2

Discrete-Time Renewal Processes

The purpose of this chapter is to provide an introduction to the theory of discrete-time renewal processes. We define renewal chains, delayed renewal chains, and associated quantities. Basic results are presented and asymptotic behavior is investigated. This is a preliminary chapter useful for the study of semi-Markov chains.

Renewal processes (RPs) provide a theoretical framework for investigating the occurrence of patterns in repeated independent trials. Roughly speaking, the reason for using the term "renewal" comes from the basic assumption that when the pattern of interest occurs for the first time, the process starts anew, in the sense that the initial situation is reestablished. This means that, starting from this "renewal instant," the waiting time for the second occurrence of the pattern has the same distribution as the time needed for the first occurrence. The process continues like this indefinitely.

Renewal processes have known a huge success in the last 60 years and much work is still being carried out in the field. The reasons for this phenomenon are manifold and we give some of them below.

- First, there are many real situations where a renewal modeling is appropriate and gives good results. It is worth mentioning that renewal theory was developed in the first place for studying system reliability, namely, for solving problems related to the failure and replacement of components.
- Second, it has become clear that fundamental renewal results are of intrinsic importance for probability theory and statistics of stochastic processes.
- Third, there are some more complex stochastic processes (called regenerative processes) in which one or more renewal processes are embedded. When this is the case, the analysis of the limit behavior of the phenomena modeled by this kind of process can be performed using the corresponding results for the embedded renewal process.
- For Markov and semi-Markov processes, under some regularity conditions, the successive times of entering a fixed state form a renewal process.

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In other words, renewal processes are very simple but nonetheless general enough. This generality is provided by the renewal phenomenon, which can be encountered in very different types of problems. As these problems can be very complex, by using their renewal feature we are at least able to answer some specific questions.

As mentioned before, a semi-Markov process has embedded renewal processes, so the results presented in this chapter will be useful for the understanding of the renewal mechanism in a semi-Markov setting and for obtaining the results derived from this mechanism. Since in the rest of the book we will be concerned only with discrete-time processes, we present here elements of a renewal theory in discrete time.

For the present chapter we have mainly used the paper of Smith (1958) and the monographs of Feller (1993, 1971), Port (1994), Karlin and Taylor (1975, 1981), Durrett (1991), and Cox (1962).

In the first part of the chapter, the (simple) renewal chain (RC) is defined, together with some related quantities of interest, and some basic results are given. We also stress a number of specific differences between renewal theory in a discrete-time and in a continuous-time framework. The second part of the chapter is devoted to the asymptotic analysis of renewal chains. Several fundamental results are provided, together with their proofs. We also introduce delayed renewal chains, which model the same type of phenomenon as do the simple renewal chains, with the only difference that there is a "delay" in the onset of observations. Associated quantities are defined and basic results are provided. We end the chapter by presenting alternating renewal chains, particularly useful in reliability analysis. Some examples spread throughout the chapter give different situations where a renewal mechanism occurs.

2.1 Renewal Chains

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of extended positive integer-valued random variables, $X_n > 0$ a.s. for $n \geq 1$ (we use the term "extended" in the sense that X_n can take the value ∞). Denote by $(S_n)_{n\in\mathbb{N}}$ the associated sequence of partial sums:

$$S_n := X_0 + X_1 + \ldots + X_n. \tag{2.1}$$

If for a certain $n \in \mathbb{N}$ we have $X_n = \infty$, then $S_n = S_{n+1} = \ldots = \infty$. Note that we have $S_0 \leq S_1 \leq \ldots$, where equality holds only for infinite S_n.

The sequence $(X_n)_{n\in\mathbb{N}^*}$ is called a *waiting time sequence* and X_n is the *nth waiting time*. The sequence $(S_n)_{n\in\mathbb{N}}$ is called an *arrival time sequence* and S_n is the *nth arrival time*.

Intuitively, $(S_n)_{n\in\mathbb{N}}$ can be seen as the successive instants when a specific event occurs, while $(X_n)_{n\in\mathbb{N}^*}$ represent the interarrival times, i.e., the times elapsed between successive occurrences of the event. See Figure 2.1, page 25, for a graphical representation of these chains.

Definition 2.1 (renewal chain).
An arrival time sequence $(S_n)_{n\in\mathbb{N}}$, *for which the waiting times* $(X_n)_{n\in\mathbb{N}^*}$ *form an i.i.d. sequence and* $S_0 = X_0 = 0$, *is called a (*simple*)* renewal chain *(RC), and every* S_n *is called a* renewal time.

Later on we will also consider the case $S_0 > 0$.

There are many real situations which could be modeled by renewal chains.

Example 2.1. Suppose, for instance, that we have a single-bulb lamp and we have replacement lightbulbs with i.i.d. lifetimes $(X_n)_{n\in\mathbb{N}^*}$. Suppose the lamp is always on as follows: at time 0 the first bulb with lifetime X_1 is placed in the lamp; when the bulb burns out at time $S_1 = X_1$, it is replaced with a second bulb with lifetime X_2, which will burn out at time $S_2 = X_1 + X_2$, and so on. In this example, the sequence $(S_n)_{n\in\mathbb{N}}$ of bulb replacements is a renewal chain.

Example 2.2. Consider the following DNA sequence of HEV (hepatitis E virus):

AGGCAGACCACATATGTGGTCGATGCCATGGAGGCCCATCAGTTTATTA
AGGCTCCTGGCATCACTACTGCTATTGAGCAGGCTGCTCTAGCAGCGGC
CATCCGTCTGGACACCAGCTACGGTACCTCCGGGTAGTCAAATAATTCC
GAGGACCGTAGTGATGACGATAACTCGTCCGACGAGATCGTCGCCGGT

Suppose that the bases $\{A, C, G, T\}$ are independent of each other and have the same probability of appearing in a location, which is equal to $1/4$. Thus the occurrences of one of them, say, C, form a renewal chain. The interarrival distribution is geometric with parameter (i.e., probability of success) equal to $1/4$.

The common distribution of $(X_n)_{n\in\mathbb{N}^*}$ is called the *waiting time distribution of the renewal chain.* Denote it by $f = (f_n)_{n\in\mathbb{N}}$, $f_n := \mathbb{P}(X_1 = n)$, with $f_0 := 0$, and denote by F the cumulative distribution function of the waiting time, $F(n) := \mathbb{P}(X_1 \leq n)$. Set $\overline{f} := \sum_{n\geq 0} f_n \leq 1 = \mathbb{P}(X_1 < \infty)$ for the probability that a renewal will ever occur.

A renewal chain is called *recurrent* if $\overline{f} = \mathbb{P}(X_1 < \infty) = 1$ and *transient* if $\overline{f} = \mathbb{P}(X_1 < \infty) < 1$. Note that if the renewal chain is transient, then X_n are improper (defective) random variables. Set $\mu := \mathbb{E}(X_1)$ for the common expected value of $X_n, n \in \mathbb{N}^*$. A recurrent renewal chain is called *positive recurrent* (resp. *null recurrent*) if $\mu < \infty$ (resp. $\mu = \infty$).

Remark 2.1. It is worth stressing here an important phenomenon which is specific to discrete-time renewal processes. As $f_0 = \mathbb{P}(X_1 = 0) = 0$, the waiting time between two successive occurrences of a renewal is at least one unit of time. Consequently, in a finite interval of time, of length, say, n, we can have at most n renewals. As will be seen in the sequel, this is the reason why many variables of interest can be expressed as finite series of basic quantities, whereas in a continuous-time setting the corresponding series are infinite. This is one advantage of a renewal theory carried out in a discrete-time setting.

Let us consider $(Z_n)_{n\in\mathbb{N}}$ the sequence of indicator variables of the events $\{$a renewal occurs at instant $n\}$, i.e.,

$$Z_n := \begin{cases} 1, & \text{if } n = S_m \text{ for some } m \geq 0 \\ 0, & \text{otherwise} \end{cases} = \sum_{m=0}^{n} \mathbf{1}_{\{S_m = n\}}.$$

Note that the previous series is finite for the reasons given in Remark 2.1. As $S_0 = 0$ by definition (time 0 is considered to be a renewal time), we have $Z_0 = 1$ a.s. Let u_n be the probability that a renewal occurs at instant n, i.e., $u_n = \mathbb{P}(Z_n = 1)$, with $u_0 = 1$. Obviously, we can have $\overline{u} := \sum_{n\geq 0} u_n = \infty$. As we will see in Theorem 2.1, this will always be the case for a recurrent renewal chain.

We can express $u_n, n \in \mathbb{N}$, in terms of convolution powers of the waiting time distribution $f = (f_n)_{n\in\mathbb{N}}$. To this end, let us first recall the definition of the convolution product and of the n-fold convolution.

Definition 2.2. *Let* $f, g : \mathbb{N} \to \mathbb{R}$. *The* discrete-time convolution product of f and g *is the function* $f * g : \mathbb{N} \to \mathbb{R}$ *defined by*

$$f * g(n) := \sum_{k=0}^{n} f(n-k)g(k), \; n \in \mathbb{N}.$$

For notational convenience, we use both $f * g(n)$ and $(f * g)_n$ for denoting the sequence convolution.

Recall that if X and Y are two independent positive-integer-valued random variables, with $f = (f_n)_{n\in\mathbb{N}}$ and $g = (g_n)_{n\in\mathbb{N}}$ the corresponding distributions, then $f * g = ((f * g)_n)_{n\in\mathbb{N}}$ is the distribution of $X + Y$.

Note that the discrete-time convolution is associative, commutative, and has the *identity element* $\delta : \mathbb{N} \to \mathbb{R}$ defined by

$$\delta(k) := \begin{cases} 1, \text{ if } k = 0, \\ 0, \text{ elsewhere.} \end{cases}$$

The power in the sense of convolution is defined by induction.

Definition 2.3. *Let* $f : \mathbb{N} \to \mathbb{R}$. *The* n-fold convolution of f *is the function* $f^{(n)} : \mathbb{N} \to \mathbb{R}$, *defined recursively by*

$$f^{(0)}(k) := \delta(k), \quad f^{(1)}(k) := f(k),$$

and

$$f^{(n)}(k) := \underbrace{(f * f * \ldots * f)}_{n-times}(k), \quad n \geq 2.$$

Note that we have $f^{(n+m)} = f^{(n)} * f^{(m)}$ for all $n, m \in \mathbb{N}$.

Recall that, if $X_1, \ldots, X_n$ are n i.i.d. positive-integer-valued random variables, with $f = (f_n)_{n \in \mathbb{N}}$ the common distribution, then $f^{(n)}$ is the distribution of $X_1 + \ldots + X_n$. In our case, for $f = (f_n)_{n \in \mathbb{N}}$ the waiting time distribution of a renewal chain, $f^{(n)}$ is the distribution of S_n, i.e., $f^{(n)}(k)$ is the probability that the $(n+1)$th occurrence of a renewal takes place at instant k, $f^{(n)}(k) = \mathbb{P}(S_n = k)$. Using this fact, we can obtain the desired expression of u_n in terms of convolution powers of f. Indeed, we have

$$u_n = \mathbb{P}(Z_n = 1) = \mathbb{E}(Z_n) = \sum_{m=0}^{n} \mathbb{P}(S_m = n) = \sum_{m=0}^{n} f_n^{(m)}, \textit{i.e.},$$

$$u_n = \sum_{m=0}^{n} f_n^{(m)}. \tag{2.2}$$

Note that the last term in (2.2) can be expressed as a finite series of convolution powers of f due to the fact that

$$f_n^{(m)} = \mathbb{P}(S_m = n) = 0, \quad m > n, \tag{2.3}$$

for the reasons described in Remark 2.1.

Let us now investigate further the relation between $(f_n)_{n \in \mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}}$. Note that, for any $n \in \mathbb{N}^*$, we have

$$\begin{aligned} u_n = \mathbb{P}(Z_n = 1) &= \mathbb{P}(S_1 = n) + \sum_{k=1}^{n-1} \mathbb{P}(S_1 = k)\mathbb{P}(Z_n = 1 \mid S_1 = k) \\ &= \mathbb{P}(X_1 = n) + \sum_{k=1}^{n-1} \mathbb{P}(X_1 = k)\mathbb{P}(Z_{n-k} = 1) \\ &= f_n + \sum_{k=1}^{n-1} f_k u_{n-k}. \end{aligned}$$

Thus, we have obtained

$$u_n = \sum_{k=1}^{n} f_k u_{n-k}, \quad n \in \mathbb{N}^*, \tag{2.4}$$

or, in convolution notation,

$$u_n = (f * u)_n, \quad n \in \mathbb{N}^*. \tag{2.5}$$

A tool frequently used in the analysis of renewal chains is the generating function. When applied to equations like (2.5) it simplifies the computations because it transforms convolutions into ordinary products. As the sequences $(f_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ are fundamental quantities associated to a renewal chain, let us denote their generating functions by

$$\Phi(s) := \sum_{n=0}^{\infty} f_n s^n, \quad U(s) := \sum_{n=0}^{\infty} u_n s^n.$$

The series are convergent at least for $| s |< 1$, while $\Phi(s)$ is convergent also for $s = 1$. Note that in the case of $\Phi(s)$, we may start the summation at index 1, since $f_0 = 0$.

Proposition 2.1. *The generating functions of* $(f_n)_{n\in\mathbb{N}}$ *and* $(u_n)_{n\in\mathbb{N}}$ *are related by*

$$\Phi(s) = \frac{U(s) - 1}{U(s)}, \quad U(s) = \frac{1}{1 - \Phi(s)}. \tag{2.6}$$

Proof. Multiplying Equation (2.5) by s^n and summing over $n = 1, 2, \ldots$ we obtain $U(s) - 1 = U(s)\Phi(s)$, where we have used the fact that the generating function of $f * u$ is the product of the corresponding generating functions, i.e., $U(s)\Phi(s)$ (see, e.g. Port, 1994). □

By means of the generating functions $U(s)$ and $\Phi(s)$ we can easily prove a necessary and sufficient condition for a renewal chain to be transient.

Theorem 2.1. *A renewal chain is transient iff* $\overline{u} := \sum_{n=0}^{\infty} u_n < \infty$. *If this is the case, the probability that a renewal will ever occur is given by* $\overline{f} = (\overline{u}-1)/\overline{u}$.

Proof. As $U(s)$ is an increasing function of s, $\lim_{\substack{s\to 1\\ s<1}} U(s)$ exists, but it can be infinite. For any $N \in \mathbb{N}$ we have

$$\sum_{n=0}^{N} u_n \leq \lim_{\substack{s\to 1\\ s<1}} U(s) \leq \overline{u} = \sum_{n=0}^{\infty} u_n.$$

As this is valid for any $N \in \mathbb{N}$, we obtain that $\lim_{\substack{s\to 1\\ s<1}} U(s) = \overline{u}$.

By definition, the chain is transient iff $\overline{f} < 1$. In this case, applying Proposition 2.1 and the fact that $\Phi(1) = \overline{f}$ we obtain that $\lim_{\substack{s\to 1\\ s<1}} U(s) = 1/(1 - \overline{f})$, thus $\overline{u} = 1/(1 - \overline{f})$. □

Note that the u_n, $n \in \mathbb{N}$, verify the equation

$$u_n = \delta(n) + (f * u)_n, \quad n \in \mathbb{N}, \tag{2.7}$$

where $\delta(0) = 1$ and $\delta(n) = 0$ for $n \in \mathbb{N}^*$. Indeed, this is exactly Equation (2.5) for $n \in \mathbb{N}^*$, whereas for $n = 0$ the equality obviously holds true, because $u_0 = 1$.

Equation (2.7) is a special case of what is called a *renewal equation in discrete time* and has the following form:

$$g_n = b_n + \sum_{k=0}^{n} f_k g_{n-k}, \quad n \in \mathbb{N}, \tag{2.8}$$

where $b = (b_n)_{n\in\mathbb{N}}$ is a known sequence and $g = (g_n)_{n\in\mathbb{N}}$ is an unknown sequence.

The following result proves that the renewal equation has a unique solution. We give two different proofs of the theorem, the first one is straightforward, whereas the second one is based on generating function technique.

Theorem 2.2 (solution of a discrete-time renewal equation).
If $b_n \geq 0, n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} b_n < \infty$, then the discrete-time renewal equation (2.8) has the unique solution

$$g_n = (u * b)_n, \quad n \in \mathbb{N}. \tag{2.9}$$

Proof (1). First, it can be immediately checked that $g_n = (u * b)_n, n \in \mathbb{N}$, is a solution of the renewal equation (2.8). Indeed, for all $n \in \mathbb{N}$, the right-hand side of (2.8) becomes

$$b_n + (f * g)_n = b_n + (f * u * b)_n = \left(b * (\delta + f * u)\right)_n = (b * u)_n,$$

where we have used Equation (2.7).

Second, for $g'_n, n \in \mathbb{N}$, another solution of the renewal equation (2.8) we obtain

$$(g - g')_n = (f * (g - g'))_n = (f^{(m)} * (g - g'))_n = \sum_{k=0}^{n} f_k^{(m)} * (g - g')_{n-k},$$

for any $m \in \mathbb{N}$. Taking $m > n$ and using Equation (2.3), we get $f_k^{(m)} = 0$ for any $k = 0, 1, \ldots, n$, so $g_n = g'_n$ for any $n \in \mathbb{N}$. □

Proof (2). Let us set $\overline{b} := \sum_{n=0}^{\infty} b_n$. From Equation (2.8) we have

$$g_n \leq \overline{b} + \max(g_0, \ldots, g_{n-1}),$$

and by induction on n we get $g_n \leq (n+1)\overline{b}$. Denote by $G(s)$ and $B(s)$ the generating functions of $(g_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, respectively.

For $0 \leq s < 1$ we have

$$G(s) = \sum_{n=0}^{\infty} g_n s^n \leq \overline{b} \sum_{n=0}^{\infty} (n+1)s^n = \overline{b}\, \frac{d\left(\sum_{n=0}^{\infty} s^n\right)}{ds} = \frac{\overline{b}}{(1-s)^2} < \infty.$$

Thus, the generating function of $(g_n)_{n \in \mathbb{N}}$ exists and is finite, for $0 \leq s < 1$. Multiplying Equation (2.8) by s^n and summing over all $n = 0, 1, \ldots$, we obtain

$$G(s) = B(s) + \Phi(s)G(s).$$

Using Proposition 2.1 we get

$$G(s) = \frac{B(s)}{1 - \Phi(s)} = B(s)U(s),$$

and equalizing the coefficients of s^n we obtain the desired result. □

There are some stochastic processes, related to a renewal chain, whose study is important for understanding the asymptotic behavior of the renewal chain. First, for all $n \in \mathbb{N}^*$, we define the counting process of the number of renewals in the time interval $[1, n]$ by

$$N(n) := \max\{k \mid S_k \leq n\} = \sum_{k=1}^{n} Z_k = \sum_{k=1}^{n} \mathbf{1}_{[0,n]}(S_k) = \sum_{k=1}^{n} \mathbf{1}_{\{S_k \leq n\}}. \tag{2.10}$$

By convention, we set $N(0) := 0$. The definition of $N(n)$ shows that $N(n) \geq k$ iff $S_k \leq n$, so the equality

$$\mathbb{P}(N(n) \geq k) = \mathbb{P}(S_k \leq n) \tag{2.11}$$

holds true for any $k \in \mathbb{N}$. As $N(n) \leq n$, the previous relation is trivially fulfilled for $k > n$, i.e., $\mathbb{P}(N(n) \geq k) = \mathbb{P}(S_k \leq n) = 0$.

Second, let us introduce the following stochastic processes:

- $U_n := n - S_{N(n)}, n \in \mathbb{N}$, called the *backward recurrence time* on time n (also called the *current lifetime* or *age*);
- $V_n := S_{N(n)+1} - n, n \in \mathbb{N}$, called the *forward recurrence time* on time n (also called the *residual* or *excess lifetime*);
- $L_n := S_{N(n)+1} - S_{N(n)} = U_n + V_n, n \in \mathbb{N}$, called the *total lifetime* on time n.

Figure 2.1 presents the counting process of the number of renewals in a renewal chain.

It can be proved that $(U_n)_{n \in \mathbb{N}}$ and $(V_n)_{n \in \mathbb{N}}$ are homogeneous Markov chains (see the following example and Exercises 2.6 and 2.7).

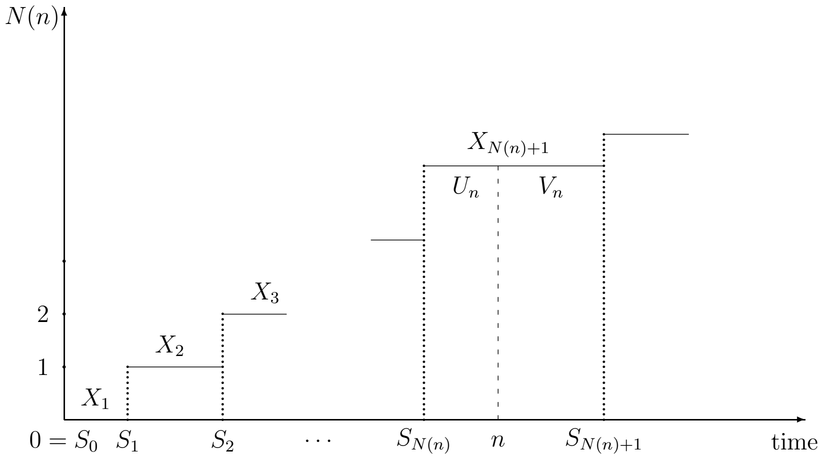

Fig. 2.1. Renewal chain

Example 2.3. Consider a recurrent renewal chain $(S_n)_{n\in\mathbb{N}}$ with waiting time distribution $(f_n)_{n\in\mathbb{N}}$. Set $m := \sup\{n \mid f_n > 0\}$ and note $E := \{1, \ldots, m\}$, if $m < \infty$, and $E := \{1, 2, \ldots\} = \mathbb{N}^*$, if $m = \infty$. Define the transition probabilities $(p_{ij})_{i,j\in E}$ by $p_{i\,i-1} := 1$, if $i > 1$, $p_{1i} := f_i$ for any $i \in E$ and $p_{ij} := 0$ for all other $i, j \in E$. It can be easily checked that $(V_n)_{n\in\mathbb{N}}$ is an irreducible and recurrent Markov chain having $(p_{ij})_{i,j\in E}$ as transition probabilities.

On the other hand, suppose that we have an irreducible and recurrent Markov chain with state space $E := \{1, \ldots, m\}$, $m < \infty$, or $E = \mathbb{N}^*$. Suppose also that the Markov chain starts in state 0 with probability 1. Then, the successive returns to state 0 form a renewal chain.

This example shows that there is a correspondence in both senses between renewal chains and Markov chains. For this reason, techniques and results developed for Markov chains can be applied to renewal chains and vice versa.

A function of great importance in renewal theory is the *renewal function* $\Psi(n), n \in \mathbb{N}$, defined as the expected number of renewals up to time n:

$$\Psi(n) := \mathbb{E}[N(n) + 1] = \sum_{k=0}^{n} \mathbb{P}(S_k \le n) = \sum_{k=0}^{n} \sum_{l=0}^{n} f_l^{(k)}, \quad n \in \mathbb{N}. \quad (2.12)$$

As $N(0) = 0$, we get $\Psi(0) = 1$.

Remark 2.2. Some authors define the renewal function without taking into account the renewal that occurs at the origin; if this were the case, we would have $\Psi(n) = \mathbb{E}(N(n))$. There is only a question of convenience for technical reasons, and we have chosen here to include the renewal at the origin.

The renewal function can be expressed in terms of $(u_n)_{n\in\mathbb{N}}$. Indeed,

$$\Psi(n) = \mathbb{E}(N(n)+1) = \mathbb{E}\left(\sum_{k=0}^{n} Z_k\right) = \sum_{k=0}^{n} u_k, \quad n \in \mathbb{N}. \tag{2.13}$$

2.2 Limit Theorems

In this section we present the basic results on the asymptotic behavior of a renewal chain. The limit theorems concern different quantities related to a renewal chain: the counting process of the number of renewals $(N(n))_{n\in\mathbb{N}}$ (Theorems 2.3 and 2.4), the expected value of the number of renewals, i.e., the renewal function $\Psi(n)$ (Theorem 2.5 and 2.6), the sequence $u = (u_n)_{n\in\mathbb{N}}$ of the probabilities that renewals will occur at instants $n, n \in \mathbb{N}$ (Theorem 2.6), and the solution of the general renewal equation given in Equation (2.8) (Theorem 2.7). In the rest of the chapter we will be using the convention $1/\infty = 0$.

Lemma 2.1.

1. *For a renewal chain we have* $\lim_{n\to\infty} S_n = \infty$ *a.s.*
2. *For a recurrent renewal chain we have* $\lim_{n\to\infty} N(n) = \infty$ *a.s.*

Proof.
1. Since $S_n \geq n$, the conclusion is straightforward.
2. The result is a direct consequence of 1. and of Relation (2.11) between $(N(n))_{n\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}}$. Indeed, for any fixed $n \in \mathbb{N}^*$ we have

$$\mathbb{P}(\lim_{k\to\infty} N(k) \leq n) = \lim_{k\to\infty} \mathbb{P}(N(k) \leq n) = \lim_{k\to\infty} \mathbb{P}(S_n \geq k),$$

where the above permutation of probability and limit is obtained applying the continuity from above of the probability (Theorem E.1) to the nonincreasing sequence of events $\{N(k) \leq n\}_{k\in\mathbb{N}}$. As the renewal chain is recurrent, $\mathbb{P}(X_n < \infty) = 1$ for any $n \in \mathbb{N}$, so $S_n = X_0 + \ldots + X_n < \infty$ a.s. for any $n \in \mathbb{N}$. Finally, we get

$$\mathbb{P}(\lim_{k\to\infty} N(k) \leq n) = \lim_{k\to\infty} \mathbb{P}(S_n \geq k) = 0,$$

which proves the result. □

The following two results investigate the asymptotic behavior of the counting chain of the number of renewals $(N(n))_{n\in\mathbb{N}}$.

Theorem 2.3 (strong law of large numbers for $N(n)$).
For a recurrent renewal chain $(S_n)_{n\in\mathbb{N}}$ *we have*

$$\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mu} \text{ a.s.}$$

Proof. We give the proof only for a positive recurrent renewal chain, i.e., $\mu = \mathbb{E}(X_1) < \infty$. Note that for all $n \in \mathbb{N}$ we have

$$S_{N(n)} \leq n < S_{N(n)+1}.$$

Using the definition of S_n given in Equation (2.1) and dividing the previous inequalities by $N(n)$ we get

$$\frac{1}{N(n)} \sum_{k=1}^{N(n)} X_k \leq \frac{n}{N(n)} \leq \frac{1}{N(n)} \sum_{k=1}^{N(n)+1} X_k,$$

or, equivalently,

$$\frac{1}{N(n)} \sum_{k=1}^{N(n)} X_k \leq \frac{n}{N(n)} \leq \frac{N(n)+1}{N(n)} \frac{1}{N(n)+1} \sum_{k=1}^{N(n)+1} X_k. \tag{2.14}$$

As $N(n) \xrightarrow[n\to\infty]{a.s.} \infty$, we have $(N(n)+1)/N(n) \xrightarrow[n\to\infty]{a.s.} 1$. Applying the SLLN to the sequence of i.i.d. random variables $(X_k)_{k\in\mathbb{N}}$ and Theorem E.5, we obtain from (2.14) that $n/N(n)$ tends to $\mathbb{E}(X_1) = \mu$, as n tends to infinity. □

Theorem 2.4 (central limit theorem for $N(n)$).
Consider a positive recurrent renewal chain $(S_n)_{n\in\mathbb{N}}$, with $\mu = \mathbb{E}(X_1) < \infty$ and $0 < \sigma^2 := Var(X_1) < \infty$. Then

$$\frac{N(n) - n/\mu}{\sqrt{n\sigma^2/\mu^3}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

Proof. We mainly follow the proof of Feller (1993) and Karlin and Taylor (1975). The main idea is to use the CLT for $S_n = X_1 + \ldots + X_n$ ($X_n, n \in \mathbb{N}^*$ i.i.d.) and Relation (2.11) between $(S_n)_{n\in\mathbb{N}}$ and $(N(n))_{n\in\mathbb{N}}$.
Set $\xi_n := \frac{\sqrt{n}}{\sqrt{\sigma^2/\mu^3}} \left[\frac{N(n)}{n} - \frac{1}{\mu} \right]$. Then we can write

$$\mathbb{P}(\xi_n \leq k) = \mathbb{P}\left(N_n \leq \frac{n}{\mu} + k\sqrt{\frac{n\sigma^2}{\mu}} \right).$$

Set $\rho_n := \lfloor \frac{n}{\mu} + k\sqrt{\frac{n\sigma^2}{\mu^3}} \rfloor$, where $\lfloor \quad \rfloor$ denotes the integer part. Note that, as $n \to \infty$, $\rho_n \sim \frac{n}{\mu} + k\sqrt{\frac{n\sigma^2}{\mu^3}}$, so $(n - \rho_n\mu) \sim -k\sqrt{\frac{n\sigma^2}{\mu^3}}$. Further, as $n \to \infty$, we have $\sigma\sqrt{\rho_n} \sim \sigma\sqrt{\frac{n}{\mu}}$, and consequently we get

$$\frac{n - \rho_n\mu}{\sigma\sqrt{\rho_n}} \sim -k, \text{ as } n \to \infty.$$

Applying the CLT to S_n, we obtain

$$\mathbb{P}(\xi_n \leq k) = \mathbb{P}(N(n) \leq \rho_n) = \mathbb{P}(S_{\rho_n} \geq n) = \mathbb{P}\left(\frac{S_{\rho_n} - \rho_n \mu}{\sigma\sqrt{\rho_n}} \geq \frac{n - \rho_n \mu}{\sigma\sqrt{\rho_n}}\right).$$

Finally, taking the limit as n tends to infinity, we get

$$\mathbb{P}(\xi_n \leq k) \xrightarrow[n\to\infty]{} 1 - \Phi(-k) = \Phi(k),$$

where $\Phi(k)$ is the cumulative distribution function of $\mathcal{N}(0,1)$. □

The standard asymptotic results in renewal theory consist in the asymptotic behavior of the renewal function. The following three theorems investigate this topic.

Theorem 2.5 (elementary renewal theorem).
For a recurrent renewal chain $(S_n)_{n\in\mathbb{N}}$ *we have*

$$\lim_{n\to\infty} \frac{\Psi(n)}{n} = \frac{1}{\mu}.$$

Proof. Consider the case where the renewal chain is positive recurrent, that is, $\mu < \infty$.

As $\mathbb{E}[N(n)] < \infty$, $n \leq S_{N(n)+1}$, and $N(n)+1$ is a stopping time for $(X_n)_{n\in\mathbb{N}}$, applying Wald's lemma (Theorem E.8) we have

$$n \leq \mathbb{E}[S_{N(n)+1}] = \mu\, \Psi(n)$$

and we obtain

$$\liminf_{n\to\infty} \frac{\Psi(n)}{n} \geq \frac{1}{\mu}. \tag{2.15}$$

Let $c \in \mathbb{N}^*$ be an arbitrary fixed integer and define $X_n^c := \min(X_n, c)$. Consider $(S_n^c)_{n\in\mathbb{N}}$ a truncated renewal chain having the interarrival times $(X_n^c)_{n\in\mathbb{N}}$. Denote by $(N^c(n))_{n\in\mathbb{N}}$ the corresponding counting process of the number of renewals in $[1,n]$. Since $X_n^c \leq c$, we obtain $n + c \geq S^c_{N^c(n)+1}$, which implies

$$n + c \geq \mathbb{E}[S^c_{N^c(n)+1}] = \mu^c\, \Psi^c(n), \tag{2.16}$$

where $\Psi^c(n) := \mathbb{E}[N^c(n)+1]$ and $\mu^c := \mathbb{E}[X_1^c]$.

Since $X_n^c \leq X_n$, we obviously have $N^c(n) \geq N(n)$ and, consequently, $\Psi^c(n) \geq \Psi(n)$. From this relation and (2.16) we immediately obtain

$$\limsup_{n\to\infty} \frac{\Psi(n)}{n} \leq \frac{1}{\mu^c}. \tag{2.17}$$

As $X_n^c \xrightarrow[c\to\infty]{a.s.} X_n$ and the sequence $(X_n^c(\omega))_{c\in\mathbb{N}^*}$ is a nondecreasing sequence in c, for any $n \in \mathbb{N}^*$ and $\omega \in \Omega$, by monotone convergence theorem (Theorem E.2) we obtain that $\lim_{c\to\infty} \mu^c = \mu$. Consequently, convergence (2.17) yields

$$\limsup_{n\to\infty} \frac{\Psi(n)}{n} \leq \frac{1}{\mu}, \tag{2.18}$$

and from Relations (2.15) and (2.18) we get the desired result.

When the renewal chain is null recurrent, i.e., $\mu = \infty$, the result is obtained by the same method, in which case we set $1/\infty = 0$. □

The asymptotic behavior of certain quantities of a renewal chain is strictly related to the notion of periodicity of the chain. The general notion of periodicity of a distribution is recalled in the following definition, while the periodicity of a renewal chain is introduced thereafter.

Definition 2.4. *A distribution* $g = (g_n)_{n\in\mathbb{N}}$ *on* $\mathbb{N}$ *is said to be* periodic *if there exists an integer* $d > 1$ *such that* $g_n \neq 0$ *only when* $n = d, 2d, \ldots$. *The greatest* d *with this property is called the* period *of* g. *If* $d = 1$, *the distribution* g *is said to be* aperiodic.

Definition 2.5. *A renewal chain* $(S_n)_{n\in\mathbb{N}}$ *is said to be* periodic of period d, $d \in \mathbb{N}^*, d > 1$, *if its waiting times distribution* $f = (f_n)_{n\in\mathbb{N}}$ *is periodic of period* d. *If* f *is aperiodic, then the renewal chain is called* aperiodic.

Theorem 2.6 (renewal theorem).

1. *For a recurrent aperiodic renewal chain* $(S_n)_{n\in\mathbb{N}}$ *we have*

$$\lim_{n\to\infty} u_n = \frac{1}{\mu}. \tag{2.19}$$

2. *For a periodic recurrent renewal chain* $(S_n)_{n\in\mathbb{N}}$ *of period* $d > 1$ *we have*

$$\lim_{n\to\infty} u_{nd} = \frac{d}{\mu} \tag{2.20}$$

and $u_k = 0$ *for all* k *not multiple of* d.

The proof of the renewal theorem will be provided at the end of next section.

Remark 2.3. The renewal theorem in a continuous-time setting states that for a continuous-time recurrent aperiodic renewal process, for any $h > 0$, we have

$$\lim_{n\to\infty} [\Psi(n+h) - \Psi(n)] = \frac{h}{\mu}. \tag{2.21}$$

The same result holds true for a continuous-time recurrent periodic renewal process of period d, provided that h is a multiple of d.

It is easy to see that in the discrete-time case Convergence (2.21) is immediately obtained from Theorem 2.6. Indeed, recall that for $n \in \mathbb{N}$ we have $\Psi(n) = \sum_{k=0}^{n} u_k$ (cf. Equation (2.13)). Consequently, for any $h \in \mathbb{N}^*$, we have

$$\Psi(n+h) - \Psi(n) = \sum_{k=n+1}^{n+h} u_k.$$

Since this sum is finite, Convergence (2.21) follows from Theorem 2.6 for both periodic and aperiodic recurrent renewal chains. As in discrete time Convergence (2.21) is not more general than Convergences (2.19) and (2.20), the form of the discrete-time version of the renewal theorem usually met in the literature is the one we gave in Theorem 2.6.

The following result is an equivalent form of the renewal theorem.

Theorem 2.7 (key renewal theorem).
Consider a recurrent renewal chain $(S_n)_{n\in\mathbb{N}}$ and a real sequence $(b_n)_{n\in\mathbb{N}}$.

1. *If the chain is aperiodic and $\sum_{n=0}^{\infty} \mid b_n \mid < \infty$, then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} b_k u_{n-k} = \frac{1}{\mu} \sum_{n=0}^{\infty} b_n. \tag{2.22}$$

2. *If the chain is periodic of period $d > 1$ and if for a certain positive integer l, $0 \leq l < d$, we have $\sum_{n=0}^{\infty} \mid b_{l+nd} \mid < \infty$, then*

$$\lim_{n\to\infty} \sum_{k=0}^{l+nd} b_k u_{l+nd-k} = \frac{d}{\mu} \sum_{n=0}^{\infty} b_{l+nd}. \tag{2.23}$$

Remark 2.4. Recall that the solution of a discrete-time renewal equation (Equation 2.8) was obtained in Theorem 2.2 as $g = b * u$. Thus, the key renewal theorem describes the asymptotic behavior of this solution g, in both periodic and aperiodic cases.

Proof.
1. Note that $\lim_{n\to\infty} u_{n-k} = 1/\mu$ (cf. Theorem 2.6 (1), with the convention $1/\infty = 0$), $\mid b_k u_{n-k} \mid \leq \mid b_k \mid$, and $\sum_{k=0}^{\infty} \mid b_k \mid < \infty$. Thus, we are under the hypotheses of Proposition E.1 and we obtain the desired result.

2. The proof is similar to that of the aperiodic case, using the second assertion of Theorem 2.6. As the chain is periodic of period $d > 1$, all the u_m with m not multiple of d are zero. Consequently, setting $k = l + id$, $i = 0, 1, \ldots, n$, the series from the left member of (2.23) becomes

$$\sum_{k=0}^{l+nd} b_k u_{l+nd-k} = \sum_{i=0}^{n} b_{l+id} u_{(n-i)d}.$$

From Theorem 2.6(2) we have $\lim_{n\to\infty} u_{(n-i)d} = d/\mu$, with the convention $1/\infty = 0$. As $\sum_{i=0}^{\infty} \mid b_{l+id} \mid < \infty$ and $\mid b_{l+id} u_{(n-1)d} \mid \leq \mid b_{l+id} \mid$, the desired result is obtained from Proposition E.1. □

Remark 2.5. In the case where $b_0 = 1$ and $b_n = 0, n \geq 1$, Convergence (2.22) becomes (2.19). Note also that we have used (2.19) in order to prove (2.22). So, the key renewal theorem and renewal theorem are equivalent in the aperiodic case, and the same remark holds true for the periodic case.

2.3 Delayed Renewal Chains

Delayed renewal chains are used for modeling the same type of phenomenon as renewal chains do, with the only difference that we do not consider the origin as the occurrence of the first renewal, that is, $S_0 = X_0 > 0$. In other words, we want to observe a normal renewal chain, but we missed the beginning and we denote by S_0 the time when we observe the first renewal, which is not identically 0 but follows a certain distribution.

Definition 2.6 (delayed renewal chain).
An arrival time sequence $(S_n)_{n\in\mathbb{N}}$ *for which the waiting times* $(X_n)_{n\in\mathbb{N}^*}$ *form an i.i.d. sequence and* X_0 *is independent of* $(X_n)_{n\in\mathbb{N}^*}$ *is called a* delayed renewal chain *and every* S_n *is called a* renewal time.

The chain $(S_n - S_0)_{n\in\mathbb{N}}$ is an ordinary renewal chain, called the *associated renewal chain.* Note that $(S_n)_{n\in\mathbb{N}}$ is a renewal chain iff $S_0 = 0$ a.s.

Figure 2.3 presents the counting process of the number of renewals in a delayed renewal chain.

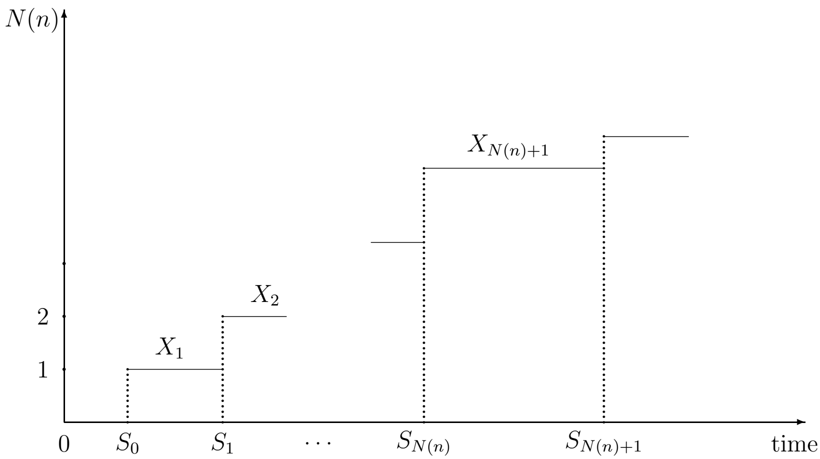

Fig. 2.2. Delayed renewal chain

Example 2.4. Consider again the Example 2.1 of the light lamp with exchange lightbulbs of lifetimes $(X_n)_{n\in\mathbb{N}}$. Suppose this time that the bulb which is on at time 0 started to function sometimes in the past, but all the others are new, with the same lifetime distribution. In other words, let $(X_n)_{n\in\mathbb{N}}$ be independent and $(X_n)_{n\in\mathbb{N}^*}$ have the same distribution. Then, the sequence $(S_n)_{n\in\mathbb{N}}$ defined by $S_n := X_0 + \ldots + X_n$, $n \in \mathbb{N}$, is an example of a delayed renewal chain.

Example 2.5. A typical example of a delayed renewal chain can be obtained as follows: consider an irreducible, finite state space Markov chain and let x be a certain state. It is easy to check that the successive returns to state x, when starting at $y \neq x$, represent a delayed renewal chain.

Example 2.6. Consider a sequence of i.i.d. Bernoulli trials. We can prove that the times of occurrence of different finite patterns form renewal chains, generally delayed. For instance, suppose we have the following sequence issued from repeated i.i.d. Bernoulli trials

$$\begin{array}{cccccccccccccccc} S & S & S & S & F & S & F & F & S & F & S & F & S & S & S & \dots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \dots \end{array}$$

and suppose that we are interested in the occurrence of the pattern SFS in this sample. Two different counting procedures can be considered.

First, suppose that the counting starts anew when the searched pattern occurs, that is, we do not allow overlapping. Note that SFS occurs in the 6th and 11th trials. Note also that SFS occurs in the 13th trial, but we do not count this occurrence, because we do not allow overlapping, so we have started anew the counting at the 12th trial (that is, after the occurrence of SFS in the 11th trial).

Second, suppose that we allow overlapping. In this case, the pattern SFS occurs in the 6th, 11th and 13th trial.

In both situations the occurrence times of the pattern SFS is a delayed renewal chain. See also Example 2.7 for more details.

Let us set out some definitions and notation for a delayed renewal chain $(S_n)_{n\in\mathbb{N}}$. The distribution $b = (b_n)_{n\in\mathbb{N}}$ of S_0 will be called the *initial distribution* (or the *delayed distribution*) of the delayed renewal chain, $b_n := \mathbb{P}(S_0 = n)$. Its generating function will be denoted by $B(s)$. Set $v_n := \sum_{k=0}^{\infty} \mathbb{P}(S_k = n)$ for the probability that a renewal will occur at instant n in the delayed renewal chain and $V(s)$ for the corresponding generating function. By u_n we denote the same probability of occurrence of a renewal at time n, but in the associated renewal chain, and we set $U(s)$ for the generating function.

As was done before, denote by $f = (f_n)_{n\in\mathbb{N}}$, with $f_0 := 0$, the common distribution of the waiting times $(X_n)_{n\in\mathbb{N}^*}$ in the delayed renewal chain. Also, let F be the cumulative distribution function of the waiting times. Since $X_n := S_n - S_{n-1} = (S_n - S_0) - (S_{n-1} - S_0)$, we have the same distribution for the waiting times considered in the delayed renewal chain or in the associated renewal chain.

A delayed renewal chain is called *aperiodic* (resp. *periodic of period* $d > 1$) if the associated renewal chain is aperiodic (resp. periodic of period $d > 1$). Similarly, a delayed renewal chain is said to be *transient* or *positive* (*null*) *recurrent* if the associated renewal chain is a such.

We can easily derive a recurrence formula for $(v_n)_{n\in\mathbb{N}}$, as we already did for $(u_n)_{n\in\mathbb{N}}$ in an ordinary renewal chain (Equation (2.4)), and obtain an expression for the generating function $V(s)$ (as we did in Proposition 2.1 for $U(s)$). Indeed, we have

$$v_n = \mathbb{P}(S_0 = n) + \sum_{k=0}^{n-1} \mathbb{P}(S_0 = k) \sum_{r=1}^{n-k} \mathbb{P}(S_r - S_0 = n-k) = b_n + \sum_{k=0}^{n-1} b_k u_{n-k}.$$

Thus we have obtained

$$v_n = \sum_{k=0}^{n} b_k u_{n-k}, \quad i.e., \quad v_n = (b * u)_n, \quad n \in \mathbb{N}. \tag{2.24}$$

Multiplying Equation (2.24) by s^n and summing over $n = 0, 1, 2, \ldots$ we get

$$V(s) = B(s)U(s) = \frac{B(s)}{1 - \Phi(s)}. \tag{2.25}$$

The following theorem describes the asymptotic behavior of $(v_n)_{n\in\mathbb{N}}$, the sequence of probabilities that a renewal will occur at time $n, n \in \mathbb{N}$.

Theorem 2.8 (renewal theorem for delayed RCs).
Consider a delayed recurrent renewal chain $(S_n)_{n\in\mathbb{N}}$ with initial distribution $b = (b_n)_{n\in\mathbb{N}}$.

1. *If the chain is aperiodic, then*

$$\lim_{n\to\infty} v_n = \frac{1}{\mu} \sum_{n=0}^{\infty} b_n. \tag{2.26}$$

2. *If the chain is periodic of period $d > 1$, then for any positive integer l, $0 \le l < d$,*

$$\lim_{n\to\infty} v_{l+nd} = \frac{d}{\mu} \sum_{n=0}^{\infty} b_{l+nd}. \tag{2.27}$$

Note that $\sum_{n=0}^{\infty} b_n = \mathbb{P}(S_0 < \infty) < 1$ means that $(b_n)_{n\in\mathbb{N}}$ is an improper distribution. Thus, the limit in (2.26) is $1/\mu$ for $(b_n)_{n\in\mathbb{N}}$ a proper distribution. The same type of remark is true for the limit in (2.27).

Proof. As $v = (b * u)$ (cf. Equation (2.24)), the proof is a direct application of the key renewal theorem (Theorem 2.7) for nondelayed renewal chains. □

Example 2.7. Let us continue Example 2.6 by considering a sequence of i.i.d. Bernoulli trials, with probability of success $\mathbb{P}(S) = p$ and probability of failure $\mathbb{P}(F) = q = 1-p$, $0 < p < 1$. We are interested in the occurrence of the pattern SFS in the case when we allow overlapping. More specifically, we want to:

1. Prove that the successive occurrence times of the pattern form an aperiodic delayed recurrent renewal chain;

2. Compute the main characteristics of the chain: the initial distribution $(b_n)_{n\in\mathbb{N}}$, the probability v_n that a renewal occurs at instant n, the common distribution $(f_n)_{n\in\mathbb{N}}$ of the waiting times $(X_n)_{n\in\mathbb{N}^*}$ and $\mu = \mathbb{E}(X_1)$, the mean waiting time between two successive occurrences of the pattern.

To answer the first question, note first that the independence of the Bernoulli trials implies the independence of the waiting times. Second, let us compute the distribution of X_m for $m \in \mathbb{N}^*$.

$$
\begin{aligned}
f_0 &= \mathbb{P}(X_m = 0) = 0 \text{ (by definition)};\\
f_1 &= \mathbb{P}(X_m = 1) = \mathbb{P}(\text{renewal at } (S_{m-1}+1)) = p^2q;\\
f_2 &= \mathbb{P}(X_m = 2) = \mathbb{P}(\text{renewal at } (S_{m-1}+2), \text{not a renewal at } (S_{m-1}+1))\\
&= p^2q(1-p^2q);\\
f_n &= \mathbb{P}(X_m = n) = p^2q(1-p^2q)^{n-1}, n \geq 2.
\end{aligned}
$$

Thus, we see that the distribution of X_m does not dependent on $m \in \mathbb{N}^*$, so we have a renewal chain. Moreover, as $f_1 = \mathbb{P}(X_m = 1) = p^2q \neq 0$, we get that the chain is aperiodic. In order to see that the renewal chain is delayed, we have to check that the distribution of X_0 is different than the distribution of $X_m, m \in \mathbb{N}^*$. Indeed, we have $b_0 = \mathbb{P}(X_0 = 0) = 0$, $b_1 = \mathbb{P}(X_0 = 1) = 0$ (because the pattern SFS cannot occur in the first two trials), and $b_n = \mathbb{P}(X_0 = n) = p^2q(1-p^2q)^{n-2}, n \geq 2$. In conclusion, the successive occurrence times of SFS form an aperiodic delayed renewal chain. As $\sum_{n\geq 0} f_n = 1$, we see that the chain is recurrent.

Concerning the characteristics of the chain, we have already computed $(f_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. Similarly, we obtain $v_0 = v_1 = 0$ (a renewal cannot occur in the first two trials) and for $n \geq 2$ we have

$$
\begin{aligned}
v_n &= \mathbb{P}(Z_n = 1)\\
&= \mathbb{P}(S \text{ at the } n\text{th trial}, F \text{ at the } (n-1)\text{th trial}, S \text{ at the } (n-2)\text{th trial})\\
&= p^2q.
\end{aligned}
$$

We can immediately see that the mean waiting time between two successive occurrences is $\mu := \mathbb{E}(X_1) = 1/p^2q$. One can also check that the relation $\lim_{n\to\infty} v_n = \frac{1}{\mu}\sum_{n=0}^{\infty} b_n$ proved in the previous theorem holds true.

The Stationary Renewal Chain

Our objective is to construct a particular case of delayed renewal chain which has important stationary or time-invariant properties. The question we want to answer is the following: For what kind of delayed renewal chain is v_n constant with respect to n ?

Let us consider $(S_n)_{n\in\mathbb{N}}$ a delayed renewal chain such that $\mathbb{P}(S_0 < \infty) = 1$, i.e., $(b_n)_{n\in\mathbb{N}}$ is a proper distribution. Suppose that $v_n = \Delta$ for all $n \in \mathbb{N}$. Consequently, the generating function $V(s)$ is given by

$$V(s) = \sum_{n=0}^{\infty} \Delta s^n = \frac{\Delta}{1-s}.$$

Using Equation (2.25) we obtain

$$B(s) = \frac{\Delta}{1-s}(1 - \Phi(s)),$$
$$\sum_{n=0}^{\infty} b_n s^n = \Delta \left(\sum_{n=0}^{\infty} s^n \right) \left(1 - \sum_{n=1}^{\infty} f_n s^n \right).$$

Equalizing the coefficients of s^n in the left-hand and right-hand side, for $n \in \mathbb{N}$, we get

$$b_0 = \Delta,$$
$$b_n = \Delta \left(1 - \sum_{k=1}^{n} f_k \right) = \Delta \mathbb{P}(X_1 > n), \quad n \in \mathbb{N}^*.$$

Taking into account the fact that $\sum_{n=0}^{\infty} b_n = 1$ and using $\sum_{n=0}^{\infty} P(X_1 > n) = \mu$ we obtain

$$\Delta = 1/\mu,$$

provided that the delayed renewal chain is positive recurrent. Thus, we have shown that if a positive recurrent delayed renewal chain satisfies $v_n = \Delta$ for all $n \in \mathbb{N}$, then $\Delta = 1/\mu$ and $b_n = \mathbb{P}(X_1 > n)/\mu$.

Now, starting with a positive recurrent delayed renewal chain $(S_n)_{n \in \mathbb{N}}$ such that $b_n = \mathbb{P}(X_1 > n)/\mu$ for all $n \in \mathbb{N}$, we want to prove that $v_n = 1/\mu$ for all $n \in \mathbb{N}$. For $0 \leq s < 1$, the generating function of the first occurrence of a renewal is

$$\begin{aligned} B(s) &= \frac{1}{\mu} \sum_{n=0}^{\infty} \mathbb{P}(X_1 > n) s^n = \frac{1}{\mu} \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \mathbb{P}(X_1 = k) s^n \\ &= \frac{1}{\mu} \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} s^n f_k = \frac{1}{\mu} \frac{1}{1-s} \left(\sum_{k=1}^{\infty} f_k - \sum_{k=1}^{\infty} s^k f_k \right) \\ &= \frac{1}{\mu} \frac{1 - \Phi(s)}{1-s}. \end{aligned}$$

From Equation (2.25) we obtain

$$V(s) = \frac{1}{\mu} \frac{1}{1-s} = \sum_{n=0}^{\infty} \frac{1}{\mu} s^n,$$

so $v_n = 1/\mu$ for all $n \in \mathbb{N}$.

This entire discussion can be summarized in the following result.

Proposition 2.2. *Let $(S_n)_{n\in\mathbb{N}}$ be a positive recurrent delayed renewal chain with waiting times $(X_n)_{n\in\mathbb{N}}$ and $\mu := \mathbb{E}(X_1) < \infty$. Then, $\mathbb{P}(S_0 = n) := \mathbb{P}(X_1 > n)/\mu$ is the unique choice for the initial distribution of the delayed renewal chain such that $v_n \equiv$ constant for all $n \in \mathbb{N}$. Moreover, this common constant value is $1/\mu$.*

The delayed renewal chain with $v_n = 1/\mu$ for all $n \in \mathbb{N}$ is called a *stationary renewal chain* and its initial distribution defined by $\mathbb{P}(S_0 = n) := \mathbb{P}(X_1 > n)/\mu$ for all $n \in \mathbb{N}$ is called the *stationary distribution of the delayed renewal chain* $(S_n)_{n\in\mathbb{N}}$.

Remark 2.6. It can be shown that for $(S_n)_{n\in\mathbb{N}}$ a simple renewal chain and m a fixed integer, $m \in \mathbb{N}$, we have the following limiting distribution of the current lifetime $U_m = m - S_{N(m)}$ and of the residual lifetime $V_m = S_{N(m)+1} - m$:

$$\lim_{m\to\infty} \mathbb{P}(U_m = n) = \mathbb{P}(X_1 > n)/\mu = \lim_{m\to\infty} \mathbb{P}(V_m = n + 1).$$

Consequently, a stationary renewal chain can be seen as a simple renewal chain which started indefinitely far in the past, such that the distribution $(b_n)_{n\in\mathbb{N}}$ of the first renewal we observe starting from time 0 is the same as the limit distribution of the current and residual lifetime in the simple renewal chain. This phenomenon explains intuitively the time-invariant property of a stationary renewal chain given in Proposition 2.2.

We end this section by proving Theorem 2.6.

Proof (of renewal theorem–Theorem 2.6).
(1) Consider $(S_n)_{n\in\mathbb{N}}$ a recurrent aperiodic renewal chain (not delayed, i.e., $S_0 = X_0 = 0$ a.s.), with $(X_n)_{n\in\mathbb{N}^*}$ the interrenewal times. We will prove the result for the positive recurrent case, i.e., for $\mu := \mathbb{E}(X_1) < \infty$, following the proof of Karlin and Taylor (1981) (proof of Theorem 1.1, pages 93–95) based on the technique of coupling random processes.

Let $(T_n)_{n\in\mathbb{N}}$ be the stationary renewal chain associated to the renewal chain $(S_n)_{n\in\mathbb{N}}$ (cf. Proposition 2.2). More specifically, $T_n := Y_0 + \ldots + Y_n$, where $(Y_n)_{n\in\mathbb{N}^*}$ i.i.d. such that Y_n has the same distribution as X_n for $n \geq 1$, and

$$\mathbb{P}(T_0 = n) = \mathbb{P}(Y_0 = n) := \mathbb{P}(X_1 > n)/\mu, \quad n \in \mathbb{N}.$$

Let us define the chain $(U_n)_{n\in\mathbb{N}}$ by induction as follows:

$$\begin{aligned} &U_0 := X_0 - Y_0 = -Y_0, \\ &U_n := U_{n-1} + (X_n - Y_n) = \ldots = (X_1 + \ldots + X_n) - (Y_0 + \ldots + Y_n), n \geq 1. \end{aligned}$$

Denote by N the first instant when the same number of renewals takes place at the same time in the chains $(S_n)_{n\in\mathbb{N}}$ and in $(T_n)_{n\in\mathbb{N}}$, i.e., $N := \min\{n \in \mathbb{N} \mid U_n = 0\}$. As $\mathbb{E}(X_n - Y_n) = 0$ for $n \geq 1$, applying Theorem D.1, we obtain that the Markov chain $(U_n - U_0)_{n\in\mathbb{N}}$ is recurrent. Consequently,

$\mathbb{P}(N < \infty) = 1$. Thus, for $n \geq N$ we have that S_n and T_n have the same distribution and we obtain

$$\begin{aligned} u_n &= \mathbb{P}(S_k = n \text{ for some } k \in \mathbb{N}) \\ &= \mathbb{P}(T_k = n \text{ for some } k \geq N) + \mathbb{P}(S_k = n \text{ for some } k < N) \\ &= \mathbb{P}(T_k = n \text{ for some } k \in \mathbb{N}) - \mathbb{P}(T_k = n \text{ for some } k < N) \\ &\quad + \mathbb{P}(S_k = n \text{ for some } k < N). \end{aligned} \tag{2.28}$$

First, note that $\mathbb{P}(T_k = n \text{ for some } k \in \mathbb{N}) = 1/\mu$ (Proposition 2.2). Second, as $\{T_k = n \text{ for some } k < N\} \subset \{T_N > n\}$ for $k < N$, we have

$$\lim_{n\to\infty} \mathbb{P}(T_k = n \text{ for some } k < N) \leq \lim_{n\to\infty} \mathbb{P}(T_N > n) = \mathbb{P}(T_N = \infty),$$

where the last equality is obtained using the continuity from above of the probability (Theorem E.1) applied to the nonincreasing sequence $\{T_N > n\}_{n\in\mathbb{N}}$. Since $\mathbb{P}(N < \infty) = 1$ and $(T_n)_{n\in\mathbb{N}}$ is a recurrent renewal chain (because $(S_n)_{n\in\mathbb{N}}$ is so), we get $\mathbb{P}(T_N = \infty) = 0$, so

$$\lim_{n\to\infty} \mathbb{P}(T_k = n \text{ for some } k < N) = 0.$$

In the same way, we get $\lim_{n\to\infty} \mathbb{P}(S_k = n \text{ for some } k < N) = 0$ and, from Equation (2.28), we obtain $\lim_{n\to\infty} u_n = 1/\mu$.

(2) The case when the recurrent renewal chain $(S_n)_{n\in\mathbb{N}}$ is periodic of period $d > 1$ can be easily reduced to the aperiodic case. Let $S'_n := S_{dn}$. The renewal chain $(S'_n)_{n\in\mathbb{N}}$ is aperiodic and we denote by μ_d its mean waiting time. Note that we have $\mu = d\mu_d$, where μ is the mean waiting time of the original renewal chain $(S_n)_{n\in\mathbb{N}}$. Using (1) we obtain

$$\lim_{n\to\infty} u_{nd} = 1/\mu_d = d/\mu,$$

which accomplishes the proof. □

2.4 Alternating Renewal Chain

We present here a particular case of a renewal chain that is important in reliability theory due to its simplicity and intuitive interpretation.

Definition 2.7 (alternating renewal chain). *Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of i.i.d. random variables, with common distribution $h = (h_n)_{n\in\mathbb{N}}$, $h_0 := 0$. Similarly, let $(Y_n)_{n\in\mathbb{N}^*}$ be a sequence of i.i.d. random variables, with common distribution $g = (g_n)_{n\in\mathbb{N}}$, $g_0 := 0$. We also suppose that the sequences $(X_n)_{n\in\mathbb{N}^*}$ and $(Y_n)_{n\in\mathbb{N}^*}$ are independent between them. Define $V_n := X_n + Y_n, n \in \mathbb{N}^*$, and $S_n := \sum_{i=1}^{n} V_i, n \in \mathbb{N}^*$, $S_0 := 0$. The sequence $(S_n)_{n\in\mathbb{N}}$ is called an* alternating renewal chain, with up-time distribution h and down-time distribution g.

One can easily check that an alternating renewal chain $(S_n)_{n\in\mathbb{N}}$ is an ordinary renewal chain with waiting times $V_n,\ n \in \mathbb{N}^*$ and waiting time distribution $f := h * g$.

We give now a reliability example where the use of an alternating renewal chain arises naturally.

Example 2.8. Consider a component of a system whose evolution in time is as follows: at time 0, a new component starts to work for a random time X_1, when it fails and is repaired during a random time Y_1 (or replaced with a new, identical component); then, it works again for a random time X_2, when it is again repaired (or replaced) for a time Y_2, and so on. Suppose that $X := (X_n)_{n\in\mathbb{N}^*}$ is a sequence of i.i.d. random variables, that is, the repair process is perfect (or the replacement components are identical to the used ones). Suppose also that $Y := (Y_n)_{n\in\mathbb{N}^*}$ is a sequence of i.i.d. random variables, i.e., all the repairing (or replacement) conditions are identical. Suppose also that sequences X and Y are independent between them. The sequence $S_n := \sum_{i=1}^{n}(X_i + Y_i), n \in \mathbb{N}^*$, $S_0 := 0$, forms an alternating renewal chain. See also Exercise 2.2.

Let us consider the system given in the previous example. Thus, we suppose that $X_n, n \in \mathbb{N}^*$, is the nth up time (working time) of the system, while $Y_n, n \in \mathbb{N}^*$, is the nth down time (repair time) of the system. First, we want to obtain the probability that the system will be working at time k. Second, we want to see what this probability will be for large k (when k tends to infinity). To answer this questions, we introduce:

- The *reliability of the system at time* $k \in \mathbb{N}$ – the probability that the system has functioned without failure in the period $[0, k]$,
- The *availability of the system at time* $k \in \mathbb{N}$ – the probability that the system will be working at time $k \in \mathbb{N}$.

In our context, the reliability at time $k \in \mathbb{N}$ is

$$R(k) = \mathbb{P}(X_1 > k) = \sum_{m\geq 1} f_{k+m}.$$

We want to obtain the availability of the system as the solution of a renewal equation. For any $k \in \mathbb{N}^*$, we can write

$$\begin{aligned}
A(k) &= \sum_{n\geq 1} \mathbb{P}(S_{n-1} \leq k < S_{n-1} + X_n) \\
&= \mathbb{P}(X_1 > k) + \sum_{n\geq 2} \mathbb{P}(S_{n-1} \leq k < S_{n-1} + X_n) \\
&= R(k) + \sum_{n\geq 2}\sum_{m=1}^{k} \mathbb{P}(S_{n-1} \leq k < S_{n-1} + X_n,\ S_1 = m)
\end{aligned}$$

$$= R(k) + \sum_{m=1}^{k} \sum_{n \geq 2} \mathbb{P}(S_{n-2} \leq k - m < S_{n-2} + X_{n-1}) \mathbb{P}(S_1 = m)$$

$$= R(k) + \sum_{m=1}^{k} A(k-m) f_m,$$

and we obtain the renewal equation associated to the availability:

$$A(k) = R(k) + f * A(k), k \in \mathbb{N}.$$

Although we proved this equality only for $k \in \mathbb{N}^*$, it is obviously satisfied for $k = 0$, because $f_0 := 0$ and the system is working at time 0, i.e., $A(0) = R(0) = 1$.

Solving this renewal equation (Theorem 2.2), we get the probability that the system will be working at time k in terms of the reliability

$$A(k) = u * R(k), \ k \in \mathbb{N},$$

where u_n, the probability that a renewal will occur at time n, represents in our case the probability that the system is just starting to function again after a repairing period.

We are interested now in the probability that the system will be working at time k, for large k. Thus, we want to obtain the limit of $A(k)$, the availability at time k, as k tends to infinity. This is called *steady-state* (or *limit*) *availability.* From the key renewal theorem (Theorem 2.7) we get

$$\lim_{k \to \infty} A(k) = \lim_{k \to \infty} u * R(k) = \frac{1}{\mu} \sum_{m=0}^{\infty} R(m),$$

where $\mu = \mathbb{E}(X_1 + Y_1)$ is the mean waiting time in the renewal chain $(S_n)_{n \in \mathbb{N}}$. Noting that

$$\sum_{m=0}^{\infty} R(m) = \sum_{m=0}^{\infty} \mathbb{P}(X_1 > m) = \mathbb{E}(X_1),$$

we get the steady-state availability

$$\lim_{k \to \infty} A(k) = \frac{\mu_X}{\mu_X + \mu_Y},$$

where we set $\mu_X := \mathbb{E}(X_1)$ for the mean lifetime and $\mu_Y := \mathbb{E}(Y_1)$ for the mean repair time.

Remark 2.7. It is worth noticing that for the results concerning availability we do not need that the sequences of random variables $(X_n)_{n \in \mathbb{N}^*}$ and $(Y_n)_{n \in \mathbb{N}^*}$ to be independent between them, but only that $(X_n + Y_n)_{n \in \mathbb{N}^*}$ is an i.i.d. sequence.

Exercises

Exercise 2.1. Show that the alternating renewal chain introduced by Definition 2.7 is an ordinary renewal chain (i.e., nondelayed) with waiting times $V_n = X_n + Y_n,\ n \in \mathbb{N}^*$, and waiting time distribution $f := h * g$.

Exercise 2.2 (binary component). Consider the binary component (or system) given in Example 2.8. The component starts to work at time $n = 0$. Consider that the lifetimes $(X_n)_{n\in\mathbb{N}^*}$ of the component have a common geometric distribution on $\mathbb{N}^*$, of parameter $p, 0 < p < 1$, denoted by $(h_n)_{n\in\mathbb{N}}$, $h_0 := 0$, $h_n := p(1-p)^{n-1}, n \geq 1$. Consider also that the repair times $(Y_n)_{n\in\mathbb{N}^*}$ have a common geometric distribution on $\mathbb{N}^*$, of parameter $q, 0 < q < 1$, denoted by $(g_n)_{n\in\mathbb{N}}$, $g_0 := 0$, $g_n := q(1-q)^{n-1}, n \geq 1$.

1. Show that the sequence $S_n := \sum_{i=1}^{n}(X_i + Y_i),\ n \in \mathbb{N}*,\ S_0 := 0$ forms a positive recurrent renewal chain and compute the characteristics of the chain: the waiting time distribution $f = (f_n)_{n\in\mathbb{N}}$, the corresponding generating function, and the sequence $(u_n)_{n\in\mathbb{N}}$ of probabilities that a renewal occurs at time n.
2. For these types of components and for a large time n, compute approximatively the number of repairs (replacements) needed during the time interval $[0; n]$.
   Numerical application: take $n = 2000, p = 0.01$, and $q = 0.1$.

Exercise 2.3 (binary component: continuation). Consider the binary component given in Exercise 2.2. Denote by 0 the working state and by 1 the failure state. Let T_n be a random variable of state space $E = \{0, 1\}$, defined as follows: $T_n = 0$ if the component is in the working state at time n and $T_n = 1$ if the component is in the failure state at time n.

1. Show that $(T_n)_{n\in\mathbb{N}}$ is a Markov chain with state space $E = \{0, 1\}$, initial distribution $\boldsymbol{\alpha} = (1, 0)$, and transition matrix
$$\mathbf{p} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}.$$
2. Show that the n step transition matrix is
$$\mathbf{p}^n = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{(1-p-q)^n}{p+q}\begin{pmatrix} p & -p \\ -q & q \end{pmatrix}.$$
3. Give the state probability vector $P(n) = (P_1(n), P_2(n))$.

Exercise 2.4 (transient or stopped discrete-time renewal processes). Let us consider a discrete-time renewal process with waiting time distribution $(f_n)_{n\in\mathbb{N}}$ such that $\sum_{n\geq 0} f_n = p$, with $0 < p < 1$. We define the lifetime of the chain by
$$T := S_N = X_1 + \ldots + X_N,$$
where $S_{N+1} = S_{N+2} = \ldots = \infty$.

1. Compute the distribution of N.
2. Show that $\mathbb{P}(T = n) = p\mathbf{1}_{\{n=0\}} + \sum_{k=1}^{n} \mathbb{P}(T = n-k) f(k), n \geq 0$.
3. Give the distribution of T.

Exercise 2.5. Let $F_n(x) = \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{X_k \leq x\}}$ be the empirical distribution function of the i.i.d. random sample $X_1, \ldots, X_n$. Denote by F the common distribution function of $X_i, i = 1, \ldots, n$. The Glivenko–Cantelli theorem tells us that

$$\sup_{x \in R} \mid F_n(x) - F(x) \mid \xrightarrow[n \to \infty]{a.s.} 0.$$

Suppose that $N(m), m \geq 0$, is a family of positive integer-valued random variables such that $N(m) \xrightarrow[m \to \infty]{a.s.} \infty$. Define another empirical distribution as follows:

$$\widetilde{F}_m(x) = \frac{1}{N(m)} \sum_{k=1}^{N(m)} \mathbf{1}_{\{X_k \leq x\}}.$$

Prove that

$$\sup_{x \in R} \mid \widetilde{F}_m(x) - F(x) \mid \xrightarrow[m \to \infty]{a.s.} 0.$$

Exercise 2.6 (forward recurrence times). Consider the settings of Example 2.3, i.e., start with $(S_n)_{n \in \mathbb{N}}$, a recurrent renewal chain, with waiting time distribution $(f_n)_{n \in \mathbb{N}}$, take $E := \{1, \ldots, m\}$, if $m < \infty$, or $E := \{1, 2, \ldots\} = \mathbb{N}^*$, if $m = \infty$, where $m := \sup\{n \mid f_n > 0\}$.

1. Show that the forward recurrence times $(V_n)_{n \in \mathbb{N}}$ form an irreducible recurrent Markov chain.
2. Show that $(V_n)_{n \in \mathbb{N}}$ has the transition matrix $(p_{ij})_{i,j \in E}$ defined by $p_{1i} := f_i$ for any $i \in E$, $p_{i\, i-1} := 1$, if $i > 1$, and $p_{ij} := 0$ for all other $i, j \in E$.

Exercise 2.7 (backward recurrence times). Let $(S_n)_{n \in \mathbb{N}}$ be a recurrent renewal chain, with waiting time distribution $(f_n)_{n \in \mathbb{N}}$.

1. Show that the backward recurrence times $(U_n)_{n \in \mathbb{N}}$ form an irreducible recurrent Markov chain.
2. As in Exercise 2.6 for the forward recurrence times, find the transition probabilities of the backward recurrence times in terms of the waiting time distribution $(f_n)_{n \in \mathbb{N}}$.

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