Chapter 2
Random Processes

2.1 Introduction

We saw in Section 1.11 on page 10 that many systems are best studied using the concept of random variables where the outcome of a random experiment was associated with some numerical value. Next, we saw in Section 1.27 on page 30 that many more systems are best studied using the the concept of multiple random variables where the outcome of a random experiment was associated with multiple numerical values. Here we study *random processes* where the outcome of a random experiment is associated with a *function of time* [1]. Random processes are also called *stochastic processes*. For example, we might study the output of a digital filter being fed by some random signal. In that case, the filter output is described by observing the output waveform at random times.

Thus a *random process* assigns a random *function of time* as the outcome of a random experiment. Figure 2.1 graphically shows the sequence of events leading to assigning a function of time to the outcome of a random experiment. First we run the experiment, then we observe the resulting outcome. Each outcome is associated with a time function $x(t)$.

A random process $X(t)$ is described by

- the *sample space* S which includes all possible outcomes s of a random experiment
- the *sample function* $x(t)$ which is the time function associated with an outcome s. The values of the sample function could be discrete or continuous
- the *ensemble* which is the set of all possible time functions produced by the random experiment
- the time parameter t which could be continuous or discrete
- the statistical dependencies among the random processes $X(t)$ when t is changed.

Based on the above descriptions, we could have four different types of random processes:

1. Discrete time, discrete value: We measure time at discrete values $t = nT$ with $n = 0, 1, 2, \ldots$. As an example, at each value of n we could observe the number of cars on the road $x(n)$. In that case, $x(n)$ is an integer between 0 and 10, say.

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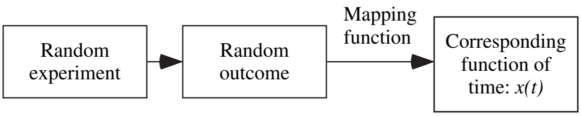

Fig. 2.1 The sequence of events leading to assigning a time function $x(t)$ to the outcome of a random experiment

Each time we perform this experiment, we would get a totally different sequence for $x(n)$.

2. Discrete time, continuous value: We measure time at discrete values $t = nT$ with $n = 0, 1, 2, \ldots$. As an example, at each value of n we measure the outside temperature $x(n)$. In that case, $x(n)$ is a real number between -30° and $+45^\circ$, say. Each time we perform this experiment, we would get a totally different sequence for $x(n)$.
3. Continuous time, discrete value: We measure time as a continuous variable t. As an example, at each value of t we store an 8-bit digitized version of a recorded voice waveform $x(t)$. In that case, $x(t)$ is a binary number between 0 and 255, say. Each time we perform this experiment, we would get a totally different sequence for $x(t)$.
4. Continuous time, continuous value: We measure time as a continuous variable t. As an example, at each value of t we record a voice waveform $x(t)$. In that case, $x(t)$ is a real number between 0 V and 5 V, say. Each time we perform this experiment, we would get a totally different sequence for $x(t)$.

Figure 2.2 shows a discrete time, discrete value random process for an observation of 10 samples where only three random functions are generated. We find that for $n = 2$, the values of the functions correspond to the random variable $X(2)$.

Therefore, random processes give rise to random variables when the time value t or n is fixed. This is equivalent to sampling all the random functions at the specified time value, which is equivalent to taking a vertical slice from all the functions shown in Fig. 2.2.

Example 1 A time function is generated by throwing a die in three consecutive throws and observing the number on the top face after each throw. Classify this random process and estimate how many sample functions are possible.

This is a discrete time, discrete value process. Each sample function will be have three samples and each sample value will be from the set of integers 1 to 6. For example, one sample function might be 4, 2, 5. Using the multiplication principle for probability, the total number of possible outputs is $6^3 = 216$. ■

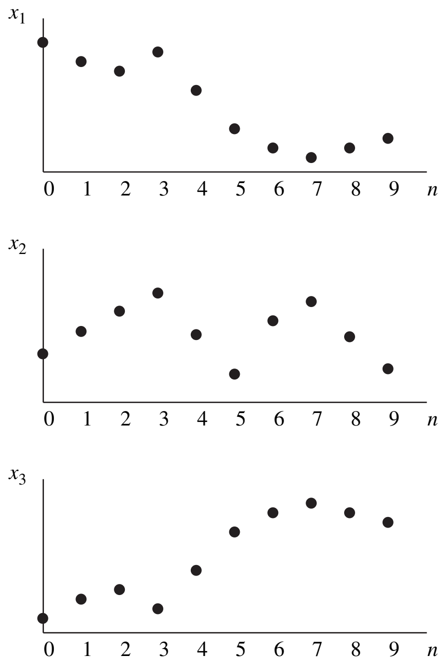

Fig. 2.2 An example of a discrete time, discrete value random process for an observation of 10 samples where only three random functions are possible.

2.2 Notation

We use the notation $X(t)$ to denote a continuous-time random process and also to denote the random variable measured at time t. When $X(t)$ is continuous, it will have a pdf $f_X(x)$ such that the probability that $x \le X \le x + \varepsilon$ is given by

$$p(X = x) = f_X(x)\, dx \tag{2.1}$$

When $X(t)$ is discrete, it will have a pmf $p_X(x)$ such that the probability that $X = x$ is given by

$$p(X = x) = p_X(x) \tag{2.2}$$

Likewise, we use the notation $X(n)$ to denote a discrete-time random process and also to denote the random variable measured at time n. That random variable is statistically described by a pdf $f_X(x)$ when it is continuous, or it is described by a pmf $p_X(x)$ when it is discrete.

2.3 Poisson Process

We shall encounter Poisson processes when we describe communication traffic. A Poisson process is a stochastic process in which the number of events occurring in a given period of time depends only on the length of the time period [2]. This number of events k is represented as a random variable K that has a Poisson distribution given by

$$p(k) = \frac{(\lambda\, t)^k e^{-\lambda\, t}}{k!} \tag{2.3}$$

where $\lambda > 0$ is a constant representing the rate of arrival of the events and t is the length of observation time.

2.4 Exponential Process

The exponential process is related to the Poisson process. The exponential process is used to model the interarrival time between occurrence of random events. Examples that lead to an interarrival time include the time between bus arrivals at a bus stop, the time between failures of a certain component, and the time between packet arrival at the input of a router.

The random variable T could be used to describe the interarrival time. The probability that the interarrival time lies in the range $t \le T \le t + dt$ is given by

$$\lambda e^{-\lambda\, t}\, dt \tag{2.4}$$

where λ is the average rate of the event under consideration.

2.5 Deterministic and Nondeterministic Processes

A *deterministic* process is one where future values of the sample function are known if the present value is known. An example of a deterministic process is the modulation technique known as quadrature amplitude modulation (QAM) for transmitting groups of binary data. The transmitted analog waveform is given by

$$v(t) = a \cos(\omega t + \phi) \tag{2.5}$$

where the signal amplitude a and phase angle ϕ change their value depending on the bit pattern that has been received. The analog signal is transmitted for the time period $0 \le t < T_0$. Since the arriving bit pattern is random, the values of the corresponding two parameters a and ϕ are random. However, once a and ϕ are determined, we would be able to predict the shape of the resulting waveform.

A *nondeterministic* random process is one where future values of the sample function cannot be known if the present value is known. An example of a nondeterministic random process is counting the number of packets that arrive at the input of a switch every one second and this observation is repeated for a certain time. We would not be able to predict the pattern even if we know the present number of arriving packets.

2.6 Ensemble Average

The random variable $X(n_1)$ represents all the possible values x obtained when time is frozen at the value n_1. In a sense, we are sampling the ensemble of random functions at this time value.

The expected value of $X(n_1)$ is called the *ensemble average* or statistical average $\mu(n_1)$ of the random process at n_1. The ensemble average is expressed as

$$\mu_X(t) = E[X(t)] \qquad \text{continuous-time process} \tag{2.6}$$

$$\mu_X(n) = E[X(n)] \qquad \text{discrete-time process} \tag{2.7}$$

The ensemble average could itself be another random variable since its value could change at random with our choice of the time value t or n.

Example 2 The modulation scheme known as frequency-shift keying (FSK) can be modeled as a random process described by

$$X(t) = a \cos \omega t$$

where a is a constant and ω corresponds to the random variable Ω that can have one of two possible values ω_1 and ω_2 that correspond to the input bit being 0 or 1, respectively. Assuming that the two frequencies are equally likely, find the expected value $\mu(t)$ of this process.

Our random variable Ω is discrete with probability 0.5 when $\Omega = \omega_1$ or $\Omega = \omega_2$. The expected value for $X(t)$ is given by

$$\begin{aligned} E[X(t)] &= 0.5\, a\ \cos \omega_1 t + 0.5\, a\ \cos \omega_2 t \\ &= a\ \cos\left[\frac{(\omega_1 + \omega_2)\, t}{2}\right] \times \cos\left[\frac{(\omega_1 - \omega_2)\, t}{2}\right] \end{aligned}$$

■

Example 3 The modulation scheme known as pulse amplitude modulation (PAM) can be modeled as a random process described by

$$X(n) = \sum_{i=0}^{\infty} g(n)\, \delta(n-i)$$

where $g(n)$ is the amplitude of the input signal at time n. $g(n)$ corresponds to the random variable G that is uniformly distributed in the range 0–A. Find the expected value $\mu(t)$ of this process.

This is a discrete time, continuous value random process. Our random variable G is continuous and the expected value for $X(n)$ is given by

$$\begin{aligned} E\left[X(n)\right] &= \frac{1}{A} \int_0^A g\, dg \\ &= \frac{A}{2} \end{aligned}$$

■

2.7 Time Average

Figure 2.2 helps us find the *time average* of the random process. The time average is obtained by finding the average value for *one* sample function such as $X_1(n)$ in the figure. The time average is expressed as

$$\overline{X} = \tfrac{1}{T} \int_0^T X(t)\, dt \qquad \text{continuous-time process} \tag{2.8}$$

$$\overline{X} = \tfrac{1}{N} \sum_0^{N-1} X(n)] \qquad \text{discrete-time process} \tag{2.9}$$

In either case we assumed we sampled the function for a period T or we observed N samples. The time average $\overline{X}$ could itself be a random variable since its value could change with our choice of the random function under consideration.

2.8 Autocorrelation Function

Assume a discrete-time random process $X(n)$ which produces two random variables $X_1 = X(n_1)$ and $X_2 = X(n_2)$ at times n_1 and n_2 respectively. The *autocorrelation function* for these two random variables is defined by the following equation:

$$r_{XX}(n_1, n_2) = E\left[X_1\ X_2\right] \tag{2.10}$$

In other words, we consider the two random variables X_1 and X_2 obtained from the same random process at the two different time instances n_1 and n_2.

Example 4 Find the autocorrelation function for a second-order finite-impulse response (FIR) digital filter, sometimes called moving average (MA) filter, whose output is given by the equation

$$y(n) = a_0 x(n) + a_1 x(n-1) + a_2 x(n-2) \tag{2.11}$$

where the input samples $x(n)$ are assumed to be zero mean independent and identically distributed (iid) random variables.

We assign the random variable Y_n to correspond to output sample $y(n)$ and X_n to correspond to input sample $x(n)$. Thus we can have the following autocorrelation function

$$r_{YY}(0) = E\left[Y_n Y_n\right] = a_0^2 E\left[X_0^2\right] + a_1^2 E\left[X_1^2\right] + a_2^2 E\left[X_2^2\right] E\left[X_0^2\right] \tag{2.12}$$

$$= \left(a_0^2 + a_1^2 + a_2^2\right)\sigma^2 \tag{2.13}$$

Similarly, we can write

$$r_{YY}(1) = E\left(Y_n Y_{n+1}\right) = 2a_0 a_1\, \sigma^2 \tag{2.14}$$

$$r_{YY}\,(2) = E\left(Y_n Y_{n+2}\right) = a_0 a_2\, \sigma^2 \tag{2.15}$$

$$r_{YY}\,(k) = 0; \qquad k > 2 \tag{2.16}$$

where σ^2 is the input sample variance. Figure 2.3 shows a plot of the autocorrelation assuming all the filter coefficients are equal. ■

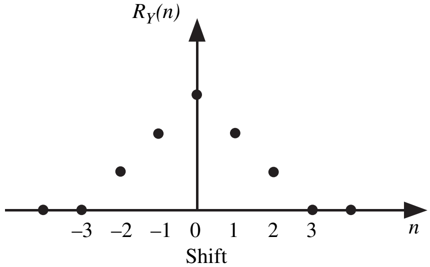

Fig. 2.3 Autocorrelation function of a second-order digital filter whose input is uncorrelated samples

2.9 Stationary Processes

A **wide-sense stationary** random process has the following two properties [3]

$$E[X(t)] = \mu = \text{ constant} \tag{2.17}$$

$$E[X(t)\, X(t+\tau)] = r_{XX}(t, t+\tau) = r_{XX}(\tau) \tag{2.18}$$

Such a process has a constant expected value and the autocorrelation function depends on the time difference between the two random variables.

The above equations apply to a continuous time random process. For a discrete-time random process, the equations for a wide-sense stationary random process become

$$E[X(n)] = \mu = \text{ constant} \tag{2.19}$$

$$E[X(n_1)\ X(n_1+n)] = r_{XX}(n_1, n_1+n) = r_{XX}(n) \tag{2.20}$$

The autocorrelation function for a wide-sense stationary random process exhibits the following properties [1].

$$r_{XX}(0) = E\left[X^2(n)\right] \geq 0 \tag{2.21}$$

$$|r_{XX}(n)| \leq r_{XX}(0) \tag{2.22}$$

$$r_{XX}(-n) = r_{XX}(n) \qquad \text{even symmetry} \tag{2.23}$$

A stationary random process is *ergodic* if all time averages equal their corresponding statistical averages [3]. Thus if $X(n)$ is an ergodic random process, then we could write

$$\overline{X} = \mu \tag{2.24}$$
$$\overline{X^2} = r_{XX}(0) \tag{2.25}$$

Example 5 The modulation scheme known as phase-shift keying (PSK) can be modeled as a random process described by

$$X(t) = a\cos(\omega t + \phi)$$

where a and ω are constant and ϕ corresponds to the random variable Φ with two values 0 and π which are equally likely. Find the autocorrelation function $r_{XX}(t)$ of this process.

The phase pmf is given by

$$p(0) = 0.5$$
$$p(\pi) = 0.5$$

The autocorrelation is found as

$$\begin{aligned} r_{XX}(\tau) &= E\,[a\cos(\omega t + \Phi)\, a\cos(\omega t + \omega\tau + \Phi)] \\ &= 0.5\, a^2\ \cos(\omega\tau)\ E\,[\cos(2\omega t + \omega\tau + 2\Phi)] \\ &= 0.5\, a^2\ \cos(\omega\tau)\ \cos(2\omega t + \omega\tau) \end{aligned}$$

We notice that this process is not wide-sense stationary since the autocorrelation function depends on t. ■

2.10 Cross-Correlation Function

Assume two discrete-time random processes $X(n)$ and $Y(n)$ which produce two random variables $X_1 = X(n_1)$ and $Y_2 = Y(n_2)$ at times n_1 and n_2, respectively. The *cross-correlation function* is defined by the following equation.

$$r_{XY}(n_1, n_2) = E\,[X_1\, Y_2] \tag{2.26}$$

If the cross-correlation function is zero, i.e. $r_{XY} = 0$, then we say that the two processes are *orthogonal*. If the two processes are *statistically independent*, then we have

$$r_{XY}(n_1, n_2) = E\,[X(n_1)] \times E\,[Y(n_2)] \tag{2.27}$$

Example 6 Find the cross-correlation function for the two random processes

$$X(t) = a\cos\omega t$$
$$Y(t) = b\sin\omega t$$

where a and b are two independent and identically distributed random variables with mean μ and variance σ^2.

The cross-correlation function is given by

$$\begin{aligned} r_{XY}(t, t+\tau) &= E\,[a\cos\omega t\ b\sin(\omega t + \omega\tau)] \\ &= 0.5[\sin\omega\tau + \sin(2\omega t + \omega\tau)]\ E[a]\ E[b] \\ &= 0.5\,\mu^2\ [\sin\omega\tau + \sin(2\omega t + \omega\tau)] \end{aligned}$$

■

2.11 Covariance Function

Assume a discrete-time random process $X(n)$ which produces two random variables $X_1 = X(n_1)$ and $X_2 = X(n_2)$ at times n_1 and n_2, respectively. The *autocovariance function* is defined by the following equation:

$$c_{XX}(n_1, n_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)] \tag{2.28}$$

The autocovariance function is related to the autocorrelation function by the following equation:

$$c_{XX}(n_1, n_2) = r_X(n_1, n_2) - \mu_1\mu_2 \tag{2.29}$$

For a wide-sense stationary process, the autocovariance function depends on the difference between the time indices $n = n_2 - n_1$.

$$c_{XX}(n) = E[(X_1 - \mu)(X_2 - \mu)] = r_{XX}(n) - \mu^2 \tag{2.30}$$

Example 7 Find the autocovariance function for the random process $X(t)$ given by

$$X(t) = a + b\cos\omega t$$

where ω is a constant and a and b are iid random variables with zero mean and variance σ^2.

We have

$$\begin{aligned} c_{XX} &= E\{(A + B\cos\omega t)[A + B\cos\omega(t+\tau)]\} \\ &= E\left[a^2\right] + E[ab]\ [\cos\omega t + \cos\omega(t+\tau)] + E\left[b^2\right]\cos^2\omega(t+\tau) \\ &= \sigma^2 + E[a]\ E[b]\ [\cos\omega t + \cos\omega(t+\tau)] + \sigma^2\cos^2\omega(t+\tau) \\ &= \sigma^2\left[1 + \cos^2\omega(t+\tau)\right] \end{aligned}$$

■

The *cross-covariance function* for two random processes $X(n)$ and $Y(n)$ is defined by

$$\begin{aligned} c_{XY}(n) &= E[(X(n_1) - \mu_X)(Y(n_1 + n) - \mu_Y)] \\ &= r_{XY}(n) - \mu_X\mu_Y \end{aligned} \tag{2.31}$$

Two random processes are called *uncorrelated* when their cross-covariance function vanishes.

$$c_{XY}(n) = 0 \tag{2.32}$$

Example 8 Find the cross-covariance function for the two random processes $X(t)$ and $Y(t)$ given by

$$X(t) = a + b\cos\omega t$$
$$Y(t) = a + b\sin\omega t$$

where ω is a constant and a and b are iid random variables with zero mean and variance σ^2.

We have

$$\begin{aligned} c_{XY}(n) &= E\left\{(A + B\cos\omega t)[A + B\sin\omega(t+\tau)]\right\} \\ &= E\left[A^2\right] + E[AB]\ [\cos\omega t + \sin\omega(t+\tau)] + E\left[B^2\right]\cos\omega t\ \sin\omega(t+\tau) \\ &= \sigma^2 + E[A]\ E[B]\ [\cos\omega t + \sin\omega(t+\tau)] + \sigma^2\cos\omega t\ \sin\omega(t+\tau) \\ &= \sigma^2\,[1 + \cos\omega t\ \sin\omega(t+\tau)] \end{aligned}$$

■

2.12 Correlation Matrix

Assume we have a discrete-time random process $X(n)$. At each time step i we define the random variable $X_i = X(i)$. If each sample function contains n components, it is convenient to construct a vector representing all these random variables in the form

$$\mathbf{x} = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}^t \tag{2.33}$$

Now we would like to study the correlation between each random variable X_i and all the other random variables. This would give us a comprehensive understanding of the random process. The best way to do that is to construct a *correlation matrix*.

We define the $n \times n$ correlation matrix $\mathbf{R}_X$, which gives the correlation between all possible pairs of the random variables as

$$\mathbf{R}_X = E\left[\mathbf{x}\,\mathbf{x}^t\right] = E\begin{bmatrix} X_1X_1 & X_1X_2 & \cdots & X_1X_n \\ X_2X_1 & X_2X_2 & \cdots & X_2X_n \\ \vdots & \vdots & \ddots & \vdots \\ X_nX_1 & X_nX_2 & \cdots & X_nX_n \end{bmatrix} \tag{2.34}$$

We can express $\mathbf{R}_X$ in terms of the individual correlation functions

$$\mathbf{R}_X = \begin{bmatrix} r_{XX}(1,1) & r_{XX}(1,2) & \cdots & r_{XX}(1,n) \\ r_{XX}(1,2) & r_{XX}(2,2) & \cdots & r_{XX}(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ r_{XX}(1,n) & r_{XX}(2,n) & \cdots & r_{XX}(n,n) \end{bmatrix} \tag{2.35}$$

Thus we see that the correlation matrix is symmetric. For a wide-sense stationary process, the correlation functions depend only on the difference in times and we get an even simpler matrix structure:

$$\mathbf{R}_X = \begin{bmatrix} r_{XX}(0) & r_{XX}(1) & \cdots & r_{XX}(n-1) \\ r_{XX}(1) & r_{XX}(0) & \cdots & r_{XX}(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ r_{XX}(n-1) & r_{XX}(n-2) & \cdots & r_{XX}(0) \end{bmatrix} \tag{2.36}$$

Each diagonal in this matrix has identical elements and our correlation matrix becomes a *Toeplitz matrix*.

Example 9 Assume the autocorrelation function for a stationary random process is given by

$$r_{XX}(\tau) = 5 + 3e^{-|\tau|}$$

Find the autocorrelation matrix for $\tau = 0$, 1, and 2.

The autocorrelation matrix is given by

$$\mathbf{R}_{XX} = \begin{bmatrix} 8 & 6.1036 & 5.4060 \\ 6.1036 & 8 & 6.1036 \\ 5.4060 & 6.1036 & 6 \end{bmatrix}$$

■

2.13 Covariance Matrix

In a similar fashion, we can define the *covariance matrix* for many random variables obtained from the same random process as

$$\mathbf{C}_{XX} = E\left[(\mathbf{x} - \overline{\mu})\,(\mathbf{x} - \overline{\mu})^t\right] \tag{2.37}$$

where $\overline{\mu} = \left[\,\mu_1\ \mu_2\ \cdots\ \mu_n\,\right]^t$ is the vector whose components are the expected values of our random variables. Expanding the above equation we can write

$$\mathbf{C}_{XX} = E\left[\mathbf{X}\mathbf{X}^{\mathbf{t}}\right] - \overline{\mu}\,\overline{\mu}^{t} \tag{2.38}$$

$$= \mathbf{R}_{X} - \overline{\mu}\,\overline{\mu}^{t} \tag{2.39}$$

When the process has zero mean, the covariance matrix equals the correlation matrix:

$$\mathbf{C}_{XX} = \mathbf{R}_{XX} \tag{2.40}$$

The covariance matrix can be written explicitly in the form

$$\mathbf{C}_{XX} = \begin{bmatrix} C_{XX}(1,1) & C_{XX}(1,2) & \cdots & C_{XX}(1,n) \\ C_{XX}(1,2) & C_{XX}(2,2) & \cdots & C_{XX}(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ C_{XX}(1,n) & C_{XX}(2,n) & \cdots & C_{XX}(n,n) \end{bmatrix} \tag{2.41}$$

Thus we see that the covariance matrix is symmetric. For a wide-sense stationary process, the covariance functions depend only on the difference in times and we get an even simpler matrix structure:

$$\mathbf{C}_{XX} = \begin{bmatrix} C_{XX}(0) & C_{XX}(1) & \cdots & C_{XX}(n-1) \\ C_{XX}(1) & C_{XX}(0) & \cdots & C_{XX}(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ C_{XX}(n-1) & C_{XX}(n-2) & \cdots & C_{XX}(0) \end{bmatrix} \tag{2.42}$$

Using the definition for covariance in (1.114) on page 35, we can write the above equation as

$$\mathbf{C}_{XX} = \sigma_X^2 \begin{bmatrix} 1 & \rho(1) & \rho(2) & \cdots & \rho(n-1) \\ \rho(1) & 1 & \rho(1) & \cdots & \rho(n-2) \\ \rho(2) & \rho(1) & 1 & \cdots & \rho(n-3) \\ \vdots & & & & \vdots \\ \rho(n-1) & \rho(n-2) & \rho(n-3) & \cdots & 1 \end{bmatrix} \tag{2.43}$$

Example 10 Assume the autocovariance function for a wide-sense stationary random process is given by

$$c_{XX}(\tau) = 5 + 3e^{-|\tau|}$$

Find the autocovariance matrix for $\tau = 0$, 1, and 2.

Since the process is wide-sense stationary, the variance is given by

$$\sigma^2 = c_{XX}(0) = 8$$

The autocovariance matrix is given by

$$\mathbf{C}_{XX} = 8 \begin{bmatrix} 1 & 0.7630 & 0.6758 \\ 0.7630 & 1 & 0.7630 \\ 0.6758 & 0.7630 & 1 \end{bmatrix}$$

■

Problems

2.1 Define deterministic and nondeterministic processes. Give an example for each type.

2.2 Let X be the random process corresponding to observing the noon temperature throughout the year. The number of sample functions are 365 corresponding to each day of the year. Classify this process.

2.3 Let X be the random process corresponding to reporting the number of defective lights reported in a building over a period of one month. Each month we would get a different pattern. Classify this process.

2.4 Let X be the random process corresponding to measuring the total tonnage (weight) of ships going through the Suez canal in one day. The data is plotted for a period of one year. Each year will produce a different pattern. Classify this process.

2.5 Let X be the random process corresponding to observing the number of cars crossing a busy intersection in one hour. The number of sample functions are 24 corresponding to each hour of the day. Classify this process.

2.6 Let X be the random process corresponding to observing the bit pattern in an Internet packet. Classify this process.

2.7 Amplitude-shift keying (ASK) can be modeled as a random process described by

$$X(t) = a \cos \omega t$$

where ω is constant and a corresponds to the random variable A with two values a_0 and a_1 which occur with equal probability. Find the expected value $\mu(t)$ of this process.

2.8 A modified ASK uses two bits of the incoming data to generate a sinusoidal waveform and the corresponding random process is described by

$$X(t) = a \cos \omega t$$

where ω is a constant and a is a random variable with four values a_0, a_1, a_2, and a_3. Assuming that the four possible bit patterns are equally likely find the expected value $\mu(t)$ of this process.

2.9 Phase-shift keying (PSK) can be modeled as a random process described by

$$X(t) = a\cos(\omega t + \phi)$$

where a and ω are constant and ϕ corresponds to the random variable Φ with two values 0 and π which occur with equal probability. Find the expected value $\mu(t)$ of this process.

2.10 A modified PSK uses two bits of the incoming data to generate a sinusoidal waveform and the corresponding random process is described by

$$X(t) = a\cos(\omega t + \phi)$$

where a and ω are constants and ϕ is a random variable Φ with four values $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$ [4]. Assuming that the four possible bit patterns occur with equal probability, find the expected value $\mu(t)$ of this process.

2.11 A modified frequency-shift keying (FSK) uses three bits of the incoming data to generate a sinusoidal waveform and the random process is described by

$$X(t) = a\cos\omega t$$

where a is a constant and ω corresponds to the random variable Ω with eight values $\omega_0, \omega_1, \ldots, \omega_7$. Assuming that the eight frequencies are equally likely, find the expected value $\mu(t)$ of this process.

2.12 A discrete-time random process $X(n)$ produces the random variable $X(n)$ given by

$$X(n) = a^n$$

where a is a uniformly distributed random variable in the range 0–1. Find the expected value for this random variable at any time instant n.

2.13 Define a wide-sense stationary random process.

2.14 Prove (2.23) on page 56.

2.15 Define an ergodic random process.

2.16 Explain which of the following functions represent a valid autocorrelation function.

$$r_{XX}(n) = a^n \quad 0 \le a < 1 \qquad\qquad r_{XX}(n) = |a|^n \quad 0 \le a < 1$$
$$r_{XX}(n) = a^{n^2} \quad 0 \le a < 1 \qquad\qquad r_{XX}(n) = |a|^{n^2} \quad 0 \le a < 1$$
$$r_{XX}(n) = \cos n \qquad\qquad r_{XX}(n) = \sin n$$

2.17 A random process described by

$$X(t) = a\cos(\omega t + \phi)$$

where a and ω are constant and ϕ corresponds to the random variable Φ which is uniformly distributed in the interval 0 to 2π. Find the autocorrelation function $r_{XX}(t)$ of this process.

2.18 Define what is meant by two random processes being orthogonal.

2.19 Define what is meant by two random processes being statistically independent.

2.20 Find the cross-correlation function for the following two random processes.

$$X(t) = a \cos \omega t$$
$$Y(t) = \alpha \; a \cos(\omega t + \theta)$$

where a and θ are two zero mean random variables and α is a constant.

2.21 Given two random processes X and Y, when are they uncorrelated?

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