5

Weak Convergence of *P*-statistics

In this chapter we shall prove a result on the weak convergence of generalized permanents which extends the results on random permanents presented in Chapter 3. Herein we return to the discussions of Chapter 1 where we introduced a matrix permanent function as a way to describe properties of perfect matchings in bipartite graphs. Consequently, the asymptotic properties which will be developed in this chapter for P-statistics can be immediately translated into the language of the graph theory as the properties of matchings in some bipartite random graphs. This will be done in Section 5.3, where we shall revisit some of the examples introduced in Chapter 1. In order to establish the main results of this chapter we will explore the path connecting the asymptotic behavior of U- and P-statistics. An important mathematical object which will be encountered here is a class of real random variables known as multiple Wiener-Itô integrals. The concept of the Wiener-Itô integral is related to that of a stochastic integral with respect to martingales introduced in Chapter 4, though its definition adopted in this chapter is somewhat different - it uses Hermité polynomial representations. It will be introduced in the next section. We shall start our discussion of asymptotics for P-statistics by first introducing the classical result for U-statistics with fixed kernel due to Dynkin and Mandelbaum, then obtaining a limit theorem for U-statistics with kernels of increasing order, and finally extending the latter to P-statistics.

The main theoretical results herein are given in Section 5.2 as Theorems 5.2.1 and 5.2.2 and describe the asymptotics for P-statistics of random matrices with independent identically distributed random entries as the number of columns and rows increases. The result of Theorem 5.2.2 is used in Section 5.3 in order to derive some asymptotic distributions for several functions of perfect matchings in random bipartite graphs under the assumption that the corresponding matrices of weights $\mathbb{X}$ have independent and identically distributed entries (e.g., the edges appear randomly or are randomly colored). Throughout this chapter we therefore strengthen the exchangeability assumptions on the entries of matrix $\mathbb{X}^{(\infty)}$ given by (A1)-(A2) in Chapter 1 to that of independence and identical distribution of the entries.

5.1 Multiple Wiener-Itô Integral as a Limit Law for U-statistics

5.1.1 Multiple Wiener-Itô Integral of a Symmetric Function

In this section we introduce the multiple Wiener-Itó integral, the object which appears as the limit in distribution for properly normalized P-statistics.

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ having the distribution ν. Let $\mathcal{J} = \{J_1(\phi)\colon \phi \in L^2(\mathbf{R}, \mathcal{B}, \nu)\}$ be a Gaussian system (possibly on a different probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathcal{P})$ with the expectation operator denoted by $\mathcal{E}$) with zero means, unit variances and covariances $\mathcal{E}(J_1(\phi)J_1(\psi)) = E(\phi(X)\psi(X))$ for any $\phi, \psi \in L^2(\mathbf{R}, \mathcal{B}, \nu)$.

For any functions $f_i : \mathcal{U}_i \to V$, where $\mathcal{U}_i$ is an arbitrary set and V is an algebraic structure with multiplication, $i = 1, \ldots, m$, the tensor product $f_1 \otimes \ldots \otimes f_m : \mathcal{U}_1 \times \ldots \times \mathcal{U}_m \to V$ is defined as

$$f_1 \otimes \ldots \otimes f_m(\mathbf{x}) = f_1(x_1) \ldots f_m(x_m)$$

for any $\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{U}_1 \times \ldots \times \mathcal{U}_m$. In particular, for $f : \mathcal{U} \to V$ we have

$$f^{\otimes m}(\mathbf{x}) = f(x_1) \ldots f(x_m)$$

for any $\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{U}^m$.

For $\phi^{\otimes m}$, where $\phi \in L^2(\mathbf{R}, \mathcal{B}, \nu)$ is such that $E\phi(X) = 0$ and $E\phi^2(X) = 1$, the m-th multiple Wiener-Itô integral J_m is defined as

$$J_m(\phi^{\otimes m}) = \frac{1}{\sqrt{m!}}\, H_m\left(J_1(\phi)\right)\ ,$$

where H_m is the m-th monic Hermité polynomial (see Definition 3.2.2).

Moreover the linearity property is imposed on J_m, i.e. for any $\phi, \psi \in L^2(\mathbf{R}, \mathcal{B}, \nu)$, such that $E\,\phi(X) = 0 = E\,\psi(X)$ and $E\,\phi^2(X) = 1 = E\,\psi^2(X)$ and for any real numbers a and b it is required that

$$J_m(a\phi^{\otimes m} + b\psi^{\otimes m}) = aJ_m(\phi^{\otimes m}) + bJ_m(\psi^{\otimes m}).$$

Consequently, J_m is linear on the space

$$\mathcal{T}_s^{(m)} = span\left(\phi^{\otimes m} :\ \phi \in L^2(\mathbf{R}, \mathcal{B}, \nu),\ E\,\phi(X) = 0,\ E\,\phi^2(X) = 1\right)$$

which is a subspace of the space of symmetric functions $L^2_s(\mathbf{R}^m, \mathcal{B}_m, \nu_m) \subset L^2(\mathbf{R}^m, \mathcal{B}_m, \nu_m)$. Here $\mathcal{B}_m$ is the Borel σ-algebra in $\mathbf{R}^m$ and $\nu_m = \nu^{\otimes m}$ is the product measure.

As noted in Chapter 3 Section 3.2.3 it is well known that Hermité polynomials are orthogonal basis in the space $L^2(\mathbf{R}, \mathcal{B}, P_{\mathcal{N}})$, where $\mathcal{N}$ is a standard normal variable and $E(H_m^2(\mathcal{N})) = m!$, $m = 0, 1, \ldots$ Note that for any $\phi, \psi \in L^2(\mathbf{R}, \mathcal{B}, \nu)$, such that $E\,\phi(X) = 0 = E\,\psi(X)$ and $E\,\phi^2(X) = 1 = E\,\psi^2(X)$

$$m!\mathcal{E}\,J_m^2\left(a\phi^{\otimes m} + b\psi^{\otimes m}\right) = m!\mathcal{E}\left(aJ_m(\phi^{\otimes m}) + bJ_m(\psi^{\otimes m})\right)^2$$

$$= a^2\,\mathcal{E}\,H_m^2(J_1(\phi)) + 2ab\,\mathcal{E}\,H_m(J_1(\phi))H_m(J_1(\psi)) + b^2\,\mathcal{E}\,H_m^2(J_1(\psi)).$$

To continue the computations, we need to recall the classical Mehler formula (see, for instance, Bryc 1995, theorem 2.4.1). It says that for a bivariate Gaussian vector (X, Y) with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 1$ and $E(XY) = \rho$ the joint density f of (X, Y) has the form

$$f(x, y) = \sum_{k=0}^{\infty} \frac{\rho^k}{k!} H_k(x)H_k(y)f_X(x)f_Y(y), \tag{5.1}$$

where $f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$ is the marginal density of X and Y. Note that (5.1) yields

$$E(H_m(X)|Y) = \rho^m H_m(Y)$$

for any $m = 1, 2, \ldots$. This is easily visible since for any bounded Borel function g by orthogonality of Hermité polynomials we have

$$\begin{aligned}
&E\,E(H_m(X)|Y)g(Y) = E\,H_m(X)g(Y)\\
&\quad= \int_{\mathbf{R}}\int_{\mathbf{R}} H_m(x)g(y)\left(\sum_{k=0}^{\infty}\frac{\rho^k}{k!}H_k(x)H_k(y)f_X(x)f_Y(y)\right)dxdy\\
&\quad= \int_{\mathbf{R}}\int_{\mathbf{R}} g(y)\frac{\rho^m}{m!}H_m^2(x)H_m(y)f_X(x)f_Y(y)\,dxdy\\
&\quad= \int_{\mathbf{R}} \rho^m H_m(y)g(y)f_Y(y)\,dy = \rho^m E\,H_m(Y)g(Y).
\end{aligned}$$

Thus

$$\mathcal{E}\,J_m^2(a\phi^{\otimes m} + b\psi^{\otimes m}) = a^2\left[E\,\phi^2(X)\right]^m + 2ab\left[E\,\phi(X)\psi(X)\right]^m + b^2\left[E\,\psi^2(X)\right]^m$$

$$= E\left(a\prod_{i=1}^{m}\phi(X_i) + b\prod_{i=1}^{m}\psi^k(X_i)\right)^2 = E\,\left[\left(a\phi^{\otimes m} + b\psi^{\otimes m}\right)(X_1, \ldots, X_m)\right]^2,$$

where $X_1, \ldots, X_m$ are independent copies of X (with distribution ν).

From the preceding computation it follows that J_m is a linear isometry between the space $\mathcal{T}_s^{(m)}$ and the subspace $J_m\left(\mathcal{T}_s^{(m)}\right)$ of $L^2(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathcal{P})$.

In our next step leading to a definition of a multiple Wiener-Itô integral we shall prove that the space $\mathcal{T}_s^{(m)}$ is dense in the symmetric subspace $L_s^2(\mathbf{R}^m, \mathcal{B}_m, \nu_m)$. To this end we need an auxiliary result which will be used in the expansion for symmetric functions given later in Proposition 5.1.1. Recall that S_m denotes the set of all possible permutations of the set $\{1, \ldots, m\}$.

Lemma 5.1.1. *Let $a_1, \ldots, a_m \in V$, where V is a linear space. Then*

$$2^m \sum_{\sigma \in S_m} a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(m)} = \sum_{\underline{\varepsilon} \in \{-1,1\}^m} \left(\prod_{i=1}^m \varepsilon_i \right) \left(\sum_{i=1}^m \varepsilon_i a_i \right)^{\otimes m}, \tag{5.2}$$

Proof. Note that the right hand side of (5.2) can be written as

$$\begin{aligned} R &= \sum_{\underline{\varepsilon} \in \{-1,1\}^m} \left(\prod_{i=1}^m \varepsilon_i \right) \sum_{\underline{i} \in \{1,\ldots,m\}^m} \left(\prod_{r=1}^m \varepsilon_{i_r} \right) (a_{i_1} \otimes \ldots \otimes a_{i_m}) \\ &= \sum_{\underline{i} \in \{1,\ldots,m\}^m} (a_{i_1} \otimes \ldots \otimes a_{i_m}) \sum_{\underline{\varepsilon} \in \{-1,1\}^m} \left(\prod_{i=1}^m \varepsilon_i \right) \left(\prod_{r=1}^m \varepsilon_{i_r} \right) \end{aligned}$$

with $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)$ and $\underline{i} = (i_1, \ldots, i_m)$. But

$$\prod_{r=1}^m \varepsilon_{i_r} = \prod_{i=1}^m \varepsilon_i^{\delta_i},$$

where

$$\delta_i = \#\{j \in \{i_1, \ldots, i_m\} : j = i\}, \qquad i = 1, \ldots, m.$$

Moreover, we have

$$\begin{aligned} \sum_{\underline{\varepsilon} \in \{-1,1\}^m} \left(\prod_{i=1}^m \varepsilon_i \right) \left(\prod_{i=1}^m \varepsilon_i^{\delta_i} \right) &= \sum_{\underline{\varepsilon} \in \{-1,1\}^m} \prod_{i=1}^m \varepsilon_i^{1+\delta_i} = \prod_{i=1}^m \left(\sum_{\varepsilon_i \in \{-1,1\}} \varepsilon_i^{1+\delta_i} \right) \\ &= \begin{cases} 2^m, \text{ if } \forall i \in \{1, \ldots, m\}\ \delta_i = 1, \\ 0, \quad \text{otherwise.} \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} R &= 2^m \sum_{\underline{i} \in \{1,\ldots,m\}^m} (a_{i_1} \otimes \ldots \otimes a_{i_m}) I(\{i_1, \ldots, i_m\} = \{1, \ldots, m\}) \\ &= 2^m \sum_{\sigma \in S_m} a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(m)}. \end{aligned}$$

□

With the result of the lemma established we may now prove that $\mathcal{T}_s^{(m)}$ is dense in $L_s^2(\mathbf{R}^m, \mathcal{B}_m, \nu_m)$.

Proposition 5.1.1. *There exists a basis (ψ_r) in $L^2(\mathbf{R}, \mathcal{B}, \nu)$ such that for any $f \in L_s^2(\mathbf{R}^m, \mathcal{B}_m, \nu_m)$*

$$f = \sum_{r=1}^\infty \alpha_r \psi_r^{\otimes m} \tag{5.3}$$

where (α_r) *is a sequence of real numbers such that the double series*

$$||f||^2 = \sum_{r,s=1}^{\infty} \alpha_r \alpha_s \rho^m(r,s) < \infty, \tag{5.4}$$

where $\rho(r,s) = E(\psi_r(X)\psi_s(X))$.

Proof. Since f is symmetric we have

$$f(x_1,\ldots,x_m) = \frac{1}{m!}\sum_{\sigma\in\Pi_m} f(x_{\sigma(1)},\ldots,x_{\sigma(m)}).$$

Now, from the theory of Hilbert spaces, it follows that there exists an orthonormal basis (ϕ_r) in $L^2(\mathbf{R},\mathcal{B},\nu)$ such that

$$\begin{aligned} f(x_1,\ldots,x_m) &= \frac{1}{m!}\sum_{\sigma\in S_m}\sum_{\underline{i}\in\mathbf{N}^m} \beta_{\underline{i}} \prod_{l=1}^{m} \phi_{i_l}(x_{\sigma(l)}) \\ &= \frac{1}{m!}\sum_{\underline{i}\in\mathbf{N}^m} \beta_{\underline{i}} \sum_{\sigma\in S_m} \prod_{l=1}^{m} \phi_{i_l}(x_{\sigma(l)}) \\ &= \frac{1}{m!}\left(\sum_{\underline{i}\in\mathbf{N}^m} \beta_{\underline{i}} \sum_{\sigma\in S_m} \phi_{i_{\sigma(1)}}\otimes\ldots\otimes\phi_{i_{\sigma(m)}}\right)(x_1,\ldots,x_m), \end{aligned}$$

with $\sum_{\underline{i}\in\mathbf{N}^m} \beta_{\underline{i}}^2 < \infty$. By Lemma 5.1.1 we have

$$\begin{aligned} f &= \frac{1}{2^m m!}\sum_{\underline{i}\in\mathbf{N}^m} \beta_{\underline{i}} \sum_{\underline{\varepsilon}\in\{-1,1\}^m} \left(\prod_{l=1}^{m}\varepsilon_l\right)\left(\sum_{l=1}^{m}\varepsilon_l\phi_{i_l}\right)^{\otimes m} \\ &= \sum_{\underline{i}\in\mathbf{N}^m,\,\underline{\varepsilon}\in\{-1,1\}^m} \frac{\beta_{\underline{i}}\prod_{l=1}^{m}\varepsilon_l}{2^m m!}\left(\sum_{l=1}^{m}\varepsilon_l\phi_{i_l}\right)^{\otimes m}. \end{aligned}$$

Define

$$c_m(\underline{i},\underline{\varepsilon}) = \left|\left|\sum_{l=1}^{m}\varepsilon_l\phi_{i_l}\right|\right|^2 = m + 2\sum_{1\le r<s\le m}\varepsilon_i\varepsilon_j\delta(i_r=i_s).$$

Then we have

$$f = \sum_{\underline{i}\in\mathbf{N}^m,\,\underline{\varepsilon}\in\{-1,1\}^m} \alpha_{(\underline{i},\underline{\varepsilon})}\,\psi_{(\underline{i},\underline{\varepsilon})}^{\otimes m},$$

where

$$\alpha_{(\underline{i},\underline{\varepsilon})} = \frac{\sqrt{c_m(\underline{i},\underline{\varepsilon})}\beta_{\underline{i}}\prod_{l=1}^{m}\varepsilon_l}{2^m m!} \qquad \text{and} \qquad \psi_{(\underline{i},\underline{\varepsilon})} = \frac{1}{\sqrt{c_m(\underline{i},\underline{\varepsilon})}}\sum_{l=1}^{m}\varepsilon_l\phi_{i_l}.$$

Note that $(\psi_{(\underline{i},\underline{\varepsilon})})_{(\underline{i},\underline{\varepsilon})}$ is a basis in $L^2(\mathbf{R},\mathcal{B},\nu)$ and the double series of $(\alpha_{(\underline{i},\underline{\varepsilon})})_{(\underline{i},\underline{\varepsilon})}$ is summable, that is (5.4) holds. □

It is obvious that in the expansion (5.3) the elements of the basis (ψ_k) can be taken as standardized, that is except of $\int_{\mathbf{R}} \psi_k(x)\,\nu(dx) = 0$ also $\int_{\mathbf{R}} \psi_k^2(x)\,\nu(dx) = 1$, $k = 1, 2, \dots$

In view of the above result the multiple Wiener-Itô integral $J_m(f)$ for any symmetric function from the space $L_s^2(\mathbf{R}^m, \mathcal{B}_m, \nu_m)$ may be now defined through the unique extension of the linear isometry J_m from the space $\mathcal{T}_s^{(m)}$ to the whole symmetric space $L_s^2(\mathbf{R}^m, \mathcal{B}_m, \nu_m)$. Thus, J_m is identified with the unique functional on the Gaussian system $\mathcal{J}$ such that

$$\mathcal{E}(J_m^2(f)) = E(f^2(X_1, \dots, X_m)) \qquad \text{for any } f \in L_s^2(\mathbf{R}^m, \mathcal{B}_m, \nu_m) \tag{5.5}$$

Moreover $J_m(f)$ can be expanded as a series of Hermité polynomials.

Proposition 5.1.2. *Let (ϕ_k) be a standardized basis in $L^2(\mathbf{R}, \mathcal{B}(\mathbf{R}), \nu)$ such that for any symmetric function $f \in L^2(\mathbf{R}^m, \mathcal{B}_m, \nu_m)$*

$$f = \sum_{r=1}^{\infty} \alpha_r \phi_r^{\otimes m}.$$

Then

$$J_m(f) = \sum_{r=1}^{\infty} \alpha_r H_m(J_1(\phi_r)), \tag{5.6}$$

Proof. To prove the result it suffices to show that (5.5) holds for $J_m(f)$ defined by (5.6). But we have

$$m!\mathcal{E}(J_m^2(f)) = \sum_{r,s=1}^{\infty} \alpha_r \alpha_s \mathcal{E}[H_m(J_1(\phi_r))H_m(J_1(\phi_s))] = m! \sum_{r,s=1}^{\infty} \alpha_r \alpha_s \rho^m(r,s),$$

where $\rho(r,s) = E(\phi_r(X)\phi_s(X))$.

On the other hand

$$E(f^2(X_1, \dots, X_m)) = \sum_{r,s=1}^{\infty} \alpha_r \alpha_s \left(E[\phi_r(X)\phi_s(X)]\right)^m.$$

Now the result follows by the definition of the quantities $\rho(r,s)$. $\square$

5.1.2 Classical Limit Theorems for *U*-statistics

Let $(X_k)_{k\geq 1}$ be a sequence of independent identically distributed random variables. Let $h \in L_s^{(m)}$ be a symmetric kernel. Consider the corresponding U-statistic

$$U_n^{(m)}(h) = \binom{n}{m}^{-1} [\pi_m^n(h)]\,(X_1, \dots, X_n) \tag{5.7}$$

with its H-decomposition

$$U_n(h) - E\,U_n(h) = \sum_{c=r}^{m} \binom{m}{c} \binom{n}{c}^{-1} [\pi_c^n(g_c)]\,(X_1, \ldots, X_n) \tag{5.8}$$

where (see (1.11))

$$g_c(x_1, \ldots, x_c) = \sum_{i=1}^{c} (-1)^{c-i} \sum_{1 \le j_1 < \ldots < j_i \le c} \tilde{h}_i(x_{j_1}, \ldots, x_{j_i}). \tag{5.9}$$

with $\tilde{h}_c = h_c - E(h)$ and

$$h_c(x_1, \ldots, x_c) = E(h(x_1, \ldots, x_c, X_{c+1}, \ldots, X_m))$$

for $c = 1, 2, \ldots, m$. We may write the complete degeneracy property of g_c (1.12) more concisely as

$$E\,g_c(x_1, \ldots, x_{c-1}, X_c) = 0, \qquad c = 1, 2, \ldots, m, \tag{5.10}$$

for any $x_1, \ldots, x_m$.

Recall that the number $r-1$ in (5.8) is the degeneration (or degeneracy) level of the U-statistic where $r = \min\{c \ge 1 : \ h_c \not\equiv 0\}$. In what follows we also refer to r as non-degeneracy level. It appears that the non-degeneracy level is essential for the limiting behavior of U-statistics.

For $r = 1$ it was proved by Hoeffding (1948) that

$$\sqrt{n}[U_n(h) - E(U_n(h))] \xrightarrow{d} m\mathcal{N}(0, Eg_1^2(Y_1)).$$

The case of $r = 2$ waited for over three decades, until Serfling (1980) showed that

$$n(U_n(h) - E(U_n(h))) \xrightarrow{d} \binom{m}{2} \sum_{k=1}^{\infty} \lambda_k(\mathcal{Z}_k - 1),$$

where $(\mathcal{Z}_k)$ is a sequence of independent chi-square with one degree of freedom random variables, and (λ_k) are defined by the decomposition of g_2:

$$g_2(x_1, x_2) = \sum_{k=1}^{\infty} \lambda_k \phi_k(x_1) \phi_k(x_2),$$

where (ϕ_k) is an orthonormal basis in $L^2(\mathbf{R}, \mathcal{B}, \nu)$, ν being the common distribution of X_i's. Dynkin and Mandelbaum (1983) considered the case of an arbitrary $r \ge 1$. They showed that

$$\binom{m}{r}^{-1} n^{r/2}(U_n(h) - E(U_n(h)) \xrightarrow{d} J_r(g_r), \tag{5.11}$$

where J_r is the r-th multiple Wiener-Itô integral as defined in in the previous section.

5.1.3 Dynkin-Mandelbaum Theorem

This section is devoted to the detailed proof of the Dynkin and Mandelbaum theorem. The proof we offer is more elementary than the original one.

Theorem 5.1.1. *Let (X_n) be a sequence of iid rv's. Let U_n, $n = m, m+1, \ldots$, be a U-statistic for the sequence (X_n) defined by (5.7) (we assume that m is fixed for all n's). Assume that the non-degeneracy level of the kernel h is r and that $E(g_r^2) < \infty$. Then the convergence (5.11) holds.*

Proof. Note that by the representation (5.8) we have

$$\binom{m}{r}^{-1} n^{r/2}\left(U_n - E(U_n)\right)$$

$$= \binom{m}{r}^{-1} n^{r/2} \sum_{k=r+1}^{m} \binom{m}{k}\binom{n}{k}^{-1} \pi_k^n(g_k) + n^{r/2}\binom{n}{r}^{-1}\pi_r^n(g_r).$$

The first term, we call it R_n, in the above sum converges in probability to zero. This is due to the following computation of its variance based on the orthogonality of g_k's and the property (5.10)

$$Var(R_n) = \binom{m}{r}^{-2} n^r \sum_{k=c+1}^{m} \binom{m}{k}^2 \binom{n}{k}^{-2} Var(\pi_k^n(g_k))$$

$$= \binom{m}{r}^{-2} n^r \sum_{k=r+1}^{m} \binom{n}{k}^{-1} E(g_k^2).$$

Due to the obvious inequality $n(n-1)\ldots(n-k+1) > (n/2)^{r+1}$ valid for $k > r$ and n sufficiently large we have

$$Var(R_n) \le \frac{A}{n}$$

with

$$A = 2^{r+1}\binom{m}{r}^{-2} \sum_{k=r+1}^{m} \binom{m}{k}^2 k! E(g_k^2).$$

Thus we have to show that

$$n^{r/2}\binom{n}{r}^{-1} \pi_r^n(g_r) \xrightarrow{d} J_r(g_r).$$

Note that we can expand g_r as in Proposition 5.1.1, i.e.

$$g_r(x_1, \ldots, x_r) = \sum_{k=1}^{\infty} \alpha_k \psi_k(x_1)\ldots\psi_k(x_r),$$

where $E(\psi_k(X_i)) = 0$ and $E(\psi_k^2(X_i)) = 1$, $k = 1, 2, \ldots$. Moreover

$$\sum_{j,k=1}^{\infty} \alpha_j \alpha_k \rho^r(j,k) = E(g_r^2) < \infty, \tag{5.12}$$

with $\rho(j,k) = \int \psi_j(x)\psi_k(x)\, dP(x)$ - see Prop. 5.1.1. Now, for any K we have

$$n^r \binom{n}{r}^{-2} E\left(\sum_{k=K+1}^{\infty} \alpha_k \sum_{1\le j_1<\ldots<j_r\le n} \psi_k(X_{j_1})\ldots\psi_k(X_{j_r})\right)^2$$
$$= n^r \binom{n}{r}^{-1} \sum_{j,k=K+1}^{\infty} \alpha_j\alpha_k\rho^r(j,k).$$

Since the first term is bounded uniformly with respect to n then by (5.12) it follows that the above quantity converges to zero as $K \to \infty$ uniformly in n. Hence

$$n^{r/2}\binom{n}{r}^{-1} \sum_{k=K+1}^{\infty} \alpha_k \sum_{1\le j_1<\ldots<j_r\le n} \psi_k(X_{j_1})\ldots\psi_k(X_{j_r}) \xrightarrow{P} 0$$

as $K \to \infty$ uniformly in n. Thus to study the limiting behaviour of $n^{r/2}\binom{n}{r}^{-1}$ $\pi_r^n(g_r)$ it suffices to consider the finite sum

$$n^{-r/2} r! \sum_{k=1}^{K} \alpha_k \sum_{1\le j_1<\ldots<j_r\le n} \psi_k(X_{j_1})\ldots\psi_k(X_{j_r}).$$

Note that the inner sum is the elementary symmetric polynomial $S_n(r)$ in variables $\psi_k(X_1), \ldots, \psi_k(X_n)$, which will be denoted by $S_n^{(k)}(r)$.

Due to the recursion formula for the elementary symmetric polynomials (see Lemma 3.2.1) which can be rewritten as

$$cS_n^{(k)}(c) = \sum_{d=0}^{c-1} (-1)^d S_n^{(k)}(c-d-1) \sum_{j=1}^{n} \psi_k^{d+1}(X_j), \qquad k = 1, \ldots, K, \tag{5.13}$$

it follows that for any $c = 1, 2, \ldots$ there exists a function F_c of c variables such that for any n

$$c! n^{-c/2} S_n^{(k)}(c) = F_c\left(n^{-j/2}\sum_{i=1}^{n} \psi_k^j(X_i),\ j = 1, \ldots, c\right), \qquad k = 1, \ldots, K.$$

Moreover, it follows from (5.13) that

$$F_c(x,1,0,\ldots,0) = xF_{c-1}(x,1,0,\ldots,0) - (c-1)F_{r-2}(x,1,0,\ldots,0), \quad c = 1, 2, \ldots,$$

with $F_{-1}(x,1,0,\dots,0)=0$ and $F_0(x,1,0,\dots,0)=1$. Consequently, see the recurrence for Hermité polynomials given in (3.7), $F_c(x,1,0,\dots,0)$ is a monic Hermité polynomial H_c, $c=0,1,\dots$.

Note that, by the classical central limit theorem the weak convergence

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_k(X_i),\ k=1,\dots,K\right)\xrightarrow{d}(J_1(\psi_k),\ k=1,\dots,K)$$

holds for any K. Additionally, by the weak law of large numbers, it follows that

$$\left(\frac{1}{n}\sum_{i=1}^{n}\psi_k^2(X_i),\ k=1,\dots,K\right)\xrightarrow{P}(1,\dots,1)$$

and for $j>2$ (see the weak law of large numbers - Theorem 3.2.4)

$$\left(\frac{1}{n^{j/2}}\sum_{i=1}^{n}\psi_k^j(X_i),\ k=1,\dots,K\right)\xrightarrow{P}(0,\dots,0).$$

Consequently, since F_r is continuous, it further follows that

$$n^{-r/2}r!\sum_{k=1}^{K}\alpha_k\sum_{1\le j_1<\dots<j_r\le n}\psi_k(X_{j_1})\dots\psi_k(X_{j_r})\xrightarrow{d}\sum_{k=1}^{K}\alpha_k H_r(J_1(\psi_k)).$$

Note that, by (5.12) the variance of the tail

$$Var\left(\sum_{k=K+1}^{\infty}\alpha_k H_r(J_1(\psi_k))\right)=\sum_{j,k=K+1}^{\infty}\alpha_j\alpha_k\rho^r(j,k)\to 0$$

as $K\to\infty$. Thus the sequence $\left(\sum_{k=1}^{K}\alpha_k H_r(J_1(\psi_k))\right)$ converges in probability to $J_r(g_r)$ as $K\to\infty$. □

5.1.4 Limit Theorem for U-statistics of Increasing Order

If the order m of the kernel $h=h_m$ of the U-statistic $U_n(h)=U_n^{(m)}$ increases with $n\to\infty$ in such a way that $m/\sqrt{n}\to\lambda\ge 0$ then under certain assumptions on the elements of the Hoeffding decomposition $g_{m,c}$ the limiting distribution of $U_n^{(m)}$ as $n\to\infty$ can be represented as the distribution of a multiple Wiener-Itô integral (in the case $\lambda=0$) or as an infinite sum of multiple Wiener-Itô integrals (in the case $\lambda>0$). Such results were originally stated by Korolyuk and Borovskikh (1990). Their derivations were based upon reducing the problem to the main theorem of Dynkin and Mandelbaum (1983) paper, which describes the limit of an infinite series of normalized elements of the Hoeffding decomposition. Below we shall prove analogous results in a simpler way, exploring the techniques which were developed in earlier chapters.

We start with the result for $\lambda=0$. Here the proof borrows a lot from the proof of the Dynkin-Mandelbaum theorem of the previous section.

Theorem 5.1.2. *Let (X_i) be a sequence of iid rv's. Let $(U_n^{(m_n)})$ be a sequence of U-statistics defined for (X_i) such that they have the same non-degeneracy level r. Assume that $m_n^2/n \to \lambda > 0$. Assume that $Eh_{m_n}^2 < \infty$ $n = 1, 2, \ldots$ and that $E(g_{m_n,r} - g_r)^2 \to 0$ as $n \to \infty$ for a function $g_r : \mathbf{R}^r \to \mathbf{R}$ satisfying (5.10). Let*

$$\sum_{k=r+1}^{m_n} \frac{\left(\frac{m_n^2}{n}\right)^{k-r}}{k!} E\, g_{m_n,k}^2 \to 0. \tag{5.14}$$

Then

$$\binom{m_n}{r}^{-1} n^{r/2} \left[U_n^{(m_n)} - E\left(U_n^{(m_n)}\right)\right] \xrightarrow{d} J_r(g_r).$$

Proof. As in the proof of Theorem 5.1.1 we decompose the normalized U-statistic as

$$\binom{m}{r}^{-1} n^{r/2} \left(U_n^{(k)} - E(U_n^{(k)}\right) = R_n + n^{r/2} \binom{n}{r}^{-1} \pi_r^n(g_{m_n,r}),$$

where

$$R_n = \binom{m_n}{r}^{-1} n^{r/2} \sum_{k=r+1}^{m_n} \binom{m_n}{k} \binom{n}{k}^{-1} \pi_k^n(g_{m_n,k}).$$

Note that

$$Var(R_n) = \binom{m_n}{r}^{-2} n^r \sum_{k=r+1}^{m_n} \binom{m_n}{k}^2 \binom{n}{k}^{-1} E\, g_{m_n,k}^2$$

as well as, for any $k = r+1, \ldots, m_n$

$$\binom{m_n}{r}^{-2} \binom{m_n}{k}^2 \le \left(\frac{r!}{k!}\right)^2 m_n^{2(k-c)}$$

and

$$\begin{aligned} n^r \binom{n}{k}^{-1} &\le \left(\frac{n}{n-r+1}\right)^r \frac{1}{(n-r)\ldots(n-k+1)} \\ &\le \left(\frac{n}{n-r+1}\right)^r \left(\frac{n}{n-m_n}\right)^{m_n} \frac{1}{n^{k-r}}. \end{aligned}$$

Since the first two terms on the right hand side of the last inequality are bounded (each converges to 1), then it follows that

$$\binom{m_n}{r}^{-2} n^r \binom{m_n}{k}^2 \binom{n}{k}^{-1} \le C\, \frac{(m_n^2/n)^{k-r}}{k!},$$

where C is a constant. Consequently, by (5.14) it follows that $Var(R_n) \to 0$.

Since, as in the proof of Theorem 5.1.1 it follows that

$$n^{r/2}\binom{n}{r}^{-1}\pi_r^n(g_r)\xrightarrow{d}J_r(g_r),$$

it suffices to show that

$$n^{r/2}\binom{n}{r}^{-1}\left(\pi_r^n(g_{m_n,r})-\pi_r^n(g_r)\right)\xrightarrow{P}0.$$

This follows from the fact that the second moment of this difference is

$$n^r\binom{n}{r}^{-1}E\left(g_{m_n,r}-g_r\right)^2\le\left(\frac{n}{n-r+1}\right)^rE\left(g_{m_n,r}-g_r\right)^2$$

and thus, the assumption of the theorem implies that it converges to 0. □

The analogous result for $\lambda>0$ is somewhat more difficult. In particular, an infinite series of multiple Wiener-Itô integrals appears in the limit.

Theorem 5.1.3. *Let (X_i) be a sequence of iid rv's. Let $(U_n^{(m_n)})$ be a sequence of U-statistics defined for (X_i) such that they have the same non-degeneracy level r. Assume that $m_n^2/n\to\lambda>0$. Assume that $Eh_{m_n}^2<\infty$ $n=1,2,\ldots$, and that there exists a sequence of functions $\mathbf{g}=(g_k)$ such that g_k is a symmetric function on $\mathbf{R}^k$ satisfying (5.10), $k=r,r+1,\ldots$. Moreover, let*

$$\sum_{k=r}^{\infty}\frac{\lambda^k}{k!}E\,g_k^2<\infty\qquad\textit{and}\qquad\sum_{k=r}^{\infty}\frac{\left(\frac{m_n^2}{n}\right)^k}{k!}E\,g_k^2<\infty\qquad\textit{uniformly in }n \tag{5.15}$$

and

$$\sum_{k=r}^{m_n}\frac{\left(\frac{m_n^2}{n}\right)^k}{k!}E\left(g_{m_n,k}-g_k\right)^2\to0. \tag{5.16}$$

Then

$$Z_n=U_n^{(m_n)}-E\left(U_n^{(m_n)}\right)\xrightarrow{d}\sum_{k=r}^{\infty}\frac{\lambda^{k/2}}{k!}J_k(g_k).$$

Proof. Consider first the random variable

$$S_{N,n}=\sum_{k=r}^{N}\binom{m}{k}\binom{n}{k}^{-1}\pi_k^{(n)}(g_k)=\sum_{k=r}^{N}\sum_{j=1}^{\infty}\alpha_{j,k}\binom{m}{k}\binom{n}{k}^{-1}\pi_k^{(n)}\psi_j^{\otimes k},$$

where $g_k=\sum_{j=1}^{\infty}\alpha_{k,j}\psi_j^{\otimes k}$, $k=r,\ldots,N$ and $m=m_n$. Note that the sequence (ψ_j) is common for all k's. This is possible by defining it for the largest space $L^2(\mathbf{R}^N,\mathcal{B}(\mathbf{R}^N),P_X^{\otimes N})$. Moreover, $E(\psi_j(X_i))=0$ and $E(\psi_j^2(X_i))=1$, $j=1,2,\ldots$.

Since the above sum with respect to k is finite, similarly as in the proof of the previous result it follows that

$$\sum_{k=r}^{N} \sum_{j=K+1}^{\infty} \alpha_{j,k} \binom{m}{k} \binom{n}{k}^{-1} \pi_k^{(n)} \psi_j^{\otimes k} \xrightarrow{P} 0$$

as $K \to \infty$ uniformly in n. Thus using the fact the $m^2/n \to \lambda$ and properties of elementary symmetric polynomials as in the previous proof we get

$$\sum_{k=r}^{N} \sum_{j=1}^{K} \alpha_{j,k} \left(\frac{m}{\sqrt{n}} \right)^k \frac{\binom{m}{k}}{m^k} \frac{(\sqrt{n})^k}{\binom{n}{k}} \pi_k^{(n)} \psi_j^{\otimes k} \xrightarrow{d} \sum_{k=r}^{N} \sum_{j=1}^{K} \alpha_{j,k} \lambda^{k/2} H_k(J_1(\psi_j))$$

Note that as $K \to \infty$ the right hand side of the above expression converges in probability (see the previous proof) to $\sum_{k=r}^{N} \lambda^{k/2} J_k(g_k)$.

Note that by orthogonality (5.10) we have

$$A_n = E\left(\sum_{k=r}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} \pi_k^{(n)} (g_{m,k} - g_k) \right)^2 = \sum_{k=r}^{m} \frac{\binom{m}{k}^2}{\binom{n}{k}} E(g_{m,k} - g_k)^2.$$

Thus the inequality $m^2/n > (m-1)^2/(n-1)$ and the condition (5.16) imply

$$A_n \le \sum_{k=r}^{m} \frac{\left(\frac{m^2}{n} \right)^k}{k!} E\left(g_{m,k} - g_k\right)^2 \to 0.$$

Moreover, the second part of (5.15) implies that

$$\sum_{k=m+1}^{\infty} \frac{\left(\frac{m^2}{n} \right)^k}{k!} E g_k^2 \to 0.$$

Consequently,

$$\sum_{k=r}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} \pi_k^{(n)} g_{m,k} \qquad \text{and} \qquad \sum_{k=r}^{\infty} \frac{\binom{m}{k}}{\binom{n}{k}} \pi_k^{n}(g_k)$$

are asymptotically (as $n \to \infty$) equivalent in distribution.

Note also that $\sum_{k=N}^{\infty} \frac{\lambda^{k/2}}{k!} J_k(g_k)$ converges in probability to 0 as $N \to \infty$: by the orthogonality of g_k's and properties of the multiple Wiener-Itô integral (see, Dynkin and Mandelbaum 1983, formula (2.2)) we get

$$E\left(\sum_{k=N}^{\infty} \frac{\lambda^{k/2}}{k!} J_k(g_k) \right)^2 = \sum_{k=N}^{\infty} \frac{\lambda^k}{k!} E\, g_k^2,$$

which by (5.15) converges to 0 with $N \to \infty$.

Finally, we consider

$$T_{N,n} = \sum_{k=N+1}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} \pi_k^n(g_k).$$

Observe that by the orthogonality of g_k's

$$E(T_{N,n}^2) = \sum_{k=N+1}^{m} \frac{\binom{m}{k}^2}{\binom{n}{k}} E(g_k^2) \le \sum_{k=N+1}^{m} \frac{\left(\frac{m^2}{n}\right)^k}{k!} E(g_k^2)$$

and thus by the second part of (5.15) it follows that $T_{N,n}$ converges in probability (uniformly in n) as $N \to \infty$ to 0.

Since for any $\varepsilon > 0$ and any $x \in \mathbf{R}$ we have

$$P(S_{N,n} \le x-\varepsilon)-P(|T_{N,n}| > \varepsilon) \le P(Z_n \le x) \le P(S_{N,n} \le x+\varepsilon)+P(|T_{N,n}| > \varepsilon),$$

the relations which has already been used in Chapter 3, the result follows because of the asymptotic properties of $S_{N,n}$ and $T_{N,n}$ derived in the course of the proof.

5.2 Asymptotics for P-statistics

Let $M_{m\times n}$ be the space of $m \times n$ matrices with real entries and let $h \in L_s^{(m)}$. Further, let $\mathbb{X}^{(m,n)}$ be a random matrix assuming values in $M_{m\times n}$ with iid entries. Let $E(|h^{(m)}|) < \infty$. Then from Theorem 2.2.1 we obtain for the associated generalized permanent function (under slightly different notation)

$$Per_{h^{(m)}}^{(m,n)} \mathbb{X}^{(m,n)} = E\left(Per_{h^{(m)}}^{(m,n)} \mathbb{X}^{(m,n)}\right) + m!\binom{n}{m} \sum_{k=1}^{m} \frac{(n-k)!}{n!} W_{g_{m,k}}^{(m,n)},$$

where

$$W_{g_{m,k}}^{(m,n)} = \sum_{1\le i_1<\ldots<i_k\le m} \sum_{1\le j_1<\ldots<j_k\le n} \sum_{\sigma\in\Pi_m} g_{m,k}\left(X_{i_{\sigma(1)},j_1}, \ldots, X_{i_{\sigma(k)},j_k}\right)$$

and

$$\begin{aligned} & g_{m,k}(w_1, \ldots, w_c) \\ & = \int_{\mathbf{R}^m} h^{(m)}(z_1, \ldots, z_m) \left(\prod_{r=1}^{k} (\delta_{w_r}(dz_r) - P_X(dz_r))\right) \left(\prod_{s=k+1}^{m} P_X(dz_s)\right) \end{aligned}$$

for P_X being the distribution of $X_{1,1}$. Recall that, similarly as for U-statistics, we say that r is non-degeneracy level of P-statistics if $g_k^{(m)} \equiv 0$ for $k < r$ and $g_r^{(m)} \not\equiv 0$.

The main object of this section is to prove limit theorems for P-statistics with $m_n/n \to \lambda \geq 0$. Similarly, as for U-statistics, two cases $\lambda = 0$ and $\lambda > 0$, differ and they are treated separately. However in both cases the asymptotic behaviour of the sequence of centered and normalized P-statistics

$$\left(\frac{Per_{h^{(m_n)}}^{(m_n,n)}(\mathbb{X}^{(m_n,n)}) - E\left(Per_{h^{(m_n)}}^{(m_n,n)}(\mathbb{X}^{(m_n,n)}) \right)}{\binom{n}{m_n} m_n!} \right)_n$$

is compared to the asymptotics of respective U-statistics.

Again, as for U-statistics, we start with the case $\lambda = 0$.

Theorem 5.2.1. *Let $(\mathbb{X}^{(m_n,n)})$ be a sequence of matrices in $M_{m_n \times n}$ embedded in an infinite matrix $\mathbb{X}^{(\infty)}$ of iid entries. Assume $m_n/n \to 0$. Consider a sequence of P-statistics $\left(Per_{h^{(m_n)}}^{(m_n,n)}(\mathbb{X}^{(m_n,n)}) \right)$ with a common level of non-degeneracy equal to r. Assume that $E\left[(h^{(m_n)})^2 \right] < \infty$, $n = 1, 2, \ldots$, and that*

$$\sum_{k=r+1}^{m_n} \frac{\left(\frac{m_n}{n} \right)^{k-r}}{k!} E\, g_{m_n,k}^2 \to 0 \, . \tag{5.17}$$

Assume also that $E(g_{m_n,r} - g_r)^2 \to 0$ as $n \to \infty$ for a symmetric function $g_r : \mathbf{R}^r \to \mathbf{R}$ satisfying (5.10).

Then

$$r! \left(\frac{n}{m_n} \right)^{r/2} \frac{Per_{h^{(m_n)}}^{(m_n,n)}(\mathbb{X}^{(m_n,n)}) - E\left(Per_{h^{(m_n)}}^{(m_n,n)}(\mathbb{X}^{(m_n,n)}) \right)}{\binom{n}{m_n} m_n!} \stackrel{d}{\to} J_r(g_r).$$

Proof. Write $m = m_n$. Consider a U-statistic $U_{mn}^{(m)}$ with the kernel h_m based on an iid sample of $m_n n$ random variables: $X_{1,1}, \ldots, X_{m,n}$. Then $m^2/(mn) = m/n \to \lambda$. Consequently, the assumptions of Theorem 5.1.2 are satisfied (with n changed into mn) and thus

$$\binom{m}{r}^{-1} (mn)^{r/2} \left(U_{mn}^{(m_n)} - E(U_{mn}^{(m)}) \right) \stackrel{d}{\to} J_r(g_r).$$

To complete the proof we will argue that

$$\begin{aligned} &\frac{r! n^{r/2} \left[Per_{h^{(m)}}^{(m,n)}(\mathbb{X}^{(m,n)}) - E\left(Per_{h^{(m)}}^{(m,n)}(\mathbb{X}^{(m,n)}) \right) \right]}{m^{r/2} \binom{n}{m} m!} \\ &- \frac{(mn)^{r/2} \left[U_{mn}^{(m)} - E(U_{mn}^{(m)}) \right]}{\binom{m}{r}} \\ &= \Omega_n(\mathbf{g}_m) \stackrel{P}{\to} 0. \end{aligned}$$

Note that symmetry of the kernels $g_{m,k}$'s entails

$$Cov\left((k!)^{-1}\binom{n}{k}^{-1}W^{(m,n)}_{g_{m,k}},\ \binom{m}{k}\binom{mn}{k}^{-1}\pi^k_{mn}(g_{m,k})\right)$$

$$= Var\left(\binom{m}{k}\binom{mn}{k}^{-1}\pi^k_{mn}(g_{m,k})\right) = \frac{\binom{m}{k}^2}{\binom{mn}{k}}E(g^2_{m,k}) \tag{5.18}$$

for any $k = r, r+1, \ldots, m$. Note also that by (2.13) it follows that

$$Var\left((k!)^{-1}\binom{n}{k}^{-1}W^{(m,n)}_{g_{m,k}}\right) = \frac{\binom{m}{k}}{\binom{n}{k}}\frac{1}{k!}E(g^2_{m,k}). \tag{5.19}$$

We will show that Ω_n converges to 0 in L_2. By orthogonality of $(g_{m,k})$ it follows that

$$Var\ \Omega_n(\mathbf{g}_m) = \left(\frac{n}{m}\right)^r\sum_{k=r}^{m} Var\left(r!\frac{W_{g_{m,k}}}{k!\binom{n}{k}} - \frac{m^r\binom{m}{k}\pi^k_{mn}(g_{m,k})}{\binom{m}{r}\binom{mn}{k}}\right).$$

Thus (5.18) and (5.19) applied to the first element of the above sum yields

$$\left(\frac{n}{m}\right)^r Var\left(r!\frac{W_{g_{m,r}}}{r!\binom{n}{r}} - \frac{m^r\binom{m}{r}\pi^r_{mn}(g_{m,r})}{\binom{m}{r}\binom{mn}{r}}\right)$$

$$= \frac{n^r}{m^r}\left[(r!)^2\frac{\binom{m}{r}}{\binom{n}{r}r!} - 2r!\frac{m^r}{\binom{m}{r}}\frac{\binom{m}{r}}{\binom{mn}{r}} + \left(\frac{m^r}{\binom{m}{r}}\right)^2\frac{\binom{m}{r}^2}{\binom{mn}{r}}\right]$$

$$= \frac{\binom{m}{r}}{m^r}\left[\frac{r!n^r}{\binom{n}{r}} - 2\frac{r!(mn)^r}{\binom{mn}{r}} + \frac{m^r}{\binom{m}{r}}\frac{(mn)^r}{\binom{mn}{r}}\right] \tag{5.20}$$

Since the term in the square brackets above converges to zero and the factor in front of it converges to $r!$ we conclude that the whole quantity converges to zero.

Note that

$$\left(\frac{n}{m}\right)^r\sum_{k=r+1}^{m} Var\left(r!\frac{W_{g_{m,k}}}{k!\binom{n}{k}} - \frac{m^r\binom{m}{k}\pi^k_{mn}(g_{m,k})}{\binom{m}{r}\binom{mn}{k}}\right)$$

$$\le \left(\frac{n}{m}\right)^r\sum_{k=r+1}^{m} Var\left(r!\frac{W_{g_{m,k}}}{k!\binom{n}{k}}\right) + \left(\frac{n}{m}\right)^r\sum_{k=r+1}^{m} Var\left(\frac{m^r\binom{m}{k}\pi^k_{mn}(g_{m,k})}{\binom{m}{r}\binom{mn}{k}}\right)$$

$$= I_1(n) + I_2(n)\ .$$

Now treat separately $I_1(n)$ and $I_2(n)$.

For $I_1(n)$ we use the second line of (5.19) to get

$$I_1(n) = \frac{(r!)^2 n^r}{m^r} \sum_{k=r+1}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} \frac{1}{k!} E(g_{m,k}^2)$$

$$\leq (r!)^2 \sum_{k=r+1}^{m} \frac{m^{k-r}}{n^{k-r}} \frac{1}{\left(1-\frac{m}{n}\right)^k} \frac{E(g_{m,k}^2)}{k!}$$

Since for n large enough $\frac{m}{n} < \frac{1}{2}$ then for such n's

$$I_1(n) \leq 2^r (r!)^2 \sum_{k=r+1}^{m} \left(\frac{m}{n}\right)^{k-r} \frac{E(g_{m,k}^2)}{k!}$$

and thus by (5.17) it follows that $I_1(n)$ converges to zero.

For $I_2(n)$ we have

$$I_2(n) = \left(\frac{n}{m}\right)^r \left(\frac{m^r}{\binom{m}{r}}\right)^2 \sum_{k=r+1}^{m} \frac{\binom{m}{k}^2}{\binom{mn}{k}} E\, g_{m,k}^2$$

$$= \left(\frac{m^r}{\binom{m}{r}}\right)^2 \sum_{k=r+1}^{m} \frac{\binom{m}{k}}{m^r} \frac{n^k \binom{m}{k}}{n^{k-r}\binom{mn}{k}} E\, g_{m,k}^2$$

$$\leq \left(\frac{m^r}{\binom{m}{r}}\right)^2 \frac{1}{\left(1-\frac{1}{n}\right)^m} \sum_{k=r+1}^{m} \frac{\left(\frac{m}{n}\right)^{k-c}}{k!} E\, g_{m,k}^2.$$

Since the expression standing just in front of the sum above is bounded then (5.17) implies that $I_2(n) \to 0$.

Thus $Var\ \Omega_n(\mathbf{g}_m) \to 0$ implying $\Omega_n(\mathbf{g}_m) \xrightarrow{P} 0$. $\square$

In our next result we consider the case $\lambda > 0$.

Theorem 5.2.2. *Let $(\mathbb{X}^{(m_n,n)})$ be a sequence of matrices in $M_{m_n \times n}$ embedded in an infinite matrix $\mathbb{X}^{(\infty)}$ of iid entries. Assume $m_n/n \to \lambda > 0$. Consider a sequence of P-statistics $\left(Per^{(m_n,n)}_{h^{(m_n)}}(\mathbb{X}^{(m_n,n)})\right)$ with a common level of non-degeneracy equal to r. Assume that $E\left[(h^{(m_n)})^2\right] < \infty$, $n = 1, 2, \ldots$, and that there exists a sequence of functions $\mathbf{g} = (g_k)$, such that g_k is a symmetric function on $\mathbf{R}^k$ satisfying (5.10), $k = c, c+1, \ldots$ as well as*

$$\sum_{k=r}^{\infty} \frac{\lambda^k}{k!} E\, g_k^2 < \infty \qquad \text{and} \qquad \sum_{k=r}^{\infty} \frac{\left(\frac{m_n}{n}\right)^k}{k!} E\, g_k^2 < \infty \qquad \text{uniformly in } n \tag{5.21}$$

and

$$\sum_{k=r}^{m_n} \frac{\left(\frac{m_n}{n}\right)^k}{k!} E\left(g_{m_n,k} - g_k\right)^2 \to 0. \tag{5.22}$$

Then

$$\frac{Per_{h^{(m_n)}}^{(m_n,n)}(\mathbb{X}^{(m_n,n)}) - E\,Per_{h^{(m_n)}}^{(m_n,n)}(\mathbb{X}^{(m_n,n)})}{\binom{n}{m_n} m_n!} \xrightarrow{d} \sum_{k=r}^{\infty} \frac{\lambda^{k/2}}{k!} J_k(g_k).$$

Proof. Consider a U-statistic $U_{m_n n}^{(m_n)}$ with the kernel h_{m_n} based on an iid sample $X_{1,1}, \ldots, X_{m_n,n}$. Then $m_n^2/(m_n n) = m_n/n \to \lambda$. Consequently, the assumptions of Theorem 5.1.3 are satisfied (with n changed into $m_n n$) and thus (writing $m = m_n$)

$$U_{mn}^{(m)} - E\,U_{mn}^{(m)} \xrightarrow{d} \sum_{k=c}^{\infty} \frac{\lambda^{k/2}}{k!} J_k(g_k).$$

To complete the proof we will argue that

$$\frac{1}{\binom{n}{m} m!}\left[Per_{h^{(m)}}^{(m,n)}(\mathbb{X}^{(m,n)}) - E\,Per_{h^{(m)}}^{(m,n)}(\mathbb{X}^{(m,n)})\right] - \left[U_{mn}^{(m)} - E\,U_{mn}^{(m)}\right]$$

$$= \sum_{k=r}^{m} (k!)^{-1} \binom{n}{k}^{-1} W_{g_{m,k}}^{(m,n)}(\mathbb{X}^{(m,n)}) - \sum_{k=r}^{m} \binom{m}{k}\binom{mn}{k}^{-1} \pi_{mn}^{k}(g_{m,k})(\mathbb{X}^{(m,n)})$$

$$= \Omega_n(\mathbf{g}_m) \xrightarrow{P} 0. \tag{5.23}$$

Firstly, note that by orthogonality of $(g_{m_n,k})_k$ and $(g_k)_k$ and due to (5.22)

$$E\left(\sum_{k=r}^{m} (k!)^{-1} \binom{n}{k}^{-1} \left[W_{g_{m,k}}^{(m,n)} - W_{g_k}^{(m,n)}\right]\right)^2 = \sum_{k=r}^{m} \frac{\binom{m}{k}}{\binom{n}{k} k!} E(g_{m,k} - g_k)^2 \to 0.$$

Similarly, by the inequality

$$\binom{mn}{k} > \binom{m}{k}\binom{n}{k} k! \tag{5.24}$$

it follows that

$$\begin{aligned} &E\left(\sum_{k=r}^{m} \binom{m}{k}\binom{mn}{k}^{-1} \left[\pi_{mn}^{k}(g_{m,k}) - \pi_{mn}^{k}(g_k)\right]\right) \\ &= \sum_{k=r}^{m} \binom{m}{k}^2 \binom{mn}{k}^{-1} E(g_{m,k} - g_k)^2 \\ &\le \sum_{k=r}^{m} \frac{\binom{m}{k}}{\binom{n}{k} k!} E(g_{m,k} - g_k)^2 \to 0. \end{aligned}$$

Thus to show the relation (5.23) it suffices to prove that

$$\Omega_n(\mathbf{g}) \stackrel{P}{\to} 0. \tag{5.25}$$

This will be accomplished by showing that the variance of the respective difference tends to zero. Note that symmetry of the kernel g_k's entails

$$Cov\left(\sum_{k=r}^{m}(k!)^{-1}\binom{n}{k}^{-1}W_{g_k}^{(m,n)},\ \sum_{k=r}^{m}\binom{m}{k}\binom{mn}{k}^{-1}\pi_{mn}^{k}(g_k)\right)$$

$$= Var\left(\sum_{k=c}^{m}\binom{m}{k}\binom{mn}{k}^{-1}\pi_{mn}^{k}(g_k)\right) = \sum_{k=r}^{m}\frac{\binom{m}{k}^2}{\binom{mn}{k}}E(g_k^2).$$

Note also that by (2.13) it follows that

$$Var\left(\sum_{k=r}^{m}(k!)^{-1}\binom{n}{k}^{-1}W_{g_k}^{(m,n)}\right) = \sum_{k=r}^{m}\frac{\binom{m}{k}}{\binom{n}{k}}\frac{1}{k!}E(g_k^2).$$

Thus

$$Var\left(\Omega_n(\mathbf{g})\right) = \sum_{k=r}^{m}\left[\frac{\binom{m}{k}}{\binom{n}{k}}\frac{1}{k!} - \frac{\binom{m}{k}^2}{\binom{mn}{k}}\right]E(g_k^2).$$

Hence, by the second part of (5.21), we have

$$\sum_{k=N}^{m}\frac{\binom{m}{k}}{\binom{n}{k}}\frac{1}{k!}E(g_k^2) \to 0$$

uniformly in n as $N \to \infty$. Similarly, using once again the inequality (5.24), we get

$$\sum_{k=N}^{m}\frac{\binom{m}{k}^2}{\binom{mn}{k}}E(g_k^2) \to 0$$

uniformly in n as $N \to \infty$. Now, fixing N sufficiently large we see that

$$\sum_{k=r}^{N}\left[\frac{\binom{m}{k}}{\binom{n}{k}}\frac{1}{k!} - \frac{\binom{m}{k}^2}{\binom{mn}{k}}\right]E(g_k^2) \to 0$$

as $n \to \infty$ since both

$$\frac{\binom{m}{k}}{\binom{n}{k}}\frac{1}{k!} \qquad \text{and} \qquad \frac{\binom{m}{k}^2}{\binom{mn}{k}}$$

converge, as $n \to \infty$ to the same limit $\frac{\lambda^k}{k!}$ and k assumes only finite number of values $1, \ldots, N$. Hence (5.25) follows and the proof of the result is complete.

$\square$

Let us note that in practice the assumptions 5.21 and 5.22 of the above theorem may be often replaced by a set of alternative conditions being easier to verify. We outline them in the remarks below.

Remark 5.2.1. Suppose that exist positive constants A, B, C and D such that

$$|g_k| \leq AB^k \qquad \text{for any } k = r, r+1, \ldots \tag{5.26}$$

and

$$|g_{m_n,k}| \leq CD^k \qquad \text{for any } k = r, r+1, \ldots, m_n \;\; n = 1, 2, \ldots, \tag{5.27}$$

as well as for any $k = r, r+1, \ldots$, a point-wise convergence

$$g_{m_n,k} \to g_k \qquad \text{as } n \to \infty \tag{5.28}$$

holds, then the assumptions (5.21) and (5.22) of Theorem 5.2.2 are satisfied.

Remark 5.2.2. Since by (5.9)

$$g_{m_n,k} = \sum_{i=1}^{k} (-1)^{k-i} \pi_i^k(\tilde{h}_{m_n,i}), \tag{5.29}$$

where

$$\tilde{h}_{m_n,i}(x_1, \ldots, x_i) = E(h_{m_n}(x_1, \ldots, x_i, X_{i+1}, \ldots, X_{m_n})) - E(h_{m_n}),$$

then it follows that if there exist positive constants a and b such that

$$|\tilde{h}_{m_n,i}| < ab^i$$

for any $i = r, r+1, \ldots$ and any $n = 1, 2, \ldots$, then

$$|g_{m_n,k}| \leq \sum_{i=1}^{k} \binom{k}{i} ab^i \leq a(1+b)^k$$

for any $k = r, r+1, \ldots, m_n$ and any $n = 1, 2, \ldots$ and therefore (5.27) is satisfied.

Remark 5.2.3. Assume that there exist positive constants A, B, C and D such that

$$E(g_k^2) \leq AB^k \qquad \text{for any } \; k = r, r+1, \ldots \tag{5.30}$$

and

$$E(g_{m_n,k}^2) \leq CD^k \quad \text{for any } k = r, r+1, \ldots, m_n \text{ and any } n = 1, 2, \ldots \tag{5.31}$$

It is worthy to note that due to the formula (1.19) written is our current notation as

$$E(g^2_{m_n,k}) = \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} E(\tilde{h}^2_{m_n,i}),$$

it follows that (5.31) is implied by the following condition: there exist positive constants c and d such that

$$E(\tilde{h}^2_{m_n,k}) \le cd^k \qquad \text{for any } k = r, r+1, \ldots, m_n \text{ and for any } n = 1, 2, \ldots. \tag{5.32}$$

It is quite easy to see that (5.30) and (5.31) (or (5.26) and (5.27)) and the point-wise convergence (5.28) imply that the assumptions of Theorem 5.2.2 hold true: The first part of (5.21) is obvious. The second part of (5.21) follows immediately from the fact that the sequence (m_n/n) is bounded. To see that (5.22) is also satisfied, let us for any $\varepsilon > 0$ choose M large enough to have

$$\sum_{k=M+1}^{\infty} \frac{\left(\frac{m_n}{n}\right)^k}{k!} \left(AB^k + CD^k\right) < \varepsilon/2.$$

Note that M does not depend on n. Then, by (5.28), we can take N large enough to have for $n \ge N$

$$\sum_{k=r}^{M} \frac{\left(\frac{m_n}{n}\right)^k}{k!} E(g_{m_n,k} - g_k)^2) < \varepsilon/2.$$

Remark 5.2.4. Note that (5.31) implies

$$\sum_{k=r+1}^{m_n} \frac{\left(\frac{m_n}{n}\right)^{k-r}}{k!} E\, g^2_{m_n,k} \le \frac{m_n}{n} CD^{r+1} \sum_{l=0}^{\infty} \frac{\left(\frac{Dm_n}{n}\right)^l}{l!} = CD^{r+1} \frac{m_n}{n} e^{\frac{Dm_n}{n}} \to 0.$$

Thus (5.17) is satisfied and Theorem 5.2.1 holds. Similarly from Remark 5.2.3 it follows that (5.32) implies (5.17).

5.3 Examples

In this section we shall illustrate the applicability of the results from the previous section by revisiting some of the examples introduced in Chapter 1. We shall simplify slightly the notation by writing $Per\,(h_m)$ for $Per^{(m_n,n)}_{h^{(m_n)}} \mathbb{X}^{(m,n)}$ also m for m_n while continuing to assume that, as in Theorem 5.2.2, $m/n \to \lambda > 0$.

Example 5.3.1 (Number of monochromatic matchings in multicolored graph). In the setting of Example 1.7.2, let as before $P(X = k) = p_k$, $k = 0, 1, \ldots, N$, where $N \leq \infty$ and $X_{i,j} \stackrel{d}{=} X$. If

$$h_m(x_1, \ldots, x_m) = \sum_{k=1}^{N} \prod_{i=1}^{m} I(x_i = k).$$

Then $Per_{h_{m,n}} = \mathcal{M}(m, n)$ counts monochromatic perfect matchings in a corresponding random bipartite graph $(G, [y_{i,j}])$.

Note that $E(h_m) = \sum_{k=1}^{N} p_k^m$ and, moreover,

$$g_{m,k}(x_1, \ldots, x_k) = \sum_{i=1}^{N} p_i^m \prod_{l=1}^{k} \left(\frac{1}{p_i} I(x_l = i) - 1 \right).$$

Consider first the case of $N < \infty$ and $p_i = p$, $i = 1, \ldots, N$. Then $E(h_m) = Np^m$ and changing h_m into $\tilde{h}_m = h_m/(Np^m)$ we get

$$\begin{aligned} & g_{m,k}(x_1, \ldots, x_k) \\ & = g_k(x_1, \ldots, x_k) = \frac{1}{N} \sum_{i=1}^{N} \prod_{l=1}^{k} \left(\frac{1}{p} I(x_l = i) - 1 \right) \\ & = \frac{1}{N} \left(\frac{1-p}{p} \right)^{k/2} \sum_{i=1}^{N} \phi_i^{\otimes k}(x_1, \ldots, x_k) , \end{aligned}$$

where

$$\phi_i(x) = \frac{I(x = i) - p}{\sqrt{p(1-p)}} , \qquad i = 1, \ldots, N,$$

are standardized. Thus if $m/n \to \lambda > 0$ then from Theorem 5.2.2 we conclude that

$$\begin{aligned} \frac{\mathcal{M}(m, n)}{Np^m \binom{n}{m} m!} & \stackrel{d}{\to} 1 + \frac{1}{N} \sum_{k=1}^{\infty} \frac{\left(\lambda \frac{1-p}{p} \right)^{k/2}}{k!} \sum_{i=1}^{N} H_k(J_1(\phi_i)) \\ & = \frac{1}{N} \exp\left(-\frac{\lambda(1-p)}{2p} \right) \sum_{i=1}^{N} \exp\left(\sqrt{\lambda} Z_i \right), \end{aligned}$$

where $(Z_1, \ldots, Z_N)$ is an N-variate centered normal random vector with covariances: $E(Z_i Z_j) = -1$ for $i \neq j$ and $EZ_i^2 = (1-p)p^{-1}$, $i, j \in \{1, \ldots, N\}$.

If $m/n \to 0$ then Remark 5.2.4 and Theorem 5.2.1 imply

$$\sqrt{\frac{n}{m}} \left(\frac{M(m, n)}{p^m \binom{n}{m} m!} - N \right) \stackrel{d}{\to} \mathcal{Z},$$

where $\mathcal{Z}$ is a zero mean Gaussian random variable with variance $Var(\mathcal{Z}) = N^2 \frac{1-p}{p}$.

In the general case, i.e. $N \leq \infty$, let $p = \max_{1 \leq i \leq N} p_i$ and let $j_1, \ldots, j_K$ be such that $p_{j_s} = p$, $s = 1, \ldots, K$. Then we define $\tilde{h}_m = h_m/(Kp^m)$ and note that $E\tilde{h}_m \to 1$ as $n \to \infty$ since

$$\left| E\tilde{h}_m - 1 \right| = \left| \frac{\sum_{i \geq 1} p_i^m}{Kp^m} - 1 \right| = \sum_{i \notin \{j_1, \ldots, j_K\}} \left(\frac{p_i}{p} \right)^m$$
$$\leq \left(\frac{\sup_{i \notin \{j_1, \ldots, j_K\}} p_i}{p} \right)^{m-1} \sum_{i \notin \{j_1, \ldots, j_K\}} \frac{p_i}{p} \to 0$$

on noting that $\sup_{i \notin \{j_1, \ldots, j_K\}} p_i < p$. Consequently,

$$g_{m,k}(x_1, \ldots, x_k) = \frac{1}{K} \sum_{i=1}^{N} \left(\frac{p_i}{p} \right)^m \prod_{l=1}^{k} \left(\frac{1}{p_i} I(x_l = i) - 1 \right)$$

Define

$$g_k(x_1, \ldots, x_k) = \frac{1}{K} \sum_{s=1}^{K} \prod_{l=1}^{k} \left(\frac{1}{p} I(x_l = j_s) - 1 \right)$$
$$= \frac{1}{K} \left(\frac{1-p}{p} \right)^{k/2} \sum_{s=1}^{K} \psi_s^{\otimes k}(x_1, \ldots, x_k),$$

where $\psi_s = \phi_{j_s}$, $s = 1, \ldots, K$, are standardized and $k = 1, 2, \ldots$. Then $|g_k| \leq C^k$ and (5.21) is satisfied. Moreover,

$$|g_{m,k}(x_1, \ldots, x_k) - g_k(x_1, \ldots, x_k)|$$
$$= \frac{1}{K} \sum_{i \notin \{0, j_1, \ldots, j_K\}} \left(\frac{p_i}{p} \right)^m \prod_{l=1}^{k} \left(\frac{1}{p_i} I(x_l = i) - 1 \right)$$
$$\leq C^k \sum_{i \notin \{j_1, \ldots, j_K\}} \left(\frac{p_i}{p} \right)^m \leq C^k \delta^{m-1} \sum_{i \notin \{j_1, \ldots, j_K\}} \frac{p_i}{p} \leq \delta^{m-1} C^k / p \to 0,$$

since $\delta = \sup_{i \notin \{j_1, \ldots, j_K\}} p_i < p$ and thus (5.22) is also satisfied. Consequently, in the case $m/n \to \lambda > 0$ Theorem 5.2.2 implies

$$\frac{\mathcal{M}(m,n)}{Kp^m \binom{n}{m} m!} \xrightarrow{d} 1 + \frac{1}{K} \sum_{k=1}^{\infty} \frac{\left(\lambda \frac{1-p}{p} \right)^{k/2}}{k!} \sum_{i=1}^{K} H_k(J_1(\phi_{j_i}))$$
$$= \frac{1}{K} \exp\left(-\frac{\lambda(1-p)}{2p} \right) \sum_{i=1}^{K} \exp\left(\sqrt{\lambda} Z_i \right),$$

where $(Z_1, \ldots, Z_K)$ is a K-dimensional centered normal random vector with covariances $E(Z_i Z_j) = -1$ for $i \neq j$ and $E(Z_i^2) = (1-p)p^{-1}$, $i, j \in \{1, \ldots, K\}$.

Alternatively, if $m/n \to 0$ then it follows from Theorem 5.2.1 (via Remark 5.2.4) that

$$\sqrt{\frac{n}{m}} \left(\frac{M(m,n)}{p^m \binom{n}{m} m!} - K \right) \xrightarrow{d} \mathcal{Z},$$

where $\mathcal{Z}$ is a zero mean Gaussian random variable with variance $Var(\mathcal{Z}) = K^2 \frac{1-p}{p}$.

In our remaining examples we consider always a set of weights in a complete bipartite graph $K(m,n)$.

Example 5.3.2 (Number of matchings with L red edges in bicolored graph). Consider a random bicolored complete bipartite graph $K(m,n)$ from Example 1.7.3 with edges which are either red ($X_{i,j} = 1$) with probability p or black ($X_{i,j} = 0$) with probability $1-p$. We are interested in the number $\mathcal{K}(n,\alpha)$ of perfect matchings with a given number L_n of red edges as $n \to \infty$, assuming that $L_n/m \to \alpha \in [0,1]$. Thus $\mathcal{K}(n,\alpha) = Per(h_m)$ for

$$h_m(x_1, \ldots, x_m) = I(x_1 + \ldots + x_m = L_n), \qquad x_j \in \{0,1\}.$$

Note that

$$E(h_m) = \binom{m}{L_n} p^{L_n} (1-p)^{m-L_n}.$$

Moreover,

$$\begin{aligned} &h_{m,i}(x_1, \ldots, x_i) \\ &= \binom{m-i}{L_n - x_1 - \ldots - x_i} p^{L_n - x_1 - \ldots - x_i} (1-p)^{m-i-L_n+x_1+\ldots+x_i}, \quad x_j \in \{0,1\}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\tilde{h}_{m,i}(x_1, \ldots, x_i) \\ &= \frac{h_{m,i}(x_1, \ldots, x_i) - E(h_m)}{E(h_m)} = \frac{\binom{m-i}{L_n - x_1 - \ldots - x_i}}{\binom{m}{L_n}} \left(\frac{1-p}{p} \right)^{x_1+\ldots+x_i} (1-p)^{-i} - 1. \end{aligned}$$

and thus by (5.9) we have

$$\begin{aligned} g_{m,k}(x_1, \ldots, x_k) &= \sum_{i=1}^{k} (-1)^{k-i} \sum_{1 \le j_1 < \ldots < j_i \le k} \tilde{h}_{m,i}(x_{j_1}, \ldots, x_{j_i}) \\ = \sum_{i=1}^{k} (-1)^{k-i} &\left(\sum_{1 \le j_1 < \ldots < j_i \le k} \frac{\binom{m-i}{L_n - x_{j_1} - \ldots - x_{j_i}}}{\binom{m}{L_n}} \left(\frac{1-p}{p} \right)^{x_{j_1}+\ldots+x_{j_i}} (1-p)^{-i} - 1 \right). \end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{\binom{m-i}{L_n-j}}{\binom{m}{L_n}} = \alpha^j(1-\alpha)^{i-j}$$

then

$$\begin{aligned}
&g_{m,k}(x_1,\ldots,x_k)\\
&\to \sum_{i=1}^{k}(-1)^{k-i}\left(\sum_{1\le j_1<\ldots<j_i\le k}\left(\frac{\alpha}{p}\right)^{x_{j_1}+\ldots+x_{j_i}}\left(\frac{1-\alpha}{1-p}\right)^{i-x_{j_1}-\ldots-x_{j_i}}-1\right)\\
&=\prod_{i=1}^{k}\left[\left(\frac{\alpha}{p}\right)^{x_i}\left(\frac{1-\alpha}{1-p}\right)^{1-x_i}-1\right]\\
&=\left(\frac{|\alpha-p|}{\sqrt{p(1-p)}}\right)^k\prod_{i=1}^{k}\phi(x_i)=g_k(x_1,\ldots,x_k),
\end{aligned}$$

with a standardized

$$\phi(x)=\frac{\sqrt{p(1-p)}}{|\alpha-p|}\left[\left(\frac{\alpha}{p}\right)^{x}\left(\frac{1-\alpha}{1-p}\right)^{1-x}-1\right],\quad x\in\{0,1\}.$$

Note that for any $k \ge c$

$$|g_k| \le \left(\max\left\{\left|\frac{\alpha}{p}-1\right|,\left|\frac{1-\alpha}{1-p}-1\right|\right\}\right)^k.$$

Also, denoting $s = x_1 + \ldots + x_i$, we have

$$|\tilde{h}_{m,i}| \le \left(\frac{L_n}{m}\right)^s\left(1-\frac{L_n}{m_n}\right)^{i-s}\frac{1}{p^s}\frac{1}{(1-p)^{i-s}}+1 \le 2\left(\frac{1}{\min\{p,1-p\}}\right)^i.$$

Thus by (5.26), (5.27) and (5.28) we conclude that the assumptions of Theorems 5.2.2 and 5.2.1 are satisfied.

Then if $m/n \to \lambda > 0$

$$\begin{aligned}
\frac{\mathcal{K}(n,\alpha)}{\binom{n}{m_n}m_n!\binom{m_n}{L_n}p^{L_n}(1-p)^{m-L_n}} &\xrightarrow{d} 1+\sum_{k=1}^{\infty}\frac{\left(\sqrt{\lambda}\frac{|\alpha-p|}{\sqrt{p(1-p)}}\right)^k}{k!}H_k(J_1(\phi))\\
&=\exp\left(\sqrt{\lambda}\mathcal{Z}-\frac{\lambda(\alpha-p)^2}{2p(1-p)}\right),
\end{aligned}$$

where $\mathcal{Z}$ is a Gaussian random variable with zero mean and variance $Var(Z) = \frac{(\alpha-p)^2}{2p(1-p)}$.

If $m/n \to 0$ then

$$\sqrt{\frac{n}{m}}\left(\frac{\mathcal{K}(n,\alpha)}{\binom{n}{m_n}m_n!\binom{m_n}{L_n}p^{L_n}(1-p)^{m-L_n}}-1\right)\xrightarrow{d}\mathcal{Z}.$$

Example 5.3.3 (Number of heavy normal matchings). Consider a complete graph $K(m,n)$ with random weights $\mathbb{X}^{(m,n)} = [X_{i,j}]$ which are iid $\mathcal{N}(0,1)$ random variables. We are interested in asymptotics of the number $\mathcal{W}(n,\alpha)$ of perfect matchings for which the average total weight exceeds given level $\alpha > 0$. More precisely, let

$$h_m(x_1,\ldots,x_m) = I(x_1+\ldots+x_m > \alpha m).$$

Then

$$W_n(\alpha) = Per(h_m).$$

Note that

$$E(h_m) = P(X_1+\ldots+X_m \geq \alpha m),$$

where (X_i) are iid $\mathcal{N}(0,1)$ random variables. Thus

$$E(h_m) = 1 - \Phi(\alpha\sqrt{m}),$$

where Φ denotes the distribution function of the standard normal distribution. Similarly

$$h_{m,i}(x_1,\ldots,x_i) = P(X_{i+1}+\ldots+X_m \geq \alpha m - s) = 1 - \Phi\left(\frac{\alpha m - s}{\sqrt{m-i}}\right),$$

where $s = x_1+\ldots+x_i$. Thus

$$\tilde{h}_{m,i} = \frac{1-\Phi\left(\frac{\alpha m - s}{\sqrt{m-i}}\right)}{1-\Phi(\alpha\sqrt{m})} - 1\,, \qquad i = 1,\ldots,m-1. \tag{5.33}$$

Now we use the classical double inequality for the tail of the distribution function of the standard normal distribution (see for instance Feller 1968, chapter 7).

$$\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\left(\frac{1}{x}-\frac{1}{x^3}\right) < 1-\Phi(x) < \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,\frac{1}{x} \tag{5.34}$$

for any $x > 0$.

Since for large n the arguments of Φ in (5.33) are positive we can use (5.34) respectively to the numerator and denominator in (5.33) to get the double inequality

$$\begin{aligned} &\exp\left[-\frac{1}{2}\left(\frac{(\alpha m - s)^2}{m-i} - \alpha^2 m\right)\right]\frac{\alpha\sqrt{m(m-i)}\,[(\alpha m-s)^2 - m + i]}{(\alpha m - s)^3} \\ &\quad < \tilde{h}_{m,i} + 1 < \exp\left[-\frac{1}{2}\left(\frac{(\alpha m - s)^2}{m-i} - \alpha^2 m\right)\right]\frac{\alpha^3 m^{3/2}\sqrt{m-i}}{(\alpha m - s)(\alpha^2 m - 1)}. \end{aligned} \tag{5.35}$$

Passing to the limit as $n \to \infty$ in (5.35) we get

$$\tilde{h}_{m,i} \to e^{\alpha s - \frac{i\alpha^2}{2}} - 1.$$

Consequently, by (5.9)

$$\begin{aligned} & g_{m,k}(x_1, \ldots, x_k) \\ & = \sum_{i=1}^{k} (-1)^{k-i} \sum_{1 \le j_1 < \ldots < j_i \le k} \tilde{h}_{m,i}(x_{j_1}, \ldots, x_{j_i}) \\ & \to \sum_{i=1}^{k} (-1)^{k-i} \sum_{1 \le j_1 < \ldots < j_i \le k} \left(e^{\alpha(x_{j_1} + \ldots + x_{j_i}) - \frac{i\alpha^2}{2}} - 1 \right) \\ & = \prod_{i=1}^{k} \left(e^{\alpha x_i - \frac{\alpha^2}{2}} - 1 \right) = \left(e^{\alpha^2} - 1 \right)^{k/2} \prod_{i=1}^{k} \phi^{\otimes k}(x_1, \ldots, x_k) = g_k(x_1, \ldots, x_k) \,, \end{aligned}$$

where

$$\phi(x) = \frac{e^{\alpha x - \frac{\alpha^2}{2}} - 1}{\sqrt{e^{\alpha^2} - 1}}$$

is standardized. Note that

$$E(g_k^2) = \left[E \left(e^{\alpha \mathcal{N} - \frac{\alpha^2}{2}} - 1 \right)^2 \right]^k .$$

and thus (5.21) is satisfied.

Since $\tilde{h}_{m,i}$ is not bounded we will use Remark 5.2.3. We need to show (5.32). Note that

$$E(\tilde{h}_{m,i}^2) = \frac{E \left[\Phi(\alpha\sqrt{m}) - \Phi\left(\frac{\alpha m - S_i}{\sqrt{m-i}} \right) \right]^2}{(1 - \Phi(\alpha\sqrt{m}))^2} \,,$$

where $S_i \sim \mathcal{N}(0, i)$, $i = 1, \ldots, m-1$. The numerator is bounded as follows

$$\begin{aligned} & E \left[\Phi(\alpha\sqrt{m}) - \Phi\left(\frac{\alpha m - S_i}{\sqrt{m-i}} \right) \right]^2 \\ & \qquad \le \frac{1}{2\pi} E \left(e^{-\min\left\{ \alpha^2 m, \frac{(\alpha m - S_i)^2}{m-i} \right\}} \left(\alpha\sqrt{m} - \frac{\alpha m - S_i}{\sqrt{m-i}} \right)^2 \right) \\ & \qquad \le \frac{1}{2\pi} \left[e^{-\alpha^2 m} E \left(\alpha\sqrt{m} + \frac{S_i - \alpha m}{\sqrt{m-i}} \right)^2 \right. \\ & \qquad\quad \left. + E \left(e^{-\frac{(S_i - \alpha m)^2}{m-i}} \left(\alpha\sqrt{m} + \frac{S_i - \alpha m}{\sqrt{m-i}} \right)^2 \right) \right] . \end{aligned}$$

Since $Z = \frac{S_i - \alpha m}{\sqrt{m-i}} \sim \mathcal{N}(-\frac{\alpha m}{\sqrt{m-i}}, \frac{i}{m-i}) = \mathcal{N}(\nu, \sigma^2)$ then

$$E(Z+\alpha\sqrt{m})^2 = \sigma^2 + \nu^2 + 2\alpha\sqrt{m}\nu + \alpha^2 m = \frac{i}{m-i}\left(1 + \frac{\alpha^2 m}{\sqrt{m}+\sqrt{m-i}}\right)$$
$$\leq (m-1)(1+\alpha^2\sqrt{m}) \leq (m-1)(1+2\alpha^2 m)e^{\alpha^2 i}$$

where the particular form of the last inequality is convenient to compare with the inequality for the second part derived below.

For the second part, we proceed as follows

$$E\left[(Z+\alpha\sqrt{m})^2 e^{-Z^2}\right] = EZ^2 e^{-Z^2} + 2\alpha\sqrt{m}EZe^{-Z^2} + \alpha^2 m E e^{-Z^2}$$

Since

$$\begin{aligned} Ee^{-Z^2} &= \frac{1}{\sqrt{1+2\sigma^2}} e^{-\frac{\nu^2}{1+2\sigma^2}} \\ EZe^{-Z^2} &= \frac{\nu}{(1+2\sigma^2)^{3/2}} e^{-\frac{\nu^2}{1+2\sigma^2}} \\ EZ^2 e^{-Z^2} &= \frac{1}{(1+2\sigma^2)^{3/2}}\left(\sigma^2 + \frac{\nu^2}{1+2\sigma^2}\right) e^{-\frac{\nu^2}{1+2\sigma^2}} \end{aligned}$$

Thus

$$E\left[(Z+\alpha\sqrt{m})^2 e^{-Z^2}\right]$$
$$= e^{-\alpha^2 m}\left(\frac{m-i}{m+i}\right)^{3/2}\left[\frac{i}{m-i} + \alpha^2 m\left(\frac{m}{m+i} - 2\sqrt{\frac{m}{m-i}} + \frac{m+i}{m-i}\right)\right] e^{\frac{\alpha^2 m}{m+i} i}$$
$$\leq e^{-\alpha^2 m}(m-1)(1+2\alpha^2 m)e^{\alpha^2 i}.$$

Consequently,

$$E\left[\Phi(\alpha\sqrt{m}) - \Phi\left(\frac{\alpha m - S_i}{\sqrt{m-i}}\right)\right]^2 \leq \frac{1}{\pi} e^{-\alpha^2 m}(1+2\alpha^2 m)e^{\alpha^2 i}.$$

Finally, we use the left inequality from (5.34) to get for large n and $i = 1, \ldots, m-1$

$$E(\tilde{h}_{m,i}^2) \leq \frac{1}{\pi} e^{-\alpha^2 m}(1+2\alpha^2 m)e^{\alpha^2 i}\sqrt{2\pi}e^{\frac{\alpha^2 m}{2}} \frac{m^{3/2}\alpha^3}{\alpha^2 m - 1} \leq C\left(e^{\alpha^2}\right)^i.$$

Additionally, for $i = m$ we have

$$E(\tilde{h}_{m,m}^2) = E\left(\frac{I(X_1 + \ldots + X_m > \alpha m)}{1 - \Phi(\alpha\sqrt{m})} - 1\right)^2 = \frac{\Phi(\alpha\sqrt{m})}{1-\Phi(\alpha\sqrt{m})}.$$

Again using the left inequality from (5.34) we get for large n that

$$E(\tilde{h}^2_{m,m}) < \sqrt{2\pi} e^{\frac{\alpha^2 m}{2}} \frac{\alpha^3 m^{3/2}}{\alpha^2 m - 1} = \sqrt{2\pi} \left[e^{-\frac{\alpha^2 m}{2}} \frac{\alpha^3 m^{3/2}}{\alpha^2 m - 1} \right] e^{\alpha^2 m} \le C \left(e^{\alpha^2} \right)^m .$$

Thus (5.32) is satisfied and we conclude that the assumptions of Theorem 5.2.2 are satisfied. Note that by (5.35)

$$E(h_m)\sqrt{2\pi}\alpha\sqrt{m} e^{\frac{\alpha^2 m}{2}} \to 1.$$

Consequently, if $m/n \to \lambda > 0$ then using Theorem 5.2.2 we obtain

$$\frac{\sqrt{m} e^{\frac{\alpha^2 m}{2}} \mathcal{W}(n,\alpha)}{\binom{n}{m} m!} \xrightarrow{d} \frac{1}{\alpha\sqrt{2\pi}} \exp\left[\sqrt{\lambda}\, \mathcal{Z} - \frac{\lambda(e^{\alpha^2} - 1)}{2} \right],$$

where $\mathcal{Z}$ is a zero mean Gaussian variable with variance $Var\,(\mathcal{Z}) = e^{\alpha^2} - 1$.
Similarly, if $m/n \to 0$ then Theorem 5.2.1 implies

$$\sqrt{\frac{n}{m}} \left(\frac{\mathcal{W}(n,\alpha)}{(1 - \Phi(\alpha\sqrt{m}))\binom{n}{m} m!} - 1 \right) \xrightarrow{d} \mathcal{Z}.$$

Example 5.3.4 (Number of perfect matchings with few light edges). Consider again $K(m,n)$ and let F denotes the common distribution function of the random weights $X_{i,j}$'s. We are interested in a number $\mathcal{R}_n(r,\alpha)$ of perfect matchings for which the r-th smallest weight (out of m weights) exceeds a given threshold α. That is

$$\mathcal{R}_n(r,\alpha) = Per(h_m)$$

for $h_m(x_1,\ldots,x_m) = I(x_{r:m} > \alpha)$, where $x_{r:m}$ denotes the rth order statistic out of m observations.

Note that

$$\begin{aligned} & I(x_{r:m} > \alpha) \qquad (5.36) \\ & = \sum_{s=0}^{r-1} \sum_{1 \le i_1 < \ldots < i_s \le m} \prod_{j \in \{i_1,\ldots,i_s\}} I(x_j \le \alpha) \prod_{l \in \{1,\ldots,m\} \setminus \{i_1,\ldots,i_s\}} I(x_l > \alpha), \end{aligned}$$

where the summand for $s = 0$ is understood as $\prod_{l=1}^m I(x_l > \alpha)$. Thus, by (5.36) we get

$$E(h_m) = \sum_{s=0}^{r-1} \binom{m}{s} F^s(\alpha) \bar{F}^{m-s}(\alpha),$$

where $\bar{F} = 1 - F$.

In order to calculate $h_{m,i}$ it is helpful to observe that the set

$$\{(x_1,\ldots,x_m):\ x_{r:m}>\alpha\}$$

can be decomposed in disjoint sets of the form

$\{(x_1,\ldots,x_m):$ exactly s elements out of $(x_{k+1},\ldots,x_m)$ are $\le \alpha$ and $x_{r-s:k}>\alpha\ \}$,

for $s=0,1,\ldots,r-1$.

We assume here that $x_{j:n}=-\infty$ if $j\le 0$ and $x_{j:n}=\infty$ if $j>n$. Since

$$h_{m,i}(x_1,\ldots,x_i)=E(h_m(x_1,\ldots,x_i,X_{i+1},\ldots,X_m))\ ,$$

where $X_{i+1},\ldots,X_m$ are iid with the df F, then by the above remark it follows that

$$h_{m,i}(x_1,\ldots,x_i)=\sum_{s=0}^{r-1}\binom{m-i}{s}F^s(\alpha)\bar{F}^{m-i-s}(\alpha)I(x_{r-s:i}>\alpha)$$

Thus

$$\begin{aligned}
&\tilde{h}_{m,i}(x_1,\ldots,x_i)\\
&=\frac{\sum_{s=0}^{r-1}\binom{m-i}{s}F^s(\alpha)\bar{F}^{m-i-s}(\alpha)I(x_{r-s:i}>\alpha)}{\sum_{s=0}^{r-1}\binom{m}{s}F^s(\alpha)\bar{F}^{m-s}(\alpha)}-1\\
&=\frac{1}{\bar{F}^i(\alpha)}\frac{\sum_{s=0}^{r-1}\binom{m-i}{s}F^s(\alpha)\bar{F}^{r-1-s}(\alpha)I(x_{r-s:i}>\alpha)}{\sum_{s=0}^{r-1}\binom{m}{s}F^s(\alpha)\bar{F}^{r-1-s}(\alpha)}-1\to\frac{I(x_{1:i}>\alpha)}{\bar{F}^i(\alpha)}-1.
\end{aligned}$$

Hence by (5.9) we get

$$\begin{aligned}
g_{m,k}(x_1,\ldots,x_k)&\to\prod_{i=1}^{k}\left(\frac{I(x_i>\alpha)}{\bar{F}(\alpha)}-1\right)\\
&=\left(\frac{F(\alpha)}{\bar{F}(\alpha)}\right)^{k/2}\prod_{i=1}^{k}\phi^{\otimes k}(x_1,\ldots,x_k)=g_k(x_1,\ldots,x_k),
\end{aligned}$$

where

$$\phi(x)=\frac{I(x>\alpha)-\bar{F}(\alpha)}{\sqrt{F(\alpha)\bar{F}(\alpha)}}$$

is standardized.

Note that $|g_k|\le\left(\max\{1,\frac{F(\alpha)}{\bar{F}(\alpha)}\}\right)^k$ and thus (5.26) is satisfied. Also

$$|\tilde{h}_{m,i}(x_1,\ldots,x_i)|\le\frac{1}{\bar{F}^i(\alpha)}\frac{\sum_{s=0}^{r-1}\binom{m-i}{s}F^s(\alpha)\bar{F}^{r-1-s}(\alpha)}{\sum_{s=0}^{r-1}\binom{m}{s}F^s(\alpha)\bar{F}^{r-1-s}(\alpha)}+1<2\left(\frac{1}{\bar{F}(\alpha)}\right)^i\ ,$$

and by Remark 5.2.2 it follows that the assumptions of Theorem 5.2.2 are satisfied. Since

$$\frac{E(h_m)}{m^{r-1}F^{r-1}(\alpha)\bar{F}^{m-r+1}(\alpha)} \to 1,$$

in the case $m_n/n \to \lambda$ Theorem 5.2.2 implies

$$\frac{\mathcal{R}_n(r,\alpha)}{\binom{n}{m}m!m^{r-1}\bar{F}^m(\alpha)} \xrightarrow{d} \left(\frac{F(\alpha)}{\bar{F}(\alpha)}\right)^{r-1} e^{\sqrt{\lambda}\mathcal{Z}-\frac{\lambda F(\alpha)}{2\bar{F}(\alpha)}},$$

where $\mathcal{Z}$ is a zero mean Gaussian variable with variance $Var(\mathcal{Z}) = \frac{F(\alpha)}{\bar{F}(\alpha)}$.

Note that the case $r = 1$ is the case of a classical permanent (Definition 1.20) since the kernel h_m is a product of the form $\prod_{i=1}^m I(x_i > \alpha)$. Thus the above asymptotics in this particular case follows from Theorem 3.4.3.

Alternatively, if $m/n \to 0$ then using Theorem 5.2.1 we get

$$\sqrt{\frac{n}{m}}\left(\frac{\mathcal{R}_n(r,\alpha)}{\binom{n}{m}m!\sum_{s=0}^{r-1}\binom{m}{s}F^s(\alpha)\bar{F}^{m-s}(\alpha)} - 1\right) \xrightarrow{d} \mathcal{Z}.$$

Example 5.3.5 (Sum of products of zero-mean weights for perfect matchings). Let $X_{i,j}$'s have zero mean and unit variance. Let $r \geq 1$ be fixed. For $K(m,n)$ consider $Per(h_m)$ with

$$h_m(x_1,\ldots,x_m) = \sum_{l=r}^{m} \sum_{1\leq j_1<\ldots<j_l\leq m} \prod_{s=1}^{l} x_{j_s}.$$

Obviously, $E(h_m) = 0$. Moreover, $h_i(x_1,\ldots,x_i) = 0$ for $i = 1,\ldots,r-1$, and

$$h_{m,i}(x_1,\ldots,x_i) = \sum_{l=r}^{i} \sum_{1\leq s_1<\ldots<s_l\leq i} \prod_{w=1}^{l} x_{s_w}$$

for $i = r, r+1, \ldots, m$, i.e. r is the common degeneracy level of all P-statistics in this example. Note that $h_{m,i}$'s do not depend on m. Using (5.9) we obtain

$$\begin{aligned} g_{m,k}(x_1,\ldots,x_k) &= \sum_{i=r}^{k}(-1)^{k-i} \sum_{1\leq j_1<\ldots<j_i\leq k} \sum_{l=r}^{i} \sum_{1\leq s_1<\ldots<s_l\leq i} \prod_{w=1}^{l} x_{j_{s_w}} \\ &= \sum_{i=r}^{k} \sum_{1\leq j_1<\ldots<j_i\leq k} \prod_{w=1}^{i} x_{j_w} \left(\sum_{l=0}^{k-i}\binom{k-i}{l}(-1)^{k-(i+l)}\right). \end{aligned}$$

Noting that the expression in the parantheses in the above formula is zero except in the case $i = k$ we obtain

$$g_{m,k}(x_1,\ldots,x_k) = \prod_{l=1}^{k} x_l = g_k(x_1,\ldots,x_k),$$

i.e. $\phi(x) = x$ (standardized). Since $E(g_{m,k}^2) = E(g_k^2) = 1$ the assumptions of Theorems 5.2.2 and 5.2.1 apparently are satisfied.

Consequently, for $m/n \to \lambda > 0$,

$$\frac{Per(h_m)}{\binom{n}{m}m!} \xrightarrow{d} e^{\sqrt{\lambda}\mathcal{N}-\frac{\lambda}{2}} - \sum_{k=0}^{r-1} \frac{\lambda^{k/2}}{k!} H_k(\mathcal{N}).$$

In particular, if $r = 2$ the limiting law is the distribution of the random variable $e^{\sqrt{\lambda}\mathcal{N}-\frac{\lambda}{2}} - 1 - \sqrt{\lambda}\mathcal{N}$.

For $m/n \to 0$

$$\sqrt{\frac{n}{m}}\frac{Per(h_m)}{\binom{n}{m}m!} \xrightarrow{d} H_c(\mathcal{N}).$$

Let us note that in the first four examples above we had:

$$\frac{h_{m,i}(x_1,\ldots,x_i)}{E(h_m)} \to \frac{1}{L}\sum_{r=1}^{L}\prod_{l=1}^{i}\psi_r(x_l)$$

for some natural number L, a positive number α and some functions ψ_r, satisfying $E(\psi_r^2(X)) = 1$, $r = 1,\ldots,L$. If this is the case then in general

$$g_{m,k}(x_1,\ldots,x_k) \to \frac{1}{L}\sum_{r=1}^{L}\prod_{l=1}^{k}(\psi_r(x_l) - 1) = g_k(x_1,\ldots,x_k).$$

Consequently, if only the assumptions of Theorem 5.2.2 are satisfied then

$$\frac{Per(h_m)}{\binom{n}{m}m!E(h_m)} \to \frac{1}{L}\sum_{r=1}^{L} e^{\sqrt{\lambda}Z_r - \frac{\lambda(E\psi_r^2(X)-1)}{2}} \tag{5.37}$$

where $(Z_1,\ldots,Z_L)$ is a zero-mean Gaussian vector with covariances

$$E(Z_rZ_s) = E(\psi_r(X)\psi_s(X)) - 1.$$

If in the situation just described the edges $[y_{i,j}]$ in a bipartite graph appear independently with the same probability q and independently of the weight matrix $\mathbb{X}^{(\infty)}$, then we have to modify the kernel of a P-statistic as follows

$$\hat{h}_m((x_1,y_1),\ldots,(x_m,y_m)) = h_m(x_1,\ldots,x_m)\prod_{l=1}^{m} I(y_l = 1),$$

where $y_l = 1$ if the respective edge is present in the graph, otherwise $y_l = 0$ (see also Section 1.7). Then, it is easy to see that

$$\frac{\hat{h}_{m,i}((x_1,y_1),\ldots,(x_i,y_i))}{E(\hat{h}_m)}$$
$$= \frac{h_{m,i}(x_1,\ldots,x_i)}{E(h_m)} \prod_{l=1}^{i} \frac{I(y_l=1)}{q} \to \frac{1}{L}\sum_{r=1}^{L}\prod_{l=1}^{i} \psi_r(x_l)\frac{I(y_l=1)}{q}.$$

Consequently,

$$g_{m,k}((x_1,y_1),\ldots,(x_k,y_k)) \to \frac{1}{L}\sum_{r=1}^{L}\prod_{l=1}^{k}\left(\psi_r(x_l)\frac{I(y_l=1)}{q}-1\right)$$
$$= g_k((x_1,y_1),\ldots,(x_k,y_k)).$$

In this situation the convergence result (5.37) has to be modified as follows. If the assumptions of Theorem 5.2.2 are satisfied then

$$\frac{Per(\hat{h}_m)}{\binom{n}{m}m!q^m E(h_m)} \to \frac{1}{L}\sum_{r=1}^{L}\exp\left(\sqrt{\lambda}\hat{Z}_r - \frac{\lambda(E\psi_r^2(X)-q)}{2q}\right), \tag{5.38}$$

where $(\hat{Z}_1,\ldots,\hat{Z}_L)$ is a zero-mean Gaussian random vector with covariances

$$E(\hat{Z}_r\hat{Z}_s) = \frac{E(\psi_r(X)\psi_s(X))}{q} - 1.$$

For instance if the edges appear at random, as described above, in the Examples 5.3.2–5.3.4 we obtain the following modifications of the limiting behavior for $m/n \to \lambda > 0$:

In Example 5.3.2 for the number of red edges $\mathcal{K}(n,\alpha)$:

$$\frac{\mathcal{K}(n,\alpha)}{\binom{n}{m}m!\binom{m}{L_n}p^{L_n}(1-p)^{m-L_n}q^m} \to \exp\left(\sqrt{\lambda}\,\mathcal{Z} - \frac{\lambda[(\alpha-p)^2+(1-q)p(1-p)]}{2qp(1-p)}\right),$$

where $\mathcal{Z}$ is a zero mean Gaussian variable with variance $Var(\mathcal{Z}) = \frac{(\alpha-p)^2}{qp(1-p)} + \frac{1-q}{q}$.

In Example 5.3.3 for the number of heavy normal matchings $\mathcal{W}(n,\alpha)$:

$$\frac{\sqrt{m}e^{\frac{\alpha^2 m}{2}}\mathcal{W}(n,\alpha)}{\binom{n}{m}m!q^m} \xrightarrow{d} \frac{1}{\alpha\sqrt{2\pi}}\exp\left[\sqrt{\lambda}\mathcal{Z} - \frac{\lambda(e^{\alpha^2}-q)}{2q}\right],$$

where $\mathcal{Z}$ is a zero mean Gaussian variable with variance $Var(\mathcal{Z}) = \frac{e^{\alpha^2}-q}{q}$.

Example 5.3.4 for the number of matchings with few light edges $\mathcal{R}_n(r,\alpha)$:

$$\frac{\mathcal{R}_n(r,\alpha)}{\binom{n}{m}m!m^{r-1}[q\bar{F}(\alpha)]^m} \xrightarrow{d} \left(\frac{F(\alpha)}{\bar{F}(\alpha)}\right)^{r-1} e^{\sqrt{\lambda}\mathcal{Z} - \frac{\lambda(1-q\bar{F}(\alpha))}{2q\bar{F}(\alpha)}},$$

where $\mathcal{Z}$ is a zero mean Gaussian variable with the variance $Var(\mathcal{Z}) = \frac{1-q\bar{F}(\alpha)}{q\bar{F}(\alpha)}$.

5.4 Bibliographic Details

Multiple Wiener-Itô integral was introduced by Itô (1951) who considerably modified the original idea of homogeneous chaos introduced by Wiener (1938). A detailed description can be found in the monograph by Major (1981) which is devoted solely to this issue. A concise and quite accessible exposition of the topic has been given recently in Kuo (2006). Here we adopted an approach which borrowed the first step from Dynkin and Mandelbaum (1983) and was especially convenient in the context of symmetric functions, which were of the primary interest.

The fundamental result on the weak convergence of non-degenerate U-statistics was obtained by Hoeffding (1948) in his seminal paper on the subject. Subsequent results for degenerate cases were obtained by Serfling (1980), Rubin and Vitale (1980) and Dynkin and Mandelbaum (1983) who were the first to introduce the idea of writing a limiting distribution in terms of multiple Wiener-Itô integrals. For a general review of the subject of multiple stochastic integration in the context of symmetric functions see e.g., monographs by Major (1981) or Lee (1990, chapter 4). There are also several other good monographs that consider, among other topics, the issue of weak convergence of U-statistics both in the case of finite order kernels as well as the sequences of kernels of increasing dimensions, like e.g., Koroljuk and Borovskich (1994). Some interesting results related to the limiting behavior of so called 'decoupled' U-statistics, which are similar to our P-statistics in the case of matrix $\mathbb{X}^{(\infty)}$ having independent and identically distributed entries, are presented in de la Peña and Giné (1999, chapter 4). For some related results on the weak convergence of both classical and decoupled U-statistics the reader may also wish to consult e.g., the papers of Giné and Zinn (1994), Latała and Zinn (2000), and Giné et al. (2001).

The material on weak convergence of P-statistics presented in this chapter is essentially new and comes from the yet unpublished manuscript Rempała and Wesołowski (2007).

Springer
http://www.springer.com/978-0-387-75145-0

Symmetric Functionals on Random Matrices and
Random Matchings Problems
Rempala, G.; Wesolowski, J.
2008, XIV, 184 p., Hardcover
ISBN: 978-0-387-75145-0