2

Lie Sphere Geometry

In this chapter, we give Lie's construction of the space of spheres and define the important notions of oriented contact and parabolic pencils of spheres. This leads ultimately to a bijective correspondence between the manifold of contact elements on the sphere S^n and the manifold Λ^{2n-1} of projective lines on the Lie quadric.

2.1 Preliminaries

Before constructing the space of spheres, we begin with some preliminary remarks on indefinite scalar product spaces and projective geometry. Finite-dimensional indefinite scalar product spaces play a crucial role in Lie sphere geometry. The fundamental result from linear algebra concerns the rank and signature of a bilinear form (see, for example, Nomizu [133, p. 108], Artin [4, Chapter 3], or O'Neill [141, pp. 46–53]).

Theorem 2.1. *Suppose that* $(,)$ *is a bilinear form on a real vector space* V *of dimension* n*. Then there exists a basis* $\{e_1, \ldots, e_n\}$ *of* V *such that*

1. $(e_i, e_j) = 0$ *for* $i \neq j$,
2. $(e_i, e_i) = 1$ *for* $1 \leq i \leq p$,
3. $(e_j, e_j) = -1$ *for* $p + 1 \leq j \leq r$,
4. $(e_k, e_k) = 0$ *for* $r + 1 \leq k \leq n$.

The numbers r and p are determined solely by the bilinear form; r is called the *rank*, $r - p$ is called the *index*, and the ordered pair $(p, r - p)$ is called the *signature* of the bilinear form. The theorem shows that any two spaces of the same dimension with bilinear forms of the same signature are isometrically isomorphic. A *scalar product* is a nondegenerate bilinear form, i.e., a form with rank equal to the dimension of V. For the sake of brevity, we will often refer to a scalar product as a "metric." Usually, we will be dealing with the scalar product space $\mathbf{R}^n_k$ with signature $(n - k, k)$ for $k = 0, 1$ or 2. However, at times we will consider subspaces of $\mathbf{R}^n_k$ on which the bilinear form is degenerate. When dealing with low-dimensional spaces, we will often indicate the signature with a series of plus and minus signs and zeroes where appropriate. For example, the signature of $\mathbf{R}^3_1$ may be written $(++-)$ instead of $(2, 1)$. If the bilinear form is nondegenerate, a basis with the properties listed in Theorem 2.1 is called an *orthonormal basis* for V with respect to the bilinear form.

A second useful result concerning scalar products is the following. Here $U^\perp$ denotes the orthogonal complement of the space U with respect to the given scalar product. (See Artin [4, p. 117] or O'Neill [141, p. 49].)

Theorem 2.2. *Suppose that* $(\,,\,)$ *is a scalar product on a finite-dimensional real vector space* V *and that* U *is a subspace of* V.

(a) *Then* $U^{\perp\perp} = U$ *and* $\dim U + \dim U^\perp = \dim V$.
(b) *The form* $(\,,\,)$ *is nondegenerate on* U *if and only if it is nondegenerate on* $U^\perp$. *If the form is nondegenerate on* U, *then* V *is the direct sum of* U *and* $U^\perp$.
(c) *If* V *is the orthogonal direct sum of two spaces* U *and* W, *then the form is nondegenerate on* U *and* W, *and* $W = U^\perp$.

Let (x, y) be the indefinite scalar product on the Lorentz space $\mathbf{R}^{n+1}_1$ defined by

$$(x, y) = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}, \tag{2.1}$$

where $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$. We will call this scalar product the *Lorentz metric*. A vector x is said to be *spacelike*, *timelike* or *lightlike*, respectively, depending on whether (x, x) is positive, negative or zero. We will use this terminology even when we are using a metric of different signature. In Lorentz space, the set of all lightlike vectors, given by the equation

$$x_1^2 = x_2^2 + \cdots + x_{n+1}^2, \tag{2.2}$$

forms a cone of revolution, called the *light cone*. Lightlike vectors are often called *isotropic* in the literature, and the cone is called the *isotropy cone*. Timelike vectors are "inside the cone" and spacelike vectors are "outside the cone."

If x is a nonzero vector in $\mathbf{R}^{n+1}_1$, let $x^\perp$ denote the orthogonal complement of x with respect to the Lorentz metric. If x is timelike, then the metric restricts to a positive definite form on $x^\perp$, and $x^\perp$ intersects the light cone only at the origin. If x is spacelike, then the metric has signature $(n-1, 1)$ on $x^\perp$, and $x^\perp$ intersects the cone in a cone of one less dimension. If x is lightlike, then $x^\perp$ is tangent to the cone along the line through the origin determined by x. The metric has signature $(n-1, 0)$ on this n-dimensional plane.

The true setting of Lie sphere geometry is real projective space $\mathbf{P}^n$, so we now briefly review some important concepts from projective geometry. We define an equivalence relation on $\mathbf{R}^{n+1} - \{0\}$ by setting $x \simeq y$ if $x = ty$ for some nonzero real number t. We denote the equivalence class determined by a vector x by $[x]$. Projective space $\mathbf{P}^n$ is the set of such equivalence classes, and it can naturally be identified with the space of all lines through the origin in $\mathbf{R}^{n+1}$. The rectangular coordinates $(x_1, \ldots, x_{n+1})$ are called *homogeneous coordinates* of the point $[x]$, and they are only determined up to a nonzero scalar multiple. The affine space $\mathbf{R}^n$ can be embedded in $\mathbf{P}^n$ as the complement of the hyperplane $(x_1 = 0)$ at infinity by the map $\phi : \mathbf{R}^n \to \mathbf{P}^n$ given by $\phi(u) = [(1, u)]$. A scalar product on $\mathbf{R}^{n+1}$, such as the Lorentz metric, determines a polar relationship between points and hyperplanes in $\mathbf{P}^n$. We will also use the notation $x^\perp$ to denote the polar hyperplane of $[x]$ in $\mathbf{P}^n$, and we will call $[x]$ the *pole* of $x^\perp$.

If x is a lightlike vector in $\mathbf{R}^{n+1}_1$, then $[x]$ can be represented by a vector of the form $(1, u)$ for $u \in \mathbf{R}^n$. Then the equation $(x, x) = 0$ for the light cone becomes $u \cdot u = 1$ (Euclidean dot product), i.e., the equation for the unit sphere in $\mathbf{R}^n$. Hence, the set of points in $\mathbf{P}^n$ determined by lightlike vectors in $\mathbf{R}^{n+1}_1$ is naturally diffeomorphic to the sphere S^{n-1}.

2.2 Möbius Geometry of Unoriented Spheres

As a first step toward Lie sphere geometry, we recall the geometry of unoriented spheres in $\mathbf{R}^n$ known as "Möbius" or "conformal" geometry. We will always assume that $n \geq 2$. In this section, we will only consider spheres and planes of codimension one, and we will often omit the prefix "hyper."

We denote the Euclidean dot product of two vectors u and v in $\mathbf{R}^n$ by $u \cdot v$. We first consider stereographic projection $\sigma : \mathbf{R}^n \to S^n - \{P\}$, where S^n is the unit sphere in $\mathbf{R}^{n+1}$ given by $y \cdot y = 1$, and $P = (-1, 0, \ldots, 0)$ is the south pole of S^n. (See Figure 2.1.) The well-known formula for $\sigma(u)$ is

$$\sigma(u) = \left(\frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u}\right).$$

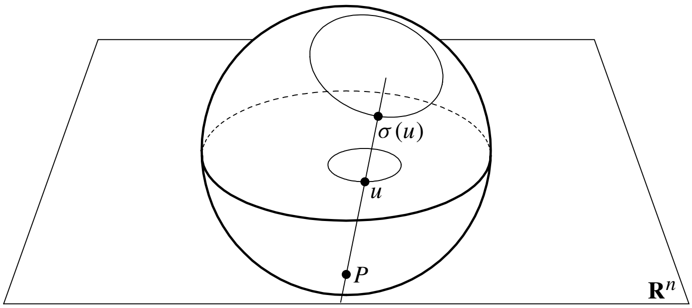

Fig. 2.1. Stereographic projection.

We next embed $\mathbf{R}^{n+1}$ into $\mathbf{P}^{n+1}$ by the embedding ϕ mentioned in the previous section. Thus, we have the map $\phi\sigma : \mathbf{R}^n \to \mathbf{P}^{n+1}$ given by

$$\phi\sigma(u) = \left[\left(1, \frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u}\right)\right] = \left[\left(\frac{1 + u \cdot u}{2}, \frac{1 - u \cdot u}{2}, u\right)\right]. \quad (2.3)$$

Let $(z_1, \ldots, z_{n+2})$ be homogeneous coordinates on $\mathbf{P}^{n+1}$ and $(\,,\,)$ the Lorentz metric on the space $\mathbf{R}^{n+2}_1$. Then $\phi\sigma(\mathbf{R}^n)$ is just the set of points in $\mathbf{P}^{n+1}$ lying on the n-sphere Σ given by the equation $(z, z) = 0$, with the exception of the *improper point* $[(1, -1, 0, \ldots, 0)]$ corresponding to the south pole P. We will refer to the points in Σ other than $[(1, -1, 0, \ldots, 0)]$ as *proper points*, and will call Σ the *Möbius sphere* or *Möbius space*. At times, it is easier to simply begin with S^n rather than $\mathbf{R}^n$ and thus avoid the need for the map σ and the special point P. However, there are also advantages for beginning in $\mathbf{R}^n$.

The basic framework for the Möbius geometry of unoriented spheres is as follows. Suppose that ξ is a spacelike vector in $\mathbf{R}^{n+2}_1$. Then the polar hyperplane $\xi^\perp$ to $[\xi]$ in $\mathbf{P}^{n+1}$ intersects the sphere Σ in an $(n-1)$-sphere S^{n-1} (see Figure 2.2). S^{n-1} is the image under $\phi\sigma$ of an $(n-1)$-sphere in $\mathbf{R}^n$, unless it contains the improper point $[(1, -1, 0, \ldots, 0)]$, in which case it is the image under $\phi\sigma$ of a hyperplane in $\mathbf{R}^n$. Hence, we have a bijective correspondence between the set of all spacelike points in $\mathbf{P}^{n+1}$ and the set of all hyperspheres and hyperplanes in $\mathbf{R}^n$.

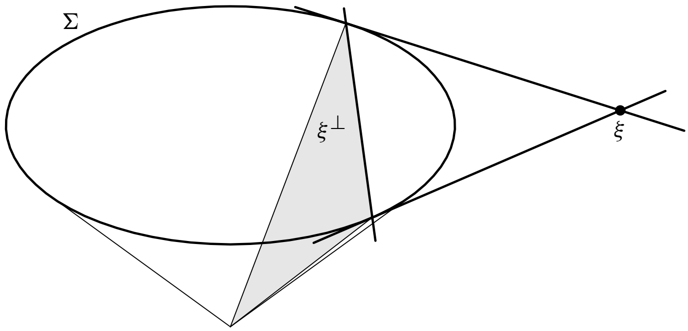

Fig. 2.2. Intersection of Σ with $\xi^\perp$.

It is often useful to have specific formulas for this correspondence. Consider the sphere in $\mathbf{R}^n$ with center p and radius $r > 0$ given by the equation

$$(u - p) \cdot (u - p) = r^2. \tag{2.4}$$

We wish to translate this into an equation involving the Lorentz metric and the corresponding polarity relationship on $\mathbf{P}^{n+1}$. A direct calculation shows that equation (2.4) is equivalent to the equation

$$(\xi, \phi\sigma(u)) = 0, \tag{2.5}$$

where ξ is the spacelike vector,

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p \right), \tag{2.6}$$

and $\phi\sigma(u)$ is given by equation (2.3). Thus, the point u is on the sphere given by equation (2.4) if and only if $\phi\sigma(u)$ lies on the polar hyperplane of $[\xi]$. Note that the first two coordinates of ξ satisfy $\xi_1 + \xi_2 = 1$, and that $(\xi, \xi) = r^2$. Although ξ is only determined up to a nonzero scalar multiple, we can conclude that $\eta_1 + \eta_2$ is not zero for any $\eta \simeq \xi$.

Conversely, given a spacelike point $[z]$ with $z_1 + z_2$ nonzero, we can determine the corresponding sphere in $\mathbf{R}^n$ as follows. Let $\xi = z/(z_1 + z_2)$ so that $\xi_1 + \xi_2 = 1$. Then from equation (2.6), the center of the corresponding sphere is the point $p = (\xi_3, \dots, \xi_{n+2})$, and the radius is the square root of (ξ, ξ).

Next suppose that η is a spacelike vector with $\eta_1 + \eta_2 = 0$. Then

$$(\eta, (1, -1, 0, \dots, 0)) = 0.$$

Thus, the improper point $\phi(P)$ lies on the polar hyperplane of $[\eta]$, and the point $[\eta]$ corresponds to a hyperplane in $\mathbf{R}^n$. Again we can find an explicit correspondence. Consider the hyperplane in $\mathbf{R}^n$ given by the equation

$$u \cdot N = h, \quad |N| = 1. \tag{2.7}$$

A direct calculation shows that (2.7) is equivalent to the equation

$$(\eta, \phi\sigma(u)) = 0, \quad \text{where } \eta = (h, -h, N). \tag{2.8}$$

Thus, the hyperplane (2.7) is represented in the polarity relationship by $[\eta]$. Conversely, let z be a spacelike point with $z_1 + z_2 = 0$. Then $(z, z) = v \cdot v$, where $v = (z_3, \dots, z_{n+2})$. Let $\eta = z/|v|$. Then η has the form (2.8) and $[z]$ corresponds to the hyperplane (2.7). Thus we have explicit formulas for the bijective correspondence between the set of spacelike points in $\mathbf{P}^{n+1}$ and the set of hyperspheres and hyperplanes in $\mathbf{R}^n$.

Of course, the fundamental invariant of Möbius geometry is the angle. The study of angles in this setting is quite natural, since orthogonality between spheres and planes in $\mathbf{R}^n$ can be expressed in terms of the Lorentz metric. Let S_1 and S_2 denote the spheres in $\mathbf{R}^n$ with respective centers p_1 and p_2 and respective radii r_1 and r_2. By the Pythagorean theorem, the two spheres intersect orthogonally (see Figure 2.3) if and only if

$$|p_1 - p_2|^2 = r_1^2 + r_2^2. \tag{2.9}$$

If these spheres correspond by equation (2.6) to the projective points $[\xi_1]$ and $[\xi_2]$, respectively, then a calculation shows that equation (2.9) is equivalent to the condition

$$(\xi_1, \xi_2) = 0. \tag{2.10}$$

A hyperplane π in $\mathbf{R}^n$ is orthogonal to a hypersphere S precisely when π passes through the center of S. If S has center p and radius r, and π is given by the equation $u \cdot N = h$, then the condition for orthogonality is just $p \cdot N = h$. If S corresponds to $[\xi]$ as in (2.6) and π corresponds to $[\eta]$ as in (2.8), then this equation for orthogonality is equivalent to $(\xi, \eta) = 0$. Finally, if two planes π_1 and π_2 are represented by $[\eta_1]$

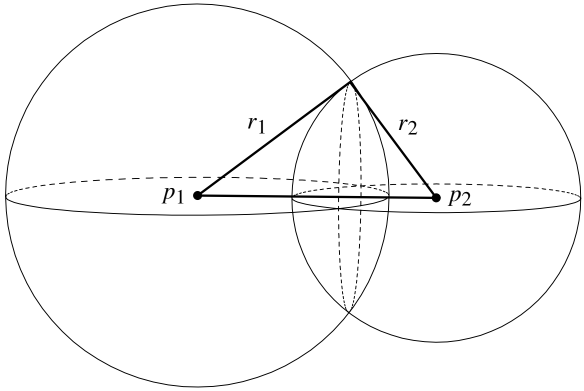

Fig. 2.3. Orthogonal spheres.

and $[\eta_2]$ as in (2.8), then the orthogonality condition $N_1 \cdot N_2 = 0$ is equivalent to the equation $(\eta_1, \eta_2) = 0$.

A *Möbius transformation* is a projective transformation of $\mathbf{P}^{n+1}$ which preserves the condition $(\eta, \eta) = 0$. By Theorem 3.1 of Chapter 3, p. 26, a Möbius transformation also preserves the relationship $(\eta, \xi) = 0$, and it maps spacelike points to spacelike points. Thus it preserves orthogonality (and hence angles) between spheres and planes in $\mathbf{R}^n$. In the next chapter, we will see that the group of Möbius transformations is isomorphic to $O(n+1, 1)/\{\pm I\}$, where $O(n+1, 1)$ is the group of orthogonal transformations of the Lorentz space $\mathbf{R}^{n+2}_1$.

Note that a Möbius transformation takes lightlike vectors to lightlike vectors, and so it induces a conformal diffeomorphism of the sphere Σ onto itself. It is well known that the group of conformal diffeomorphisms of the sphere is precisely the Möbius group.

2.3 Lie Geometry of Oriented Spheres

We now turn to the construction of Lie's geometry of oriented spheres and planes in $\mathbf{R}^n$. Let W^{n+1} be the set of vectors in $\mathbf{R}^{n+2}_1$ satisfying $(\zeta, \zeta) = 1$. This is a hyperboloid of revolution of one sheet in $\mathbf{R}^{n+2}_1$. If α is a spacelike point in $\mathbf{P}^{n+1}$, then there are precisely two vectors $\pm\zeta$ in W^{n+1} with $\alpha = [\zeta]$. These two vectors can be taken to correspond to the two orientations of the oriented sphere or plane represented by α, although we have not yet given a prescription as to how to make the correspondence. To do this, we need to introduce one more coordinate. First, embed $\mathbf{R}^{n+2}_1$ into $\mathbf{P}^{n+2}$ by the embedding $z \mapsto [(z, 1)]$. If $\zeta \in W^{n+1}$, then

$$-\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_{n+2}^2 = 1,$$

so the point $[(\zeta, 1)]$ in $\mathbf{P}^{n+2}$ lies on the quadric Q^{n+1} in $\mathbf{P}^{n+2}$ given in homogeneous coordinates by the equation

$$\langle x, x\rangle = -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - x_{n+3}^2 = 0. \tag{2.11}$$

The manifold Q^{n+1} is called the *Lie quadric*, and the scalar product determined by the quadratic form in (2.11) is called the *Lie metric* or *Lie scalar product*. We will let $\{e_1, \ldots, e_{n+3}\}$ denote the standard orthonormal basis for the scalar product space $\mathbf{R}_2^{n+3}$ with metric $\langle , \rangle$. Here e_1 and e_{n+3} are timelike and the rest are spacelike.

We shall now see how points on Q^{n+1} correspond to the set of oriented hyperspheres, oriented hyperplanes and point spheres in $\mathbf{R}^n \cup \{\infty\}$. Suppose that x is any point on the quadric with homogeneous coordinate $x_{n+3} \neq 0$. Then x can be represented by a vector of the form $(\zeta, 1)$, where the Lorentz scalar product $(\zeta, \zeta) = 1$. Suppose first that $\zeta_1 + \zeta_2 \neq 0$. Then in Möbius geometry $[\zeta]$ represents a sphere in $\mathbf{R}^n$. If as in equation (2.6), we represent $[\zeta]$ by a vector of the form

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p\right),$$

then $(\xi, \xi) = r^2$. Thus ζ must be one of the vectors $\pm\xi/r$. In $\mathbf{P}^{n+2}$, we have

$$[(\zeta, 1)] = [(\pm\xi/r, 1)] = [(\xi, \pm r)].$$

We can interpret the last coordinate as a signed radius of the sphere with center p and unsigned radius $r > 0$. In order to be able to interpret this geometrically, we adopt the convention that a positive signed radius corresponds to the orientation of the sphere represented by the inward field of unit normals, and a negative signed radius corresponds to the orientation given by the outward field of unit normals. Hence, the two orientations of the sphere in $\mathbf{R}^n$ with center p and unsigned radius $r > 0$ are represented by the two projective points,

$$\left[\left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p, \pm r\right)\right] \tag{2.12}$$

in Q^{n+1}. Next if $\zeta_1 + \zeta_2 = 0$, then $[\zeta]$ represents a hyperplane in $\mathbf{R}^n$, as in equation (2.8). For $\zeta = (h, -h, N)$, with $|N| = 1$, we have $(\zeta, \zeta) = 1$. Then the two projective points on Q^{n+1} induced by ζ and $-\zeta$ are

$$[(h, -h, N, \pm 1)]. \tag{2.13}$$

These represent the two orientations of the plane with equation $u \cdot N = h$. We make the convention that $[(h, -h, N, 1)]$ corresponds to the orientation given by the field of unit normals N, while the orientation given by $-N$ corresponds to the point $[(h, -h, N, -1)] = [(-h, h, -N, 1)]$.

Thus far we have determined a bijective correspondence between the set of points x in Q^{n+1} with $x_{n+3} \neq 0$ and the set of all oriented spheres and planes in $\mathbf{R}^n$. Suppose now that $x_{n+3} = 0$, i.e., consider a point $[(z, 0)]$, for $z \in \mathbf{R}_1^{n+2}$. Then $\langle x, x\rangle = (z, z) = 0$, and $[z] \in \mathbf{P}^{n+1}$ is simply a point of the Möbius sphere Σ. Thus we have the following bijective correspondence between objects in Euclidean space and points on the Lie quadric:

$$
\begin{array}{cc}
\textbf{Euclidean} & \textbf{Lie} \\
\text{points: } u \in \mathbf{R}^n & \left[\left(\frac{1+u\cdot u}{2}, \frac{1-u\cdot u}{2}, u, 0\right)\right] \\
\infty & [(1,-1,0,0)] \\
\text{spheres: center } p, \text{ signed radius } r & \left[\left(\frac{1+p\cdot p-r^2}{2}, \frac{1-p\cdot p+r^2}{2}, p, r\right)\right] \\
\text{planes: } u\cdot N = h, \text{ unit normal } N & [(h,-h,N,1)]
\end{array}
\tag{2.14}
$$

In Lie sphere geometry, points are considered to be spheres of radius zero, or point spheres. From now on, we will use the term *Lie sphere* or simply "sphere" to denote an oriented sphere, oriented plane or a point sphere in $\mathbf{R}^n \cup \{\infty\}$. We will refer to the coordinates on the right side of equation (2.14) as the *Lie coordinates* of the corresponding point, sphere or plane. In the case of $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively, these coordinates were classically called *pentaspherical* and *hexaspherical* coordinates (see [10]). At times it is useful to have formulas to convert Lie coordinates back into Cartesian equations for the corresponding Euclidean object. Suppose first that $[x]$ is a point on the Lie quadric with $x_1 + x_2 \neq 0$. Then $x = \rho y$, for some $\rho \neq 0$, where y is one of the standard forms on the right side of the table above. From the table, we see that $y_1 + y_2 = 1$, for all proper points and all spheres. Hence if we divide x by $x_1 + x_2$, the new vector will be in standard form, and we can read off the corresponding Euclidean object from the table. In particular, if $x_{n+3} = 0$, then $[x]$ represents the point $u = (u_3, \ldots, u_{n+2})$, where

$$u_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n+2. \tag{2.15}$$

If $x_{n+3} \neq 0$, then $[x]$ represents the sphere with center $p = (p_3, \ldots, p_{n+2})$ and signed radius r given by

$$p_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n+2; \quad r = x_{n+3}/(x_1 + x_2). \tag{2.16}$$

Finally, suppose that $x_1 + x_2 = 0$. If $x_{n+3} = 0$, then the equation $\langle x, x\rangle = 0$ forces x_i to be zero for $3 \leq i \leq n+2$. Thus $[x] = [(1, -1, 0, \ldots, 0)]$, the improper point. If $x_{n+3} \neq 0$, we divide x by x_{n+3} to make the last coordinate 1. Then if we set $N = (N_3, \ldots, N_{n+2})$ and h according to

$$N_i = x_i/x_{n+3}, \quad 3 \leq i \leq n+2; \quad h = x_1/x_{n+3}, \tag{2.17}$$

the conditions $\langle x, x\rangle = 0$ and $x_1 + x_2 = 0$ force N to have unit length. Thus $[x]$ corresponds to the hyperplane $u \cdot N = h$, with unit normal N and h as in equation (2.17).

2.4 Geometry of Hyperspheres in S^n and H^n

In some ways it is simpler to use the sphere S^n rather than $\mathbf{R}^n$ as the base space for the study of Möbius or Lie sphere geometry. This avoids the use of stereographic projection and the need to refer to an improper point or to distinguish between spheres and planes. Furthermore, the correspondence in the table in equation (2.14) can be reduced to a single formula (2.21) below.

As in Section 2.2, we consider S^n to be the unit sphere in $\mathbf{R}^{n+1}$, and then embed $\mathbf{R}^{n+1}$ into $\mathbf{P}^{n+1}$ by the canonical embedding ϕ. Then $\phi(S^n)$ is the Möbius sphere Σ, given by the equation $(z, z) = 0$ in homogeneous coordinates. First we find the Möbius equation for the unoriented hypersphere in S^n with center $p \in S^n$ and spherical radius ρ, $0 < \rho < \pi$. This hypersphere is the intersection of S^n with the hyperplane in $\mathbf{R}^{n+1}$ given by the equation

$$p \cdot y = \cos\rho, \quad 0 < \rho < \pi. \tag{2.18}$$

Let $[z] = \phi(y) = [(1, y)]$. Then

$$p \cdot y = \frac{-(z, (0, p))}{(z, e_1)}.$$

Thus equation (2.18) can be rewritten as

$$(z, (\cos\rho, p)) = 0. \tag{2.19}$$

Therefore, a point $y \in S^n$ is on the hyperplane determined by equation (2.18) if and only if $\phi(y)$ lies on the polar hyperplane in $\mathbf{P}^{n+1}$ of the point

$$[\xi] = [(\cos\rho, p)]. \tag{2.20}$$

To obtain the two oriented spheres determined by equation (2.18) note that

$$(\xi, \xi) = -\cos^2\rho + 1 = \sin^2\rho.$$

Noting that $\sin\rho \neq 0$, we let $\zeta = \pm\xi/\sin\rho$. Then the point $[(\zeta, 1)]$ is on the quadric Q^{n+1}, and

$$[(\zeta, 1)] = [(\xi, \pm\sin\rho)] = [(\cos\rho, p, \pm\sin\rho)].$$

We can incorporate the sign of the last coordinate into the radius and thereby arrange that the oriented sphere S with signed radius $\rho \neq 0$, $-\pi < \rho < \pi$, and center p corresponds to a point in Q^{n+1} as follows:

$$S \longleftrightarrow [(\cos\rho, p, \sin\rho)]. \tag{2.21}$$

The formula still makes sense if the radius $\rho = 0$, in which case it yields the point sphere $[(1, p, 0)]$. This one formula (2.21) plays the role of all the formulas given in equation (2.14) in the preceding section for the Euclidean case.

As in the Euclidean case, the orientation of a sphere S in S^n is determined by a choice of unit normal field to S in S^n. Geometrically, we take the positive radius in (2.21) to correspond to the field of unit normals which are tangent vectors to geodesics from p to $-p$. Each oriented sphere can be considered in two ways, with center p and signed radius ρ, $-\pi < \rho < \pi$, or with center $-p$ and the appropriate signed radius $\rho \pm \pi$.

Given a point $[x]$ in the quadric Q^{n+1}, we now determine the corresponding hypersphere in S^n. Multiplying by -1, if necessary, we may assume that the first homogeneous coordinate x_1 of x satisfies $x_1 \geq 0$. If $x_1 > 0$, then we see from (2.21) that the center p and signed radius ρ, $-\pi/2 < \rho < \pi/2$, satisfy

$$\tan \rho = x_{n+3}/x_1, \qquad p = (x_2, \ldots, x_{n+2})/(x_1^2 + x_{n+3}^2)^{1/2}. \tag{2.22}$$

If $x_1 = 0$, then $x_{n+3} \neq 0$, so we can divide by x_{n+3} to obtain a point of the form $(0, p, 1)$. This corresponds to the oriented hypersphere with center p and signed radius $\pi/2$, which is a great sphere in S^n.

To treat oriented hyperspheres in hyperbolic space H^n, we let $\mathbf{R}_1^{n+1}$ be the Lorentz subspace of $\mathbf{R}_1^{n+2}$ spanned by the orthonormal basis $\{e_1, e_3, \ldots, e_{n+2}\}$. Then H^n is the hypersurface

$$\{y \in \mathbf{R}_1^{n+1} \mid (y, y) = -1,\ y_1 \geq 1\},$$

on which the restriction of the Lorentz metric $(\,,)$ is a positive definite metric of constant sectional curvature -1 (see [95, Vol. II, pp. 268–271] for more detail). The distance between two points p and q in H^n is given by

$$d(p, q) = \cosh^{-1}(-(p, q)).$$

Thus the equation for the unoriented sphere in H^n with center p and radius ρ is

$$(p, y) = -\cosh \rho. \tag{2.23}$$

As before with S^n, we first embed $\mathbf{R}_1^{n+1}$ into $\mathbf{P}^{n+1}$ by the map

$$\psi(y) = [y + e_2].$$

Let $p \in H^n$ and let $z = y + e_2$ for $y \in H^n$. Then we have

$$(p, y) = (z, p)/(z, e_2).$$

Hence equation (2.23) is equivalent to the condition that $[z] = [y + e_2]$ lies on the polar hyperplane in $\mathbf{P}^{n+1}$ to

$$[\xi] = [p + \cosh \rho\, e_2].$$

Following exactly the same procedure as in the spherical case, we find that the oriented hypersphere S in H^n with center p and signed radius ρ corresponds to a point in Q^{n+1} as follows:

$$S \longleftrightarrow [p + \cosh \rho\, e_2 + \sinh \rho\, e_{n+3}]. \tag{2.24}$$

There is also a stereographic projection τ with pole $-e_1$ from H^n onto the unit disk D^n in $\mathbf{R}^n = \mathrm{Span}\{e_3, \ldots, e_{n+2}\}$ given by

$$\tau(y_1, y_3, \ldots, y_{n+2}) = (y_3, \ldots, y_{n+2})/(y_1 + 1). \tag{2.25}$$

The metric g induced on D^n in order to make τ an isometry is the usual Poincaré metric.

In Section 3.5, we will see that from the point of view of Klein's Erlangen Program, all three of these geometries, Euclidean, spherical and hyperbolic, are subgeometries of Lie sphere geometry.

2.5 Oriented Contact and Parabolic Pencils of Spheres

In Möbius geometry, the principal geometric quantity is the angle. In Lie sphere geometry, the corresponding central concept is that of oriented contact of spheres. Two oriented spheres S_1 and S_2 in $\mathbf{R}^n$ are in *oriented contact* if they are tangent to each other and they have the same orientation at the point of contact. (See Figures 2.4 and 2.5 for the two possibilities.)

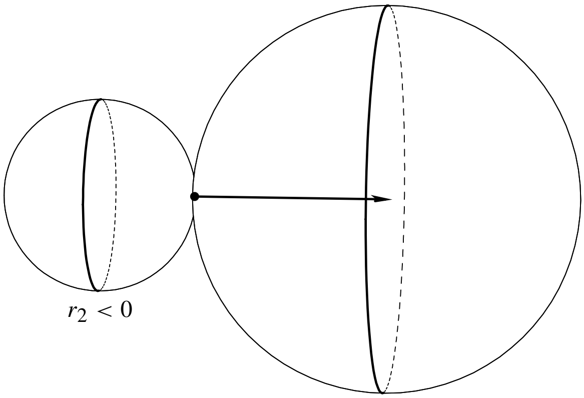

Fig. 2.4. Oriented contact of spheres, first case.

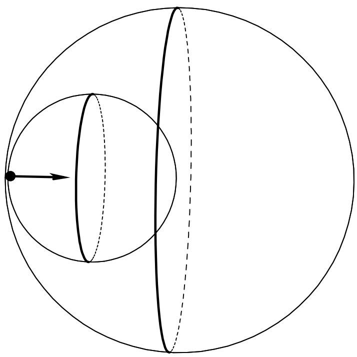

Fig. 2.5. Oriented contact of spheres, second case.

If p_1 and p_2 are the respective centers of S_1 and S_2, and r_1 and r_2 are their respective signed radii, then the analytic condition for oriented contact is

$$|p_1 - p_2| = |r_1 - r_2|. \tag{2.26}$$

An oriented sphere S with center p and signed radius r is in oriented contact with an oriented hyperplane π with unit normal N and equation $u \cdot N = h$ if π is tangent to S and their orientations agree at the point of contact. Analytically, this is just the equation

$$p \cdot N = r + h. \tag{2.27}$$

Two oriented planes π_1 and π_2 are in oriented contact if their unit normals N_1 and N_2 are the same. Two such planes can be thought of as two oriented spheres in oriented contact at the improper point.

A proper point u in $\mathbf{R}^n$ is in oriented contact sphere or plane if it lies on the sphere or plane. Finally, the improper point is in oriented contact with each plane, since it lies on each plane.

Suppose that S_1 and S_2 are two Lie spheres which are represented in the standard form given in equation (2.14) by $[k_1]$ and $[k_2]$. One can check directly that in all cases, the analytic condition for oriented contact is equivalent to the equation

$$\langle k_1, k_2 \rangle = 0. \tag{2.28}$$

Next we do some linear algebra to establish the important fact that the Lie quadric contains projective lines but no linear subspaces of higher dimension. We then show that the set of oriented spheres in $\mathbf{R}^n$ corresponding to the points on a line on Q^{n+1} forms a so-called *parabolic pencil* of spheres (see Figure 2.6). We also show that each parabolic pencil contains exactly one point sphere. Furthermore, if this point sphere is a proper point$p$ in $\mathbf{R}^n$, then the pencil contains exactly one hyperplane π. The pencil consists of all oriented hyperspheres in oriented contact with π at the point p.

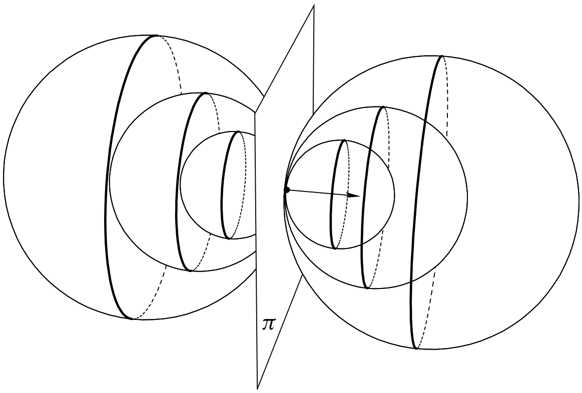

Fig. 2.6. Parabolic pencil of spheres.

The fundamental result needed from linear algebra is the following. Note that a subspace of a scalar product space is called *lightlike* if it consists of only lightlike vectors.

Theorem 2.3. *Let* $(,)$ *be a scalar product of signature* $(n-k,k)$ *on a real vector space* V*. Then the maximal dimension of a lightlike subspace of* V *is the minimum of the two numbers* k *and* $n-k$*.*

Proof. First, note that the theorem holds for scalar products having signature $(n-k,k)$ if and only if it holds for scalar products of signature $(k,n-k)$, since changing the signs of the quantities (e_i,e_i) for an orthonormal basis does not change the set of lightlike vectors.

Thus, we now assume that $k \leq n-k$ and do the proof by induction on the index k. The theorem is clearly true for scalar products of index 0, since the only lightlike vector is 0 itself. Assume now that the theorem holds for all spaces with a scalar product of index $k-1$, and let V be a scalar product space of index $k \geq 1$. Let W be a lightlike subspace of V of maximal dimension, and let v be a timelike vector in V. Then the scalar product restricts to a scalar product of index $k-1$ on the hyperplane $U = v^{\perp}$, and $W \cap U$ is a lightlike subspace of U. By the induction hypothesis, $\dim W \cap U \leq k-1$ and therefore, $\dim W \leq k$, as desired. On the other hand, it is easy to exhibit a lightlike subspace of V of dimension k. Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis for V with $\{e_1,\ldots,e_k\}$ timelike and the rest spacelike. For $1 \leq i \leq k$, let $v_i = e_i + e_{k+i}$. Then the span of $\{v_1,\ldots,v_k\}$ is a lightlike subspace of dimension k. □

Corollary 2.4. *The Lie quadric contains projective lines but no linear subspaces of higher dimension.*

Proof. This follows immediately from Theorem 2.3, since a linear subspace of $\mathbf{P}^{n+2}$ of dimension $k-1$ that lies on the quadric corresponds to a lightlike vector subspace of dimension k in $\mathbf{R}^{n+3}_2$. □

Theorem 2.3 also implies the following result concerning the orthogonal complement of a line on the quadric. This was pointed out by Pinkall [146, p. 24].

Corollary 2.5. *Let* ℓ *be a line on* $\mathbf{P}^{n+2}$ *that lies on the quadric* Q^{n+1}*.*

(a) *If* $[x] \in \ell^{\perp}$ *and* $[x]$ *is lightlike, then* $[x] \in \ell$*.*
(b) *If* $[x] \in \ell^{\perp}$ *and* $[x]$ *is not on* ℓ*, then* $[x]$ *is spacelike.*

Proof.

(a) Suppose that $[x]$ is a lightlike point in $\ell^{\perp}$ but not on ℓ. Then the two-dimensional linear lightlike subspace spanned by $[x]$ and ℓ lies on the quadric, contradicting Corollary 2.4.

(b) Suppose that $[x]$ is in $\ell^{\perp}$ but not on ℓ. From (a) we know that $[x]$ is either spacelike or timelike. Suppose that $[x]$ is timelike. Then the Lie metric $\langle,\rangle$ has signature $(n+1,1)$ on the vector space $x^{\perp}$, and $x^{\perp}$ contains the two-dimensional lightlike vector space that projects to ℓ. This contradicts Theorem 2.3. □

The next result establishes the relationship between the points on a line in Q^{n+1} and the corresponding parabolic pencil of spheres in $\mathbf{R}^n$.

Theorem 2.6.
(a) *The line in* $\mathbf{P}^{n+2}$ *determined by two points* $[k_1]$ *and* $[k_2]$ *of* Q^{n+1} *lies on* Q^{n+1} *if and only if the the spheres corresponding to* $[k_1]$ *and* $[k_2]$ *are in oriented contact, i.e.,* $\langle k_1, k_2 \rangle = 0$.
(b) *If the line* $[k_1, k_2]$ *lies on* Q^{n+1}*, then the parabolic pencil of spheres in* $\mathbf{R}^n$ *corresponding to points on* $[k_1, k_2]$ *is precisely the set of all spheres in oriented contact with both* $[k_1]$ *and* $[k_2]$.

Proof.
(a) The line $[k_1, k_2]$ consists of the points of the form $[\alpha k_1 + \beta k_2]$, where α and β are any two real numbers, at least one of which is not zero. Since $[k_1]$ and $[k_2]$ are on Q^{n+1}, we have

$$\langle \alpha k_1 + \beta k_2, \alpha k_1 + \beta k_2 \rangle = 2\alpha\beta \langle k_1, k_2 \rangle.$$

Thus the line is contained in the quadric if and only if $\langle k_1, k_2 \rangle = 0$.

(b) Let $[\alpha k_1 + \beta k_2]$ be any point on the line. Since $\langle k_1, k_2 \rangle = 0$ by (a), we easily compute that $[\alpha k_1 + \beta k_2]$ is orthogonal to both $[k_1]$ and $[k_2]$. Hence, the corresponding sphere is in oriented contact with the spheres corresponding to $[k_1]$ and $[k_2]$. Conversely, suppose that the sphere corresponding to a point $[k]$ on the quadric is in oriented contact with the spheres corresponding to $[k_1]$ and $[k_2]$. Then $[k]$ is orthogonal to every point on the line $[k_1, k_2]$, and so $[k]$ is on the line $[k_1, k_2]$ by Corollary 2.5(a). $\square$

As we have noted in the proofs of the previous results, given any timelike point $[z]$ in $\mathbf{P}^{n+2}$, the scalar product $\langle , \rangle$ has signature $(n+1, 1)$ on $z^{\perp}$. Hence, $z^{\perp}$ intersects Q^{n+1} in a Möbius space. We now show that any line on the quadric intersects such a Möbius space at exactly one point.

Corollary 2.7. *Let* $[z]$ *be a timelike point in* $\mathbf{P}^{n+2}$ *and* ℓ *a line that lies on* Q^{n+1}. *Then* ℓ *intersects* $z^{\perp}$ *at exactly one point.*

Proof. Any line in projective space intersects a hyperplane in at least one point. We simply must show that ℓ is not contained in $z^{\perp}$. But this follows from Theorem 2.3, since $\langle , \rangle$ has signature $(n+1, 1)$ on $z^{\perp}$, and therefore $z^{\perp}$ cannot contain the two-dimensional lightlike vector space that projects to ℓ. $\square$

As a consequence, we obtain the following corollary.

Corollary 2.8. *Every parabolic pencil contains exactly one point sphere. Furthermore, if the point sphere is a proper point, then the pencil contains exactly one plane.*

Proof. The point spheres are precisely the points of intersection of Q^{n+1} with $e_{n+3}^{\perp}$. Thus each parabolic pencil contains exactly one point sphere by Corollary 2.7. The hyperplanes correspond to the points in the intersection of Q^{n+1} with $(e_1 - e_2)^{\perp}$. The line ℓ on the quadric corresponding to the given parabolic pencil intersects this hyperplane at exactly one point unless ℓ is contained in the hyperplane. But ℓ is contained in $(e_1 - e_2)^\perp$ if and only if the improper point $[e_1 - e_2]$ is in $\ell^\perp$. By Corollary 2.5(a), this implies that the point $[e_1 - e_2]$ is on ℓ. Hence, if the point sphere of the pencil is not the improper point, then the pencil contains exactly one hyperplane. $\square$

By Corollary 2.8 and Theorem 2.6, we see that if the point sphere in a parabolic pencil is a proper point p in $\mathbf{R}^n$, then the pencil consists precisely of all spheres in oriented contact with a certain oriented plane π at p. Thus, one can identify the parabolic pencil with the point (p, N) in the unit tangent bundle to $\mathbf{R}^n$, where N is the unit normal to the oriented plane π. If the point sphere of the pencil is the improper point, then the pencil must consist entirely of planes. Since these planes are all in oriented contact, they all have the same unit normal N. Thus the pencil can be identified with the point (∞, N) in the unit tangent bundle to $\mathbf{R}^n \cup \{\infty\} = S^n$.

It is also useful to have this correspondence between parabolic pencils and elements of the unit tangent bundle $T_1 S^n$ expressed in terms of the spherical metric on S^n. Suppose that ℓ is a line on the quadric. From Corollary 2.7 and equation (2.21), we see that ℓ intersects both $e_1^\perp$ and $e_{n+3}^\perp$ at exactly one point. So the corresponding parabolic pencil contains exactly one point sphere and one great sphere, represented respectively by the points,

$$[k_1] = [(1, p, 0)], \quad [k_2] = [(0, \xi, 1)].$$

The fact that $\langle k_1, k_2 \rangle = 0$ is equivalent to the condition $p \cdot \xi = 0$, i.e., ξ is tangent to S^n at p. Hence the parabolic pencil of spheres corresponding to ℓ can be identified with the point (p, ξ) in $T_1 S^n$. The points on the line ℓ can be parametrized as

$$[K_t] = [\cos t \; k_1 + \sin t \; k_2] = [(\cos t, \cos t \; p + \sin t \; \xi, \sin t)].$$

From equation (2.21), we see that $[K_t]$ corresponds to the sphere in S^n with center

$$p_t = \cos t \; p + \sin t \; \xi, \tag{2.29}$$

and signed radius t. These are precisely the spheres through p in oriented contact with the great sphere corresponding to $[k_2]$. Their centers lie along the geodesic in S^n with initial point p and initial velocity vector ξ.

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