Existence Theorems for the 3D–Navier–Stokes System Having as Initial Conditions Sums of Plane Waves

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We study the 3-dimensional Navier–Stokes system on $\mathbb{R}^3$ without external forcing and prove local and global existence theorems for initial conditions which are the Fourier transforms of finite linear combinations of δ–functions. Bibliography: 8 titles.

1. Formulation of the result. We consider the 3D Navier–Stokes system on $\mathbb{R}^3$ for incompressible fluids. The viscosity is taken to be 1, and no external forcing is assumed. After the Fourier transform the system takes the form

$$v(k,t) = \exp\{-t|k|^2\}v(k,0) + i\int_0^t \exp\{-(t-s)|k|^2\}ds \int_{\mathbb{R}^3} \langle v(k-k',s),k\rangle P_k v(k',s)d^3k', \tag{1}$$

where $v(k,0)$ is the initial condition, P_k is the orthogonal projection to the subspace orthogonal to k, i.e., $P_k v = v - \frac{\langle v,k\rangle}{\langle k,k\rangle}k$. The values $v(k,s) \in \mathbb{C}^3$, the incompressibility means that $\langle v(k,t),k\rangle = 0$. If $v(k,t)$ is the Fourier transform of a real-valued function $u(x,t)$, then

$$v(-k,t) = \overline{v(k,t)}. \tag{2}$$

However, in this paper, we consider arbitrary $v(k,t)$ not assuming (2) (see also [**5**]).

Beginning with the works of Leray [**1**], Hopf [**3**], Kato [**4**], people considered the problem of local and global existence of solutions of (1) with initial conditions having finite energy

$$E(0) = \int_{\mathbb{R}^3} \langle v(k,0), \overline{v(k,0)}\rangle d^3k < \infty.$$

In this paper, we deal with another class of initial conditions having infinite energy. Namely, we consider the initial condition of the form

$$v(k,0) = \sum_{j=1}^{n} B_j \delta(k-k_j), \tag{3}$$

where the sum is taken over a finite set $\{k_j\}$ of points $k_j \neq 0$, $\langle B_j, k_j\rangle = 0$ for any j, and $\delta(k-k_j)$ is the delta-function concentrated at k_j. Since $\delta(k-k_j) \notin L^2(\mathbb{R}^3)$, the initial condition $v(k,0)$ has infinite energy.

Denote by $G(k_1,\dots,k_n)$ the semigroup generated by a finite set $\{k_j\}$, i.e., $k \in G(k_1,\dots,k_n)$ if and only if $k = \sum_{j=1}^{n} p_j k_j$ for some nonnegative integers p_j. The main results of this paper are the following theorems.

Theorem 1. *Let $v(k,0) = \sum_{j=1}^{n} B_j\delta(k-k_j)$. There exists $T > 0$ depending on $\{k_j\}$, $\{B_j\}$ such that there exists a solution $v(k,t)$ of (1)* *on the interval* $0 \leqslant t \leqslant T$ *which can be written as a signed measure*

$$v(k,t) = \sum_{g \in G(k_1,\dots,k_n),\ g \neq 0} B_g(t)\delta(k-g). \tag{4}$$

The coefficients $B_g(t)$ *satisfy the inequalities:*

$$|B_g(t)| < t^{\frac{p(g)}{2}} C^{p(g)} \left(\sum_{j=1}^{n} |B_j|\right)^{p(g)} \left(1 - t^{\frac{1}{2}} C \sum_{j=1}^{n} |B_j|\right)^{-1}, \tag{5}$$

where $p(g) = \min\limits_{\sum p_j k_j = g} \sum\limits_{i=1}^{n} p_i$, C *depends only on* k_j.

Theorem 2. *Let* k_j, $1 \leqslant j \leqslant n$, *belong to some cone with angle less than* π, *i.e., the angle between any* k_{j_1} *and* k_{j_2} *is less than* π. *Then for a sufficiently small* B *and an initial condition* $v(k,0)$ *such that* $\sum\limits_{j=1}^{n} |B_j| < B$ *there exists a global solution of* (1) *having the form* (4) *for which*

$$\sum_{g \in G(k_1,\dots,k_n),\ g \neq 0} |B_g(t)| < \infty.$$

Some existence results for a similar class of initial conditions can be found in [**2, 8**].

2. Proof of Theorems 1 and 2. The proofs of both theorems are based on the method of power series introduced in [**6, 7**]. We consider a one-parameter family of initial conditions $Av(k,0) = v_A(k,0)$ and write down the solution of (1) as a power series with respect to the complex parameter A:

$$v_A(k,t) = Ah_1(k,t) + \sum_{p>1} A^p \int_0^t \exp\{-(t-s)|k|^2\} h_p(k,s)\,ds, \tag{6}$$

where $h_1(k,t) = \exp\{-t|k|^2\}v(k,0)$. Substituting (6) into (1), we obtain the following system of recurrent relations between the functions $h_p(k,t)$:

$$h_2(k,t) = i \int_{\mathbb{R}^3} \langle v(k-k',0), k\rangle P_k v(k',0) \exp\{-t|k-k'|^2 - t|k'|^2\} d^3k'$$

and for $p > 2$

$$h_p(k,t) = i \int_0^t ds_2 \int_{\mathbb{R}^3} \langle v(k-k',0), k\rangle P_k h_{p-1}(k',s_2)$$

$$\times \exp\{-t|k-k'|^2-(t-s_2)|k'|^2\}d^3k'$$
$$+\, i \sum_{\substack{p_1+p_2=p \\ p_1,\, p_2>1}} \int_0^t ds_1 \int_0^t ds_2 \int_{\mathbb{R}^3} \langle h_{p_1}(k-k',s_1),k\rangle P_k h_{p_2}(k',s_2)$$
$$\times \exp\{-(t-s_1)|k-k'|^2-(t-s_2)|k'|^2\}d^3k'$$
$$+\, i \int_0^t ds_1 \int_{\mathbb{R}^3} \langle h_{p-1}(k-k',s_1),k\rangle P_k v(k',0)$$
$$\times \exp\{-(t-s_1)|k-k'|^2-t|k'|^2\}d^3k'. \tag{7}$$

In our case of solutions of type (4), the convolutions given by the integrals in (7) are well defined, and we can write

$$h_1(k,t)=\sum_{j=1}^{n}\exp\{-t|k_j|^2\}B_j\delta(k-k_j).$$

Recall that B_j are 3–dimensional vectors $\langle B_j,k_j\rangle=0$. Further,

$$h_2(k,t)=i\int_{\mathbb{R}^3}\sum_{j_1,j_2=1}\langle B_{j_1},k\rangle\left(B_{j_2}-\frac{\langle B_{j_2},k\rangle k}{\langle k,k\rangle}\right)$$
$$\times \exp\{-t|k-k'|^2-t|k'|^2\}\delta(k-k'-k_{j_1})\delta(k'-k_{j_2})d^3k'$$
$$=\sum_{j_1,j_2=1}^{n}B_{j_1,j_2}(t)\delta(k-(k_{j_1}+k_{j_2}))$$

with

$$B_{j_1,j_2}(t)=i\langle B_{j_1},k_{j_2}\rangle\exp\{-t|k_{j_1}|^2-t|k_{j_2}|^2\}$$
$$\times\left(B_{j_2}-\frac{\langle B_{j_2},(k_{j_1}+k_{j_2})\rangle(k_{j_1}+k_{j_2})}{\langle (k_{j_1}+k_{j_2}),(k_{j_1}+k_{j_2})\rangle}\right).$$

If $k_{j_1}+k_{j_2}=0$, then the corresponding term in the last sum is zero.

From the last formula it easily follows that

$$|B_{j_1,j_2}(t)|\leqslant C_1|B_{j_1}||B_{j_2}|, \tag{8}$$

$C_1=\max\limits_{1\leqslant j\leqslant n}|k_j|$, and

$$\sum_{j_1,j_2=1}^{n}|B_{j_1,j_2}(t)|\leqslant C_1\left(\sum_{j=1}^{n}|B_j|\right)^2. \tag{9}$$

The proof of Theorem 1 is based on the following assertion.

Lemma 1. *Assume that for any initial condition* (3) *and* $0 < t < 1$ *the functions* $h_q(k,t)$, $2 \leqslant q < p$, *can be written in the form*

$$h_q(k,t) = \sum_{1 \leqslant j_1,\ldots,j_q \leqslant n} B_{j_1\ldots j_q}(t)\delta\Big(k - \sum_{j=1}^{q} k_j\Big), \tag{10}$$

where $B_{j_1\ldots j_q}(t)$ *are continuous functions of* t *and*

$$|B_{j_1\ldots j_q}(t)| \leqslant C_2^{q-1} t^{\frac{q-2}{2}} \prod_{l=1}^{q} |B_{j_l}|, \tag{11}$$

C_2 *is another constant which depends only on the vectors* k_j, $1 \leqslant j \leqslant n$. *Then* (8) *and* (9) *are valid for* $q = p$.

Using Lemma 1, we derive Theorem 1. We have

$$\begin{aligned} v(k,t) &= \sum_{j=1}^{n} B_j \exp\{-t|k|^2\}\delta(k - k_j) \\ &\quad + \sum_{p>1} \sum_{j_1,\ldots,j_p=1}^{n} \tilde{B}_{j_1\ldots j_p}(t)\delta\Bigg(k - \sum_{l=1}^{p} k_{j_l}\Bigg) \end{aligned} \tag{12}$$

and

$$\tilde{B}_{j_1\ldots j_p}(t) = \int_0^t \exp\{-(t-s)|k|^2\} B_{j_1\ldots j_p}(s) ds.$$

If $B_{j_1\ldots j_p}(t)$ satisfies (8), then from Lemma 1 it follows that

$$|\tilde{B}_{j_1\ldots j_p}(t)| \leqslant \frac{2}{p} C_2^{p-1} t^{\frac{p}{2}} \prod_{l=1}^{p} |B_{j_l}|$$

and

$$\sum_{j_1,\ldots,j_p=1}^{n} |\tilde{B}_{j_1\ldots j_p}(t)| \leqslant \frac{2}{p} C_2^{p-1} t^{\frac{p}{2}} \Bigg(\sum_{j=1}^{n} |B_j|\Bigg)^{p}. \tag{13}$$

Therefore, the series $\sum\limits_{p>1} \sum\limits_{j_1,\ldots,j_p=1}^{n} |\tilde{B}_{j_1\ldots j_p}(t)|$ converges absolutely if

$$t^{\frac{1}{2}} < \min\Bigg(1, \Big(C_2 \sum_{j=1}^{n} |B_j|\Big)^{-1}\Bigg).$$

Taking together all terms with the same value of the sum $k_{j_1}+\cdots+k_{j_p}$, we get the representation of the solution $v(k,t)$ as

$$\sum_{g\in G(k_{j_1},\dots,k_{j_p})} B_g(t)\delta(k-g),$$

where for each $B_g(t)$ we have the estimate (5). Theorem 1 is proved. $\square$

PROOF OF LEMMA 1. For $p=2$ the statement of the lemma is already proved (see (8), (9)). For $p\geqslant 3$ we use the recurrent relation (7). Denote by $h_p^{(p_1,p_2)}(k,t)$ the term in (7) which corresponds to p_1, p_2. Consider the case p_1, $p_2\geqslant 2$. We have

$$\begin{aligned}
&h_p^{(p_1,p_2)}(k,t)=i\int_0^t ds_1\int_0^t ds_2\int_{\mathbb{R}^3}\langle h_{p_1}(k-k',s_1),k\rangle\\
&\times P_k h_{p_2}(k',s_2)\exp\{-(t-s_1)|k-k'|^2-(t-s_2)|k'|^2\}d^3k'\\
&=i\int_0^t ds_1\int_0^t ds_2\sum_{j_1\dots j_p=1}^n\langle B_{j_1\dots j_{p_1}}(s_1),k\rangle\\
&\times\left(B_{j_{p_1+1}\dots j_p}(s_2)-\frac{\langle B_{j_{p_1+1}\dots j_p}(s_2),k\rangle k}{|k|^2}\right)\\
&\times\biggl[\Bigl(\exp\{-(t-s_1)|k|^2\}\delta\Bigl(k-\sum_{l=1}^{p_1}k_{j_l}\Bigr)\Bigr)\\
&\circledast\Bigl(\exp\{-(t-s_2)|k|^2\}\delta\Bigl(k-\sum_{l=p_1+1}^{p}k_{j_l}\Bigr)\Bigr)\biggr]\\
&=\sum_{j_1\dots j_p}^n B_{j_1\dots j_p}^{(p_1,p_2)}(s)\delta\Bigl(k-\sum_{l=1}^p k_{j_l}\Bigr),
\end{aligned}$$

where $\circledast$ is the convolution and

$$\begin{aligned}
B_{j_1\dots j_p}^{(p_1,p_2)}(t)=i\int_0^t ds_1\int_0^t ds_2\Bigl\langle B_{j_1\dots j_{p_1}}(s_1),\sum_{l=p_1+1}^{p}k_{j_l}\Bigr\rangle\\
\times\exp\Bigl\{-(t-s_1)\Bigl|\sum_{l=1}^{p_1}k_{j_l}\Bigr|^2-(t-s_2)\Bigl|\sum_{l=p_1+1}^{p}k_{j_l}\Bigr|^2\Bigr\}
\end{aligned}$$

$$\times\left(B_{j_{p_1+1}\ldots j_p}(s_2)-\frac{\langle B_{j_{p_1+1}\ldots j_p}(s_2),\sum\limits_{l=1}^{p}k_{j_l}\rangle\sum\limits_{l=1}^{p}k_{j_l}}{|\sum\limits_{l=1}^{p}k_{j_l}|^2}\right).$$

Here, we used the incompressibility condition

$$\Big\langle B_{j_1\ldots j_{p_1}}(s_1),\sum_{l=1}^{p_1}k_{j_l}\Big\rangle=0.$$

It is clear that $B_p^{(p_1,p_2)}(t)$ is continuous as a function of $t\in[0,1]$. By the assumption of Lemma 1,

$$\begin{aligned}\left|B^{(p_1,p_2)}_{j_1\ldots j_p}(s)\right| &\leqslant \int\limits_0^s s_1^{\frac{p_1-2}{2}}\exp\Big\{-(s-s_1)\Big|\sum_{l=1}^{p_1}k_{j_l}\Big|^2\Big\}ds_1\\ &\times\int\limits_0^s s_2^{\frac{p_2-2}{2}}\exp\Big\{-(s-s_2)\Big|\sum_{l=p_1+1}^{p}k_{j_l}\Big|^2\Big\}\Big|\sum_{l=p_1+1}^{p}k_{j_l}\Big|ds_2\\ &\times C^{p_1-1}\prod_{l=1}^{p_1}|B_{j_l}|C^{p_2-1}\prod_{l=p_1+1}^{p}|B_{j_l}|.\end{aligned}$$

In the integral with respect to s_1, we replace the exponent by 1, and we estimate the integral with respect to s_2 with the help of the Cauchy–Schwarz inequality:

$$\begin{aligned}&\int\limits_0^s s_2^{\frac{p_2-2}{2}}\exp\{-(s-s_2)|\overline{k}|^2\}|\overline{k}|ds_2\\ &\leqslant\left(|;\int\limits_0^s s_2^{p_2-2}ds_2\int\limits_0^s\exp\{-2(s-s_2)|\overline{k}|^2\}|\overline{k}|^2ds_2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, we get

$$|B^{(p_1,p_2)}_{j_1\ldots j_p}(s)|\leqslant\frac{s^{\frac{p_1}{2}}}{p_1}\frac{s^{\frac{p_2-1}{2}}}{\sqrt{\frac{p_2-1}{2}}}C^{p-2}\prod_{l=1}^{p}|B_{j_l}|\leqslant\frac{s^{\frac{p-2}{2}}}{p_1\sqrt{\frac{p_2-1}{2}}}C^{p-2}\prod_{l=1}^{p}|B_{j_l}|.$$

At the last step, we used the fact that $0\leqslant s\leqslant 1$. The first and last terms in (7) are estimated in a similar way:

$$|B^{(1,p-1)}_{j_1\ldots j_p}(s)| \leqslant \frac{s^{\frac{p-2}{2}}}{\sqrt{\frac{p-2}{2}}} C^{p-2}\prod_{l=1}^{p}|B_{j_l}|$$

and

$$|B^{(p-1,1)}_{j_1\ldots j_p}(s)| \leqslant \frac{s^{\frac{p-2}{2}}}{\sqrt{\frac{p-2}{2}}} C^{p-2}\prod_{l=1}^{p}|B_{j_l}|.$$

For $h_p(k,s)$ we have the representation:

$$h_p(k,s) = \sum_{p_1=1}^{p-1} h_p^{(p_1,p-p_1)}(s) = \sum_{p_1=1}^{p-1}\sum_{j_1\ldots j_p=1}^{n} B^{(p_1,p-p_1)}_{j_1\ldots j_p}(s)\delta\Bigl(k-\sum_{l=1}^{n}k_{j_l}\Bigr)$$

$$= \sum_{j_1\ldots j_p=1}^{n} B_{j_1\ldots j_p}(s)\delta\Bigl(k-\sum_{l=1}^{n}k_{j_l}\Bigr),$$

where

$$B_{j_1\ldots j_p}(s) = \sum_{p_1=1}^{p-1} B^{(p_1,p-p_1)}_{j_1\ldots j_p}(s).$$

For $B_{j_1\ldots j_p}(s)$ we have the estimate

$$|B_{j_1\ldots j_p}(s)| \leqslant \sum_{p_1=1}^{p-1}|B^{(p_1,p-p_1)}_{j_1\ldots j_p}(s)|$$

$$\leqslant s^{\frac{p-2}{2}}C^{p-2}\prod_{l=1}^{p}|B_{j_l}|\Bigl(\sum_{p_1=2}^{p-2}\frac{\sqrt{2}}{p_1\sqrt{p-p_1}} + \frac{2\sqrt{2}}{\sqrt{p-2}}\Bigr).$$

It is easy to see that for $p \geqslant 3$ the sum

$$\sum_{p_1=2}^{p-2}\frac{\sqrt{2}}{p_1\sqrt{p-p_1}} + \frac{2\sqrt{2}}{\sqrt{p-2}}$$

is bounded by some constant C_1. Taking $C_2 = \max(C_1, \max\limits_{1\leqslant j\leqslant n}|k_j|)$, we get the required inequality for $q = p$. The lemma is proved. □

Proof of Theorem 2. We use the following lemma.

Lemma 2. *Let $\{k_j\}$ be contained in a cone whose angle is less than π. Assume that for $2 \leqslant q < p$ the functions $h_q(k,s)$ have the representation*

(10) *and for some constant D depending on initial conditions*

$$|B_{j_1\dots j_q}(s)| \leqslant D^{q-1}\prod_{l=1}^{q}|B_{j_l}|.$$

Then the same representation is valid for $q = p$.

PROOF. Using the notation from Lemma 1, we can write

$$|B^{(p_1,p_2)}_{j_1\dots j_p}(s)| \leqslant \int\limits_0^s ds_1 \int\limits_0^s ds_2 \Big|\sum_{l=p_1}^{p} k_{j_l}\Big| D^{p_1-1}\prod_{l=1}^{p_1}|B_{j_l}| D^{p_2-1}\prod_{l=p_1+1}^{p}|B_{j_l}|$$

$$\times \exp\Big\{-(s-s_1)\Big|\sum_{l=1}^{p_1}k_{j_l}\Big|^2 - (s-s_2)\Big|\sum_{l=p_1+1}^{p}k_{j_l}\Big|^2\Big\}$$

$$< D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\frac{\Big|\sum\limits_{l=p_1+1}^{p}k_{j_l}\Big|}{\Big|\sum\limits_{l=1}^{p_1}k_{j_l}\Big|^2\Big|\sum\limits_{l=p_1+1}^{p}k_{j_l}\Big|^2}.$$

By the assumption of the lemma, $\Big|\sum\limits_j \alpha_j k_j\Big| \geqslant d\sum\alpha_j$, where $\alpha_j \geqslant 0$ and $d > 0$ is some constant depending on the initial vectors k_j. Indeed, under some rotation of $\mathbb{R}^3$, the vectors k_j can be brought inside a cone of angle less than π belonging to the subspace $k^{(1)} > 0$. Then

$$\Big|\sum_j \alpha_j k_j\Big| = \Big|\sum_j \alpha_j \overline{k}_j\Big| \geqslant \sum_j \alpha_j \min_j \overline{k}_j^{(1)},$$

where $\overline{k}_j$ is the image of k_j under the above-mentioned rotation.

Put $d = \min\limits_j |\overline{k}_j^{(1)}|$. Now, we can write

$$|B^{(p_1,p_2)}_{j_1\dots j_p}(s)| \leqslant D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\frac{p_2\max\limits_j|k_j|}{p_1^2p_2^2d^4}.$$

Similarly, for $p_1 = 1$

$$|B^{(1,p-1)}_{j_1\dots j_p}(s)| \leqslant D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\frac{(p-1)\max\limits_j|k_j|}{(p-1)^2d^2}$$

and for $p_2 = 1$

$$|B^{(p-1,1)}_{j_1\dots j_p}(s)| \leqslant D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\frac{(p-1)\max\limits_{j}|k_j|}{(p-1)^2d^2}.$$

As in the proof of Lemma 1, we have

$$\begin{aligned}|B_{j_1\dots j_p}(s)| &\leqslant \sum_{p_1=1}^{p-1}|B^{(p_1,p-p_1)}_{j_1\dots j_p}(s)|\\ &< D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\max_{1\leqslant j\leqslant n}|k_j|\left[\frac{2}{(p-1)d^2}+\sum_{p_1=2}^{p-2}\frac{1}{p_1^2(p-p_1)d^4}\right].\end{aligned}$$

The expression in the square brackets is bounded by some constant D_0 depending on initial conditions. Taking $D \geqslant D_0 \max\limits_{1\leqslant j\leqslant n}|k_j|$, we get the required inequality for $q = p$. The lemma is proved. □

Now, we derive Theorem 2 from Lemma 2. Using the notation from the proof of Theorem 1, we can write

$$|\tilde{B}_{j_1\dots j_p}(t)| < D^{p-1}\frac{\prod_{l=1}^{p}|B_{j_l}|}{\left|\sum\limits_{l=1}^{p}k_{j_l}\right|} < \frac{D^{p-1}\prod_{l=1}^{p}|B_{j_l}|}{(pd)^2}$$

and

$$\sum_{j_1\dots j_p=1}^{n}|\tilde{B}_{j_1\dots j_p}(t)| < \frac{D^{p-1}}{(pd)^2}\left(\sum_{j=1}^{n}|B_j|\right)^p.$$

Therefore, if

$$\sum_{j=1}^{n}|B_j| < D^{-1},$$

then the series

$$\sum_{p>0}\sum_{j_1\dots j_p=1}^{n}|B_{j_1\dots j_p}(t)|$$

converges absolutely for all t. The end of the proof is the same as in Theorem 1. □

3. Example. We consider $n = 2$ and the initial condition

$$v(k,0) = B_1\delta(k-k_1) + B_2\delta(k-k_2), \tag{14}$$

where the vectors k_1 and k_2 are linearly independent and B_1 is orthogonal to k_1 and k_2.

Theorem 3. *For the initial condition* (14) *the system* (1) *has a global solution.*

Proof. In this case,

$$h_1(k,s) = \sum_{j=1}^{2} \exp\{-s|k_j|^2\} B_j \delta(k-k_j),$$

$$\begin{aligned} h_2(k,s) &= i \sum_{j_1,j_2=1}^{2} \langle k, B_{j_1}\rangle \left(B_{j_2} - \frac{\langle k, B_{j_2}\rangle k}{|k|^2} \right) \\ &\quad \times \exp\{-s|k_{j_1}|^2 - s|k_{j_2}|^2\} \delta(k - (k_{j_1}+k_{j_2}) \\ &= i \sum_{j_1,j_2=1}^{2} \langle k_{j_1}+k_{j_2}, B_{j_1}\rangle \left(B_{j_2} - \frac{\langle k_{j_1}+k_{j_2}, B_{j_2}\rangle (k_{j_1}+k_{j_2})}{|k_{j_1}+k_{j_2}|^2} \right) \\ &\quad \times \exp\{-s|k_{j_1}|^2 - s|k_{j_2}|^2\} \delta(k - (k_{j_1}+k_{j_2}). \end{aligned}$$

It is easy to see that only the term with $j_1 = 2$, $j_1 = 1$ is different from zero. Therefore,

$$h_2(k,s) = \langle k_1, B_2\rangle B_2 \exp\{-s|k_1|^2 - s|k_2|^2\} \delta(k-(k_1+k_2)).$$

From the recurrent relation it easily follows that, in the sum (7), only the term with $p_1 = p-1$ is different from zero. If

$$h_q(k,s) = B^{(q)}(s)\delta(k-(k_1+(q-1)k_2)$$

and

$$|B^{(q)}(s)| < \frac{C^{q-1}}{\sqrt{(q-2)!}} |B_1||B_2|^{q-1},$$

then the same inequality is valid for $q = p$ (see the proof of Lemma 1). Theorem 3 is proved. □

Remark. Theorem 3 is equivalent to the estimates of simple diagrams in [**7**].

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References

1. J. Leray, *Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique*, J. Math. Pures Appl. Sér. IX **12** (1933), 1–82.

2. Y. Giga, K. Innui, A. Makhalov, and J. Saal, *Global Solvability of the Navier–Stokes Equations in Spaces Based on Some Closed Frequency Set*, Preprint, Hokkaido Univ. Ser. Math. no. 795 (2006). [To appear in *Ann. Math. Stud.*]
3. E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nach. **4** (1950/51), 213-231.
4. T. Kato, *Strong L^p solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions*, Math. Z. **187** (1984), 471–480.
5. D. Li and Ya. G. Sinai, *Blow ups of complex solutions of the 3D Navier–Stokes system and renormalization group method*, 2006. [In preparation]
6. Ya. G. Sinai, *Power series for solutions of the 3D Navier–Stokes system on* $\mathbb{R}^3$, J. Stat. Phys. **121** (2005), no. 5-6, 779-803.
7. Ya. G. Sinai, *Diagramatic approach to the 3D Navier–Stokes system*, Russ. Math. Surv. **60** (2005), no. 5, 47-70.
8. M. I. Vishik and A. V. Fursikov, *Mathematical Problems of Statistical Hydromechanics,* Kluwer Acad. Publ., Dordrecht–Boston–London, 1988.

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