifolds to gain the advantages stated above (in this new setting).

We refer the reader to [94] for fundamental properties and notations of Riemannian manifolds. Instead of working with Frobenius norm of rank-two tensors, we use "operator norm" of linear transformations on each tangent space.

We need the definitions:

Definition 2.12.1. *Let $S_z \colon T_z M \to T_z M$ be a linear operator. Define*

$$\|S_z\|_{op} = \sup\{\|S_z y\|, \, y \in T_z M, \, \|y\| = 1\} \tag{2.12.8}$$

Definition 2.12.2. *Let D be an open and convex subset of M, and let G be a C^1 vector field defined on D. We say: covariant derivative ∇G is Lipschitz if there exist a constant L for any geodesic γ, and $a, b \in R$ with $\gamma([a, b]) \subseteq D$ such that*

$$\left\| P(\gamma)_b^a \nabla G_{\gamma(b)} P(\gamma)_a^b - \nabla G_{\gamma(a)} \right\| \leq L \int_a^b \|\gamma'(t)\| \, dt, \tag{2.12.9}$$

where $P(\gamma)$ is the parallel transport along γ [94].

We use the notation $\nabla G \in \operatorname{Lip}_L(D)$, and for the corresponding center-Lipschitz condition for $z_0 \in D$ fixed $\nabla G_{z_0} \in \operatorname{Lip}_{L_0}(D)$.

Note that in general

$$L_0 \leq L \tag{2.12.10}$$

holds. Moreover $\frac{L}{L_0}$ can be arbitrarily large (see Section 2.2).

We can now show the following extension of the Newton-Kantorovich theorem on Riemannian manifolds using method (2.12.7):

Theorem 2.12.3. *Let D be an open and convex subset of a complete Riemannian manifold M. Let G be a continuous vector field defined on $\overline{D}$ that is C^1 on D with $\nabla G \in \operatorname{Lip}_L(D)$, and for $z_0 \in D$ fixed $\nabla G_{z_0} \in \operatorname{Lip}_{L_0}(D)$.*

Assume:

$$\nabla G_{z_0} \textit{ is invertible;}$$

there exist constants c_0 and c_1 such that

$$\left\| \nabla G_{z_0}^{-1} \right\| \leq c_0, \left\| \nabla G_{z_0}^{-1} G(z_0) \right\| \leq c_1 \tag{2.12.11}$$

$$h_0 = c_0 c_1 \subset \leq \tfrac{1}{2}, \; c = \tfrac{L_0 + L}{2}, \tag{2.12.12}$$

and

$$U\left(z_0, t^*\right) \subseteq D \tag{2.12.13}$$

where,

$$t^* = \lim_{n \to \infty} t_n, \tag{2.12.14}$$

$$t_0 = 0, \; t_1 = c_1, \; t_{n+2} = t_{n+1} + \tfrac{L(t_{n+1} - t_n)^2}{2(1 - L_0 t_{n+1})} \quad (n \geq 0). \tag{2.12.15}$$

Then

(a) sequence $\{t_n\}$ $(n \geq 0)$ is monotonically increasing and converges to t^* with

$$t^* \leq 2n; \tag{2.12.16}$$

(b) sequence $\{z_n\}$ $(n \geq 0)$ generated by Newton's method (2.12.7) is well defined, remains in $U(z_0, t^*)$ for all $n \geq 0$, and converges to z^*, which is the unique singularity of G in $\overline{U}(z_0, t^*)$. Moreover if strict inequality holds in (2.12.10), z^* is the unique singularity of G in $U(z_0, 2n)$. Furthermore the following error bounds hold:

$$d(z_{n+1}, z_n) \leq t_{n+1} - t_n; \tag{2.12.17}$$

and

$$d\left(z_n, z^*\right) \leq t^* - t_n \quad (n \geq 0). \tag{2.12.18}$$

Proof. Simply use L_0 instead of L where the use of the center-Lipschitz (and not L) suffices in the proof of Theorem 3.1 in [94] (e.g., in the computation of an upper bound on $\left\| \nabla G_{z_n}^{-1} \right\|$).

Remark 2.12.4. If equality holds in (2.12.10) then Theorem 2.12.3 reduces to Theorem 3.1 in [94]. Denote the corresponding Newton-Kantorovich-type hypothesis there by:

$$h = c_0 c_1 \leq \tfrac{1}{2}. \tag{2.12.19}$$

By (2.12.10), (2.12.12), and (2.12.18) we see

$$h \leq \tfrac{1}{2} \Longrightarrow h_0 \leq \tfrac{1}{2} \tag{2.12.20}$$

but not vice versa unless if equality holds in (2.12.10).

The rest of the claims made at the introduction can now follow along the same lines of our work in Section 2.2 [39].

2.13 Computation of shadowing orbits

In this section, we are concerned with the problem of approximating shadowing orbits for dynamical systems. It is well-known in the theory of dynamical systems that actual computations of complicated orbits rarely produce good approximations to the trajectory. However under certain conditions, the computed section of an orbit lies in the shadow of a true orbit. Hence using product spaces and a recent result of ours (Section 2.2) [39], we show that the sufficient conditions for the convergence of Newton's method to a true orbit can be weakened under the same computational cost as in the elegant work by Hadeller in [108]. Moreover the information on the location of the solutions is more precise and the corresponding error bounds are finer.

Let f be a Fréchet-differentiable operator defined on an open convex subset D of a Banach space X with values in X.

The operator f defines a local dynamical system as follows:

$$x_{n+1} = f(x_k) \quad (x_0 \in D) \tag{2.13.1}$$

as long as $x_k \in D$.

A sequence $\{x_m\}_{i=0}^{N}$ in D with $x_{m+1} = f(x_m)$, $i = 0, ..., N-1$ is called an orbit. Any sequence $\{x_m\}_{i=0}^{N}$, $x_m \in D$, $m = 0, ..., N$ is called a pseudo-orbit of length N.

We can now pass to product spaces. Let $y = X^{N+1}$ equipped with maximum norm. The norm $\mathbf{x} = (x_0, ..., x_N) \in Y$ is given by

$$\|\mathbf{x}\| = \max_{0 \leq m \leq N} \|x_m\|.$$

Set $S = D^{N+1}$. Let $F\colon S \to Y$ be an operator associated with $\dot{f}$:

$$F(\mathbf{x}) = F\begin{bmatrix} x_o \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} f(x_0) - x_1 \\ \vdots \\ f(x_N) \end{bmatrix}.$$

Assume there exist constants l_0, l, L_0, L such that:

$$\begin{aligned}
\left\| f'(u) - f'(x_0) \right\| &\leq l_0 \left\| u - x_0 \right\| \\
\left\| f'(u) - f'(v) \right\| &\leq l \left\| u - v \right\| \\
\left\| F'(\mathbf{u}) - F'(\mathbf{x}_0) \right\| &\leq L_0 \left\| \mathbf{u} - \mathbf{x}_0 \right\| \\
\left\| F'(\mathbf{u}) - F'(\mathbf{v}) \right\| &\leq L \left\| \mathbf{u} - \mathbf{v} \right\|
\end{aligned}$$

for all $u, v \in D$, $\mathbf{u}, \mathbf{v} \in S$.

From now on we assume: $l_0 = L_0$ and $l = L$.

For $y \in X$ define an operator

$$F_y : S \to Y$$

by

$$F_y(\mathbf{x}) = \left\| \begin{matrix} f(x_0) - x_1 \\ \vdots \\ f(x_N) - y \end{matrix} \right\|.$$

It follows that $F_y'(\mathbf{x}) = F'(\mathbf{x})$.

As in [108], define the quantities

$$a(\mathbf{x}) = \max_{0 \leq i \leq N} \sum_{j=i}^{N} \left\| f'(x_i) \ldots f'(x_j) \right\|^{-1},$$

$$b(\mathbf{x}) = \max_{0 \leq i \leq N-1} \left\| \sum_{j=i}^{N-1} f'(x_i) \ldots f'(x_j) \left(f(x_j) - x_{j+1} \right) \right\|,$$

and

$$b_y(\mathbf{x}) = b(\mathbf{x}) + \left\| f'(x_N) \left(f(x_N) - y \right) \right\|$$

for $x \in Y$.

That is $a(x)$ is the operator norm of $F'(x)^{-1}$ and by (x) is the norm of the Newton convection $F'(\mathbf{x})^{-1} F_y(\mathbf{x})$.

Remark 2.13.1. The interpretation to the measures studied in [61], [108] is given by:

(a) The dilation measures $a(\mathbf{x})$ and by $(\mathbf{x})$ are the norm of $F'(x)^{-1}$ and the norm of the Newton-correction $F'(x)^{-1} F_y(x)$, respectively;

(b) the solution of equation

$$F_y(\mathbf{x}) = 0$$

yields a section $(x_0, ..., x_N)$ of length $N + 1$ of a true orbit that meets the prescribed point y at the Nth iteration step [108].

Using a weak variant of the Newton-Kantorovich theorem, we recently showed in [39] (see Section 2.2) we obtain the following existence and uniqueness result for a true orbit:

Theorem 2.13.2. *Let* $\mathbf{x} \in Y$, $y \in X$ $a(\mathbf{x})$, $by(\mathbf{x})$ *be as above and* $\{x_i\}_{i=0}^N$ *be a pseudo-orbit.*

Assume:

$$h_0 = \bar{L}, \quad a(\mathbf{x})\, by(\mathbf{x}) \leq \tfrac{1}{2}, \quad \bar{L} = \tfrac{L_0+L}{2} \tag{2.13.2}$$

and

$$\overline{U}\left(\mathbf{x}_0 = \mathbf{x}, \quad r^* = 2by(\mathbf{x}) = \left\{\mathbf{z} \in Y \mid \|\mathbf{x}-\mathbf{z}\| \leq r^*\right\}\right) \subseteq S.$$

Then there is a unique true orbit $\mathbf{x}^* = \left(x_0^*, ..., x_N^*\right)$ *inside* $U(\mathbf{x}, r^*)$ *satisfying* $f\left(x_N^*\right) = y$.

We also have a more neutral form of Theorem 2.13.2:

Theorem 2.13.3. *Let* $\{x_i\}_{i=0}^N$ *be a pseudo-orbit of length* $N+1$. *Assume:*

$$h_0^1 = \bar{L}\, a(\mathbf{x})\, b(\mathbf{x}) \leq \tfrac{1}{2} \tag{2.13.3}$$

and

$$\overline{U}\left(\mathbf{x}, r_1^* = 2b(x)\right) \subseteq S,$$

where $a(\mathbf{x})$, $b(\mathbf{x})$, r^*, $\bar{L}$ *are as defined above. Then there is a unique true orbit*

$$\mathbf{x}^* = \left(x_0^*, ..., x_N^*\right) \in U\left(\mathbf{x}, r_1^*\right)$$

satisfying $f\left(x_N^*\right) = f(x_N)$.

Remark 2.13.4. If

$$L_0 = L, \tag{2.13.4}$$

then Theorems 2.13.2 and 2.13.3 reduce to Theorems 1 and 2 in [108], respectively. However in general

$$L_0 \leq L. \tag{2.13.5}$$

The conditions corresponding with (2.13.2) and (2.13.3), respectively, in Theorem 1 and 2 in [108] are given by

$$h = La(\mathbf{x})\, by(\mathbf{x}) \leq \tfrac{1}{2} \tag{2.13.6}$$

and

$$h^1 = L\, a(\mathbf{x})\; b(\mathbf{x}) \leq \tfrac{1}{2}. \tag{2.13.7}$$

It follows from (2.13.2), (2.13.3), (2.13.5), (2.13.6), and (2.13.7) that:

$$h \leq \tfrac{1}{2} \Longrightarrow h_0 \leq \tfrac{1}{2} \tag{2.13.8}$$

$$h^1 \leq \tfrac{1}{2} \Longrightarrow h_0^1 \leq \tfrac{1}{2} \tag{2.13.9}$$

but not vice versa unless if (2.13.4) holds. Hence we managed to weaken the sufficient convergence conditions given in [108], and under the same computational cost, as the evaluation of L requires in precise the evaluation of L_0.

Moreover the information on the location of the true orbit is more precise and the corresponding error bounds are finer [39] (see also Section 2.2).

2.14 Computation of continuation curves

In this study, we are concerned with approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{2.14.1}$$

where F is a continuously Fréchet-differentiable operator defined on an open convex subset D of R^m (a positive integer) in to R^m.

In recent years, a number of approaches have been proposed for the numerical computation of continuation curves, and with techniques for overcoming turning points [175], [205]. It turns out that all numerical continuation methods are of the predictor-corrector type. That is, information on the already computed portion of the curve is used to calculate an extrapolation approximating an additional curve portion. At the end, a point on the so constructed curve is chosen as the initial guess for the corrector method to converge to some point of the continuation curve.

Consider the system of n equations

$$F(x) = F\left(x^0\right), x \in \mathcal{R}(F), x_0 \in \mathcal{R}(F), \tag{2.14.2}$$

together with the (popular) choice [77], [176]

$$u^T x = z, \tag{2.14.3}$$

where u is derived from fixing the value of one of the variables. For example, set $u = e^i$, where e^i is the ith unit-basis vector of R^{n+1}.

System (2.14.2)–(2.14.3) can now be rewritten as

$$G(x) = 0, \tag{2.14.4}$$

where

$$G(x) = \begin{bmatrix} F(x) - F\left(x^0\right) \\ \left(e^i\right)^T x - z \end{bmatrix} \tag{2.14.5}$$

with z a not known yet constant.

Clearly, for $T(x) = u$:

$$\begin{aligned} \det G'(x) &= \det \begin{bmatrix} F'(x) \\ \left(e^i\right)^T \end{bmatrix} = \begin{bmatrix} F'(x) \\ (T(x))^T \end{bmatrix} \left[I + T(x) \left(e^i - T(x)\right)^T \right] \\ &= \left[T(x)^T e^i \right] \det \begin{bmatrix} F'(x) \\ (T(x))^T \end{bmatrix}. \end{aligned} \tag{2.14.6}$$

Therefore i should be chosen so that $\left| T(x)^T e^i \right|$ is as large as possible.

Here we address the length of the step-size. In particular, we show that under the same hypotheses and computational cost as before, we can enlarge the step size of the iteration process [39] (see Section 2.2). This observation is important in computational mathematics.

As in the elegant paper by Rheinboldt [176], we use NK method as the corrector method.

We need the following local convergence result of ours concerning the radius of convergence for NK method [39]:

Lemma 2.14.1. *Let G: $D \subseteq R^m \to R^m$ be a Fréchet-differentiable operator. Assume: there exists a solution x^* of equation $G(x) = 0$ such that $G'(x^*)^{-1}$ is irrevertible;*

$$\left\| G'\left(x^*\right)^{-1}\left[G'(x) - G'(y)\right]\right\| \leq \ell \|x - y\| \tag{2.14.7}$$

$$\left\| G'\left(x^*\right)^{-1}\left[G'(x) - G'\left(x^*\right)\right]\right\| \leq \ell_0 \left\|x - x^*\right\| \tag{2.14.8}$$

for all $x, y \in D$;
and

$$\overline{U}\left(x^*, r_A\right) \subseteq D, \tag{2.14.9}$$

where

$$r_A = \tfrac{2}{\ell_0 + \ell}. \tag{2.14.10}$$

Then NK method applied to G is well defined, remains in $U(x^*, r_A)$, and converges to x^* provided that $x_0 \in U(x^*, r_A)$.

Moreover the following error bounds hold for all $n \geq 0$:

$$\left\|x_{n+1} - x^*\right\| \leq \tfrac{\ell}{[1 - \ell_0 \|x_n - x^*\|]} \left\|x_n - x^*\right\|^2. \tag{2.14.11}$$

Remark 2.14.2. In general

$$\ell_0 \leq \ell \tag{2.14.12}$$

holds.

The corresponding radius r_R given by Rheinboldt [175]:

$$r_R = \tfrac{2}{3\ell} \tag{2.14.13}$$

is smaller than r_A if strict inequality holds in (2.14.12). Consequently, the step-size used with Newton's method as corrector can be increased (as it depends on r_A).

Indeed as in [176], the Lipschitz conditions (2.14.7) and (2.14.8) hold in compact subset C of R^{n+1}. We can have:

$$\left\| G'(x)^{-1}\right\| \leq \left(1 + \tfrac{2}{|T(x)^T e^i|}\right)\sqrt{1 + b(x)}, \tag{2.14.14}$$

(see (4.9) in [176]),
where,

$$b(x) = \|F'(x)(F'(x)^T)^{-1}\|_2. \tag{2.14.15}$$

If $x^* \in \mathcal{R}(F)$ is n solution of equation (2.14.5), let

$$\tau\left(x^*\right) = \max_{j=1,\ldots,n} \left|T\left(x^*\right)e^j\right| \delta\left(x^*, C\right) = \operatorname{dist}\left(x^*, \delta C\right). \tag{2.14.16}$$

For $\theta \in (0, 1) \cup (x^*, r_A(x^*))$ with

$$r_A\left(x^*\right) = \min\left\{\delta\left(x^*, C\right), \frac{2\theta\tau(x^*)}{(2+\tau(x^*))(2\ell_0+\ell)\sqrt{1+b(x^*)^2}}\right\} \geq r_R\left(x^*\right), \tag{2.14.17}$$

where ℓ_0, ℓ depend on C (not D) but we used the same symbol, and $r_R(x^*)$ is defined as r_A (with $\ell_0 = \ell$ in (2.14.17)) (clearly $Ux^*, r_A(x^*) \subseteq \mathcal{R}(F)$). By Lemma 2.14.1, NK iteration for (2.14.5) converges to x^*.

Continuing as in [176], let

$$x: I \to R \to \mathcal{R}(F), \left\|x'(s)\right\|_2 = 1 \text{ for all } x \in I, x(s_0) = x^0, s_0 \in I \tag{2.14.18}$$

be the unique C^1 operator—parameterized in terms of the path length—that solves equation (2.14.2). We use the Euler-line

$$y_E(s) = x(s_0) + T(x)(s_0)(s - s_0) \quad s \in I \tag{2.14.19}$$

as predictor with Newton's method as corrector. Let $x^k \in \mathcal{R}(F)$ be n known approximation to $x(s_k)$, $s_k \in I$; then one step of the process is:

1. Compute $T\left(x^k\right)$;
2. Determine i such that $\left|\left(e^i\right)^T\left(x^k\right)\right| = \max\limits_{j=1,\ldots,n+1}\left|\left(e^j\right)^T(T(x^*))\right|$;
3. Choose the step-size $h_{k+1} > 0$;
4. Compute the predicted point $y = x^k + h_{k+1}\left(T\left(x^k\right)\right)$;
5. Apply Newton's method to (2.14.5) with $z = \left(e^i\right)^T(y)$ with y as starting point;
6. If "satisfactory convergence," then $x^{k+1} =$ last iterate;

$$\text{else replace } h_{k+1} \text{ by } qh_k \tag{2.14.20}$$

   for some $q \in (0, 1)$ and go to step 4.
7. $S_{k+1} = S_k + \left\|x^{k+1} - x^k\right\|_2$.

(A) Assume: We want to compute $x: \overline{I}_0 \to \mathcal{R}(F)$ of (2.14.18) for $\overline{I}_0 = \left[\underline{s}, \overline{s}\right] \subset I$, $\underline{s} < \overline{s}$. There exists $\delta > 0$ such that

$$C = \left\{x \in R^{n+1} / \operatorname{dist}\left(x, x\left(\overline{I}_0\right)\right) \leq \delta\right\} \subseteq \mathcal{R}(F). \tag{2.14.21}$$

We can have:

$$\begin{aligned} r_A(x) \geq r_A^0 &= \min\left(\delta, \frac{\theta\tau_0}{(2+\tau_0)(2\ell_0+\ell)\sqrt{1+\left(\overline{b}\right)^2}}\right) \\ \geq r_R^0 &= \min\left(\delta, \frac{2\theta\tau_0}{3(2+\tau_0)\sqrt{1+\left(\overline{b}\right)^2}}\right) > 0 \end{aligned} \tag{2.14.22}$$

for all $s \in \overline{I}_0, \theta \in (0, 1)$,
where

$$\overline{b} = \sup\left\{b\left(x\left(s\right)\right), s \in \overline{I}_0\right\} < \infty,$$
$$\tau_0 = \inf\left\{\max_{j=1,\ldots,n+1}\left|T\left(x\right)^T e^j\right|, x \in C\right|\right\} > 0. \tag{2.14.23}$$

(B) Moreover assume:
approximation x^k of $x\left(s_k\right), s_k \in \overline{I}_0$ satisfies:

$$\left(e^i\right)^T\left(x^k - x\left(s_k\right)\right) = 0, \quad \left\|x^k - x\left(s_k\right)\right\|_2 \leq \frac{\min\left(\delta, r_A^0\right)}{2} \tag{2.14.24}$$

(C) Furthermore assume:

$$n_k = \min\left\{\overline{s} - s_k, \tfrac{1}{2}\tau_0\delta, \tfrac{\tau_0}{2\ell_1}\right\} > 0, \tag{2.14.25}$$

where ℓ_1 is the Lipschitz constant of T on C. For any point $x\left(s_k + \sigma\right)$ on (2.14.18) with $\sigma \in \overline{I}_k = \left[s_k, s_k + n_k\right]$ there exists $y = x^k + g\left(\sigma\right) T\left(x^k\right)$ on the Euler line with the same ith component, i.e., point y with

$$g\left(\sigma\right) = \frac{\left(e^i\right)^T\left(x\left(s_k+\sigma\right) - x^k\right)}{\left(e^i\right)^T T\left(x^k\right)}. \tag{2.14.26}$$

By Rheinboldt [176] we have:

$$\left|g\left(\sigma\right)\right| \leq \tfrac{1}{2}\delta, \tag{2.14.27}$$

$$x^k + g\left(\sigma\right) T\left(x^k\right) \in C \text{ for all } \sigma \in \overline{I}_k \tag{2.14.28}$$

and

$$y = x^k + h_{k+1} T\left(x^k\right) \in U\left(x\left(s_k + \sigma_k\right), r_0\right) \tag{2.14.29}$$

with

$$h_{k+1}^A = g\left(\sigma_k^A\right) \geq g\left(\sigma_k^R\right) \tag{2.14.30}$$

$$0 < \sigma_k^A = \min\left(n_k, \left[\frac{\tau_0 r_A^0}{\ell_1\left(1+\tau_0\right)}\right]^{1/2}\right) \geq \sigma_k^R = \min\left(n_k, \left[\frac{\tau_0 r_R^o}{\ell_1\left(1+\tau_0\right)}\right]^{1/2}\right). \tag{2.14.31}$$

Hence the convergence of NK method for (2.14.5) from y to $x\left(s_k + \sigma_k^A\right)$ is ensured.
Define

$$\sigma_*^A = \min\left(\tfrac{1}{2}\tau_0\delta, \tfrac{\tau_0}{2\ell_1}, \left[\frac{\tau_0 r_0^A}{\ell_1\left(1+\tau_0\right)}\right]^{1/2}\right)$$
$$\geq \min\left(\tfrac{1}{2}\tau_0\delta, \tfrac{\tau_0}{2\ell_1}, \left[\frac{\tau_0 r_0^R}{\ell_1\left(1+\tau_0\right)}\right]^{1/2}\right) = \sigma_*^R. \tag{2.14.32}$$

Then we get

$$\sigma_*^A = \sigma_*^A \text{ for } 0 \leq s_k \leq \overline{s} - \sigma_* \tag{2.14.33}$$

and for $s_k \in \left[\overline{s} - s^*, \overline{s}\right]$ we reach $\overline{s}$ in one step, whereas interval $\overline{I}_0$ is traversed in finitely many steps.

Hence we showed as in Theorem 4.2 in [176]:

Theorem 2.14.3. *Under hypotheses (A)–(C) there exists* $s_k \in \overline{I}_0$*, a step* $h_{k+1} > 0$ *along the euler line such that Newton's method of step 5 is well defined and converges to some* $x\left(s_k + \sigma_k^A\right), \sigma_k^A > 0$*. Starting from* $s_0 = \underline{s}$*, we can choose* h_k^A $k = 0, 1, \ldots$ *such that* $s_k = \underline{\sigma} + k\sigma_*^A, k = 0, 1, \ldots, M^A, s_{m+1} = \overline{s}$ *with a constant* $\sigma_*^A > 0$ *for which*

$$M^A \sigma_*^A \leq \overline{\sigma} - \underline{\sigma} \leq \left(M^A + 1\right)\sigma_*^A. \tag{2.14.34}$$

Remark 2.14.4. Under hypotheses of Theorem 2.14.3 and Theorem 4.2 in [176], because of (2.14.17), (2.14.22), (2.14.31), and (2.14.32) (if strict inequality holds in (2.14.12) for C instead of D), we conclude:

$$h_k^R \leq h_k^A \tag{2.14.35}$$

$$\sigma_*^R \leq \sigma_*^A \tag{2.14.36}$$

$$\sigma_k^R \leq \sigma_k^A \tag{2.14.37}$$

and

$$M^A \leq M^R. \tag{2.14.38}$$

Estimates (2.14.35)–(2.14.38) justify the claims made in the introduction about the improvements on the step-size. Note also that strict inequalities will hold in (2.14.35)–(2.14.38) if the "minimum" is expressed in terms of r_0^A in the definition of the above quantities (see (2.14.22)).

Some comments on a posteriori, asymptotic estimates are given next:

Remark 2.14.5. Rheinboldt also showed [176, p. 233] that if the solution (2.14.18) of equation (2.14.2) is three times continuously Fréchet-differentiable on the open interval I, then σ should be chosen by

$$\sigma^R = \theta \sqrt{\frac{\rho_R}{\left\| w^k - \gamma_k T\left(x^k\right) \right\|_2}} \tag{2.14.39}$$

where w^k, γ_k are given (4.27) and (4.29) in [176, p. 233], $\theta \in (0, 1)$ and ρ_R is a "safe" radius of convergence of NK method at $x\left(s_k + \sigma^R\right)$. Because again our corresponding radius of convergence ρ_A is such that

$$\rho_R < \rho_A \tag{2.14.40}$$

we deduce (if strict inequality holds in (2.14.12)):

$$\sigma^R < \sigma^A, \tag{2.14.41}$$

where σ^A is given by (2.14.39) for ρ_R replaced by ρ_A.

2.15 Gauss-Newton method

In this section, we are concerned with the problem of approximating a point x^* minimizing the objective operator

$$Q(x) = \frac{1}{2}\|F(x)\|_2^2 = \frac{1}{2}F^T(x)F(x) \tag{2.15.1}$$

where F is a Fréchet-differentiable regression operator defined on an open subset D of R^j with values in R^m $(j \le m)$.

It is well-known that for x^* to be a local minimum, it is necessary to be a zero of the gradient ∇Q of Q, too.

That is why Ben-Israel [46] suggested the so called Gauss-Newton method:

$$x_{n+1} = x_n - J^+(x_n)F(x_n) \quad (n \ge 0), \tag{2.15.2}$$

where, $J(x) = F'(x)$, the Fréchet derivative of F. Here M^+ denotes the pseudo inverse of a matrix M satisfying:

$$\left(M^+M\right)^T = M^+M,\ \left(MM^+\right)^T = MM^+,\ M^+MM^+ = M^+, MM^+M = M. \tag{2.15.3}$$

Moreover, if rank-(m, j) matrix M is of full rank, then its pseudo inverse becomes

$$M^+ = \left(M^TM\right)^{-1}M^T. \tag{2.15.4}$$

A semilocal convergence analysis for method (2.15.2) has already been given in the elegant paper in [110]. However, we noticed that under weaker hypotheses, we can provide a similar analysis with the following advantages over the ones in [110], and under the same computational cost:

(a) our results apply whenever the ones in [110] do but not vice versa;

(b) error bounds $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ $(n \ge 0)$ are finer;

(c) the information on the location of the solution x^* is more precise.

The results obtained here can be naturally extended to hold in arbitrary Banach spaces using outer or generalized inverses [59] (see also Chapter 8).

We need the following result on majorizing sequences for method (2.15.2).

Lemma 2.15.1. *Let $a \ge 0$, $b > 0$, $c \ge 0$, $L_0 \ge 0$, $L \ge 0$ be given parameters. Assume there exists $d \in [0, 1)$ with $c \le d$ such that for all $k \ge 0$*

$$\left[\frac{1}{2}bL(1-d)d^k + dbL_0\left(1 - d^{k+1}\right)\right]a + (c-d)(1-d) \le 0, \tag{2.15.5}$$

and

$$\frac{bL_0a}{1-d}\left(1 - d^k\right) < 1. \tag{2.15.6}$$

Then, iteration $\{s_n\}$ $(n \ge 0)$ given by

$$s_0 = 0, s_1 = a, s_{n+2} = s_{n+1} + \frac{\frac{1}{2}bL(s_{n+1}-s_n)+c}{1-bL_0s_{n+1}} \cdot (s_{n+1} - s_n) \tag{2.15.7}$$

is nondecreasing, bounded above by $s^{**} = \frac{a}{1-d}$, *and converges to some* s^* *such that*

$$0 \leq s^* \leq s^{**}. \tag{2.15.8}$$

Moreover, the following estimates hold for all $n \geq 0$*:*

$$0 \leq s_{n+2} - s_{n+1} \leq d\,(s_{n+1} - s_n) \leq d^{n+1} a. \tag{2.15.9}$$

Proof. We shall show using induction that for all $k \geq 0$

$$\tfrac{1}{2} bL\,(s_{k+1} - s_k) + dbL_0 s_{k+1} + c \leq d, \tag{2.15.10}$$

$$s_{k+1} - s_k \geq 0, \tag{2.15.11}$$

and

$$1 - bL_0 s_{k+1} > 0. \tag{2.15.12}$$

Using (2.15.5)–(2.15.7), estimates (2.15.10)–(2.15.12) hold. But then (2.15.7) gives

$$0 \leq s_2 - s_1 \leq d\,(s_1 - s_0)\,.$$

Let us assume (2.15.9)–(2.15.12) hold for all $k \leq n+1$.
We can have in turn

$$\begin{aligned} &\tfrac{1}{2} bL\,(s_{k+2} - s_{k+1}) + dbL_0 s_{k+2} + c \leq \qquad (2.15.13) \\ &\leq \tfrac{1}{2} bLd^{k+1} + dbL_0 \left[s_1 + d\,(s_1 - s_0) + d^2\,(s_1 - s_0) + \cdots + d^{k+1}\,(s_1 - s_0) \right] + c \\ &\leq \tfrac{1}{2} bLd^{k+1} + dbL_0 \tfrac{1-d^{k+2}}{1-d} a + c \leq d \quad \text{(by (2.15.5))}. \end{aligned}$$

Moreover we show:

$$s_k \leq s^{**}. \tag{2.15.14}$$

For $k = 0, 1, 2$, $s_0 = 0 \leq s^{**}$, $s_1 = a \leq s^{**}$, $s_2 \leq a + da = (1+d)\,a \leq s^{**}$.
It follows from (2.15.9) that for all $k \leq n+1$

$$\begin{aligned} s_{k+2} &\leq s_{k+1} + d\,(s_{k+1} - s_k) \leq \cdots \leq s_1 + d\,(s_1 - s_0) + \cdots + d\,(s_{k+1} - s_k) \qquad (2.15.15) \\ &\leq \left[1 + d + d^2 + \cdots + d^{k+1} \right] a = \tfrac{1-d^{k+2}}{1-d} a \leq s^{**}. \end{aligned}$$

Furthermore, we get

$$bL_0 s_{k+1} \leq bL_0 \tfrac{1-d^{k+1}}{1-d} a < 1. \tag{2.15.16}$$

Finally (2.15.9), (2.15.11) hold by (2.15.7), (2.15.13)–(2.15.16).
The induction is now complete.
Hence, sequence $\{s_n\}$ $(n \geq 0)$ is nondecreasing and bounded above by s^{**}, and as such it converges to some s^* satisfying (2.15.8).
That completes the proof of the Lemma.

We can show the following main semilocal convergence result for method (2.15.2).

Theorem 2.15.2. *Let $F\colon D_0 \subseteq D \subseteq R^j \to R^m$ be a Fréchet-differentiable operator, where D_0 is a convex set. Assume:*
there exists $x_0 \in D_0$ with $\operatorname{rank}(J(x_0)) = r \leq m$, $r \geq 1$ *and* $\operatorname{rank}(J(x)) \leq r$ *for all $x \in D_0$;*

$$\left\| J^+(x_0)\, F(x_0) \right\| \leq a, \tag{2.15.17}$$

$$\| J(x) - J(x_0) \| \leq L_0 \| x - x_0 \|, \tag{2.15.18}$$

$$\| J(x) - J(y) \| \leq L \| x - y \|, \tag{2.15.19}$$

$$\left\| J^+(x_0) \right\| \leq b, \tag{2.15.20}$$

$$\left\| J^+(y)\, q(x) \right\| \leq c(x) \| x - y \| \tag{2.15.21}$$

with $q(x) = \left(I - J(x) J^+(x)\right) F(x)$, and $q(x) \leq c < 1$, for all $x, y \in D_0$;
conditions (2.15.5) and (2.15.6) hold;
and

$$\overline{U}\left(x_0, s^*\right) \subseteq D_0, \tag{2.15.22}$$

where s^* is defined in Lemma 2.15.1.

Then,

(a) sequence $\{x_n\}$ $(n \geq 0)$ generated by method (2.15.2) is well defined, remains in $U(x_0, s^*)$ for all $n \geq 0$, and converges to a solution $x^* \in \overline{U}(x_0, s^*)$ of equation $J^+(x)\, F(x) = 0$;
(b) $\operatorname{rank}(J(x)) = r$ *for all $x \in U(x_0, s^*)$;*
(c) $\operatorname{rank}\left(J\left(x^0\right)\right) = r$ *if strict inequality holds in (2.15.5) or equality and $c > 0$.*

Moreover the following estimates hold for all $n \geq 0$

$$\| x_{n+1} - x_n \| \leq s_{n+1} - s_n, \tag{2.15.23}$$

and

$$\left\| x_n - x^* \right\| \leq s^* - s_n. \tag{2.15.24}$$

Furthermore, if

$$\operatorname{rank}(J(x_0)) = m, \quad \text{and } F\left(x^*\right) = 0, \tag{2.15.25}$$

then x^* is the unique solution of equation $F(x) = 0$ in $U(x_0, s^{**})$, and the unique zero of equation $J^+(x)\, F(x) = 0$ in $U(x_0, s^*)$.

Proof. We shall show $\{s_n\}$ $(n \geq 0)$ is a majorizing sequence for $\{x_n\}$ so that estimate (2.15.23) holds, and iterates $s_n \in U(x_0, s^*)$ $(n \geq 0)$.

It follows from the Banach Lemma, and the estimate

$$\| J(x) - J(x_0) \| \leq L_0 \| x - x_0 \| \leq L_0 s^* < 1 \quad \text{(by (2.15.6))}$$

for all $x \in U(x_0, s^*)$ that (b) and (c) above hold, with

$$\left\| J^{+}(x) \right\| \leq \tfrac{b}{1-bL\|x-x_0\|} \text{ for all } x \in U\left(x_0, s^*\right). \tag{2.15.26}$$

Consequently, operator

$$P(x) = x - J^{+}(x)\, F(x) \tag{2.15.27}$$

is well defined on $U(x_0, s^*)$. If x, $P(x) \in U(x_0, s^*)$ using (2.15.2), (2.15.17)–(2.15.21) we can obtain in turn:

$$\begin{aligned}
&\|P(P(x)) - P(x)\| = \\
&= \left\| J^{+}(P(x)) \int_0^1 \{J(x + t(P(x) - x)) - J(x)\}(P(x) - x)\, dt \right. \qquad (2.15.28)\\
&\quad \left. + J^{+}\left(P(x)\left(I - J(x)\, J^{+}(x)\right)\right) F(x) \right\| \\
&\leq \tfrac{1}{1-bL_0\|P(x)-x_0\|} \left(\tfrac{1}{2} bL \|P(x) - x\| + c\right) \|P(x) - x\|.
\end{aligned}$$

Estimate (2.15.23) holds for $n = 0$ by the initial conditions. Assuming by induction: $\|x_i - x_{i-1}\| \leq s_i - s_{i-1}$ $(i = 1, 2, ..., k)$ it follows

$$\|x_i - x_0\| \leq s_k - s_0 \text{ for } i = 1, 2, ..., k. \tag{2.15.29}$$

Hence, we get $\{x_n\} \subset (x_0, s^*)$.

It follows from (2.15.7) and (2.15.29) that (2.15.23) holds for all $n \geq 0$.

That is $\{x_n\}$ $(n \geq 0)$ is a Cauchy sequence in R^m and as such it converges to some $x^* \in \overline{U}(x_0, s^*)$ (because $\overline{U}(x_0, s^*)$ is a closed set).

Using the continuity of $J(x)$, $F(x)$, and the estimate

$$\begin{aligned}
\left\| J^{+} F(x_k) \right\| &\leq \left\| J^{+}\left(x^*\right)\left(I - J(x_k)\, J^{+}(x_k)\, F(x_k)\right) \right\| \\
&\quad + \left\| J^{+}\left(x^*\right) \right\| \cdot \left\| J(x_k)\, J^{+}(x_k)\, F(x_k) \right\| \\
&\leq c \left\| x_k - x^* \right\| + \left\| J^{+}\left(x^*\right) \right\| \|J(x_k)\| \|x_{k+1} - x_k\|
\end{aligned} \tag{2.15.30}$$

we conclude $J^{+}(x^*)\, F(x^*) = 0$.

The uniqueness part follows exactly as in Theorem 2.4 in [110] (see also [39] or Section 2.2, or Theorem 12.5.5 in [154]).

Remark 2.15.3. Conditions (2.15.5), (2.15.6) are always present in the study of Newton-type methods. We wanted to leave conditions (2.15.5) and (2.15.6) as uncluttered as possible. We may replace (2.15.5) and (2.15.6) by the stronger

$$\left[\tfrac{1}{2} bL(1 - d) + dbL_0\right] a + (c - d)(1 - d) \leq 0 \tag{2.15.31}$$

and

$$\tfrac{bL_0 a}{1-d} < 1, \tag{2.15.32}$$

respectively. Clearly conditions (2.15.5) and (2.15.6) are weaker than the Newton-Kantorovich-type hypothesis

$$h = \frac{abL}{(1-c)^2} \le \frac{1}{2} \tag{2.15.33}$$

used in Theorem 2.4 in [110, p. 120].

Indeed first of all

$$L_0 \le L \tag{2.15.34}$$

holds in general. If equality holds in (2.15.35), then iteration $\{s_n\}$ reduces to $\{t_n\}$ $(n \ge 0)$ in [110] (simply set $L_0 = L$ in (2.15.7)), and Theorem 2.15.2 reduces to Theorem 2.4 in [110]. However, if strict inequality holds in (2.15.34), then our estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ are more precise than the ones in [110]. Indeed we immediately get

$$s_{n+1} - s_n < t_{n+1} - t_n \qquad (n \ge 1), \tag{2.15.35}$$

$$s^* - s_n \le t^* - t_n \qquad (n \ge 0) \tag{2.15.36}$$

and

$$s^* \le t^*. \tag{2.15.37}$$

For $c = 0$ and $d = \frac{1}{2}$, conditions (2.15.5) and (2.15.6) hold provided that

$$h_1 = abL_1 \le \frac{1}{2} \tag{2.15.38}$$

where,

$$L_1 = \frac{L_0 + L_1}{2}. \tag{2.15.39}$$

Corresponding condition (e) in Theorem 2.4 in [110] becomes the famous Newton-Kantorovich hypothesis

$$h_2 = abL \le \frac{1}{2}. \tag{2.15.40}$$

Note that (2.15.39) is weaker than (2.15.41) if strict inequality holds in (2.15.35). Hence, we have

$$h_2 \le \frac{1}{2} \Longrightarrow h_1 \le \frac{1}{2} \tag{2.15.41}$$

but not necessarily vice versa unless if $L_0 = L$.

Remark 2.15.4. Along the lines of our comments above, the corresponding results in [110, pp. 122–124] can now be improved (see also Section 2.2).

2.16 Exercises

2.16.1. Show that f defined by $f(x, y) = |\sin y| + x$ satisfies a Lipschitz condition with respect to the second variable (on the whole xy-plane).

2.16.2. Does f defined by $f(t, x) = |x|^{1/2}$ satisfy a Lipschitz condition?

2.16.3.

(a) Let $F: D \subseteq X \to X$ be an analytic operator. Assume:

- there exists $\alpha \in [0, 1)$ such that

$$\|F'(x)\| \le \alpha \quad (x \in D); \tag{2.16.1}$$

- $$\gamma = \sup_{\substack{k>1 \\ x\in D}} \left\| \frac{1}{k!} F^{(k)}(x) \right\|^{\frac{1}{k-1}} \qquad \text{is finite;}$$

- there exists $x_0 \in D$ such that
$$\|x_0 - F(x_0)\| \le \eta \le \frac{3-\alpha-2\sqrt{2-\alpha}}{\gamma}, \quad \gamma \ne 0;$$

- $\bar{U}(x_0, r_1) \subseteq D$, where, r_1, r_2 with $0 \le r_1 \le r_2$ are the two zeros of function f, given by
$$f(r) = \gamma(2-\alpha)r^2 - (1+\eta\gamma-\alpha)r + \eta.$$

Show: method of successive substitutions is well defined, remains in $\bar{U}(x_0, r_1)$ for all $n \ge 0$ and converges to a fixed point $x^* \in \bar{U}(x_0, r_1)$ of operator F. Moreover, x^* is the unique fixed point of F in $\bar{U}(x_0, r_2)$. Furthermore, the following estimates hold for all $n \ge 0$:

$$\|x_{n+2} - x_{n+1}\| \le \beta \|x_{n+1} - x_n\|$$

and
$$\|x_n - x^*\| \le \frac{\beta^n}{1-\beta}\eta,$$

where
$$\beta = \frac{\gamma\eta}{1-\gamma\eta} + \alpha.$$

The above result is based on the assumption that the sequence

$$\gamma_k = \left\| \frac{1}{k!} F^{(k)}(x) \right\|^{\frac{1}{k-1}} \quad (x \in D), \quad (k > 1)$$

is bounded above by γ. This kind of assumption does not always hold. Let us then not assume sequence $\{\gamma_k\}$ $(k > 1)$ is bounded and define "function" f_1 by

$$f_1(r) = \eta - (1-\alpha)r + \sum_{k=2}^{\infty} \gamma_k^{k-1} r^k.$$

(b) Let $F: D \subseteq X \to X$ be an analytic operator. Assume (2.16.1) holds and for $x_0 \in D$ function f_1 has a minimum positive zero r_3 such that

$$\bar{U}(x_0, r_3) \subseteq D.$$

Show: method of successive substitutions is well defined, remains in $\bar{U}(x_0, r_3)$ for all $n \ge 0$ and converges to a unique fixed point $x^* \in \bar{U}(x_0, r_3)$ of operator F. Moreover the following estimates hold for all $n \ge 0$

$$\|x_{n+2} - x_{n+1}\| \le \beta_1 \|x_{n+1} - x_n\|$$

and

$$\|x_n - x^*\| \le \tfrac{\beta_1^n}{1-\beta_1}\eta,$$

where,

$$\beta_1 = \sum_{k=2}^{\infty} \gamma_k^{k-1}\eta^{k-1} + \alpha.$$

2.16.4.

(a) It is convenient to define:

$$\gamma = \sup_{k>1} \left\| \tfrac{1}{k!} F^{(k)}(x^*) \right\|^{\frac{1}{k-1}}$$

with $\gamma = \infty$, if the supremum does not exist. Let $F\colon D \subseteq X \to X$ be an analytic operator and $x^* \in D$ be a fixed point of F. Moreover, assume that there exists α such that

$$\|F'(x^*)\| \le \alpha, \tag{2.16.2}$$

and

$$\bar{U}(x^*, r^*) \subseteq D,$$

where,

$$r^* = \begin{cases} \infty, & \text{if } \gamma = 0 \\ \frac{1}{\gamma}\cdot\frac{1-\alpha}{2-\alpha}, & \text{if } \gamma \ne 0. \end{cases}$$

Then, if

$$\beta = \alpha + \tfrac{\gamma r^*}{1-\gamma r^*} < 1,$$

show: the method of successive substitutions remains in $\bar{U}(x^*, r^*)$ for all $n \ge 0$ and converges to x^* for any $x_0 \in U(x^*, r^*)$. Moreover, the following estimates hold for all $n \ge 0$:

$$\|x_{n+1} - x^*\| \le \beta_n \|x_n - x^*\| \le \beta\|x_n - x^*\|,$$

where,

$$\beta_0 = 1, \quad \beta_{n+1} = \alpha + \tfrac{\gamma r^* \beta_n}{1-\gamma r^* \beta_n} \quad (n \ge 0).$$

The above result was based on the assumption that the sequence

$$\gamma_k = \left\| \tfrac{1}{k!} F^{(k)}(x^*) \right\|^{\frac{1}{k-1}} \quad (k \ge 2)$$

is bounded by γ. In the case where the assumption of boundedness does not necessarily hold, we have the following local alternative.

(b) Let $F\colon D \subseteq X \to X$ be an analytic operator and $x^* \in D$ be a fixed point of F. Moreover, assume: $\max\limits_{r>0} \sum\limits_{k=2}^{\infty} (\gamma_k r)^{k-1}$ exists and is attained at some $r_0 > 0$. Set

$$p = \sum_{k=2}^{\infty} (\gamma_k r_0)^{k-1};$$

there exist α, δ with $\alpha \in [0, 1)$, $\delta \in (\alpha, 1)$ such that (2.16.2) holds,

$$p + \alpha - \delta \leq 0$$

and

$$\bar{U}(x^*, r_0) \subseteq D.$$

Show: the method of successive substitutions $\{x_n\}$ $(n \geq 0)$ remains in $\bar{U}(x^*, r_0)$ for all $n \geq 0$ and converges to x^* for any $x_0 \in \bar{U}(x^*, r_0)$. Moveover the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x^*\| \leq \alpha \|x_n - x^*\| + \sum_{k=2}^{\infty} \gamma_k^{k-1} \|x_n - x^*\|^k \leq \delta \|x_n - x^*\|.$$

2.16.5. Let x^* be a solution of Equation (2.1.1). If the linear operator $F'(x^*)$ has a bounded inverse, and $\lim_{\|x - x^*\| \to 0} \left\| F'(x) - F'(x^*) \right\| = 0$, then show NK method converges to x^* if x_0 is sufficiently close to x^* and

$$\left\| x_n - x^* \right\| \leq d \varepsilon^n \quad (n \leq 0),$$

where ε is any positive number; d is a constant depending on x_0 and ε.

2.16.6. The above result cannot be strengthened, in the sense that for every sequence of positive numbers c_n such that: $\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = 0$, there is an equation for which NK converges less rapidly than c_n. Define

$$s_n = \begin{cases} c_{n/2}, & \text{if } n \text{ is even} \\ \sqrt{c_{(n-1)/2} c_{(n+1)/2}}, & \text{if } n \text{ is odd.} \end{cases}$$

Show: $s_n \to 0$, $\frac{s_{n+1}}{s_n} \to 0$, and $\lim_{n \to \infty} \frac{c_n}{s_{n+k}} = 0$, $(k \geq 1)$.

2.16.7. Assume operator $F'(x)$ satisfies a Hölder condition

$$\left\| F'(x) - F'(y) \right\| \leq a \left\| x - y \right\|^b,$$

with $0 < b < 1$ and $U(x_0, R)$. Define $h_0 = b_0 a \eta_0^b \leq c_0$, where c_0 is a root of

$$\left(\tfrac{c}{1+b} \right)^b = (1 - c)^{1+b} \quad (0 \leq c \leq 1)$$

and let $R \geq \frac{\eta_0}{1 - d_0} = r_0$, where $d_0 = \frac{h_0}{(1+b)(1-h_0)}$. Show that NK method converges to a solution x^* of Equation $F(x) = 0$ in $U(x_0, r_0)$.

2.16.8. Let K, B_0, η_0 be as in Theorem 2.2.4. If $h_0 = b_0 \eta_0 K < \frac{1}{2}$, and

$$r_0 = \tfrac{1 - \sqrt{1 - 2h_0}}{h_0} \eta_0 \leq r.$$

Then show: modified Newton's method (2.1.5) converges to a solution $x^* \in U(x_0, r_0)$ of Equation (2.1.1). Moreover, if

$$r_0 \leq r < \tfrac{1 + \sqrt{1 - 2h_0}}{h_0} \eta_0,$$

then show: Equation (2.1.1) has a unique solution x^* in $U(x_0, r)$. Furthermore show: $\bar{x}_{n+1} = \bar{x}_n - F'(x_0)^{-1} F(\bar{x}_n)$ $(n \geq 0)$ converges to a solution x^* of Equation (2.1.1) for any initial guess $\bar{x}_0 \in U(x_0, r)$.

2.16.9. Under the hypotheses of Theorem 2.2.4, let us introduce $\bar{U} = \bar{U}\,(x_1, r_0 - \eta)$, sequence $\{t_n\}$ $(n \geq 0)$, $t_0 = 0$, $t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$, $f(t) = \frac{1}{2}Kt^2 - t + \eta$, $\Delta = r^* - r_0$, $\theta = \frac{r_0}{r^*}$, $\nabla t_{n+1} = t_{n+1} - t_n$, $d_n = \|x_{n+1} - x_n\|$, $\Delta_n = \|x_n - x_0\|$, $\bar{U}_0 = \bar{U}$, $\bar{U}_n = \bar{U}\,(x_n, r_0 - t_n)$ $(n \geq 1)$, $K_0 = L_0 = K$,

$$K_n = \sup_{\substack{x,y\in\bar{U}_n\\ x\neq y}} \frac{\|F'(x_n)^{-1}\big(F'(x)-F'(y)\big)\|}{\|x-y\|}\ (n \geq 1)\,,$$

$$L_n = \sup_{\substack{x,y\in\bar{U}\\ x\neq y}} \frac{\|F'(x_n)^{-1}\big(F'(x)-F'(y)\big)\|}{\|x-y\|}\ (n \geq 1)\,,$$

$$\underline{\lambda}_n = \frac{2d_n}{1+\sqrt{1+2L_nd_n}}\ \ (n \geq 0)\,,$$

$$\lambda_n = \frac{2d_n}{1+\sqrt{1-2L_nd_n}}\ \ (n \geq 0)\,,\ \ \underline{\kappa}_n = \frac{2d_n}{1+\sqrt{1+2K_nd_n}},$$

$$k_n = \frac{2d_n}{1+\sqrt{1-2K_nd_n}}\ \ (n \geq 0)\,,$$

$$s_0 = 1,\ \ s_n = \frac{s_{n-1}^2}{2^{n-1}\sqrt{1-2h}+s_{n-1}\big(1-\sqrt{1-2h}\big)^{2^n-1}}\ \ (n \geq 0)\,.$$

With the notation introduced above show (Yamamoto [206]):

$$\begin{aligned}
\big\|x^* - x_n\big\| &\leq \mathcal{K}_n\ (n \geq 0) \leq \lambda_n\ (n \geq 0)\\
&\leq \frac{2d_n}{1 + \sqrt{1 - 2K\,(1 - K\Delta_n)^{-1}\,d_n}}\ \ (n \geq 0)\\
&\leq \frac{2d_n}{1 + \sqrt{1 - 2K\,(1 - Kt_n)^{-1}\,d_n}}\ \ (n \geq 0)\\
&= \frac{2d_n}{1 + \sqrt{1 - 2KB_nd_n}}\ \ (n \geq 0)\\
&= \begin{cases} \dfrac{2d_n}{1+\sqrt{1-\frac{4}{\Delta}\cdot\frac{1-\theta^{2^n}}{1+\theta^{2^n}}d_n}} & (2h < 1)\\[2ex] \dfrac{2d_n}{1+\sqrt{1-\frac{2^n}{\eta}d_n}} & (2h = 1)\ \ (n \geq 0)\end{cases}\\
&\leq \frac{r_0 - t_n}{\nabla t_{n+1}}d_n\quad (n \geq 0)\\
&= \frac{2d_n}{1 + \sqrt{1 - 2h_n}}\quad (n \geq 0)\\
&\leq \frac{KB_nd_{n-1}^2}{1+\sqrt{1-2h_n}}\quad (n \geq 0)\\
&= \frac{r_0 - t_n}{(\nabla t_n)^2}d_{n-1}^2\quad (n \geq 0)\\
&= \begin{cases} \frac{1-\theta^{2^n}}{\Delta}d_{n-1}^2 & (2h < 1)\\ \frac{2^{n-1}}{\eta}d_{n-1}^2 & (2h = 1)\ (n \geq 1)\end{cases}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{K d_{n-1}^2}{\sqrt{1-2h} + \sqrt{1-2h+(K d_{n-1})^2}} \quad (n \ge 1) \\
&\le \frac{K \eta_{n-1} d_{n-1}}{\sqrt{1-2h} + \sqrt{1-2h+(K \eta_{n-1})^2}} \quad (n \ge 1) \\
&= e^{-2^{n-1}\varphi} d_{n-1} \quad (n \ge 1) \\
&= \theta^{2^{n-1}} d_{n-1} \quad (n \ge 1) \\
&= \frac{r_0 - t_n}{\bigtriangledown t_n} d_{n-1} \quad (n \ge 1) \\
&\le r_0 - t_n \quad (n \ge 0) \\
&= \frac{2\eta_n}{1+\sqrt{1-2h_n}} \quad (n \ge 0) \\
&= \begin{cases} e^{-2^{n-1}\varphi \frac{\sinh\varphi}{\sinh 2^{n-1}\varphi}\eta} & (2h < 1) \\ 2^{1-n}\eta & (2h < 1) \end{cases} \\
&= \begin{cases} \dfrac{\Delta\theta^{2^n}}{1-\theta^{2^n}} & (2h < 1) \\ 2^{1-n}\eta & (2h = 1)\ (n \ge 0) \end{cases} \\
&= \frac{s_n}{2^n K}\left(\frac{2h}{1+\sqrt{1-2h}}\right)^{2^n} \quad (n \ge 0) \\
&\le \frac{1}{2^n} K \left(\frac{2h}{1+\sqrt{1-2h}}\right)^{2^n} \quad (n \ge 0) \\
&\le \frac{1}{2^{n-1}} (2h)^{2n-1} \eta \quad (n \ge 0)\,,
\end{aligned}
$$

$$
\begin{aligned}
\left\|x^* - x_n\right\| &\le \lambda_n \quad (n \ge 0) \\
&\le \frac{L_n d_{n-1}^2}{1+\sqrt{1-(L_n d_{n-1})^2}} \quad (n \ge 1) \\
&\le \frac{L_{n-1} d_{n-1}^2}{1 - L_{n-1} d_{n-1} + \sqrt{1-2L_{n-1} d_{n-1}}} \quad (n \ge 1) \\
&\le \frac{L_{n-1} d_{n-1}^2}{1 - L_{n-1} d_{n-1}} \quad (n \ge 1)\,,
\end{aligned}
$$

$$
\begin{aligned}
\left\|x^* - x_n\right\| &\le \lambda_n \\
&\le \frac{2 d_n}{1+\sqrt{1-2L_0\,(1-L_0\Delta_n)^{-1}\, d_n}}
\end{aligned}
$$

$$\leq \frac{2\|F'(x_0)^{-1}F(x_n)\|}{1 - L_0\Delta_n + \sqrt{(1 - L_0\Delta_n)^2 - 2L_0\|F'(x_0)^{-1}F(x_n)\|}}$$

$$\leq \frac{L_0 d_{n-1}^2}{1 - L_0\Delta_n + \sqrt{(1 - L_0\Delta_n)^2 - (L_0 d_{n-1})^2}},$$

$$\begin{aligned}
\|x^* - x_n\| &\geq \underline{\kappa}_n \ (n \geq 0) \geq \underline{\lambda}_n \ (n \geq 0) \\
&\geq \frac{2d_n}{1 + \sqrt{1 + 2K(1 - K\Delta_n)^{-1} d_n}} \quad (n \geq 0) \\
&\geq \frac{2d_n}{1 + \sqrt{1 + 2K(1 - Kt_n)^{-1} d_n}} \quad (n \geq 0) \\
&= \frac{2d_n}{1 + \sqrt{1 + 2KB_n d_n}} \quad (n \geq 0) \\
&= \frac{2d_n}{1 + \sqrt{1 + 4 \cdot \frac{r_0 - t_{n+1}}{(r_0 - t_n)^2} d_n}} \quad (n \geq 0) \\
&= \frac{2d_n}{1 + \sqrt{1 + 4 \cdot \frac{\nabla t_{n+1}}{(\nabla t_n)^2} d_n}} \quad (n \geq 0) \\
&= \frac{2d_n}{1 + \sqrt{1 + \frac{2Kd_n}{\sqrt{1 - 2h + (K\eta_{n-1})^2}}}} \quad (n \geq 0) \\
&\geq \frac{2d_n}{1 + \sqrt{1 + \frac{2Kd_n}{\sqrt{1 - 2h + (Kd_{n-1})^2}}}} \quad (n \geq 0) \\
&= \frac{2d_n}{1 + \sqrt{1 + \frac{2d_n}{\sqrt{a^2 + d_{n-1}^2}}}} \quad \left(a = \sqrt{1 - 2h}/K, n \geq 1\right) \\
&\geq \frac{2d_n}{1 + \sqrt{1 + \frac{2d_n}{d_n + \sqrt{a^2 + d_n^2}}}} \quad (n \geq 0) \\
&\geq \frac{2d_n}{1 + \sqrt{1 + 2h_n}} \quad (n \geq 0) \\
&= \frac{2d_n}{1 + \sqrt{1 + \frac{4\theta^{2^n}}{(1 + \theta^{2^n})^2}}} \quad (n \geq 0),
\end{aligned}$$

$$\|x^* - x_{n+1}\| \leq \kappa_{n+1} \leq \kappa_n - d_n \leq \frac{r_0 - t_{n+1}}{\nabla t_{n+1}} d_n,$$

$$\|x^* - x_{n+1}\| \leq \lambda_{n+1} \leq \lambda_n - d_n \leq \frac{r_0 - t_{n+1}}{\nabla t_{n+1}} d_n,$$

$$
\begin{aligned}
d_n &\le \tfrac{1}{2} K_n d_{n-1}^2 \\
&\le \tfrac{1}{2} L_n d_{n-1}^2 \\
&\le \tfrac{1}{2} K \left(1 - K\Delta_n\right)^{-1} d_{n-1}^2 \\
&\le \tfrac{1}{2} K \left(1 - K\Delta_{n-1} - K d_{n-1}\right)^{-1} d_{n-1}^2 \\
&\le \tfrac{1}{2} K \left(1 - K t_n\right)^{-1} d_{n-1}^2 \\
&= \tfrac{1}{2} K B_n d_{n-1}^2 \\
&= \frac{r_0 - t_{n+1}}{(r_0 - t_n)^2} d_{n-1}^2 \\
&= \frac{\nabla t_{n+1}}{(\nabla t_n)^2} d_{n-1}^2 \\
&= \frac{d_{n-1}^2}{2\sqrt{a^2 + \eta_{n-1}^2}} \\
&\le \frac{d_{n-1}^2}{2\sqrt{a^2 + d_{n-1}^2}} \\
&\le \frac{\nabla t_{n+1}}{\nabla t_n} d_{n-1} \\
&= \frac{\eta_n}{\eta_{n-1}} d_{n-1} \\
&= \frac{1}{2\cosh 2^{n-1}\varphi} d_{n-1} \\
&\le \tfrac{1}{2} d_{n-1} \\
&\le \tfrac{1}{2} \eta_{n-1} = \tfrac{1}{2} \nabla t_n ,
\end{aligned}
$$

and

$$
\begin{aligned}
d_n &\le \eta_n \\
&= \nabla t_{n+1} \\
&= (r_0 - t_{n+1})\,\theta^{-2^n} \\
&= (r_0 - t_{n+1})\,e^{2^n}\varphi \\
&= \begin{cases} \dfrac{\sinh\varphi}{\sinh 2^n\varphi}\eta & (2h < 1) \\ 2^{-n}\eta & (2h = 1) \end{cases} \\
&= \frac{\Delta\theta^{2^n}}{1 - \theta^{2^{n+1}}} \quad (2h < 1)\,.
\end{aligned}
$$

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