2

Finding Roots of Numbers

COMMENTARY

The scope of Chapter 2 is square roots of numbers: how to find them and how to compute with them. (Cube roots are discussed in Chapter 5.) Not only are the readers instructed on how to find square roots of up to six digit numbers, each clearly exemplified, but they are returned to the world of Pisan units of measurement and shown how to find the roots of square units of rod, feet, and inches, not to overlook a digression into the astronomical world of square degrees, minutes, seconds, and thirds (!). Fibonacci offered two ways of proving that a root is what it claims to be, the obvious method of multiplying a number by itself and one that suggest modular arithmetic [11]. Noteworthy is the fact that he made no reference to any of the geometric theorems discussed at length in Chapter 1, even if the reader can see where some are applied. In grappling with large numbers such as 9876543 he advised the reader to find first the root of the last five digits (98765). Next and after joining any remainder to the remaining digits (43), find its root. And then simply put the two roots together as one root. Finally, and after a practical definition of *radix*, the readers are instructed on how to add, subtract, multiply, and divide radicals.

Three remarks may make the reading of the root procedures easier. First, the method is essentially the tremendously tedious technique of the Hindus.[1] It is apparently based on Euclid's *Elements*, II.4, say applied to a three-digit whole number $n = 100a^2 + 20ab + b^2$. Each term on the right is subtracted successively from n beginning with $100a^2$, any remainder being equal to or greater than zero but less than (10a + b). Secondly, subtraction is always performed from the left, another Arabic characteristic.[2] Because the technique depends on division, the medieval format is followed that requires placing digits of numbers above and below the number being rooted.[3] The following example is an enlargement of [4] in which the square root of 864 is sought, the numbers in parentheses indicating the sequence of steps in finding the square root.

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[1] Datta and Singh (1935–1938) I, 169–175.
[2] Levey and Petruck (1982), 50.
[3] D.E. Smith (1953) II, 136–139.

(1)	(2)	(3)	(4)	(5)	(6)
				1	**1**
	4	4	4	4 0	4 0
8 6 4	**8 6 4**	**8 6 4**	**8 6 4**	**8 6 4**	**8 6 4**
2	2	2	2 9	2 9	2 9
		4	4	4	4

[23]

(1) The root will be a two-digit number whose square is immediately less than the third digit 8. So place 2 under the 6.
(2) Subtract 2^2 from 8 and put the remainder 4 over the 8.
(3) Double 2 and put it under 4.
(4) Readers forced to learn the Hindu Method in elementary or primary school may recall that the critical step was finding the unit's digit for a two-digit root. Fibonacci's method for "guessing" this number follows. On a diagonal you can see 46. Divide it by the 4 from (3) to get 11. The largest single digit less than 11 is 9, probably the unit's digit of the root. Put 9 in the first place under 4.
(5) Multiply 9 by 4 (under 2) and subtract 36 from 46 to get 10. Place 10 on the diagonal.
(6) Square 9, subtract 81 from 104 (on the diagonal again), to get the remainder 23.
(7) The square root of 864 is 29 with the remainder marked as [23].

Fibonacci's curious method for proving that a derived root is correct is well met in a nearly literal translation; see [11].[4] A modern representation of the method showing that the root of the 12345 is 111 with remainder 24 suggests an operation in mod 7:

[1] $111 \div 7$ leaves remainder 6. [2] $6^2 = 36$.
[3] $36 \div 7$ leaves remainder 1. [4] $24 \div 7$ leaves remainder 3.
[5] (From [3] and [4]) $1 + 3 = 4$. [6] $12345 \div 7$ leaves remainder 4.
[7] Because the results of steps [5] and [6] are identical, the root is correct.

Fibonacci closed the section on finding square roots with a discussion on finding the square roots of numbers that cannot be expressed as rational numbers but can be found as lines. The method is based on Euclid's *Elements*, II.13, "To find a mean proportional between two straight lines." In Figure 2.a let AB $= 67$, BD $= 1$, then BD $= \sqrt{67}$. We turn now to operations on radical numbers.

If nothing else, the sections on operating with roots make us eternally grateful for modern notation. We have symbols; they had only words. Addition of some roots (and their subtraction, *mutatis mutandis*) is based on the ratio of squares, the squares hidden within the radicands. For instance, to add the root of 27 to

[4] The method also appears in *Liber abaci*, 39.35–41; Sigler (2002), 67.

the root of 48, notice first that they have the ratio of 9 to 16 whose roots are 3 and 4. The sum of these last numbers is 7 whose square is 49. Because 27 is thrice 9 and 48 is three times 16, multiply 49 by three to get 147 whose root is the required sum. Multiplication and division of roots are performed "within the radical sign;" for example, $(4\sqrt{5})(3\sqrt{2}) = (\sqrt{50})(\sqrt{18}) = \sqrt{(50)(18)} = \sqrt{900}$. Fibonacci accompanied all operations in the margins of the manuscript with geometric procedures, if not proofs. Fibonacci announced two ways to find the roots of fractions, "as accurately as possible." Inasmuch as the second way is marred by manuscript errors (*q.v.*), his example of the first method follows, to find the square root of two-thirds in modern dress but retaining Fibonacci's auxiliary number 60:

$$\sqrt{\frac{2}{3}} \rightarrow \frac{2}{3}(60) = 40 \rightarrow 40(60) = 2400 \rightarrow \sqrt{2400} \approx 49 \rightarrow 49 \div 60 = 0.816667,$$

a very reasonable approximation.

The method of finding square roots is discussed further under Sources. Here I want to consider why Fibonacci included the method at all. Not that it was not needed, but rather he might have discovered a simpler, actually a very old, technique, then verbal but now in symbols:
To find

$$\sqrt{N} \text{ let } N = a^2 + r.$$

Then

$$\sqrt{N} = a + \frac{r}{2a+1}.$$

My thinking was prompted by his original method for finding cube roots as expressed in a modern formula:
To find

$$\sqrt[3]{N} \text{ let } N = a^3 + r.$$

Then

$$\sqrt[3]{N} = a + \frac{r}{3(a[a+1]) + 1}.$$

The analogy is so close: 3 represents three dimensions for the cube; so 2 would be introduced for the square. The additional term $[a + 1]$ contributes nothing to the square root, so it would be dropped. Thus Fibonacci might have made a universally accepted boon to all who extracted square roots down to modern times.

Unlike Chapter 5 on finding cube roots, which for the most part is an excerpt from Chapter 14 from *Liber abaci*, the present chapter reflects only the method and two of the several examples of the first part of Chapter 14. For this chapter, Fibonacci apparently rewrote nearly the entire section on square roots with a tighter explanation and more examples. Compare, for example, how he found the square root of 12345 here and in *Liber abaci* where the technique is not so clear.[5] Because *De practica geometrie* was written before the revision of *Liber abaci* in 1228, one may wonder why Fibonacci did not incorporate the clearer, more succinct version into the revision. The material on binomina and computing with them as studied in *Liber abaci* is not discussed here.

TEXT[6]

2.1 Finding Roots

Because we want to measure fields by geometric rules, we will have to find the roots of five numbers in certain sizes. Consequently I now describe how to find these roots efficiently enough for the purposes of this work.

Integral Roots

[1] The root of a number is the side of its square, because when you multiply a side by itself, you obtain the area of the square. Whence the sum[7] of the multiplication of the side is properly called a square because it encloses a squared surface. Note further that the root of a single or double digit number is a single digit number. There are two **{p. 19}** digits for the root of a three or four digit number, three digits for the root of a five or six digit number, four digits for the root of a seven or eight digit number, and so on. It is quite necessary to know by heart the roots of numbers of one and two digits, for only thereby can we proceed efficiently to find the roots of numbers with more digits. Hence, 1 is the root of 1 because one times one makes one: 2 is the root of 4, 3 the root of 9, 4 the root of 16, 5 the root of 25, 6 the root of 36, 7 the root of 49, 8 the root of 64, 9 the root of 81, and 10 the root of 100. The other numbers below these square numbers do not have roots. If necessary, however, we can approximate them closely by many methods that I shall show in their own place.

[2] If you want to find the integral root of a three digit number, you begin with the root of the last digit.[8] Place this under the digit in the second place.[9] You do this because you know that the root of a three digit number has two digits. Thus one digit is under the second digit and the other under the first.

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5 *Liber abaci* 354; Sigler (2002), 492.

6 "Distinctio secunda, incipit capitulum de inuencione radicum" (18.35–36, *f.* 11r.35–11v.3).

7 This is probably an Arabicism. The word comes from *jamaʿa* which means "what is gathered," and is often used to speak of the result of a multiplication (Oaks, 2 February 2006).

8 Often a modern reader needs to accommodate one's thinking to Fibonacci's right-to-left description of the digits of any number. We may consider the 3 in 369 as the first digit because we write it first; 3 is the last digit for Fibonacci because he wrote it last, after the Arabic fashion.

9 This is a in the expansion of the binomial $(10a + b)^2$ which governs the search for the root.

Whatever remains [from subtracting the square] from the third digit, you put above the third digit. Now, join the remainder with the second digit [to form a new number]. Put the other digit [of the root] under the first digit, that is, before the digit that you put under the second digit. Now the product of this[10] by twice the digit[11] you found as a root is so close to the new number that from the difference of the new number and the first, you can subtract the square of the first digit.[12] And nothing remains beyond twice the root you found. If nothing remains from [the subtraction of] the last digit, understand that it was due to the union of the remainder from the third digit with the second digit.

[3] For example, we wish to find the root of 153. You will find the root of 1 in the third place to be 1. Place it under the 5 and put 2 before it under the 3. Multiply 2 by 2, which is twice the root that you found, to get 4 [see Figure 2.1]. Subtract this from 5 and put the remainder 1 above the 5. Join the 1 to the 3 in the first place to form 13. Subtract from this the square of 2 to get 9 that is less than 24, twice the root which has been found. Consequently, the whole root of 153 is 12 with remainder 9.

			(9
	1		
1	5	3	
	1	2	
		2	

Figure 2.1

[4] If you wish to find the root of 864, put 2 under the 6 because 2 is the whole root of 8. Put the remainder 4 above the 8. Then double 2 to get 4 placing it under the 2. Form 46 from the 4 above the 8 and the 6 in the second place. Now divide the new number 46 by 4 to get 11 [see Figure 2.2]. From this division we get an idea[13] of the following first digit[14] which must be multiplied by twice the digit[15] you already found. Afterwards, square it.[16] The digit is a little less or exactly as much as what comes from the division. Practice with

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10 b.

11 $20a$.

12 Subtract b^2.

13 "possumus habere arbitrium" (19.28).

14 b. The phrase Fibonacci uses here is *figura ponenda* (19.29). He means the single digit that will be put in the unit's place to complete the finding of the square root. I uniformly translate this as *first digit*.

15 $20a$.

16 b^2.

this procedure will perfect you. So we choose 9 since it is less than 11, and put it under the first digit. Multiply 9 by 4 (twice the second term) and subtract the product from 46. The remainder is 10. Put 0 over the 6 and 1 above the 4. Join 10 with 4 in the first place to make 104. Subtract the square of 9 from it to get 23. This is less than 29 the root that has been found.

			(23
1			
4	0		
8	6	4	
	2	9	
	4		

Figure 2.2

[5] Again, if you wish to find the root of 960, put 3 the root of 9 under the 6. Double 3 to get 6 and place it under the 3. Now multiply 6 by the first digit. Subtract the product from the upper 6 from which a digit remains. Join it with 0 [see Figure 2.3]. You can subtract the square of the digit. There will be no remainder beyond twice the root that had been found. And that digit is 0. Now the product of 0 with 6 (twice 3) subtracted from 6 leaves the same **{p. 20}** 6, Join this with 0 in the first place to make 60. Subtracting the square of 0 from this leaves 60, twice 30 the root that was found.

			(60
9	6	0	
	3	0	
	6		

Figure 2.3

[6] If you wish to know the root of a four digit number, find first the root of the last two digits. Join the remainder with the remaining two digits and proceed according to what we said above for three digit numbers. For instance, we want to find the root of 1234 [see Figure 2.4]. Put 3 the root of the square less than 12 under the 3 and the remainder 3 over the 2 of 12. Join 3 with the following digits to make 334. Then double the root which you found namely 6 and place it under the root 3. Now think: how many multiplications are to be made according to the instructions given above,

and how many digits will there be in the number from which we must subtract those multiplications? There will be two multiplications. The first is the product of the first digit by twice the [partial] root or 6. The second is the square of the first digit. Both of these products are to be subtracted from 33 in order to finish the last multiplication under the digit in the first place. Hence, the first product will be subtracted from 33 the union of the last two digits, and the other product from the union of the remainder and the 4 under the first digit. Consequently, put 5 under the first digit because 33 divided by 6 leaves 5. Multiply 5 by 6 and subtract the product from 33 for a remainder of 3 in the second place. Join 3 with 4 in the first place to make 34. From this subtract the square of 5 to leave 9. Thus, you have 35 as the root of 1234 with remainder 9.

					(9
	3				
1	2	3	4		
		3	5		
		6			

Figure 2.4

[7] Again, if you wish to find the root of 6142, find the root of 61. It is 7 with remainder 12. Put the 7 under the 4 and the 12 over 61. Double 7 to get 14. Put 4 under the 7 and 1 after the 7. You know about the union of the remainder 12 with 42 to make 1242. As a four digit number three subtractions are required [see Figure 2.5]. The first of these is the first digit by the [last][17] digit. The second is [the first digit] by 4 that is before the 1. The third is the square [of the first digit] itself. We must gradually subtract the three products from the four given digits so that the final product falls in the first place. Because the number of digits exceeds the number of products in the first figure, one digit must be joined to the last digits of 1242 with what follows, namely 12. From this 12 subtract the first product. Then the second product falls under the second place, and the last under the first. Whence put 8 before 7. Multiply 8 by the given 1 and subtract [the product] from 12 for a remainder of 4. Join this with the 4 that follows in the second place to get 44. Subtract from this the product of 8 and 4 (the 4 being under the 7) to yield a remainder of 12. Place this over the 44 and join it with 2 in the first place to make 122. Subtract from this 64 the

[17] The text and manuscript have *per positum vnum* (20.26, *f.* 12v.9). However, the procedure requires one times eight, as seen six lines below. Hence, *vnum* should be *ultimum*.

square of 8 for a remainder of 58. And thus you have 78 as the root of 6142 with remainder 58.

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        (72
8 1 7 2
    9 0
    1 8
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Figure 2.5

[8] If you wish to find the root of 8172,[18] put the root of 81 or 9 under the 7. Double the 9 and put the 8 under the 9 and the 1 after it to the left. Now the 1 and 8 must be multiplied by the first digit, one at a time. Then square the first digit [see Figure 2.6]. And thus there are three products to be subtracted gradually from 72, the remainder from the 81 after finding of the root of 81. Whence, as we obviously know, nothing comes after it except 0. Since a step is lacking, it is the first product that can be subtracted. Because if the first product is subtracted from 7, the second needs be **{p. 21}** subtracted from 2. But then there is no place from where to subtract the third product. Or in another way: because the first place is a factor with any step, that step arises from the multiplication. Since the product of the digit in the first place and the digit in the third place, namely by 1, fills the third place, there is no place for 72. Therefore the root of 8172 is 90 and the remainder is 72.

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           (58
1 2 1 2
6 1 4 2
    7 8
  1 4
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Figure 2.6

[9] If you wish to find the root of any number of 5 digits, find the root by the foregoing instructions for the last three digits. If anything is left over, put the excess over that place or the place where the excess occurred. Then join the excess with

[18] The text and manuscript show 8172 (20.37, *f.* 12v.20) but the accompanying illustration has 81.71 which is obviously incorrect.

the two remaining digits. Put twice the root you found under that root, like places under like places. By studying the preceding instructions, put some other digit in the first place. Because the first figure in the root of a five digit number belongs in the third place, the root of a five digit number has three digits.

[10] For example: we wish to find the root of 12345 [see Figure 2.7]. First, find the root of 123 which is 11 with remainder 2. Put the first 1 of 11 under the 3 and the other 1 under the 4. Double 11 to get 22 and place it under the 11. Place that remaining 2 above the 3. Join it to the following digits to make 245, as a three-digit number. There will be the usual three multiplications. There will be a single multiplication from each single digit. Hence, there needs be placed such a digit before the 11 already in position. This will be multiplied by the first binomial, then by the second, and finally by itself. The first multiplication can be taken from the 2. Put what remains over the 3. And then another from the 4 and finally another from the 5 which is in the first position. And that digit is 1. Having subtracted the product of 1 by the first binomium and the product by the 2 over the 3, nothing remains. Likewise, having multiplied 1 by the following binomium and subtracted it from 4, what remains is 2 over the 4. Having joined the 2 with the 5 in the first place, 25 is made. Having squared 1 and subtracted it from 25, 24 remains. And thus you have 111 as the root of 12345 with remainder 24.

					(24
		2			
1	2	3	4	5	
		1	1	1	
		2	3		

Figure 2.7

[11] There is a way to prove what you did is correct. Simply multiply 111 by itself. Accept the product as the proof. Add 24 to the product. Thus you have the proof that what you did for 12345 was correct. Another example: the remainder[19] from 111 divided by 7 is 6. The quotient of 111 by 7 has a remainder of 6 that multiplied by itself gives 36. This divided by 7 produces a remainder of 1. Add this to 3, the remainder of 24 [divided by 7], and you have 4. This is the proof [that $\frac{24}{111}$ 111 is the root of] 12345, because 12345 divided by 7 has a remainder of 4. This is what we want. By this technique you can always prove your answers in finding roots.

[19] Fibonacci used the word *proba*, which means "a sample taken for an examination" (Latham [1965], 373), whereas in *Liber abaci* he used *pensa*, a unit of weight (*Liber abaci*, 356), which Sigler translated "residue" adding "modulo seven" for *per septemnarium* ([2002], 494).

[12] Again, if you wish to find the root of 98765, find first the root of 987. It is 31 with a remainder of 26. Put the 3 under the 7, the 1 under the 6, and the remainder 26 over the 87, the 2 above the 8 and the 6 above the 7. Join 26 with the other two digits to make 2665. Double the 31 and put the 6 under the 3 and the 2 under the 1 [see Figure 2.8]. And because four digits remain in the number, we must gradually delete them by three multiplications that have to be done with the first digit before 31. The multiplications are first by 6, second by 2, and the third is the square of the first digit. We must therefore take the first product from 26. Whence we divide 26 by 6 (twice the 3 found in the partial root and placed under the 3) to get 4. Therefore the first digit is 4 to be placed before the 31. So, put 4 before 31. Since it can be done, multiply 4 by 6. Subtract the product from 26 for a remainder of 2 that you put over the 6 above the 7. Join it with the 6 in the second place to make 26. Then subtract from this number the product of 4 by 2 **{p. 22}** which placed under the 1 leaves 18, 1 over the 2 and 8 over the 6. Join the 18 with the 5 in the first place to form 185. From this subtract the square of 4 to leave 169. Thus you have 314 for the root of 98765 with a remainder of 169.

					(169
		1			
		2			
	2	6	8		
9	8	7	6	5	
		3	1	4	
		6	2		

Figure 2.8

[13] If you wish to find the root of a number with six digits, first find the root of the last four digits and join the remainder with the following two digits, and continue as before. For example, if we want to find the root of 123456, first find the root of 1234, which is 35 with remainder 9. Put 35 under 45 and the remainder 9 above[20] the 4 [see Figure 2.9]. Join 9 to the 56 to make 956. Because there are three digits here and there are three multiplications. The first must be by 7, the second by zero, and the third the square of the number itself. Then we know that the first multiplication must come out of the 9. So divide 9 by 7 to produce 1, the first digit before 35. Multiply 1 by 7 and subtract the product from 9 to leave the remainder 2 above the 9. Join the 2 to the 5 to make 25. Multiply 1 by 0, subtract the product from 25, and 25 remains. Join this to the following digit to make 256. From this subtract the square of 1 to leave 255. Thus we have 351 for the root of 123456 with remainder 255.

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[20] The text has *sub* but the procedure requires *supra*, as the diagram in the margin shows.

```
           (255
      2
      9
1 2 3 4 5 6
      3 5 1
      7 0
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Figure 2.9

[14] Again, if you wish to find the root of 987654, find first the root of 9876 which is 99 with a remainder of 75. Put 99 under 65 and 75 above 76 [see Figure 2.10]. Double the root you found to get 198. From this number, put 8 under the 9 in the second place, 9 under the 9 in the third place, and 1 after the last. Join 75 with 54 to make 7554. There are four digits in this number which we must delete gradually by four multiplications by the three digits 1, 9, and 8 with the first digit and finally by the square of [1] itself. Whence the first product must be subtracted only from 7. If we know that the first digit is 3, the product of 3 and 1 in 198 subtracted from 7 leaves 4 above 7 itself. Join it to 5 in the third place to make 45. Now subtract from this the product of 3 and the 9 in 198, to have 18 remain over the fourth and third places. Join 18 to 5 in the second place to make 185. Subtract from this the product of 3 and the 8 in 198. 161 remains above the fourth, third, and second places. Join 161 to 4 in the first place to make 1614. Subtract the square of 3 to give a remainder of 1605. And so we have 993 as the root.

```
          (1605
    1 6
    4 8
    7 5
9 8 7 6 5 4
      9 9 3
    1 0 8
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Figure 2.10

[15] Likewise, if you wish to find the root of a seven digit number, find first the root of the last five digits. Join any remainder with the remaining two digits, and proceed as before. For example: we want to find the root of 9876543 [see Figure 2.11]. Since the root of the seven digit number has four digits, we must put the first digit of the root under the fourth place, the following digit under the third, and the next one under the second place. You will find these three digits by finding the root of the last five digits. Their root is 314 with remainder 169 that you place above the fifth, fourth, and third places. Join 169 to 43 the

remaining two digits to make 16943. Then double the 314 that you found and put the 8 under the 4, the 2 under the 1, and the 6 under the 3. You will multiply the first digit by those three digits, and then square the first digit. Consequently there are four products which we must subtract step-by-step from the five digit figure 16943. So the first product to be subtracted is from the last two figures, namely **{p. 23}** from 16. Hence divide 16 by 6 the first product to get 2. Put the 2 in the first place of the root. Then multiply it by the 6 of 628 to get 12 that you subtract from 16 for a remainder of 4. Put this over the 6. Join 4 with 9 to make 49. From this subtract the product of 2 by 2 to leave 45 over the 49. Join this with the 4 in the second place to make 454. Subtract from this the product of 2 by 8 to leave 438. Put the 4 over the fourth place, 3 above the third, and 8 above the second. Join 438 with the 3 in the first place to make 4383. From this subtract the square of 2 for a remainder of 4379. This is less than twice the root you found. Consequently, the root is 3142.

```
              (4379

      1 6 9
  9 8 7 6 5 4 3
        3 1 4 2
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Figure 2.11

[16] Similarly, if you want to find the root of an eight digit number, find the root of the last six digits, join its remainder with the remaining two digits, and proceed as has been said. And so on with finding the roots of numbers with an unlimited number of digits. For if you want to know how many digits are in the root of any number of many digits, give thought to whether the number is odd or even. If it is even, take half its number of digits. The number of digits in the half is the number of roots of the number itself. If however the number of digits is odd, make it even by adding one to it. Consequently the number of digits in the half is the number of digits in the root. Then you begin by putting the first digit under the place where it had fallen.

[17] Now in dealing with the fractions which are left over from finding the whole number root, if you want to get close to the truth, you have to decide whether they are rods for measurements in geometry or degrees for measurements in astronomy. The roots of fractional parts of a rod must be reduced to feet, inches and parts of inches. The roots of fractional parts of a degree are to be reduced to minutes, seconds, and parts of seconds. As noted before, there are 6 feet in a rod, 18 inches in a foot. Further, an inch is $\frac{1}{18}$ of a foot, a foot is $\frac{1}{6}$ of a rod, and one whole rod is 108 inches.

Likewise a degree can be divided into 60 minutes, a minute into 60 seconds, and similarly a second into 60 thirds. Whence a whole degree is 3600 seconds and 21600 thirds, all of which must be memorized. If you wish to be able to find the root of any number of rods, multiply the number of rods by the product of 108 times 108 (11664) and you will find the root of the product. You will have the number of inches that are in the desired root. Inches divided by 18 yield feet for the root, and the number of feet divided by 6 yield rods.

[18] For instance, we want to find the root of 67 rods. Multiply 67 by 11664 to get 781488 the root of which is 884 with a remainder of 32. Divide the 32 by twice 884 or half of 32 (16) by 884 to get nearly $\frac{1}{55}$. Consequently, the root of 67 rods is 884 and $\frac{1}{55}$ inches. Now if the inches are divided by 18, we have 49 feet, 2 and $\frac{1}{55}$ inches. If we divide the 49 feet by 6, we have 8 rods and 1 foot. Consequently, the root of 67 rods is 8 rods, 1 foot, and 2 and $\frac{1}{55}$ inches. Wherefore you must learn how to do this for all similar cases.

[19] If you want to find the root of a number of degrees, square the number of seconds (3600) in one degree to get 12960000. This is the number by which you will multiply the number of degrees whose root you seek. You will find the root in that product. Divide the excess by twice the root you found and you will have the number of seconds that are in the desired root. Dividing the seconds by 60 yields minutes. Minutes divided by 60 produces degrees. And thus you have the degrees, minutes, seconds and parts of one second which are **{p. 24}** in the root of however so many degrees.

[20.1] If you wish to find step by step the rods, feet, inches, and finally the smallest parts of an inch that are in the root of however so many rods, first find the root of the rods, as has been said. Then change any remaining rods to halves of soldi or feet. Divide them by twice the number of rods you found for the root, and you will have the feet. From what remains from these, change to deniers. From these subtract the product of the square of the number of feet which you found. Triple what remains, that is, change them to inches that you divide again by twice the number of rods and feet that were found in the root. Thus you will have rods, feet, inches, and parts of inches. This is the root of however so many rods.

[20.2] For example: we wish to find the root of 67 rods. First, the integral root of 67 rods is 8 whole rods with a remainder of 3 rods. These are halves of soldi or 18 feet. Divide them by 16 (twice the 8) to get 1 foot with a remainder of 2 feet that make 12 deniers. Subtract from them the square of one foot or 1 denier, because the square of one foot is one denier, as said above. There remain 11 deniers. Multiply these by 3 to get 33 inches. Divide them by twice 8 rods and 1 foot ($\frac{1}{3}$ 16) for 2 inches and a remainder of $\frac{1}{3}$ of one inch. If we subtract the square of two inches from $\frac{1}{3}$, what remains from that third of an inch is about $\frac{1}{18}$ of one inch, because we divided what was left over by twice the root that we had found.

[21] Again, you wish to use this technique for finding the root of 111 rods. Take 10 as the integral root, double it to get 20 by which you are to divide 66 feet (the 11 rods left over). You get 3 feet with a remainder of 6 feet or 36 deniers. From this subtract the 9 deniers that came from squaring 3 feet. 27 deniers remain. Triple these to 81 that you divided by 21 or twice the root you had found to get $\frac{6}{7}$ 3 inches. There is no remainder from which you would subtract the square of the inches. Simply call it a little less than $\frac{6}{7}$ of an inch. Thus you have for the root of 111 rods: 10 rods, 3 feet, and $\frac{5}{6}$ inches. Do remember what we said above about multiplying rods, feet and inches into rods, feet and inches. Keep this in mind as you proceed to find feet and inches in the roots of rods.

[22] Likewise, if you wish to find the root of 1234 rods, take the integral root as 35 rods with a remainder of 9 rods. Now 9 rods make 54 feet. Since these are less than twice the number of integral roots (70), we know that for this root there are no feet. Hence, make 972 inches from the 54 feet. Divide these by 70 to get 13 inches and $\frac{7}{8}$. Again, there is not enough here from which the square of the $\frac{7}{8}$ 13 inches can be subtracted. So, take some fraction less than $\frac{7}{8}$ such as $\frac{6}{7}$ or $\frac{5}{6}$ or $\frac{4}{5}$ or $\frac{3}{4}$, whatever you choose, although by doing so you deviate a bit from absolute accuracy. By this method you can come very close to the number of roots of rods, or rods and feet, or rods, feet, and inches.

Irrational Roots

[23] In my book *On Calculations* we discussed thoroughly how to find minutes and seconds, the fractional parts of degrees.[21] Now the roots of numbers that are not squares can be expressed as lines but not as numbers. The method of expressing the roots as lines is to draw a line representing the length of such a quantity or number, the root of which **{p. 25}** you wish to have, and add an extension of one unit. Find the middle of the extended line to serve as a center about which you construct a circle. Through the point between the original line and its extension erect at right angles another line to the circle. This last line is the true root of the number you seek. For example: we wish to find the root of 67. Draw straight line *ab* of 67 units. Add line *ag*, an extension of 1 unit, so that the full length is 68. Divide line *gb* in two equal parts at point *d*. Using the distance *dg* or .*db*, draw the circle *gebz*. Further, draw the line *ae* at right angles. I say that the line *ae* is the true root of line *ab*, the root of 67.

[24] Draw line *ea* directly toward point *z*. Because line *ba* meets line *ez*, it makes either two right angles or two angles equal to two right angles. Right angle *bae* equals right angle *baz*. And because straight line *ag* through the center cuts a certain straight line *ez* at two right angles, it necessarily cuts that straight line into equal parts [see Figure 2.12]. Therefore line *ea* equals

[21] For a table of the 59 sixtieth parts of sixty, see *Liber abaci*, 79, Sigler (2002), 121.

line az. Again: because in circle $gebz$ the two straight lines bg and ez intersect one another at point a, the product of ba by ag is as the product of ea by az. But the product of ae by az is as the square on ea. Whence the product of ba by ag is as the square on ae. But the product of ba by ag is as the product of 67 by 1. And thus the square on ae is the same as 67, which we wanted.

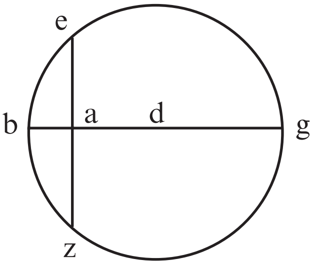

Figure 2.12

2.2. Multiplication of Roots[22]

Now we wish to show how to multiply and add roots of numbers by other roots, how to subtract smaller roots from larger ones, and finally how to divide roots.

[25] Suppose you wish to multiply the root of 16 by the root of 25. Since both 16 and 25 are square numbers, take their roots (4 and 5) and multiply them together to get the desired product, 20. Suppose you wish to multiply the root of 16 by the root of 30. Since the root of 30 is a surd, multiply 16 by 30 to get 480. Then find the root of their product as closely as possible. This gives you the root of your multiplication. Again, if you wish to multiply the root of 20 by the root of 30, multiply 20 by 30 to get 600. Find its root and you have what you wanted to find. For example, let a be 20, b be 30, g the root of 20, and d the root of 30. The product of a by b is e or 600. The product of g by d is z. I say that z is the root of e or 600. First it must be noted that as one number is to another number, so is the product of one to the product of the other as the part of one is to the part of the other. We shall prove this with 12 and 24. The ratio of 12 to 24 is as twice 12 to twice 24, as thrice 12 is to thrice 24. Similarly, the quadruple and the others. Likewise, the same ratio holds for 12 to 24 as for half of 12 to half of 24, the third to the third and so on. With this established, we return to the problem [see Figure 2.13]. Because the number g is the root of number a or 20, the square of g is a. And so the product of g by d is the number z. Therefore a is a multiple of g as is z of d. Whence g is to d as a is to z. Again: because d is the root of b or 30, then the square of d is b. But the product of

[22] "De multiplicatione radicum" (25.19, *f.* 15v.4).

d by *g* is *z*. Consequently *z* is a multiple of *g*; so *b* is a multiple of *d*. Whence *g* is to *d* as *z* **{p. 26}** is to *b*. Then as *g* is to *d* so *a* is to *z*. But *per equale* as *a* is to *z* so *z* is to *b*. Whence the product of *a* by *b* is as the square of *z*. But the product of *a* by *b* is *e* or 600. And the square of *z* is similarly *e*. Therefore *z* is the root of *e*, as required. Note that when the roots of numbers are multiplied together, the numbers themselves have among themselves the ratios of squares of the numbers. The product is the square of the number. Hence the product of roots themselves is a rational number.

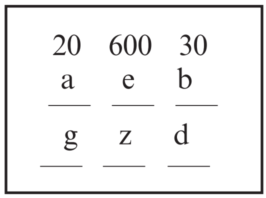

Figure 2.13

[26] For example: we wish to multiply the root of 8 by the root of 18. Their ratio is as 4 to 9, the square of one number to the square of another. I say that the product of 8 by 18 produces a square number, namely 144. Its root or 12 equals the product of the root of 8 and the root of 18. Likewise if you wish to multiply 10 by the root of 20, multiply 100 the square of 10 by 20 to obtain 2000. Its root is the product of the stated multiplication. Note further that what happens when multiplying 10 by the root of 20. The product equals 10 roots of 20. Whence we can reduce 10 roots of 20 to the root of one number by multiplying the square of 10 by 20 to get 2000. The root of this is found from the 10 roots. The same must be understood for similar cases.

[27] So if you wish to reduce the root of some number to many roots of another, divide the number by some square number. As many units as there are in the root of the squared number, so many roots belong to the number resulting from the division. For example, you wish to reduce the root of 1200 to several roots of another number. Divide 1200 by some square number such as 16 to get 75. Then you have four roots of 75 for the root of 1200. If you were to divide 1200 by 25, then you would have five roots of 48 for the one root of 1200. Thus you can reduce the root of 1200 to several roots by different divisions.

2.3 Addition of Roots[23]

[28] When the roots of numbers are added to the roots of other numbers, their sum is either a rational number or the root of another number. When they cannot be added, then either a number arises from their sum or another root.

[23] "De addictione (*sic*) radicum" (26.26, *f.* 16r.16).

When we wish to join squares, then a number results from their sum. For example, the root of 16 added to the root of 25 unites 4 with 5 (the roots of 16 and 25) make 9 their sum.

[29] When we wish to add roots that have among themselves the ratio of their squares, then the sum is the root of some rational number. We shall demonstrate this in two ways. We wish to add the root of 27 to root of 48. Their squares have the same ratio as 9 and 16. Now the sum of their roots is 7, and the square of 7 is 49. Since 27 and 48 are thrice 9 and 16, triple 49 to get 147. Its root is the sum of the addition of the root of 27 to root of 48 [see Figure 2.14]. This rule can be proved by the ratios of similar triangles thus: let *ab* and *bg* be the parts of the same straight line, with *ab* the root of 27 and *bg* the root of 48. From point *a* create an angle by drawing straight lines *ad* and *de* with *ad* equal to 3 and *de* equal to 4. Join *eg* **{p. 27}** and *db*. Thus the square on line *ab* is to thrice the square on line *ad* as the square on line *bg* is to thrice the square on line *de*. Therefore as line *ab* is to line *ad*, so is line *bg* to *de*. Whence line *db* is equidistant from line *eg*. Hence triangle *adb* is similar to triangle *age*, with angle *a* in common. Angle *adb* equals angle *aeg* and angle *abd* equals angle *age*, the exterior angles being equal to the interior angles. Therefore the square on line *ab* is to the square on line *ad*, as the square on line *ag* is to the square on line *ae*. But the square on line *ab* is thrice the square on line *ad*. Hence, the square on line *ag* is similarly three times the square on line *ae*. But the square on line *ae* is 49. Whence the square on line *ag* is triple this or 147. Therefore, by joining lines *ab* and *bg* the root of 147 appears, as had to be shown.

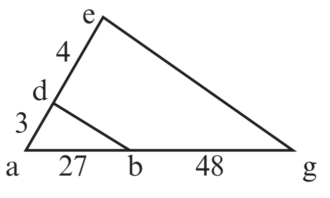

Figure 2.14

[30] Note that all roots of numbers having ratios among themselves can be reduced to the roots of one number, by dividing the antecedent number by the antecedent square, and the consequent number by the consequent square. For example, if you wish to reduce the roots of 27 and 48 to the roots of one number: divide 27 the first number by 9 the first square, and the following number 48 by its accompanying square 16. The root of 3 results from each division [see Figure 2.15]. Consequently, there are as many units in the root of 9 as there are roots of 3 in the root of 27. As many units as there are in the root of 16, so many roots of 3 are there in the root of 48. Thus for the one root of 27 you have three roots of 3. For the one root of 48 you have four roots of 3. If you add them together, you have 7 roots of 3 in sum. If you wish

to reduce them to the root of one number, multiply the square of 7 by 3 and take the root of the product. This gives you the root of 147 for their sum, as above. Another way: first add 27 with 48 to get 75. Multiply 27 by 48. Doubling the root of the product yields 72. Add this to 75 and you have 147 for the square of the stated addition.

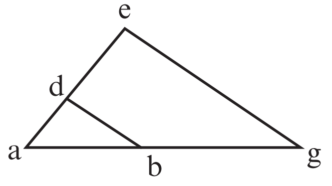

Figure 2.15

[31] We will demonstrate this by an example. Extend straight line *ab* by *bg*. Let *ab* equal the root of 27 and *bg* be the root of 48. We want to know the measure of line *ag*. Because line *ag* is divided in two parts at point *b*, the two squares on parts *ab* and *bg* with twice the product of *ab* by *bg* equals the square on the whole line *ag* [see Figure 2.16]. But the sum of the squares on *ab* and *bg* (27 and 48) is 75. Further, the product of line *ab* and *bg* gives the root of the product of 27 and 48. Twice this plus 75 is 147 for the measure of the square on line *ag*, as we predicted.

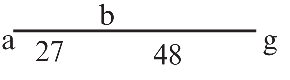

Figure 2.16

[32] It is not possible to add the root of 20 to the root of 30, because 20 and 30 do not have a ratio between themselves as the square of one number to the square of another. We know this because the product of 20 and 30 is a number that does not have a root. Therefore the sum of their roots does not result in a number nor in the root of a number. For example: [Figure 2.17] let straight line *ba* be the root of 20 and line *ag* be the root of 30. Therefore the square on line *ba* with the square on line *ag* is 50. And the product of *ba* by *ag* is the root of 600. Whence doubling the product of *ba* by *ag* produces the root of four times 600, namely the root of 2400. Since this does not have a root, we know that we cannot have a number or the root of a number from the addition of the foregoing. Rather their addition produces the root of a number and of a root, namely the root of 50 and the root of 2400. So that we can have something, let us find the root of 2400. Its nearest approximation is 49 less $\frac{1}{98}$. After adding 50, we have 99 **{p. 28}** less $\frac{1}{98}$. We can find the root of this, as we wanted. Or, we may find the individual roots of 20 and 30 as close as possible, add them, and thus have what we wanted.

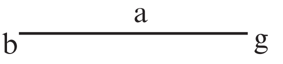

Figure 2.17

2.4 Subtraction of Roots[24]

[33] Suppose you want to subtract one root from another, such as the root of 16 from the root of 49. Then subtract 4 (the root of 16) from 7 (the root of 49) for the remainder 3 the residual of the subtraction. Suppose you wish to subtract the root of 45 from the root of 125. Since their ratio is as the ratio of squares (9 to 25), subtract the root of 9 from the root of 25 for a remainder of 2. Square this to get 4. Then multiply it by 5 to get 20, because 45 and 125 are each five times 9 and 25. The root of 20 remains from the subtraction.

[34] For example: let straight line *ab* be the root of 125. At point *b* attach the straight line *ag* (equal to 5 the root of 25) to make angle *bag*. Take from *ag* line *gd* (3) leaving *da* (2). Through point *d* draw straight line *de* equidistant from side *gb* [see Figure 2.18]. Because line *de* in triangle *agb* was constructed equidistant from line *bg*, sides *ab* and *ag* were cut proportionally at points *d* and *e*. That is, as *ae* is to *eb*, so is *ad* to *dg*. By composition therefore, as *ba* is to *be*, so is *ga* to *gd*. By alternation we have *ba* is to *ag* as *be* is to *gd*. Whence as the square on side *ba* is to the square on side *ag*, so the square on line *bd* is to the square on line *gd*. But the square on line *ba* is five times the square on side *ag*. Whence the square on line *be* is five times the square on line *gd*. Since the square on line *gd* is 9, the square on line *be* is 45. Therefore line *be* is the root of 45. We have taken this from *ab* (the root of 125) to find the remainder *ae*. The ratio of the square [on *ae*] to the square on line *ad* is as the ratio of the square on line *be* to the square on line *gd*. Whence the square on line *ae* is five times the square on line *ad*. But the square on line *ad* is 4. Therefore the square on line *ae* the desired residue is 20. And thus *ae* has been found as the root of 20, as above.

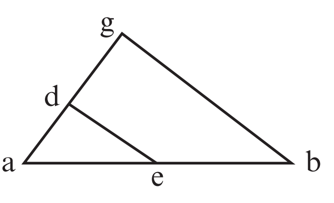

Figure 2.18

[24] "De extractione radicum" (28.3, *f.* 17r.6).

[35] Or, if we reduce the roots of 45 and 125 into [so many] roots of 5, then we have for the root of 45 three roots of 5 and for the root of 125 five roots of 5. So if 3 roots of 5 are removed from 5 roots of 5, then 2 roots of 5 remain. This equals one root of 20, as we found.

[36] Another way: add 45 to 125 to get 170. From this subtract 150 twice the root of the product of 45 and 125 to get 20 for the square of the desired remainder. For example, let line *ab* be the root of 125. To this add line *bg* the root of 45 [see Figure 2.19]. I say that line *ga* is the root of 20. Since line *ab* is divided at point *g*, the two squares on lines *ab* and *gb* are equal to the square on *ag* and twice the product of *gb* and *ab*. For the two squares on *ab* and *gb* are 170. This equals the square on line *ag* and twice the product of *bg* and *ba*. But *bg* by *ba* is the root of the product of 45 and 125 or 75. Twice 75 is 150. Subtract this from 170 to leave 20 for the square on line *ag*, as required.

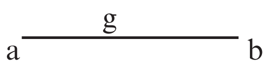

Figure 2.19

[37] If you wish to subtract the root of 20 from the root of 30, even if they are not in the ratio of their squares, find the root of 20 and the root of 30 as accurately as possible. Then subtract the root of 20 from the root of 30 [see Figure 2.20]. While the remainder is not perfect, it is nearly so. Or if you want, add 20 and 30 to get 50. Then multiply them to get 600. From this take the two roots, namely twice its root, that is the root of four times **{p. 29}** 600 which is nearly 49 less $\frac{1}{98}$. Subtract this from 50 and you have $\frac{1}{98}$ 1. Find the root of this and you will have a close approximation of the desired remainder.

Figure 2.20

2.5 Division of Roots[25]

[38] If you wish to divide the root of 600 by the root of 40, divide 600 by 40 to get 15. The root is the result of the division. For example: let *a* be 40, *b* 600, *g* 15, *d* the root of *a*, and *e* the root of *b*. Divide *e* by *d* (the root of 600 by the root of 40) to get *z*. I say that *z* is the root of *g* or 15. Since the quotient of *e* by *d* is *z*, then the product of *d* by *z* is *e*. Now the square of *d* is *a*. Therefore the product of *a* by *z* equals the product of *e* by *d*.[26] Consequently as *d* is to *z*

[25] "De diuisione radicvm" (29.3, *f.* 17v.13).

[26] Text has "*a* ex *d*… *e* ex *z*"; context requires "*a* ex *z*… *e* ex *d*" (29.9–10).

so is a to e [see Figure 2.21]. Again: z times d makes e. And z squared makes i. Therefore, as d is to z so is e to i. But as d is to z, so a to e is met again. Whence as a is to e, so is e to i. Therefore a, e, and i are in continued proportion. So the product of a by i is as the square of e. But the square of e is b. Therefore the product of a by I is likewise b or 600. But b divided by a yields g. Therefore the product of a by g is b. And the product of a by i is b. So the number i equals the number g. But the square of z is i. Consequently squaring z produces in like fashion g. Therefore z is the root of g, as we had to show.

40	15	600
a	g	b
d	z	e

Figure 2.21

[39] In another way: d times z makes e. The square on e is b. Therefore the product of d by z multiplied by g produces b. Therefore the square on d times g is b. Whence the product of d by z times e equals the square on d times g. By removing the common factor d from both products, the product of z by e remains equal to the product of d by g. Hence, as d is to z, so is e to g. But as d is to z, so is e to I. Therefore e to g and to I is the same ratio. Consequently g equals I. But z is the root of I. Therefore z is the root of g. So if you wish to divide the root of 40 by the root of 600, dividing 40 by 600 gives $\frac{1}{15}$. Find this root and you have what you proposed.

[40] Now we wish to explain how to find as accurately as possible the roots of fractions. First observe that when the roots of squared numbers are divided among themselves or by their ratios, a rational number is always the result. For example: we wish to divide the root of 64 by the root of 16. Sixty-four divided by 16 is 4. Its root is 2, the result of the division. Now 8 (the root of 64) divided by 4 (the root of 16) also gives the root 2. Note that you get the same results from dividing the roots of all the numbers having the same ratio as 16 to 64. If we want to divide the root of 80 by the root of 20, the division produces 2.

[41] If you want to find the root of some fraction or group of fractions, there are two ways to do this. The first way is to take the part or parts of some large number and multiply however many [fractions] you have by that number, and you will find the root of the sum of the products. Then divide by that same number, and you will have what you wanted. For example, you want to find the root of $\frac{2}{3}$, so take $\frac{2}{3}$ of some large number, such as 60. The larger the number you chose, so much closer will you get to the root. Now $\frac{2}{3}$ of 60 is 40. Multiply it by 60 to get 2400 whose root is 49 less **{p. 30}** $\frac{1}{98}$.

Divide this by 60, and you have what you wanted. Now if you want to have this in feet and inches, then take $\frac{2}{3}$ of 108 the number of inches in one rod to get 72. Multiply this by 108 to obtain 7776. The root of this is $\frac{2}{11}$ 88. That is how many inches there are in the root of $\frac{2}{3}$ of one rod. In a similar way, suppose you want the answer in minutes and seconds, say of the root of $\frac{4}{5}$ of one degree. Take $\frac{4}{5}$ of the seconds of one degree, that is, 2880 of 3600. The product of these two numbers is a fourth of 10 368 000,[27] the root of which will give you the number of seconds in the root of $\frac{4}{5}$.

[42] Another way: we want to find the root of $\frac{2}{3}$ of one rod, remembering that multiplying a rod by a rod produces a denier. Thus there are 24 deniers in $\frac{2}{3}$ of one rod. The root of this is 4 feet and 8 inches. These equal 144 eighteenths of one denier. Divide this by 8, twice the root you found, to obtain 16 inches[28] with a remainder of $\frac{16}{18}$. From these subtract the square of 16 inches ($\frac{256}{324}$) to leave $\frac{8}{81}$ of one denier. Divide this by twice the root you found to produce about $\frac{2}{11}$ of one inch. Or alternately: take the root of 24 deniers that is 5 feet less one denier. Make 18 eighteenths of this which you will divide by twice the root that you found (10) to yield $\frac{4}{5}$ of an inch. Subtract this from 5 feet and what remains is 4 feet and $\frac{1}{5}$ 16 inches.

[27] "2880; que multiplica per 3600 erunt quarta 10368000" (30.6, *f.* 18r.19). Although the reading is correct, there seems to be too much information here. If there are 2880 seconds in $\frac{4}{5}$ of a degree, then the root of that number is easily found, a bit less than $\frac{2}{3}$ 53 seconds.

[28] Text has "16"; context requires "18" (30.12). The correction vitiates all that follows up to the alternate method. The difference of the two answers makes one wonder if this example [42] were not an addition by someone other than Fibonacci.

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