
Homogenization for ODEs and SDEs

11.1 Introduction

In this chapter we continue our study of systems of SDEs with two widely separated characteristic time scales. The setting is similar to the one considered in the previous chapter. The difference is that in this chapter we seek to derive an effective equation describing dynamics on the longer, *diffusive time scale*. This is the time scale of interest when the effective drift $F(x)$ defined in Equation (10.3.1) vanishes due, for example, to the symmetries of the problem. The vanishing of the effective drift is captured in the centering condition; see Equation (11.2.5). In contrast to the case considered in the previous chapter, in the diffusive time scale the effective equation is stochastic, even when noise does not act directly on the slow variables, that is, even when $\alpha(x, y) \equiv 0$ in Equation (11.2.1).

In Section 11.2 we present the SDEs that we will analyze in this chapter. Section 11.3 contains the simplified equations, which we derive in Section 11.4. In Section 11.5 we describe various properties of the simplified equations. The derivation assumes that the fast process to be eliminated is stochastic. In Section 11.6 we show how the deterministic case can be handled. In Section 11.7 we present various applications of the theory developed in this chapter: the case where the fast process is of Ornstein–Uhlenbeck type is in Section 11.7.1 and the case where the fast process is a chaotic deterministic process is in Section 11.7.2. Deriving the Stratonovich stochastic integral as the limit of smooth approximations to white noise is considered in Section 11.7.3; Stokes’ law is studied in Section 11.7.4. The Green–Kubo formula from statistical mechanics is derived in Section 11.7.5. The case where the stochastic integral in the limiting equation can be interpreted in neither the Itô nor the Stratonovich sense is considered in Section 11.7.6. Lévy area corrections are studied in Section 11.7.7. Various extensions of the results presented in this chapter, together with bibliographical remarks, are presented in Section 11.8.

11.2 Full Equations

Consider the SDEs

$$\frac{dx}{dt} = \frac{1}{\varepsilon} f_0(x, y) + f_1(x, y) + \alpha(x, y) \frac{dU}{dt}, \quad x(0) = x_0, \quad (11.2.1a)$$

$$\frac{dy}{dt} = \frac{1}{\varepsilon^2} g(x, y) + \frac{1}{\varepsilon} \beta(x, y) \frac{dV}{dt}, \quad y(0) = y_0. \quad (11.2.1b)$$

Here U and V are independent standard Brownian motions. Both the x and y equations contain fast dynamics, but the dynamics in y is an order of magnitude faster than in x . As discussed in Sections 4.1 and 6.1 $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $\mathcal{X} \oplus \mathcal{Y} = \mathcal{Z}$.

For Equation (11.2.1), the backward Kolmogorov Equation (6.3.4) with $\phi = \phi(x)$ is¹

$$\frac{\partial v}{\partial t} = \frac{1}{\varepsilon^2} \mathcal{L}_0 v + \frac{1}{\varepsilon} \mathcal{L}_1 v + \mathcal{L}_2 v, \quad \text{for } (x, y, t) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^+, \quad (11.2.2a)$$

$$v = \phi(x), \quad \text{for } (x, y, t) \in \mathcal{X} \times \mathcal{Y} \times \{0\}, \quad (11.2.2b)$$

where

$$\mathcal{L}_0 = g \cdot \nabla_y + \frac{1}{2} B : \nabla_y \nabla_y, \quad (11.2.3a)$$

$$\mathcal{L}_1 = f_0 \cdot \nabla_x, \quad (11.2.3b)$$

$$\mathcal{L}_2 = f_1 \cdot \nabla_x + \frac{1}{2} A : \nabla_x \nabla_x, \quad (11.2.3c)$$

with

$$A(x, y) := \alpha(x, y) \alpha(x, y)^T,$$

$$B(x, y) := \beta(x, y) \beta(x, y)^T.$$

By using the method of multiple scales we eliminate the y dependence in this Kolmogorov equation, to identify a simplified equation for the dynamics of x alone.

In terms of the generator \mathcal{L}_0 , which is viewed as a differential operator in y , in which x appears as a parameter, the natural ergodicity assumption to make for variable elimination is the statement that \mathcal{L}_0 has one-dimensional null space characterized by

$$\mathcal{L}_0 1(y) = 0, \quad (11.2.4a)$$

$$\mathcal{L}_0^* \rho^\infty(y; x) = 0. \quad (11.2.4b)$$

Here $1(y)$ denotes constants in y and $\rho^\infty(y; x)$ is the density of an ergodic measure $\mu_x(dy) = \rho^\infty(y; x) dy$. We also assume that $f_0(x, y)$ averages to zero under this measure, so that the **centering condition**

¹ For simplicity we will take the initial condition of the backward Kolmogorov equation to be independent of y . This is not necessary. See the discussion in Section 11.8

$$\int_{\mathcal{Y}} f_0(x, y) \mu_x(dy) = 0 \quad \forall x \in \mathcal{X} \quad (11.2.5)$$

holds. It can then be shown that the term involving f_0 in the x equation will, in the limit $\varepsilon \rightarrow 0$, give rise to $\mathcal{O}(1)$ effective drift and noise contributions in an approximate equation for x .

As in the previous chapter, in the case where $\mathcal{Y} = \mathbb{T}^d$, the operators \mathcal{L}_0 and \mathcal{L}_0^* are equipped with periodic boundary conditions. Then, assuming that $B(x, y)$ is strictly positive definite, uniformly in $(x, y) \in \mathcal{X} \times \mathbb{T}^d$, Theorem 6.16 justifies the statement that the null space of \mathcal{L}_0^* is one-dimensional. In more general situations, such as when $\mathcal{Y} = \mathbb{R}^d$, or $B(x, y)$ is degenerate, similar rigorous justifications are possible, but the functional setting is more complicated, typically employing weighted L^p -spaces that characterize the decay of the invariant density at infinity.

When $\mathcal{Y} = \mathbb{T}^d$ and $B(x, y)$ is strictly positive definite, Theorem 7.9 also applies, and we have a solvability theory for Poisson equations of the form

$$-\mathcal{L}_0 \phi = h. \quad (11.2.6)$$

In particular, the equation has a solution if and only if the right-hand side of the preceding equation is centered with respect to the invariant measure of the fast process $\mu_x(dy)$:

$$\int_{\mathbb{T}^d} h(x, y) \mu_x(dy) = 0 \quad \forall x \in \mathcal{X}. \quad (11.2.7)$$

When (11.2.7) is satisfied, the solution of (11.2.6) is unique up to a constant in the null space of \mathcal{L}_0 . We can fix this constant by requiring that

$$\int_{\mathbb{T}^d} \phi(x, y) \mu_x(dy) = 0 \quad \forall x \in \mathcal{X}.$$

In more general situations, such as when $\mathcal{Y} = \mathbb{R}^d$ or $B(x, y)$ is degenerate, the question of existence and uniqueness of solutions to the Poisson Equation (11.2.6) becomes more complicated; however, analogous results are possible in function space settings that enforce appropriate decay properties at infinity. See the remarks and references to the literature in Section 11.8.

11.3 Simplified Equations

We assume that the operator \mathcal{L}_0 satisfies the Fredholm alternative, Theorem 2.42, and has one-dimensional null space characterized by (11.2.4). We define the **cell problem**² as follows:

$$-\mathcal{L}_0 \Phi(x, y) = f_0(x, y), \quad \int_{\mathcal{Y}} \Phi(x, y) \rho^\infty(y; x) dy = 0. \quad (11.3.1)$$

² The word "cell" here refers to the periodic unit cell, which sets the scale for the fast variable in the case $\mathcal{Y} = \mathbb{T}^d$. The terminology comes from the theory of periodic homogenization for PDEs.

This is viewed as a PDE in y , with x a parameter. By the Fredholm alternative, (11.3.1) has a unique solution, since f_0 satisfies (11.2.5). We may then define a vector field F by

$$\begin{aligned} F(x) &= \int_{\mathcal{Y}} (f_1(x, y) + (\nabla_x \Phi(x, y)) f_0(x, y)) \rho^\infty(y; x) dy \\ &= F_1(x) + F_0(x) \end{aligned} \quad (11.3.2)$$

and a diffusion matrix $A(x)$ by

$$A(x)A(x)^T = A_1(x) + \frac{1}{2} \left(A_0(x) + A_0(x)^T \right), \quad (11.3.3)$$

where

$$A_0(x) := 2 \int_{\mathcal{Y}} f_0(x, y) \otimes \Phi(x, y) \rho^\infty(y; x) dy, \quad (11.3.4)$$

$$A_1(x) := \int_{\mathcal{Y}} A(x, y) \rho^\infty(y; x) dy. \quad (11.3.5)$$

To make sure that $A(x)$ is well defined, it is necessary to prove that the sum of $A_1(x)$ and the symmetric part of $A_0(x)$ are positive semidefinite. This is done in Section 11.5.

Result 11.1. *For $\varepsilon \ll 1$ and times t up to $\mathcal{O}(1)$, the process $x(t)$, the solution of (11.2.1), is approximated by the process $X(t)$, the solution of*

$$\frac{dX}{dt} = F(X) + A(X) \frac{dW}{dt}, \quad X(0) = x_0. \quad (11.3.6)$$

Remark 11.2. Notice that knowledge of AA^T is not sufficient to determine A uniquely. As a result, Equation (11.3.3) does not determine the limiting SDE (11.3.6) uniquely. This is a consequence of the fact that there may be many SDEs that have the same generator. This in turn relates to the fact that the approximation of the solution to (11.2.1) by the solution to (11.3.6) is only valid in the sense of weak convergence of probability measures; see Chapter 18. \square

11.4 Derivation

We seek a multiscale expansion for the solution of (11.2.2) with the form

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots. \quad (11.4.1)$$

Here $v_j = v_j(x, y, t)$. Substituting this expansion into (11.2.2) and equating powers of ε gives a hierarchy of equations, the first three of which are

$$\mathcal{O}(1/\varepsilon^2) \quad -\mathcal{L}_0 v_0 = 0, \quad (11.4.2a)$$

$$\mathcal{O}(1/\varepsilon) \quad -\mathcal{L}_0 v_1 = \mathcal{L}_1 v_0, \quad (11.4.2b)$$

$$\mathcal{O}(1) \quad -\mathcal{L}_0 v_2 = -\frac{\partial v_0}{\partial t} + \mathcal{L}_1 v_1 + \mathcal{L}_2 v_0. \quad (11.4.2c)$$

By (11.2.4) Equation (11.4.2a) implies that the first term in the expansion is independent of y , $v_0 = v_0(x, t)$. We proceed now with Equation (11.4.2b). The solvability condition is satisfied for this equation since, by assumption (11.2.5), $f_0(x, y)$ is centered with respect to the invariant measure for $\varphi_x^t(\cdot)$ and, from (11.2.3b),

$$\mathcal{L}_1 v_0 = f_0(x, y) \cdot \nabla_x v_0(x, t).$$

Equation (11.4.2b) becomes

$$-\mathcal{L}_0 v_1 = f_0(x, y) \cdot \nabla_x v_0(x, t). \quad (11.4.3)$$

Since \mathcal{L}_0 is a differential operator in y alone with x appearing as a parameter, the general solution of (11.4.3) has the form

$$v_1(x, y, t) = \Phi(x, y) \cdot \nabla_x v_0(x, t) + \Phi_1(x, t). \quad (11.4.4)$$

The function Φ_1 plays no role in what follows so we set it to zero. Thus we represent the solution v_1 as a linear operator acting on v_0 . As our aim is to find a closed equation for v_0 , this form for v_1 is a useful representation of the solution. Substituting for v_1 in (11.4.3) shows that Φ solves the cell problem (11.3.1). Condition (11.2.5) ensures that there is a solution to the cell problem and the normalization condition makes it unique. Turning now to Equation (11.4.2c) we see that the right-hand side takes the form

$$-\left(\frac{\partial v_0}{\partial t} - \mathcal{L}_2 v_0 - \mathcal{L}_1(\Phi \cdot \nabla_x v_0)\right).$$

Hence solvability of (11.4.2c) for each fixed x requires

$$\begin{aligned} \frac{\partial v_0}{\partial t} &= \int_{\mathcal{Y}} \rho^\infty(y; x) \mathcal{L}_2 v_0(x, t) dy + \int_{\mathcal{Y}} \rho^\infty(y; x) \mathcal{L}_1(\Phi(x, y) \cdot \nabla_x v_0(x, t)) dy \\ &= I_1 + I_2. \end{aligned} \quad (11.4.5)$$

We consider the two terms on the right-hand side separately. The first is

$$\begin{aligned} I_1 &= \int_{\mathcal{Y}} \rho^\infty(y; x) (f_1(x, y) \cdot \nabla_x + \frac{1}{2} A(x, y) : \nabla_x \nabla_x) v_0(x, t) dy \\ &= F_1(x) \cdot \nabla_x v_0(x, t) + \frac{1}{2} A_1(x) : \nabla_x \nabla_x v_0(x, t). \end{aligned}$$

Now for the second term I_2 , note that

$$\mathcal{L}_1(\Phi \cdot \nabla_x v_0) = f_0 \otimes \Phi : \nabla_x \nabla_x v_0 + (\nabla_x \Phi f_0) \cdot \nabla_x v_0.$$

Hence $I_2 = I_3 + I_4$ where

$$I_3 = \int_{\mathcal{Y}} \rho^\infty(y; x) (\nabla_x \Phi(x, y) f_0(x, y)) \cdot \nabla_x v_0(x, t) dy$$

and

$$I_4 = \int_{\mathcal{Y}} \rho^\infty(y; x) (f_0(x, y) \otimes \Phi(x, y) : \nabla_x \nabla_x v_0(x, t)) dy.$$

Thus

$$I_2 = F_0(x) \cdot \nabla_x v_0(x, t) + \frac{1}{2} A_0(x) : \nabla_x \nabla_x v_0(x, t).$$

Combining our simplifications of the right-hand side of (11.4.5) we obtain, since by (2.2.2) only the symmetric part of A_0 is required to calculate the Frobenius inner product with another symmetric matrix, the following expression:

$$\frac{\partial v_0}{\partial t} = F(x) \cdot \nabla_x v_0 + \frac{1}{2} A(x) A(x)^T : \nabla_x \nabla_x v_0.$$

This is the backward equation corresponding to the reduced dynamics given in (11.3.6).

11.5 Properties of the Simplified Equations

The effective SDE (11.3.6) is only well defined if $A(x)A(x)^T$ given by (11.3.3), (11.3.5) is nonnegative definite. We now prove that this is indeed the case.

Theorem 11.3. *Consider the case where $\mathcal{Y} = \mathbb{T}^d$ and \mathcal{L}_0 is equipped with periodic boundary conditions. Then*

$$\langle \xi, A_1(x)\xi + A_0(x)\xi \rangle \geq 0 \quad \forall x \in \mathcal{X}, \xi \in \mathbb{R}^l.$$

Hence the real-valued matrix function $A(x)$ is well defined by (11.3.3) since $A(x)A(x)^T$ is nonnegative definite.

Proof. Let $\phi(x, y) = \xi \cdot \Phi(x, y)$. Then ϕ solves

$$-\mathcal{L}_0 \phi = \xi \cdot f_0.$$

By Theorem 6.12 we have

$$\begin{aligned} & \langle \xi, A_1(x)\xi + A_0(x)\xi \rangle \\ &= \int_{\mathcal{Y}} \left(|\alpha(x, y)^T \xi|^2 - 2(\mathcal{L}_0 \phi(x, y)) \phi(x, y) \right) \rho^\infty(y; x) dy \\ &= \int_{\mathcal{Y}} \left(|\alpha(x, y)^T \xi|^2 + |\beta(x, y)^T \nabla_y \phi(x, y)|^2 \right) \rho^\infty(y; x) dy \\ &\geq 0. \end{aligned}$$

Thus

$$\begin{aligned}\langle \xi, AA^T \xi \rangle &= \langle \xi, A_1 \xi \rangle + \frac{1}{2} \langle \xi, (A_0 + A_0^T) \xi \rangle \\ &= \langle \xi, (A_1 + A_0) \xi \rangle \geq 0. \square\end{aligned}$$

Two important remarks are in order.

Remark 11.4. Techniques similar to those used in the proof of the previous theorem, using (6.3.11) instead of the Dirichlet form itself, show that

$$\frac{1}{2} (A_0(x) + A_0(x)^T) = \int_{\mathcal{Y}} \left(\nabla_y \Phi(x, y) \beta(x, y) \otimes \nabla_y \Phi(x, y) \beta(x, y) \right) \rho^\infty(y; x) dy. \quad (11.5.1)$$

Remark 11.5. By virtue of Remark 6.13 we see that the proceeding theorem can be extended to settings other than $\mathcal{Y} = \mathbb{T}^d$. \square

11.6 Deterministic Problems

As in the previous chapter, it is useful to have representations of the effective equation in terms of time averages, both for numerical purposes and for deterministic problems. To this end, a second representation of $A_0(x)$ and $F_0(x)$ is as follows. Let $\varphi_\xi^t(y)$ solve (10.5.1) and let \mathbb{E}^{μ_x} be the product measure formed from use of $\mu_x(\cdot)$ on initial data and standard independent Wiener measure on driving Brownian motions. Using this notation we may now employ a time integral to represent the solution of the cell problem, leading to the following representation formulae. Derivation is given at the end of the section.

Result 11.6. *Alternative representations of the vector field $F_0(x)$ and diffusion matrix $A_0(x)$ can be found through the following integrals over time and \mathbb{E}^{μ_x} :*

$$A_0(x) = 2 \int_0^\infty \mathbb{E}^{\mu_x} \left(f_0(x, y) \otimes f_0(x, \varphi_x^t(y)) \right) dt \quad (11.6.1)$$

and, if the generator \mathcal{L}_0 is independent of x , then

$$F_0(x) = \int_0^\infty \mathbb{E}^{\mu_x} \left(\nabla_x f_0(x, \varphi_x^t(y)) f_0(x, y) \right) dt. \quad (11.6.2)$$

All these representations hold for any y , by ergodicity.

The integral over t in this result enables us to express the effective equations without explicit reference to the solution of the cell problem Φ and requires sufficiently fast *decay of correlations* in order to be well-defined.

Another pair of alternative representations of $F(x)$ and $A(x)A(x)^T$ may be found by using time averaging (over s) to replace the expectations in the previous result. The expressions for A_0 and F_0 then involve two time integrals: the integral over

s is an ergodic average, replacing averaging with respect to the stationary measure on path space; the integral over t expresses the effective equations without reference to the solution of the cell problem Φ and, again, requires sufficiently fast *decay of correlations* in order to be well-defined. In fact the well posedness of the cell problem (11.3.1) implies the decay of correlations property.

Result 11.7. *Alternative representations of the vector field F and diffusion matrix A can be found through the following integrals over time:*

$$F_1(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_1(x, \varphi_x^s(y)) ds,$$

$$A_1(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T A(x, \varphi_x^s(y)) ds;$$

and

$$A_0(x) = 2 \int_0^\infty \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_0(x, \varphi_x^s(y)) \otimes f_0(x, \varphi_x^{t+s}(y)) ds \right) dt, \quad (11.6.3)$$

where $\varphi_x^t(y)$ solves (10.5.1). Furthermore, if the generator \mathcal{L}_0 is independent of x , then

$$F_0(x) = \int_0^\infty \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \nabla_x f_0(x, \varphi_x^{t+s}(y)) f_0(x, \varphi_x^s(y)) ds \right) dt.$$

All these representations hold for any y , by ergodicity.

The following result will be useful to us in deriving the alternate representations of $A_0(x)$ and $F_0(x)$ in the two preceding results. It uses ergodicity to represent the solution of the cell problem, and related Poisson equations, as time integrals.

Result 11.8. *Let \mathcal{L} be the generator of the ergodic Markov process $y(t)$ on \mathcal{Y} which satisfies the SDE*

$$\frac{dy}{dt} = g(y) + \beta(y) \frac{dV}{dt}, \quad y(t) = y \quad (11.6.4)$$

and let $\mu(dy)$ denote the unique invariant measure. Assume that h is centered with respect to μ :

$$\int_{\mathcal{Y}} h(y) \mu(dy) = 0.$$

Then the solution $f(y)$ of the Poisson equation

$$-\mathcal{L}f = h, \quad \int_{\mathcal{Y}} f(y) \mu(dy) = 0$$

admits the representation formula

$$f(y) = \int_0^\infty (e^{\mathcal{L}t} h)(y) dt. \quad (11.6.5)$$

Proof. We apply the Itô formula to $f(y(t))$ to obtain

$$\begin{aligned} f(y(t)) - f(y) &= \int_0^t \mathcal{L}f(y(s)) ds + \int_0^t \langle \nabla_y f(y(s)), \beta(y(s)) dW(s) \rangle \\ &= \int_0^t -h(y(s)) ds + \int_0^t \langle \nabla_y f(y(s)), \beta(y(s)) dW(s) \rangle. \end{aligned}$$

We take expectation with respect to the Wiener measure and use the martingale property of stochastic integrals and the fact that $\mathbb{E}h(y(s)|y(0) = y)$ solves the backward Kolmogorov equation to conclude that

$$f(y) = \mathbb{E}f(y(t)) + \int_0^t (e^{\mathcal{L}s}h)(y) ds.$$

We take the limit $t \rightarrow \infty$ and use the ergodicity of the process $y(t)$, together with the fact that $f(y)$ is centered with respect to the invariant measure with density $\rho^\infty(y; x)$, to deduce that

$$\begin{aligned} f(y) &= \lim_{t \rightarrow \infty} \mathbb{E}f(y(t)) + \int_0^\infty (e^{\mathcal{L}t}h)(y) dt \\ &= \int_{\mathcal{Y}} f(y) \mu(dy) + \int_0^\infty (e^{\mathcal{L}t}h)(y) dt \\ &= \int_0^\infty (e^{\mathcal{L}t}h)(y) dt \end{aligned}$$

and the proof is complete. \square

Remark 11.9. Notice that the preceding result implies that we can write, at least formally,

$$\mathcal{L}^{-1} = - \int_0^\infty e^{\mathcal{L}t} dt$$

when applied to functions centered with respect to μ . Furthermore, the result is also valid for the case where the coefficients in (11.6.4) depend on a parameter x . \square

We complete the section by deriving the alternative expressions for $A(x)$ and $F(x)$ through time integration, given in Results 11.7 and 11.6. The expressions for $F_1(x)$ and $A_1(x)$ in Result 11.7 are immediate from ergodicity, simply using the fact that the time average equals the average against ρ^∞ . By use of Result 11.8, the solution to the cell problem can be written as

$$\Phi(x, y) = \int_0^\infty (e^{\mathcal{L}_0 t} f_0)(x, y) dt = \int_0^\infty \mathbb{E} f_0(x, \varphi_x^t(y)) dt \quad (11.6.6)$$

where \mathbb{E} denotes expectation with respect to the Wiener measure. Now

$$F_0(x) = \int_{\mathcal{Y}} \rho^\infty(y; x) \nabla_x \Phi(x, y) f_0(x, y) dy.$$

In the case where \mathcal{L}_0 is x -independent so that $\varphi_x^t(\cdot) = \varphi^t(\cdot)$ is also x -independent, as are $\mu_x = \mu$ and $\rho^\infty(\cdot; x) = \rho^\infty(\cdot)$, we may use (11.6.6) to see that

$$F_0(x) = \int_{\mathcal{Y}} \rho^\infty(y; x) \int_0^\infty \mathbb{E} \nabla_x f_0(x, \varphi^t(y)) f_0(x, y) dt dy,$$

where \mathbb{E} is expectation with respect to Wiener measure. Recall that \mathbb{E}^{μ_x} denotes the product measure formed from distributing y in its invariant measure, together with the Brownian motion driving the equation for $\varphi^t(y)$. Changing the order of integration we find that

$$F_0(x) = \int_0^\infty \mathbb{E}^{\mu_x} \left(\nabla_x f_0(x, \varphi_x^t(y)) f_0(x, y) \right) dt \quad (11.6.7)$$

as required for the expression in Result 11.6. Now we replace averages over \mathbb{E}^{μ_x} by time averaging to obtain, for all y ,

$$F_0(x) = \int_0^\infty \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \nabla_x f_0(x, \varphi_x^{t+s}(y)) f_0(x, \varphi_x^s(y)) ds \right) dt,$$

and so we obtain the desired formula for Result 11.7.

A similar calculation to that yielding (11.6.7) gives (11.6.1) for $A_0(x)$ in Result 11.6. Replacing the average against \mathbb{E}^{μ_x} by time average we arrive at the desired formula for $A_0(x)$ in Result 11.7.

11.7 Applications

We give a number of examples illustrating the wide applicability of the ideas in this chapter.

11.7.1 Fast Ornstein-Uhlenbeck Noise

Consider the equations

$$\frac{dx}{dt} = \frac{1}{\varepsilon} (1 - y^2)x, \quad (11.7.1)$$

$$\frac{dy}{dt} = -\frac{\alpha}{\varepsilon^2} y + \sqrt{\frac{2\alpha}{\varepsilon^2}} \frac{dV}{dt}, \quad (11.7.2)$$

where $V(t)$ is a standard one-dimensional Brownian motion. Here

$$f_0(x, y) = (1 - y^2)x \text{ and } f_1(x, y) = 0.$$

Recall that the equation for y is a time-rescaling of the OU process from Example 6.19, with $\lambda = \alpha$. Furthermore, these equations arise from the first application in Section 10.6, in the case where $\lambda = \alpha$ and after time rescaling to produce nonzero effects.

We have that

$$\int_{-\infty}^{\infty} (1 - y^2)x\rho^{\infty}(y) dy = 0,$$

where $\rho^{\infty}(y)$ is the invariant density of the Ornstein–Uhlenbeck process, namely a standard unit normal distribution. Thus the theory put forward in this chapter applies.

The generator of the process $\varphi_{\xi}^t(\cdot) = \varphi^t(\cdot)$ is

$$\mathcal{L}_0 = -\alpha y \frac{\partial}{\partial y} + \alpha \frac{\partial^2}{\partial y^2} \quad (11.7.3)$$

and the cell problem (Poisson equation) (11.3.1) becomes

$$\alpha y \frac{\partial \Phi}{\partial y} - \alpha \frac{\partial^2 \Phi}{\partial y^2} = (1 - y^2)x.$$

The unique centered solution to this equation is

$$\Phi(y, x) = \frac{1}{2\alpha}(1 - y^2)x.$$

Under the standard normal distribution, the fourth and second moments take values 3 and 1, respectively. Hence, the coefficients in the limiting Equation (11.3.6) are

$$F(x) = \int_{-\infty}^{\infty} \left(-\frac{1}{2\alpha}y^2(1 - y^2)x \right) \rho^{\infty}(y) dy = \frac{1}{\alpha}x$$

and

$$A^2(x) = 2 \int_{-\infty}^{\infty} \left(-\frac{1}{2\alpha}y^2x(1 - y^2)x \right) \rho^{\infty}(y) dy = \frac{2}{\alpha}x^2.$$

The homogenized SDE is thus

$$\frac{dX}{dt} = \frac{X}{\alpha} + \sqrt{\frac{2}{\alpha}}X \frac{dW}{dt}. \quad (11.7.4)$$

This is the geometric Brownian motion studied in Example 6.4. The solution is

$$X(t) = X(0) \exp\left(\sqrt{\frac{2}{\alpha}}W(t)\right).$$

It converges neither to 0 nor to ∞ , but subsequences in time attain both limits. This should be compared with the behavior found in the first example in Section 10.6, which gives rise to decay (resp. growth) if $\lambda > \alpha$ (resp. $\lambda < \alpha$). Our example corresponds to the case $\lambda = \alpha$ with time rescaled to see nontrivial dynamics. It thus lies between decay and growth. Notice that we could have also taken the function in front of the white noise with a minus sign; see Remark 11.2.

Let us now obtain the coefficients of the homogenized equation by using the alternative representations (11.6.1) and (11.6.2). To this end we need to study the

variable $\varphi^t(y)$ solving (10.5.1). From the calculations presented in Example 6.19 we have that

$$\begin{aligned}\varphi^t(y) &= e^{-\alpha t}y + \sqrt{2\alpha} \int_0^t e^{-\alpha(t-s)} dV(s), \\ \varphi^t(y)^2 &= e^{-2\alpha t}y^2 + \sqrt{2\alpha}ye^{-\alpha t} \int_0^t e^{-\alpha(t-s)} dV(s) + 2\alpha \left(\int_0^t e^{-\alpha(t-s)} dV(s) \right)^2.\end{aligned}\tag{11.7.5}$$

In addition, by the Itô isometry,

$$\begin{aligned}\mathbb{E} \left(\int_0^t e^{-\alpha(t-s)} dV(s) \right)^2 &= \int_0^t e^{-2\alpha(t-s)} ds, \\ &= \frac{1}{2\alpha} (1 - e^{-2\alpha t}).\end{aligned}$$

To construct the measure \mathbb{E}^{μ^x} we take the initial condition y to be a standard unit Gaussian distribution and an independent driving Brownian motion V . (The measure is, in fact, independent of x in this particular example.) Thus, by stationarity under this initial Gaussian distribution,

$$\int \rho^\infty(y) y^2 dy = 1, \quad \mathbb{E}^{\mu^x} \varphi^t(y)^2 = 1.$$

Furthermore

$$\begin{aligned}\mathbb{E}^{\mu^x} \left(\int \rho^\infty(y) y^2 \varphi^t(y)^2 dy \right) &= e^{-2\alpha t} \int \rho^\infty(y) y^4 dy \\ &\quad + 2\alpha \mathbb{E}^{\mu^x} \left(\int_0^t e^{-\alpha(t-s)} dV(s) \right)^2 \\ &= 3e^{-2\alpha t} + 1 - e^{-2\alpha t} \\ &= 1 + 2e^{-2\alpha t}.\end{aligned}$$

Since $f_0(x, y) = (1 - y^2)x$, combining these calculations in (11.6.2) gives

$$\begin{aligned}F_0(x) &= x \int_0^\infty \mathbb{E}^{\mu^x} \left((1 - \varphi^t(y)^2)(1 - y^2) \right) dt \\ &= x \int_0^\infty 2e^{-2\alpha t} dt \\ &= \frac{x}{\alpha}.\end{aligned}\tag{11.7.6}$$

Similarly from (11.6.1) we obtain

$$A_0(x) = \frac{2x^2}{\alpha}.$$

This confirms that the effective equation is (11.7.4).

11.7.2 Fast Chaotic Noise

We now consider an example that is entirely deterministic but behaves stochastically when we eliminate a fast chaotic variable. In this context it is essential to use the representation of the effective diffusion coefficient given in Result 11.7. This representation uses time integrals, and makes no reference to averaging over the invariant measure (which does not have a density with respect to Lebesgue measure in this example; see Example 4.16). Consider the equations

$$\begin{aligned}\frac{dx}{dt} &= x - x^3 + \frac{\lambda}{\varepsilon} y_2, \\ \frac{dy_1}{dt} &= \frac{10}{\varepsilon^2} (y_2 - y_1), \\ \frac{dy_2}{dt} &= \frac{1}{\varepsilon^2} (28y_1 - y_2 - y_1 y_3), \\ \frac{dy_3}{dt} &= \frac{1}{\varepsilon^2} (y_1 y_2 - \frac{8}{3} y_3).\end{aligned}\tag{11.7.7}$$

The vector $y = (y_1, y_2, y_3)^T$ solves the Lorenz equations, at parameter values where the solution is ergodic (see Example 4.16). In the invariant measure the component y_2 has mean zero. Thus the centering condition holds. The equation for x is a scalar ODE driven by a chaotic signal with characteristic time ε^2 . Because $f_0(x, y) \propto y_2$, with invariant measure shown in Figure 4.2, and because $f_1 = (x, y) = f_1(x)$ only, the candidate equation for the approximate dynamics is

$$\frac{dX}{dt} = X - X^3 + \sigma \frac{dW}{dt},\tag{11.7.9}$$

where σ is a constant. Now let $\psi^t(y) = e_2 \cdot \varphi^t(y)$. Then the constant σ can be found by use of (11.6.3) giving

$$\sigma^2 = 2\lambda^2 \int_0^\infty \frac{1}{T} \left(\lim_{T \rightarrow \infty} \int_0^T \psi^s(y) \psi^{t+s}(y) ds \right) dt.$$

This is the integrated autocorrelation function of y_2 . By ergodicity we expect the value of σ^2 to be independent of y and to be determined by the SRB measure for the Lorenz equations alone. Notice that the formula is expected to make sense, even though the cell problem is not well-posed in this case because the generator of the fast process is not elliptic.

Another way to derive this result is as follows. Gaussian white noise $\sigma \dot{W}$, the time derivative of Brownian motion, may be thought of as a delta-correlated stationary process. The integral of its autocorrelation function on $[0, \infty)$ gives $\sigma^2/2$. On the assumption that y_2 has a correlation function that decays in time, and noting that this has time scale ε^2 , the autocorrelation of $\frac{\lambda}{\varepsilon} \psi^{s/\varepsilon^2}(y)$ at timelag t may be calculated and integrated from 0 to ∞ ; matching this with the known result for Gaussian white noise gives the desired result for σ^2 .

11.7.3 Stratonovich Corrections

When white noise is approximated by a smooth process this often leads to Stratonovich interpretations of stochastic integrals, at least in one dimension. We use multiscale analysis to illustrate this phenomenon by means of a simple example. Consider the equations

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\varepsilon} f(x)y, \\ \frac{dy}{dt} &= -\frac{\alpha y}{\varepsilon^2} + \sqrt{\frac{2\alpha}{\varepsilon^2}} \frac{dV}{dt},\end{aligned}\tag{11.7.10}$$

with V being a standard one-dimensional Brownian motion.

Assume for simplicity that $y(0) = 0$. Then

$$\mathbb{E}(y(t)y(s)) = e^{-\frac{\alpha}{\varepsilon^2}|t-s|}$$

and, consequently,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left(\frac{y(t)}{\varepsilon} \frac{y(s)}{\varepsilon} \right) = \frac{2}{\alpha} \delta(t-s),$$

which implies the heuristic

$$\lim_{\varepsilon \rightarrow 0} \frac{y(t)}{\varepsilon} = \sqrt{\frac{2}{\alpha}} \frac{dV}{dt}.\tag{11.7.11}$$

Another way of seeing this is by solving (11.7.10) for y/ε :

$$\frac{y}{\varepsilon} = \sqrt{\frac{2}{\alpha}} \frac{dV}{dt} - \frac{\varepsilon}{\alpha} \frac{dy}{dt}.\tag{11.7.12}$$

If we neglect the $\mathcal{O}(\varepsilon)$ term on the right-hand side, then we arrive, again, at the heuristic (11.7.11).

Both of these arguments lead us to conjecture a limiting equation of the form

$$\frac{dX}{dt} = \sqrt{\frac{2}{\alpha}} f(X) \frac{dV}{dt}.\tag{11.7.13}$$

We will show that, as applied, *the heuristic gives the incorrect limit*: this is because, in one dimension, whenever white noise is approximated by a smooth process, the limiting equation should be interpreted in the Stratonovich sense, giving

$$\frac{dX}{dt} = \sqrt{\frac{2}{\alpha}} f(X) \circ \frac{dV}{dt}\tag{11.7.14}$$

in this case. We now derive this limit equation by the techniques introduced in this chapter.

The cell problem is

$$-\mathcal{L}_0 \Phi(x, y) = f(x)y$$

with \mathcal{L}_0 given by (11.7.3). The solution is readily seen to be

$$\Phi(x, y) = \frac{1}{\alpha} f(x)y, \quad \nabla_x \Phi(x, y) = \frac{1}{\alpha} f'(x)y.$$

The invariant density is

$$\rho^\infty(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right),$$

which is in the null space of \mathcal{L}_0^* and corresponds to a standard unit Gaussian $\mathcal{N}(0, 1)$ random variable.

From Equation (11.3.2) we have

$$\begin{aligned} F(x) &= \int_{\mathbb{R}} \frac{1}{\alpha} f'(x)f(x)y^2 \rho^\infty(y) dy \\ &= \frac{1}{\alpha} f'(x)f(x). \end{aligned}$$

Also (11.3.3) gives

$$\begin{aligned} A(x)^2 &= \int_{\mathbb{R}} \frac{2}{\alpha} f(x)^2 y^2 \rho^\infty(y) dy \\ &= \frac{2}{\alpha} f(x)^2. \end{aligned}$$

The limiting equation is therefore the Itô SDE

$$\frac{dX}{dt} = \frac{1}{\alpha} f'(X)f(X) + \sqrt{\frac{2}{\alpha}} f(X) \frac{dV}{dt}.$$

This is the Itô form of (11.7.14), by Remark 6.2. Hence, the desired result is established.

11.7.4 Stokes' Law

The previous example may be viewed as describing the motion of a massless particle with position x in a velocity field proportional to $f(x)y$, with y an OU process. If the particle has mass m then it is natural to study the generalized equation

$$m \frac{d^2 x}{dt^2} = \frac{1}{\varepsilon} f(x)y - \frac{dx}{dt}, \quad (11.7.15a)$$

$$\frac{dy}{dt} = -\frac{\alpha y}{\varepsilon^2} + \sqrt{\frac{2\alpha}{\varepsilon^2}} \frac{dV}{dt}. \quad (11.7.15b)$$

(Note that setting $m = 0$ gives the previous example.) Equation (11.7.15a) is *Stokes' law*, stating that the force on the particle is proportional to a drag force,

$$\frac{1}{\varepsilon} f(x)y - \frac{dx}{dt},$$

which is equal to the difference between the fluid velocity and the particle velocity. As in the previous example, y is a fluctuating OU process. For simplicity we consider the case of unit mass, $m = 1$.

Using the heuristic argument from the previous section it is natural to conjecture the limiting equation

$$\frac{d^2 X}{dt^2} = \sqrt{\frac{2}{\alpha}} f(X) \frac{dV}{dt} - \frac{dX}{dt}. \quad (11.7.16)$$

In contrast to the previous application, the conjecture that this is the limiting equation turns out to be correct. The reason is that, here, x is smoother and the Itô and Stratonovich integrals coincide; there is no Itô correction to the Stratonovich integral. (To see this it is necessary to first write (11.7.16) as a first-order system; see Exercise 2a). We verify the result by using the multiscale techniques introduced in this chapter.

We first write (11.7.15) as the first-order system

$$\begin{aligned} \frac{dx}{dt} &= r, \\ \frac{dr}{dt} &= -r + \frac{1}{\varepsilon} f(x)y, \\ \frac{dy}{dt} &= -\frac{1}{\varepsilon^2} \alpha y + \frac{1}{\varepsilon} \sqrt{2\alpha} \frac{dV}{dt}. \end{aligned}$$

Here (x, r) are slow variables (x in (11.2.1)) and y the fast variables (y in (11.2.1)). The cell problem is now given by

$$\mathcal{L}_0 \Phi(x, r, y) = -f_0(x, r, y) = \begin{pmatrix} 0 \\ -f(x)y \end{pmatrix},$$

with \mathcal{L}_0 given by (11.7.3). The solution is

$$\Phi(x, r, y) = \begin{pmatrix} 0 \\ \frac{1}{\alpha} f(x)y \end{pmatrix}, \quad \nabla_{(x,r)} \Phi(x, y) = \begin{pmatrix} 0 & 0 \\ \frac{1}{\alpha} f'(x)y & 0 \end{pmatrix}.$$

Notice that f_0 is in the null space of $\nabla_{(x,r)} \Phi$, and hence (11.3.2) gives

$$F(X, R) = F_1(X, R) = \begin{pmatrix} R \\ -R \end{pmatrix}. \quad (11.7.17)$$

From (11.3.3) we have

$$A(X, R)A(X, R)^T = \int_{\mathbb{R}} 2 \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\alpha} f(X)^2 y^2 \end{pmatrix} \rho^\infty(y) dy.$$

Recall that $\rho^\infty(y)$ is the density of an $\mathcal{N}(0, 1)$ Gaussian random variable. Evaluating the integral gives

$$A(X, R)A(X, R)^T = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{\alpha} f(X)^2 \end{pmatrix}.$$

Hence a natural choice for $A(x)$ is

$$A(X, R) = \begin{pmatrix} 0 \\ \sqrt{\frac{2}{\alpha}} f(X) \end{pmatrix}.$$

Thus from (11.7.17) and (11.7.18) we obtain the limiting equation

$$\begin{aligned} \frac{dX}{dt} &= R, \\ \frac{dR}{dt} &= -R + \sqrt{\frac{2}{\alpha}} f(X) \frac{dW}{dt}, \end{aligned}$$

which, upon elimination of R , is seen to coincide with the conjectured limit (11.7.16).

11.7.5 Green–Kubo Formula

In the previous application we encountered the equation of motion for a particle with significant mass, subject to Stokes drag. Here we study the same equation of motion, but where the velocity field is steady. We also assume that the particle is subject to molecular diffusion. The equation of motion is thus

$$\frac{d^2 x}{dt^2} = f(x) - \frac{dx}{dt} + \sigma \frac{dU}{dt}. \quad (11.7.18)$$

Here U is a standard unit Brownian motion. We will study the effective diffusive behavior of the particle x on large length and time scales, under the assumption that $f(x)$ is a mean zero periodic function. We show that, on appropriate large length and time scales, the particle performs an effective Brownian motion, and we calculate its diffusion coefficient.

To this end we rescale the equation of motion by setting $x \rightarrow x/\varepsilon$ and $t \rightarrow t/\varepsilon^2$ to obtain

$$\varepsilon^2 \frac{d^2 x}{dt^2} = \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right) - \frac{dx}{dt} + \sigma \frac{dU}{dt}.$$

Introducing the variables $y = \varepsilon dx/dt$ and $z = x/\varepsilon$ we obtain the system

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{\varepsilon} y, \\ \frac{dy}{dt} &= -\frac{1}{\varepsilon^2} y + \frac{1}{\varepsilon^2} f(z) + \frac{\sigma}{\varepsilon} \frac{dW}{dt}, \\ \frac{dz}{dt} &= \frac{1}{\varepsilon^2} y. \end{aligned}$$

The process (y, z) is ergodic, with characteristic time scale ε^2 , and plays the role of y in (11.2.1); x plays the role of x in (11.2.1). The operator \mathcal{L}_0 is the generator of the process (y, z) . Furthermore

$$f_1(x, y, z) = 0, \quad f_0(x, y, z) = y.$$

Thus, since the evolution of (y, z) is independent of x , $\Phi(x, y, z)$, the solution of the cell problem, is also x -independent. Hence (11.3.2) gives $F(x) = 0$. Turning now to the effective diffusivity we find that, since $\alpha(x, y) = A(x, y) = 0$, (11.3.3) gives $A(x)^2 = A_0(x)$. Now define $\psi^t(y, z)$ to be the component of $\varphi^t(y, z)$ projected onto the y coordinate. By Result 11.7 we have that

$$A_0(x) = 2 \int_0^\infty \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \psi^s(y) \psi^{s+t}(y) ds \right) dt.$$

The expression

$$C(t) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \psi^s(y) \psi^{s+t}(y) ds$$

is the *velocity autocorrelation function*. Thus the effective equation is

$$\frac{dX}{dt} = \sqrt{2D} \frac{dW}{dt},$$

a Brownian motion with diffusion coefficient

$$D = \int_0^\infty C(t) dt.$$

Thus, the effective diffusion coefficient is given by the *integrated velocity autocorrelation*. This is an example of the **Green–Kubo formula**.

11.7.6 Neither Itô nor Stratonovich

We again use Stokes' law (11.7.15a), now for a particle of small mass $m = \tau_0 \varepsilon^2$ where $\tau_0 = \mathcal{O}(1)$, and neglecting molecular diffusion. If we also assume that the velocity field of the underlying fluid is of the form $\frac{1}{\varepsilon} f(x) \eta$ where η solves an SDE, then we obtain

$$\tau_0 \varepsilon^2 \frac{d^2 x}{dt^2} = -\frac{dx}{dt} + \frac{1}{\varepsilon} f(x) \eta, \quad (11.7.19a)$$

$$\frac{d\eta}{dt} = \frac{1}{\varepsilon^2} g(\eta) + \frac{1}{\varepsilon} \sqrt{2\sigma(\eta)} \frac{dW}{dt}. \quad (11.7.19b)$$

We interpret equations (11.7.19b) in the Itô sense. We assume that $g(\eta)$, $\sigma(\eta)$ are such that there exists a unique stationary solution of the Fokker-Planck equation for (11.7.19b), so that η is ergodic.

We write (11.7.19) as a first-order system,

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{\varepsilon \sqrt{\tau_0}} v, \\ \frac{dv}{dt} &= \frac{f(x) \eta}{\varepsilon^2 \sqrt{\tau_0}} - \frac{v}{\tau_0 \varepsilon^2}, \\ \frac{d\eta}{dt} &= \frac{g(\eta)}{\varepsilon^2} + \frac{\sqrt{2\sigma(\eta)}}{\varepsilon} \frac{dW}{dt}. \end{aligned} \quad (11.7.20)$$

Equations (11.7.20) are of the form (11.2.1) and, under the assumption that the fast process (v, η) is ergodic, the theory developed in this chapter applies. In order to calculate the effective coefficients we need to solve the stationary Fokker–Planck equation

$$\mathcal{L}_0^* \rho(x, v, \eta) = 0$$

and the cell problem

$$-\mathcal{L}_0 h = \frac{v}{\sqrt{\tau_0}}, \quad (11.7.21)$$

where

$$\mathcal{L}_0 = g(\eta) \frac{\partial}{\partial \eta} + \sigma(\eta) \frac{\partial^2}{\partial \eta^2} + \left(\frac{f(x)\eta}{\sqrt{\tau_0}} - \frac{v}{\tau_0} \right) \frac{\partial}{\partial v}.$$

Equation (11.7.21) can be simplified considerably: we look for a solution of the form

$$h(x, v, \eta) = \left(\sqrt{\tau_0} v + f(x) \hat{h}(\eta) \right). \quad (11.7.22)$$

Substituting this expression in the cell problem we obtain, after some algebra, the equation

$$-\mathcal{L}_\eta \hat{h} = \eta.$$

Here \mathcal{L}_η denotes the generator of η . We assume that the unique invariant measure for $\eta(t)$ has density $\rho_\eta(\eta)$ with respect to Lebesgue measure; the centering condition that ensures the well-posedness of the Poisson equation for \hat{h} is

$$\int_{\mathbb{R}} \eta \rho_\eta(\eta) d\eta = 0.$$

We assume that this holds. The homogenized SDE is

$$\frac{dX}{dt} = F(X) + \sqrt{D(X)} \frac{dW}{dt}, \quad (11.7.23)$$

where

$$F(x) := \int_{\mathbb{R}^2} \left(\frac{v}{\sqrt{\tau_0}} \hat{h}(\eta) f'(x) \right) \rho(x, v, \eta) dv d\eta$$

and

$$D(x) := 2 \int_{\mathbb{R}^2} \left(v^2 + \frac{v}{\sqrt{\tau_0}} \hat{h}(\eta) f(x) \right) \rho(x, v, \eta) dv d\eta.$$

In the case where $\eta(t)$ is the Ornstein–Uhlenbeck process,

$$\frac{d\eta}{dt} = -\frac{\alpha}{\varepsilon^2} \eta + \sqrt{\frac{2\lambda}{\varepsilon^2}} \frac{dW}{dt}, \quad (11.7.24)$$

we can compute the homogenized coefficients $D(X)$ and $B(X)$ explicitly. The effective SDE is

$$\frac{dX}{dt} = \frac{\lambda}{\alpha^2(1 + \tau_0\alpha)} f(X) f'(X) + \sqrt{\frac{2\lambda}{\alpha^2}} f(X) \frac{dW}{dt}. \quad (11.7.25)$$

Note that in the limit $\tau_0 \rightarrow \infty$ we recover the Itô stochastic integral, as in Subsection 11.7.4, whereas in the limit $\tau_0 \rightarrow 0$ we recover the Itô interpretation of the Stratonovich stochastic integral as in Subsection 11.7.3. For $\tau_0 \in (0, \infty)$ the limiting equation is of neither the Itô nor the Stratonovich form. In fact, Equation (11.7.25) can be written in the form

$$X(t) = x_0 + \int_0^t \frac{2\lambda}{\alpha^2} f(X) \widehat{\circ} dW(t),$$

where the definition of the stochastic integral through Riemann sums depends on the value of τ_0 . The fact that we recover this interesting limit is very much tied to the scaling of the mass as $\mathcal{O}(\varepsilon^2)$. This scaling ensures that the time scale of the ergodic process η and the relaxation time of the particle are the same. Resonance between these time scales gives the desired effect.

11.7.7 The Lévy Area Correction³

In Section 11.7.3 we saw that smooth approximation to white noise in one dimension leads to the Stratonovich stochastic integral. This is not true in general, however, in the multidimensional case: an additional drift can appear in the limit. This extra drift contribution is related to the properties of the Lévy area of the limit process (see the discussion in Section 11.8).

Consider the fast-slow system

$$\dot{x}_1 = \frac{1}{\varepsilon} y_1, \quad (11.7.26a)$$

$$\dot{x}_2 = \frac{1}{\varepsilon} y_2, \quad (11.7.26b)$$

$$\dot{x}_3 = \frac{1}{\varepsilon} (x_1 y_2 - x_2 y_1), \quad (11.7.26c)$$

$$\dot{y}_1 = -\frac{1}{\varepsilon^2} y_1 - \alpha \frac{1}{\varepsilon^2} y_2 + \frac{1}{\varepsilon} \dot{W}_1, \quad (11.7.26d)$$

$$\dot{y}_2 = -\frac{1}{\varepsilon^2} y_2 + \alpha \frac{1}{\varepsilon^2} y_1 + \frac{1}{\varepsilon} \dot{W}_2, \quad (11.7.26e)$$

where $\alpha > 0$. Here W_1, W_2 are standard independent Brownian motions.

Notice that Equations (11.7.26d) and (11.7.26e) may be written in the form

$$\dot{y} = -\frac{1}{\varepsilon^2} y + \frac{1}{\varepsilon^2} \alpha J y + \frac{1}{\varepsilon} \dot{W},$$

where $y = (y_1, y_2)$, $W = (W_1, W_2)$, and J is the antisymmetric (symplectic) matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

³ This section was written in collaboration with M. Hairer.

Applying the heuristic that

$$y \approx \varepsilon(I - \alpha J)^{-1} \frac{dW}{dt}$$

leads to the conjectured limiting equations

$$\dot{x}_1 = \frac{1}{1 + \alpha^2} \left(\dot{W}_1 - \alpha \dot{W}_2 \right), \quad (11.7.27a)$$

$$\dot{x}_2 = \frac{1}{1 + \alpha^2} \left(\dot{W}_2 + \alpha \dot{W}_1 \right), \quad (11.7.27b)$$

$$\dot{x}_3 = \frac{1}{1 + \alpha^2} \left((\alpha x_1 - x_2) \dot{W}_1 + (\alpha x_2 + x_1) \dot{W}_2 \right). \quad (11.7.27c)$$

We know from Subsections 11.7.3 and 11.7.6 that we must take care in conjecturing such a limit as typically smooth approximations of white noise give rise to the Stratonovich stochastic integral. However, in this case Itô and Stratonovich coincide so this issue does not arise. Nonetheless, the conjectured limit equation is wrong.

Multiscale techniques, as described in this chapter, lead to the correct homogenized system:

$$\dot{x}_1 = \frac{1}{1 + \alpha^2} \left(\dot{W}_1 - \alpha \dot{W}_2 \right), \quad (11.7.28a)$$

$$\dot{x}_2 = \frac{1}{1 + \alpha^2} \left(\dot{W}_2 + \alpha \dot{W}_1 \right), \quad (11.7.28b)$$

$$\dot{x}_3 = \frac{1}{1 + \alpha^2} \left((\alpha x_1 - x_2) \dot{W}_1 + (\alpha x_2 + x_1) \dot{W}_2 \right) + \frac{\alpha}{1 + \alpha^2}. \quad (11.7.28c)$$

Notice the additional constant drift that appears in Equation (11.7.28c). It is the antisymmetric part in the equation for the fast process y that is responsible for the presence of the additional drift in the homogenized equation. In particular, when $\alpha = 0$ the homogenized equation becomes

$$\dot{x}_1 = \dot{W}_1,$$

$$\dot{x}_2 = \dot{W}_2,$$

$$\dot{x}_3 = -x_2 \dot{W}_1 + x_1 \dot{W}_2,$$

which agrees with the original (in general incorrect) conjectured limit (11.7.27).

11.8 Discussion and Bibliography

The perturbation approach adopted in this chapter, and more general related ones, is covered in a series of papers by Papanicolaou and co-workers – see [244, 241, 242, 240], building on original work of Khasminkii [165, 166]. See [155, 154, 31, 205, 244, 242, 240, 155, 154] for further material. We adapted the general analysis to the

simple case where $\mathcal{Y} = \mathbb{T}^d$. This may be extended to, for example \mathbb{R}^d , by working in the appropriate functional setting; see [249, 250, 251].

The basic perturbation expansion outlined in this chapter can be rigorously justified and weak convergence of x to X proved as $\varepsilon \rightarrow 0$; see Kurtz [181] and Chapter 18. The perturbation expansion that underlies the approach is clearly exposed in [241]; see also [117; ch. 6; 321; 291]. Similar problems are analyzed in [27, ch. 8], by using eigenfunction expansions for the Fokker–Planck operator of the fast process. Projection operator techniques are also often employed in the physics literature as a method for eliminating fast variables. See [117, ch. 6] and the references therein.

Studying the derivation of effective stochastic models when the original system is an ODE is a subject investigated in some generality in [242]. The specific example in Section 11.7.2 relies on the ergodicity of the Lorenz equations, something established in [318, 319]. Use of the integrated autocorrelation function to calculate the effective diffusion coefficient numerically is highlighted in [322]; a different approach to finding the effective diffusion coefficient is described in [125]. The program described is carried out in discrete time by Beck [31], who uses a skew-product structure to facilitate an analysis; the ideas can then be rigorously justified in some cases. A skew-product setup is also employed in [322] and [125]. A rigorous limit theorem for ODEs driven by a fast mixing system is proved in [225], using the large deviation principle for dynamical systems developed in [224]. In the paper [208], the idea that fast chaotic motion can introduce noise in slow variables is pursued for an interesting physically motivated problem where the fast chaotic behavior arises from the Burgers bath of [204]. Further numerical experiments on the Burgers bath are reported in [209].

Related work can be found in [124], and similar ideas in continuous time are addressed in [155, 154] for differential equations; however, rather than developing a systematic expansion in powers of ε , they find the exact solution of the Fokker–Planck equation, projected into the space \mathcal{X} , by use of the Mori–Zwanzig formalism [65], and then make power series expansions in ε of the resulting problem.

In Section 11.7.5 we derived a formula for the effective diffusion coefficient in terms of the integral of the velocity autocorrelation function, giving the Green–Kubo formula. This calculates a *transport coefficient* via the time integral of an autocorrelation function. The Green–Kubo formula, and other transport coefficients, are studied in many books on statistical mechanics; see, for example, [28, ch. 11, 269].

Applications of multiscale analysis to climate models, where the atmosphere evolves quickly relative to the slow oceanic variations, are surveyed in Majda et al. [205, 202]. Further applications to the atmospheric sciences may be found in [206, 207]; see also [78]. Stokes’ law, Equation (11.7.15a) is a phenomenological model for the motion of inertial particles in fluids; see [217]. Models of the form (11.7.15), where the velocity field of the fluid in which the particles are immersed is taken to be a Gaussian Markovian random field, were developed in [288, 289] and analyzed further in [254]. Similar Gaussian models for passive tracers were studied in [55, 56].

The fact that smooth approximations to white noise in one dimension lead, in the limit as we remove the regularization, to Stratonovich stochastic integrals (see Section 11.7.3) is often called the Wong–Zakai theorem after [332]. Whether one should interpret the stochastic integral in the sense of Itô or Stratonovich is usually called the Itô-versus-Stratonovich problem. In cases where more than one fast time scale is present, as in the example considered in Section 11.7.6, the correct interpretation of the stochastic integral in the limiting SDE depends on the order with which we take the limits; see [109, 280]. As was shown in Section 11.7.6, there are instances where the stochastic integral in the limiting SDE can be interpreted in neither the Itô nor the Stratonovich sense; see [129, 180, 255]. A similar phenomenon for the case where the fast process is a discrete deterministic chaotic map was observed in [124]. An interesting setup to consider in this context is Stokes’ law (11.7.15) in the case where the mass is small:

$$\begin{aligned}\varepsilon^a \frac{d^2 x}{dt^2} &= \frac{1}{\varepsilon} f(x)y - \frac{dx}{dt} + \sigma \frac{dU}{dt}, \\ \frac{dy}{dt} &= -\frac{\alpha y}{\varepsilon^2} + \sqrt{\frac{2\alpha}{\varepsilon^2}} \frac{dV}{dt}.\end{aligned}$$

Setting $\varepsilon = 0$ in the first equation, and invoking a white noise approximation for y/ε leads to the conjecture that the limit X of x satisfies a first-order SDE. The question then becomes the interpretation of the stochastic integral. In [180] multiscale expansions are used to derive the limiting equation satisfied by x in the cases $a = 1, 2$, and 3. The case $a = 1$ leads to the Itô equation in the limit, the case $a = 3$ to the Stratonovich equation, and $a = 2$ to an intermediate limit between the two.

In higher dimensions smooth approximations to white noise result (in general, and depending of the type of regularization) in an additional drift – apart from the Stratonovich stochastic integral – which is related to the commutator between the row vectors of the diffusion matrix; see [151]. A rigorous framework for understanding examples such as that presented in Section 11.7.7, based on the theory of rough paths, can be found in [198].

In this chapter we have considered equations of the form (11.2.1), where U and V are independent Brownian motions. Frequently applications arise where the noise in the two processes are correlated. We will cover such situations in Chapter 13, where we study homogenization for parabolic PDEs. The structure of the linear equations considered will be general enough to subsume the form of the backward Kolmogorov equation, which arises from (11.2.1) when U and V are correlated – in fact they are identical. The main change over the derivation in this chapter is that the operator \mathcal{L}_1 has additional terms arising from the correlation in the noises; see Exercises 5 and 1.

When writing the backward Kolmogorov equation for the full system, Equation (11.2.2), we assumed that the initial conditions depended only on the slow variable x . This assumption simplifies the analysis but is not necessary. If the initial condition is a function of both x and y , then an initial (or boundary) layer appears that has to be resolved. This can be achieved by adding appropriate terms in the two-scale

expansion that decay exponentially fast in time. This is done in [336] for continuous-time Markov chains and in [167] for SDEs. In this case the initial conditions for the homogenized SDE are obtained by averaging the initial conditions of the original SDE with respect to the invariant measure of the fast process.

In this chapter we have studied homogenization for finite-dimensional stochastic systems. Similar results can be proved for infinite-dimensional stochastic systems, SPDEs. See [40] for an application of the techniques developed in this chapter to the stochastic Burgers equation.

The use of the representations in Result 11.1 is discussed in [241]. The representations in Results 11.7 and 11.6 for the effective drift and diffusion can be used in the design of coarse time-stepping algorithms; see [322]. In general, the presence of two widely separated characteristic time scales in the SDEs (11.2.1) renders their numerical solution a formidable task. New numerical methods have been developed that aim at the efficient numerical solution of such problems. In the context of averaging for Hamiltonian systems the subject is described in [116]; the subject is revisited, in a more general setting, in [93]. Many of these methods exploit the fact that for ε sufficiently small the solution of (11.2.1a) can be approximated by the solution of the homogenized Equation (11.3.6). The homogenized coefficients are computed through formulae of the form (11.6.3) or (11.6.1), integrating Equation (11.2.1b) over short time intervals; see [322, 81, 84, 123]. An ambitious program to numerically compute a subset of variables from a (possibly stochastic) dynamical system is outlined in [162]; this approach does not use scale separation explicitly and finds application in a range of different problems; see [163, 164, 149, 30, 190, 278, 334]. Numerical methods for multiscale problems are overviewed in [83]. For work on parameter estimation for multiscale SDEs see [258]. For other (partly computational) work on dimension reduction in stochastic systems see [59, 148, 273].

11.9 Exercises

1. Find the homogenized equation for the SDEs

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\varepsilon} f_0(x, y) + f_1(x, y) + \alpha_0(x, y) \frac{dU}{dt} + \alpha_1(x, y) \frac{dV}{dt}, & x(0) &= x_0, \\ \frac{dy}{dt} &= \frac{1}{\varepsilon^2} g(x, y) + \frac{1}{\varepsilon} g_1(x, y) + \frac{1}{\varepsilon} \beta(x, y) \frac{dV}{dt}, & y(0) &= y_0,\end{aligned}$$

assuming that f_0 satisfies the centering condition and that U and V are independent Brownian motions.

2. a. Let \mathcal{Y} denote either \mathbb{T}^d or \mathbb{R}^d . What is the generator \mathcal{L} for the process $y \in \mathcal{Y}$ given by

$$\frac{dy}{dt} = g(y) + \frac{dV}{dt}?$$

In the case where $g(y) = -\nabla \Psi(y)$ find a function in the null space of \mathcal{L}^* .

- b. Find the homogenized SDE arising from the system

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\varepsilon} f(x, y), \\ \frac{dy}{dt} &= \frac{1}{\varepsilon^2} g(y) + \frac{1}{\varepsilon} \frac{dV}{dt},\end{aligned}$$

in the case where $g = -\nabla\Psi(y)$.

- c. Define the cell problem, giving appropriate conditions to make the solution unique in the case $\mathcal{Y} = \mathbb{T}^d$. State clearly any assumptions on f that are required in the preceding derivation.
3. Use the Itô formula to derive the solution to the SDE (11.7.4). Convert this SDE into Stratonovich form. What do you observe?
4. a. Let \mathcal{Y} be either \mathbb{T}^d or \mathbb{R}^d . Write down the generator \mathcal{L}_0 for the process $y \in \mathcal{Y}$ given by:

$$\frac{dy}{dt} = g(y) + \frac{dV}{dt}.$$

In the case where g is divergence-free, find a function in the null space of \mathcal{L}_0^* .

- b. Find the averaged SDE arising from the system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y), \\ \frac{dy}{dt} &= \frac{1}{\varepsilon} g(y) + \frac{1}{\sqrt{\varepsilon}} \frac{dV}{dt},\end{aligned}$$

in the case where g is divergence-free.

- c. Find the homogenized SDE arising from the system

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\varepsilon} f(x, y), \\ \frac{dy}{dt} &= \frac{1}{\varepsilon^2} g(y) + \frac{1}{\varepsilon} \frac{dV}{dt},\end{aligned}$$

in the case where g is divergence-free.

- d. Define the cell problem, giving appropriate conditions to make the solution unique in the case $\mathcal{Y} = \mathbb{T}^d$. Clearly state any assumptions on f that are required in the preceding derivation.
5. Consider the equation of motion

$$\frac{dx}{dt} = f(x) + \sigma \frac{dW}{dt},$$

where $f(x)$ is divergence-free and periodic with mean zero. It is of interest to understand how x behaves on large length and time scales. To this end, rescale the equation of motion by setting $x \rightarrow x/\varepsilon$ and $t \rightarrow t/\varepsilon^2$ and introduce $y = x/\varepsilon$. Write down a pair of coupled SDEs for x and y . Use the methods developed in Exercise 1 to enable elimination of y to obtain an effective equation for x .

6. Carry out the analysis presented in Section 11.7.6 in arbitrary dimensions. Does the limiting equation have the same structure as in the one-dimensional case?
7. Derive Equation (11.7.25) from (11.7.23) when $\eta(t)$ is given by (11.7.24).
8. (The Kramers to Smoluchowski limit) Consider the Langevin equation

$$\varepsilon^2 \frac{d^2 x}{dt^2} = b(x) - \frac{dx}{dt} + \sqrt{2\sigma} \frac{dW}{dt}, \quad (11.9.1)$$

where the particle mass is assumed to be small, $m = \varepsilon^2$.

- a. Write (11.9.1) as a first-order system by introducing the variable $y = \varepsilon \dot{x}$.
- b. Use multiscale analysis to show that, when $\varepsilon \ll 1$ the solution of (11.9.1) is well approximated by the solution of the Smoluchowski equation

$$\frac{dX}{dt} = b(X) + \sqrt{2\sigma} \frac{dW}{dt}.$$

- c. Calculate the first correction to the Smoluchowski equation.
9. Write Equations (11.7.16) as a first-order system and show that the Itô and Stratonovich forms of the equation coincide.



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