Chapter 2
Combinatorial Properties of (Pre)-Semirings

1. Introduction

Many results of classical linear algebra, such as the well-known Cayley–Hamilton theorem, first established in the context of vector spaces on fields, do not actually require all the properties of these structures. We show in this chapter that many known results of this type are deduced from purely combinatorial properties which are valid in more elementary algebraic structures such as semirings and pre-semirings. We will not even require the dioid structure since there is no need to assume the presence of a canonical order relation.

In the present chapter we will thus consider matrices, polynomials and formal series with elements or coefficients in a pre-semiring or in a semiring.

The basic definitions concerning matrices, polynomials and formal series are introduced in Sects. 2 and 3.

Definitions and basic properties for permutations are recalled in Sect. 4.1, and the concepts of a *bideterminant* and of the *characteristic bipolynomial* of a matrix are introduced in Sects. 4.2 and 4.3.

Section 5 presents a combinatorial proof of the extended version of the classical identity for the determinant of the product of two matrices. Section 6 provides a combinatorial proof of the Cayley–Hamilton theorem generalized to commutative pre-semirings.

In Sect. 7, we focus on the links between the bideterminant of a matrix and the arborescences of the associated directed graph. An extension to semirings of the classical "Matrix Tree Theorem" is first established in Sects. 7.1 and 7.2. A more general form of this result is then studied in Sect. 7.4, which may be considered as an extension to semirings, of the so-called "All Minors Matrix Tree Theorem".

Finally, a version of the well-known Mac Mahon identity, generalized to commutative pre-semirings, is presented in Sect. 8.

In order to derive each of the identities discussed in this chapter, a superficial analysis might lead one to believe that it is enough to start from the corresponding classical result (usually stated in the field of real numbers) and to simply rewrite it by moving all the negative terms to the other side to make them appear positively.

The result of Sect. 5 (about the bideterminant of the product of two matrices), as well as the generalization of the classical "All-Minors Matrix Tree Theorem", which is studied in Sect. 7.4, provide concrete examples where such an approach would lead to a wrong result; this indeed confirms the necessity of new direct proofs, different from those previously known for the standard case.

2. Polynomials and Formal Series with Coefficients in a (Pre-) Semiring

2.1. Polynomials

Let $(E, \oplus, \otimes)$ be a pre-semiring or a semiring with neutral elements ε and e (for $\oplus$ and $\otimes$ respectively).

Definition 2.1.1. *A polynomial* P *of degree* n *in the variable* x *is defined by specifying a mapping* $f: \{0, 1, \ldots . n\} \to E$ *where,* $\forall k, 0 \le k \le n, f(k) \in E$ *is called the coefficient of* x^k *in the polynomial* P. P *can thus be represented by the sum:*

$$P(x) = \sum_{k=0}^{n} f(k) \otimes x^k$$

where the sum is to be understood in the sense of the operation $\oplus$ *(by convention* $x^0 = e$ *and,* $\forall k$: $\varepsilon \otimes x^k = \varepsilon$).

In accordance with classical notation, we denote $E[x]$ the set of polynomials in x with coefficients in E.

Let P and Q be two polynomials of $E[x]$ defined as:

$$P(x) = \sum_{k=0}^{p} f(k) \otimes x^k$$

$$Q(x) = \sum_{k=0}^{q} g(k) \otimes x^k$$

The sum of P and Q, denoted $S = P \oplus Q$, is the polynomial of degree at most $s = \mathrm{Max}\{p, q\}$ defined as:

$$S(x) = \sum_{k=0}^{s} (f(k) \oplus g(k)) \otimes x^k$$

(we agree to set $f(j) = \varepsilon$ for $j > p$ and $g(j) = \varepsilon$ for $j > q$).

The product of P and Q, denoted $T = P \otimes Q$ is the polynomial of degree $r = p + q$ defined as:

$$T(x) = \sum_{k=0}^{r} t(k) \otimes x^k$$

with, $\forall k = 0 \ldots . r$:

$$t(k) = \sum_{\substack{0 \le i \le p \\ 0 \le j \le q \\ i+j=k}} f(i) \otimes g(j)$$

ε being the neutral element of $\oplus$, E[x] has, as neutral element for $\oplus$, the polynomial denoted $\varepsilon(x)$, of degree 0, defined as: $\varepsilon(x) = \varepsilon \otimes x^0 = \varepsilon$. Likewise, e being the neutral element of $\otimes$, E[x] has as neutral element for $\otimes$, the polynomial denoted e(x) of degree 0 defined as: $e(x) = e \otimes x^0 = e$.

Proposition 2.1.2. *(i) If* $(E, \oplus, \otimes)$ *is a pre-semiring, then* $(E[x], \oplus, \otimes)$ *is a pre-semiring*
(ii) If $(E, \oplus, \otimes)$ *is a semiring, then* $(E[x], \oplus, \otimes)$ *is a semiring*
(iii) If $(E, \oplus, \otimes)$ *is a dioid, then* $(E[x], \oplus, \otimes)$ *is a dioid.*

Proof. It follows from the fact that the elementary properties of $\oplus$ and $\otimes$ on E induce the same properties on E[x]. Let us just show that, in case (iii), the canonical preorder relation on E[x] defined as:

$$P \le Q \Leftrightarrow \exists\, R \in E[x] \quad \text{such that:} \quad Q = P \oplus R$$

is an order relation.

$$\text{If } P(x) = \sum_{k=0}^{p} f(k) \otimes x^k$$

$$Q(x) = \sum_{k=0}^{q} g(k) \otimes x^k$$

then $P \le Q \Rightarrow \exists\, R$ with: $R(x) = \sum_{k=0}^{r} h(k) \otimes x^k$, such that: $Q = P \oplus R$

Similarly $Q \le P \Rightarrow \exists\, R'$ with: $R'(x) = \sum_{k=0}^{r'} h'(k) \otimes x^k$ such that: $P = Q \oplus R'$

Set $K = \text{Max}\{p, q, r, r'\}$ and let us agree that:

$$\begin{array}{llll}
\text{If} \quad K > p, & f(j) = \varepsilon & \text{for every} & j \in [p+1, K] \\
\text{If} \quad K > q, & g(j) = \varepsilon & \text{for every} & j \in [q+1, K] \\
\text{If} \quad K > r, & h(j) = \varepsilon & \text{for every} & j \in [r+1, K] \\
\text{If} \quad K > r', & h'(j) = \varepsilon & \text{for every} & j \in [r'+1, K]
\end{array}$$

We deduce $\forall k = 0, \ldots . K$:

$$\exists\, r(k): \quad g(k) = f(k) \oplus r(k)$$
$$\exists\, r'(k): \quad f(k) = g(k) \oplus r'(k)$$

in other words:

$$f(k) \leq g(k), \quad \text{and} \quad g(k) \leq f(k)$$

Since $(E, \oplus, \otimes)$ is a dioid, we deduce $\forall k$: $f(k) = g(k)$ and therefore $P = Q$. $(E[x], \oplus, \otimes)$ is thus clearly a dioid in this case. □

The above is easily generalized to *multivariate* polynomials in several commutative indeterminates $x_1, x_2, \ldots . x_m$, the set of these polynomials being denoted $E[x_1, x_2, \ldots . x_m]$.

2.2. Formal Series

Let $(E, \oplus, \otimes)$ be a pre-semiring or a semiring with neutral elements ε and e (for $\oplus$ and $\otimes$, respectively).

Definition 2.2.1. *A formal series* F *in* m *commutative indeterminates* $x_1, x_2, \ldots x_m$ *is defined by specifying a mapping* $f: N^m \to E$*, where:* $\forall (k_1, k_2, \ldots . k_m) \in N^m$, $f(k_1, k_2, \ldots . k_m)$ *is the coefficient of the term* $x_1^{k_1} \otimes x_2^{k_2} \otimes \cdots \otimes x_m^{k_m}$

Formally, we represent F *by the (infinite) sum:*

$$F = \sum_{\substack{(k_1, k_2, \ldots, k_m) \\ \in N^m}} f(k_1, k_2, \ldots k_m) \otimes x_1^{k_1} \otimes \cdots \otimes x_m^{k_m}$$

Let us consider two formal series with coefficients $f(k_1, k_2 \cdots k_m)$ and $g(k_1, \ldots, k_m)$. The sum is the formal series of coefficients $s(k_1, \ldots k_m)$ defined as:

$$\forall (k_1, k_2 \cdots k_m) \in N^m\colon s(k_1 \cdots . k_m) = f(k_1 \cdots k_m) \oplus g(k_1 \cdots . k_m).$$

The product is the formal series of coefficients $t(k_1 \cdots k_m)$ defined as: $\forall (k_1 \ldots, k_m) \in N^m$: $t(k_1, k_2 \ldots, k_m) = \Sigma f(i_1, i_2 \ldots i_m) \otimes g(j_1, \ldots j_m)$ where the sum extends to all the pairs of m-tuples $(i_1, \ldots i_m) \in N^m, (j_1, j_2, \ldots j_m) \in N^m$ such that:

$$i_1 + j_1 = k_1, i_2 + j_2 = k_2, \ldots, i_m + j_m = k_m.$$

Proposition 2.1.2 of Sect. 2.1 easily extends to formal series as defined above.

3. Square Matrices with Coefficients in a (Pre)-Semiring

Let $(E, \oplus, \otimes)$ be a pre-semiring or a semiring. We denote $M_n(E)$ the set of square $n \times n$ matrices with elements in E.

Given two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of $M_n(E)$

- The sum, denoted $A \oplus B$, is the matrix $S = (s_{ij})$ defined as:

$$\forall i, j\colon s_{ij} = a_{ij} \oplus b_{ij}$$

- The product, denoted $A \otimes B$, is the matrix $T = (t_{ij})$ defined as:

$$\forall i, j\colon t_{ij} = \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \quad \text{(sum in the sense of } \oplus).$$

If E has a neutral element ε for ⊕, the matrix:

$$\sum = \begin{bmatrix} \varepsilon, \varepsilon, \ldots, \varepsilon \\ \vdots \\ \varepsilon, \varepsilon, \ldots, \varepsilon \end{bmatrix}$$

is the neutral element of $M_n(E)$ for ⊕.

If, moreover, E has unit element e, and ε is absorbing for ⊗, then the matrix:

$$I = \begin{bmatrix} e & & & \\ & e & & \varepsilon \\ & \varepsilon & \ddots & \\ & & & e \end{bmatrix}$$

is the unit element of $M_n(E)$ for ⊗.

It is then easy to prove the following:

Proposition 3.1. *(i) If* $(E, \oplus, \otimes)$ *is a pre-semiring then* $(M_n(E), \oplus, \otimes)$ *is a pre-semiring*

(ii) If $(E, \oplus, \otimes)$ *is a semiring, then* $(M_n(E), \oplus, \otimes)$ *is a semiring (in general noncommutative)*

(iii) If $(E, \oplus, \otimes)$ *is a dioid, then* $(M_n(E), \oplus, \otimes)$ *is a dioid (in general noncommutative)*

In the subsequent sections, we study properties of square $n \times n$ matrices with elements in a commutative pre-semiring $(E, \oplus, \otimes)$. For some of the properties considered, we will have to assume that $(E, \oplus, \otimes)$ has a semiring structure.

4. Bideterminant of a Square Matrix. Characteristic Bipolynomial

In this section we introduce the concept of *bideterminant* for matrices with coefficients in a pre-semiring.

4.1. Reminder About Permutations

Let π be a permutation of $X = \{1, 2, \ldots, n\}$ where, $\forall i \in X$, $\pi(i) \in X$ denotes the element corresponding to i through π. The *graph associated with* π is the directed graph G_π having X as set of vertices and n arcs of the form $(i, \pi(i))$. This graph can contain loops (when $\pi(i) = i$).

It is well-known that the permutation graph decomposes into *disjoint elementary circuits* (each connected component is an elementary circuit). If a connected component is reduced to a single vertex i, the corresponding circuit is the loop (i, i).

Figure 1 below represents the permutation graph of $\{1, \ldots 7\}$ defined as:

$$\pi(1) = 7,\ \pi(2) = 4,\ \pi(3) = 5,\ \pi(4) = 2,\ \pi(5) = 1,\ \pi(6) = 6,\ \pi(7) = 3.$$

The *parity* of a permutation π, is defined as the parity of the number of transpositions necessary to transform the permutation π into the identity permutation.

Thus, in the above example, a possible sequence of transpositions would be:

$$\begin{pmatrix}7\\4\\5\\2\\1\\6\\3\end{pmatrix} \to \begin{pmatrix}1\\4\\5\\2\\7\\6\\3\end{pmatrix} \to \begin{pmatrix}1\\2\\5\\4\\7\\6\\3\end{pmatrix} \to \begin{pmatrix}1\\2\\3\\4\\7\\6\\5\end{pmatrix} \to \begin{pmatrix}1\\2\\3\\4\\5\\6\\7\end{pmatrix}$$

The permutation of Fig. 1 is therefore *even*.

More generally, we can prove:

Property 4.1.1. The parity of a permutation π is equal to the parity of the number of circuits of even length of the graph G_π associated with the permutation.

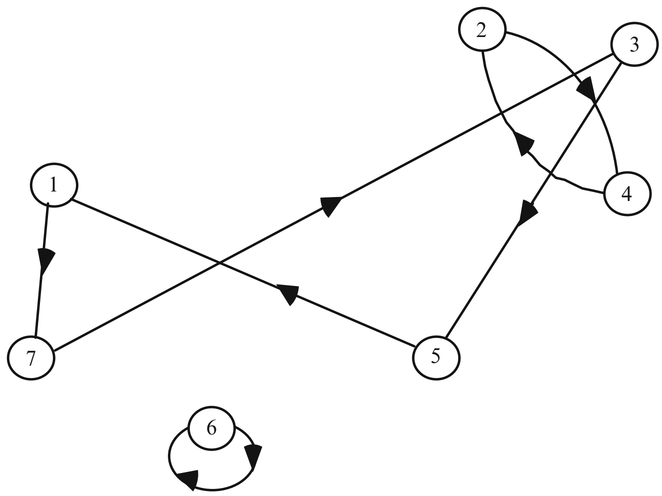

Fig. 1 Permutation graph

Example. The graph of Fig. 1 contains two circuits of even length (1, 7, 3, 5) and (2, 4), the corresponding permutation is therefore *even*. ||

We call *signature* of a permutation π, the quantity sign (π) defined as:

$$\begin{aligned} \mathrm{sign}(\pi) &= +1 \quad \text{if } \pi \text{ is even} \\ \mathrm{sign}(\pi) &= -1 \quad \text{if } \pi \text{ is odd} \end{aligned}$$

It is easy to see that the signature of a permutation π can be calculated as:

$$\mathrm{sign}(\pi) = \prod_{\text{C circuit of } G_\pi} (-1)^{|C|-1}$$

(where $|C|$ is the cardinality of the circuit, and where the product extends to the set of the circuits of G_π).

In the example of Fig. 1 we have three circuits: $C_1 = (6)$ of odd length and $C_2 = (2, 4)$; $C_3 = (1, 3, 5, 7)$ of even length. We clearly have:

$$\begin{aligned} \mathrm{sign}(\pi) &= (-)^{|C_1|-1} \times (-1)^{|C_2|-1} \times (-1)^{|C_3|-1} \\ &= +1 \end{aligned}$$

Hereafter we denote:

Per(n) the set of all the permutations of $\{1, 2, \ldots, n\}$
$\mathrm{Per}^+(n)$ the set of all the even permutations of $\{1, 2, \ldots, n\}$ (the set of the permutations of signature +1)
$\mathrm{Per}^-(n)$ the set of *odd* permutations of $\{1, 2 \ldots, n\}$ (of signature − 1)

We will also make use of the concept of *partial permutation*: a *partial permutation* of $X = \{1, \ldots, n\}$ is simply a permutation of a subset S of X.

Example. If $X = \{1, \ldots, 7\}$ $S = \{2, 3, 5, 7\}$ then σ defined as:

$$\sigma(2) = 3; \quad \sigma(3) = 7; \quad \sigma(5) = 5; \quad \sigma(7) = 2.$$

is a permutation of S and a partial permutation of X. The domain of definition of σ, denoted dom (σ), is $S = \{2, 3, 5, 7\}$

With every partial permutation σ of $X = \{1, \ldots, n\}$ we can associate the permutation $\hat{\sigma}$ of $\{1, \ldots, n\}$ defined as:

$$\begin{cases} \hat{\sigma}(i) = \sigma(i) & \text{if} \quad i \in \mathrm{dom}(\sigma) \\ \hat{\sigma}(i) = i & \text{if} \quad i \in X \backslash (\mathrm{dom}(\sigma)) \end{cases}$$

$\hat{\sigma}$ will be referred to as the *extension* of σ.

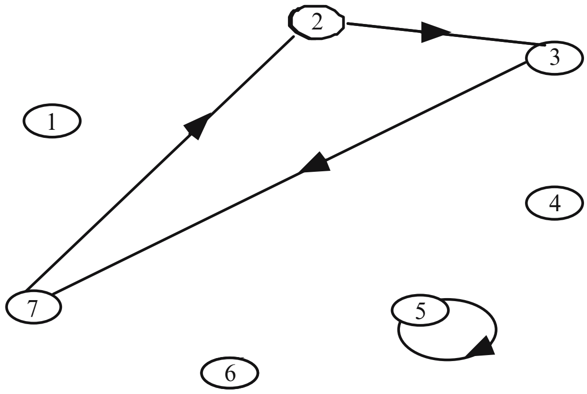

Fig. 2 Graph associated with a partial permutation σ of characteristic +1: $\sigma \in \mathrm{Part}^{+}(7)$

The *parity* (resp. *signature*) of a partial permutation σ is the *parity* (resp. *signature*) of its extension $\hat{\sigma}$.

The *characteristic* of a partial permutation σ, denoted char (σ), is defined as:

$$\mathrm{char}(\sigma) = \mathrm{sign}(\sigma) \times (-1)^{|\sigma|}$$

$|\sigma|$ denoting the cardinality of dom(σ).

We observe that, if σ is a partial permutation of order k (i.e. $|\sigma| = |\mathrm{dom}(\sigma)| = k$) and *cyclic* (i.e. such that the associated graph contains a single circuit covering all the vertices of dom (σ)) then: $\mathrm{sign}(\sigma) = \mathrm{sign}(\hat{\sigma}) = (-1)^{k-1}$, hence:

$$\mathrm{char}(\sigma) = (-1)^{2k+1} = -1.$$

From the above, we deduce:

Property 4.1.2. For every partial permutation σ, $\mathrm{char}(\sigma) = (-1)^{r}$ where r is the number of circuits in the graph associated with σ.

Example. For the partial permutation of {1, ..., 7} defined as:

$$\sigma(2) = 3; \quad \sigma(3) = 7; \quad \sigma(5) = 5; \quad \sigma(7) = 2.$$

the associated graph (see Fig. 2) contains two circuits, therefore: $\mathrm{char}(\sigma) = +1$. ||

Hereafter, we denote Part(n) the set of all the partial permutations of {1, ..., n} (Observe that $\mathrm{Per}(n) \subset \mathrm{Part}(n)$).

The set of partial permutations of characteristic +1, (resp. of characteristic −1), will be denoted $\mathrm{Part}^{+}(n)$ (resp. $\mathrm{Part}^{-}(n)$).

4.2. Bideterminant of a Matrix

For a square matrix of order n, $A = (a_{ij})$ with elements in $\mathbb{R}$ endowed with the standard operations, the determinant det (A) is classically defined as:

$$\det(A) = \sum_{\pi \in \mathrm{Per}(n)} \mathrm{sign}(\pi) \left(\prod_{i=1}^{n} a_{i,\pi(i)} \right) \tag{1}$$

or equivalently, with the notation of Sect. 4.1., as:

$$\det(A) = \sum_{\pi \in \mathrm{Per}^+(n)} \left(\prod_{i=1}^{n} a_{i,\pi(i)} \right) - \sum_{\pi \in \mathrm{Per}^-(n)} \left(\prod_{i=1}^{n} a_{i,\pi(i)} \right) \tag{2}$$

(the above sums should be understood in the sense of the addition of reals). This notation is possible given that $(\mathbb{R}, +)$ is a *group*.

If one wishes to generalize the concept of determinant to algebraic structures featuring fewer properties, where addition does not induce a group structure, one must introduce the concept of *bideterminant*.

Definition 4.2.1. *(Bideterminant)*

Let $A = (a_{ij})$ *be a square* $n \times n$ *matrix with elements in a commutative pre-semiring* $(E, \oplus, \otimes)$. *We call bideterminant of* A *the pair* $(\det^+(A),\ \det^-(A))$ *where the values* $\det^+(A) \in E$ *and* $\det^-(A) \in E$ *are defined as:*

$$\det^+(A) = \sum_{\pi \in \mathrm{Per}^+(n)} \left(\prod_{i=1}^{n} a_{i,\pi(i)} \right) \tag{3}$$

$$\det^-(A) = \sum_{\pi \in \mathrm{Per}^-(n)} \left(\prod_{i=1}^{n} a_{i,\pi(i)} \right) \tag{4}$$

(the above sums and products should be understood in the sense of the operations $\oplus$ and $\otimes$ of the pre-semiring).

4.3. Characteristic Bipolynomial

In the case of a real $n \times n$ matrix A, the characteristic polynomial is defined as the polynomial in the variable λ equal to the determinant of the matrix $\lambda I - A$ where I is the $n \times n$ unit matrix:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I - A) \\ &= \sum_{\pi \in \mathrm{Per}(n)} \mathrm{sign}(\pi) \left(\prod_{i=1}^{n} b_{i,\pi(i)} \right) \end{aligned}$$

$$\text{where, } \forall i, j: \begin{cases} b_{ij} = -a_{ij} & \text{if} \quad i \neq j \\ b_{ij} = \lambda - a_{ij} & \text{if} \quad i = j \end{cases}$$

We observe that, for every q, $1 \le q \le n$, the coefficient of the term involving λ^{n-q} in the above expression can be expressed as:

$$\begin{aligned}&\sum_{\substack{\sigma\in\mathrm{Part}(n)\\|\sigma|=q}} \mathrm{sign}(\sigma)\left(\prod_{i\in\mathrm{dom}(\sigma)}(-a_{i,\sigma(i)})\right)\\&=\sum_{\substack{\sigma\in\mathrm{Part}\ (n)\\|\sigma|=q}}(-1)^{|\sigma|}\cdot\mathrm{sign}(\sigma)\left(\prod_{i\in\mathrm{dom}(\sigma)}^{n}(a_{i,\sigma(i)})\right)\end{aligned} \tag{5}$$

For $q = 0$, the λ^n term has coefficient equal to 1. Observing that $(-1)^{|\sigma|}$ sign(σ) is none other than the characteristic char(σ) (see Sect. 4.1), (5) is rewritten:

$$\sum_{\substack{\sigma\in\mathrm{Part}(n)\\|\sigma|=q}} \mathrm{car}(\sigma)\left(\prod_{i\in\mathrm{dom}(\sigma)}^{n}(a_{i,\sigma(i)})\right) \tag{6}$$

By denoting (see Sect. 4.1) $\mathrm{Part}^+(n)$ (resp. $\mathrm{Part}^-(n)$) the set of partial permutations of $\{1, \ldots, n\}$ with characteristic $+1$ (resp. with characteristic -1) then the above sum becomes:

$$\sum_{\substack{\sigma\in\mathrm{Part}^+(n)\\|\sigma|=q}}\left(\prod_{i\in\mathrm{dom}(\sigma)}^{n}(a_{i,\sigma(i)})\right)-\sum_{\substack{\sigma\in\mathrm{Part}^-(n)\\|\sigma|=q}}\left(\prod_{i\in\mathrm{dom}(\sigma)}^{n}(a_{i,\sigma(i)})\right) \tag{7}$$

Now, when $A = (a_{ij})$ is a matrix with coefficients in a pre-semiring $(E, \oplus, \otimes)$, one is then naturally lead to define the *characteristic bipolynomial* as follows.

Definition 4.3.1. ***(characteristic bipolynomial)*** *Let $A = (a_{ij})$ be a square $n \times n$ matrix with elements in a commutative pre-semiring $(E, \oplus, \otimes)$. We call characteristic bipolynomial the pair $(P_A^+(\lambda), P_A^-(\lambda))$ where $P_A^+(\lambda)$ and $P_A^-(\lambda)$ are two polynomials of degree n in the variable λ, defined as:*

$$P_A^+(\lambda)=\sum_{q=1}^{n}\left(\sum_{\substack{\sigma\in\mathrm{Part}^+(n)\\|\sigma|=q}}\left(\prod_{i\in\mathrm{dom}(\sigma)}^{n}(a_{i,\sigma(i)})\right)\right)\otimes\lambda^{n-q}\oplus\lambda^n \tag{8}$$

and:

$$P_A^-(\lambda)=\sum_{q=1}^{n}\left(\sum_{\substack{\sigma\in\mathrm{Part}^-(n)\\|\sigma|=q}}\left(\prod_{i\in\mathrm{dom}(\sigma)}^{n}(a_{i,\sigma(i)})\right)\right)\otimes\lambda^{n-q} \tag{9}$$

(the sums and the products above are to be understood in the sense of the addition $\oplus$ and the multiplication $\otimes$ of the pre-semiring $(E, \oplus, \otimes)$).

We observe that, in the case where $(E, \oplus, \otimes)$ is a semiring, ε the neutral element of $\oplus$, is absorbing and the formulae (8)–(9) give:

$$P_A^+(\varepsilon) = \sum_{\substack{\sigma\in \mathrm{Part}^+(n)\\ |\sigma|=n}} \left(\prod_i a_{i,\sigma(i)}\right)$$

$$P_A^-(\varepsilon) = \sum_{\substack{\sigma\in \mathrm{Part}^-(n)\\ |\sigma|=n}} \left(\prod_i a_{i,\sigma(i)}\right)$$

Since, for $|\sigma| = n$, $\mathrm{char}(\sigma) = (-1)^n \,\mathrm{sign}(\sigma)$, we see that for *even n*, $\mathrm{Part}^+(n) = \mathrm{Per}^+(n)$ and consequently:

$$P_A^+(\varepsilon) = \det{}^+(A), P_A^-(\varepsilon) = \det{}^-(A)$$

For odd n, we have $\mathrm{Part}^+(n) = \mathrm{Per}^-(n)$ and consequently:

$$P_A^+(\varepsilon) = \det{}^-(A), P_A^-(\varepsilon) = \det{}^+(A).$$

We thus find again the analogue of the classical property for the characteristic polynomial:

$$P_A(0) = \det(-A) = (-1)^n \det(A).$$

5. Bideterminant of a Matrix Product as a Combinatorial Property of Pre-Semirings

Given two square $n \times n$ real matrices, a classical result of linear algebra is the identity:

$$\det(A \times B) = \det(A) \times \det(B)$$

In the present section we study the generalization of this result to square matrices with elements in a commutative *pre-semiring* $(E, \oplus, \otimes)$.

$$\text{If} \quad A = (a_{ij}) \quad B = (b_{ij}) \quad \text{and} \quad C = A \otimes B = (c_{ij})$$

with:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \quad \text{(sum in the sense of the operation } \oplus\text{)}$$

Then, by definition (see Sect. 4.2):

$$\det{}^+(A \otimes B) = \sum_{\pi\in \mathrm{Per}^+(n)} \left(\prod_{i=1}^{n} c_{i,\pi(i)}\right) \tag{10}$$

For $\pi \in \mathrm{Per}^+(n)$ fixed, we can write:

$$\prod_{i=1}^{n} c_{i,\pi(i)} = \prod_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} \otimes b_{k,\pi(i)} \right) \tag{11}$$

By using distributivity, each term in the expansion of expression (11) is obtained by choosing, for each value of $i (1 \le i \le n)$, a value of $k \in \{1, \ldots, n\}$. In other words, each term in the expanded expression is associated with a mapping $f: \{1, \ldots, n\} \to \{1, \ldots, n\}$, and the value of the corresponding term in (11) is:

$$\prod_{i=1}^{n} \left(a_{i,f(i)} \otimes b_{f(i),\pi(i)} \right)$$

By denoting $F(n)$ the set of mappings: $\{1, \ldots, n\} \to \{1, \ldots, n\}$, (10) can therefore be rewritten:

$$\det{}^+(A \otimes B) = \sum_{f \in F(n)} \sum_{\pi \in \mathrm{Per}^+(n)} \prod_{i=1}^{n} (a_{i,f(i)} \otimes b_{f(i),\pi(i)}) \tag{12}$$

We would similarly obtain:

$$\det{}^-(A \otimes B) = \sum_{f \in F(n)} \sum_{\pi \in \mathrm{Per}^-(n)} \prod_{i=1}^{n} (a_{i,f(i)} \otimes b_{f(i),\pi(i)}) \tag{13}$$

Among the mappings of $F(n)$, we find (even and odd) permutations, i.e.:

$$F(n) = \mathrm{Per}^+(n) \cup \mathrm{Per}^-(n) \cup F'(n)$$

where $F'(n)$ denotes the set of all the mappings of $F(n)$ which are not permutations.

Expression (12) therefore decomposes into the sum of three sub-expressions:

$$\alpha^+ = \sum_{f \in \mathrm{Per}^+(n)} \sum_{\pi \in \mathrm{Per}^+(n)} \prod_{i=1}^{n} (a_{i,f(i)} \otimes b_{f(i),\pi(i)}) \tag{14}$$

$$\beta^+ = \sum_{f \in \mathrm{Per}^-(n)} \sum_{\pi \in \mathrm{Per}^+(n)} \prod_{i=1}^{n} (a_{i,f(i)} \otimes b_{f(i),\pi(i)}) \tag{15}$$

$$\gamma^+ = \sum_{f \in F'(n)} \sum_{\pi \in \mathrm{Per}^+(n)} \prod_{i=1}^{n} (a_{i,f(i)} \otimes b_{f(i),\pi(i)}) \tag{16}$$

In cases where f is a permutation, let g be the permutation $\pi \circ f^{-1}$. In the expressions (14) and (15) above, we can rewrite the term:

$$\left(\prod_{i=1}^{n} a_{i,f(i)} \right) \otimes \left(\prod_{i=1}^{n} b_{f(i),\pi(i)} \right) \text{ as: } \left(\prod_{i=1}^{n} a_{i,f(i)} \right) \otimes \left(\prod_{i=1}^{n} b_{i,g(i)} \right)$$

Let us then consider the expression α^+.

f being an even permutation, f^{-1} is even and g, as the product of two even permutations is even. Then α^+ can be rewritten:

$$\begin{aligned}\alpha^+ &= \left(\sum_{f\in \mathrm{Per}^+(n)} \prod_{i=1}^{n} a_{i,f(i)}\right) \otimes \left(\sum_{g\in \mathrm{Per}^+(n)} \prod_{i=1}^{n} b_{i,g(i)}\right) \\ &= \det{}^+(A) \otimes \det{}^+(B)\end{aligned} \tag{17}$$

Let us now consider the expression β^+.

f being odd, f^{-1} is odd and g, as the product of an even permutation and an odd permutation, is odd. Then β^+ can be rewritten:

$$\begin{aligned}\beta^+ &= \left(\sum_{f\in \mathrm{Per}^-(n)} \prod_{i=1}^{n} a_{i,f(i)}\right) \otimes \left(\sum_{g\in \mathrm{Per}^-(n)} \prod_{i=1}^{n} b_{i,g(i)}\right) \\ &= \det{}^-(A) \otimes \det{}^-(B)\end{aligned} \tag{18}$$

From the above, we deduce:

$$\det{}^+(A \otimes B) = \det{}^+(A) \otimes \det{}^+(B) \oplus \det{}^-(A) \otimes \det{}^-(B) \oplus \gamma^+ \tag{19}$$

Through similar reasoning, we would prove that:

$$\det{}^-(A \otimes B) = \det{}^+(A) \otimes \det{}^-(B) \oplus \det{}^-(A) \otimes \det{}^+(B) \oplus \gamma^- \tag{20}$$

with:

$$\gamma^- = \sum_{f\in F'(n)} \sum_{\pi\in \mathrm{Per}^-(n)} \prod_{i=1}^{n} (a_{i,f(i)} \otimes b_{f(i),\pi(i)}) \tag{21}$$

Now we prove:

Lemma 5.1. *The two expressions* γ^+*, given by* (16)*, and* γ^-*, given by* (21)*, take the same value.*

Proof. Let us consider an arbitrary term of the sum (16) whose value is:

$$\theta = \prod_{i=1}^{n} a_{i,f(i)} \otimes b_{f(i),\pi(i)}$$

with $f \in F'(n)$ and $\pi \in \mathrm{Per}^+(n)$.

We are going to show that we associate it with a term θ' of expression (21) such that $\theta' = \theta$.

Since $f \in F'(n)$, f is not a permutation of $X = \{1, \ldots, n\}$, which therefore implies that there exists $i_0 \in X$, $i'_0 \in X$, $i'_0 \neq i_0$, $k \in X$ such that:

$$f(i_0) = k = f(i'_0) \tag{22}$$

If there exist several ordered triples (i_0, i'_0, k) satisfying (22) we choose the smallest possible value of k and, for this value of k, the two smallest possible values for i_0 and i'_0.

From the permutation π, let us define the following permutation π':

$$\begin{cases} \pi'(j) = \pi(j) \forall j \in X \backslash \{i_0, i'_0\}, \\ \pi'(i_0) = \pi(i'_0), \\ \pi'(i'_0) = \pi(i_0) \end{cases}$$

We observe that π' is deduced from π by transposition of the elements i_0 and i'_0, consequently $\pi' \in \mathrm{Per}^-(n)$. Furthermore, we observe that the same construction that obtains (f, π') from (f, π) enables one to obtain (f, π) from (f, π').

Finally, we have:

$$\begin{aligned} \theta' &= \prod_{i=1}^{n} \left(a_{i,f(i)} \otimes b_{f(i),\pi'(i)}\right) \\ &= \left(\prod_{\substack{i=1 \\ i \neq i_0 \\ i \neq i'_0}}^{n} a_{i,f(i)} \otimes b_{f(i),\pi'(i)} \right) \otimes a_{i_{0,k}} \otimes b_{k,\pi'(i_0)} \otimes a_{i'_{0,k}} \otimes b_{k,\pi'(i'_0)} \\ &= \left(\prod_{\substack{i=1 \\ i \neq i_0 \\ i \neq i'_0}}^{n} a_{i,f(i)} \otimes b_{f(i),\pi(i)} \right) \otimes a_{i_{0,k}} \otimes b_{k,\pi(i_0)} \otimes a_{i'_{0,k}} \otimes b_{k,\pi(i'_0)} \\ &= \theta \end{aligned}$$

which completes the proof. □

We have therefore obtained:

Theorem 1. *Let* A *and* B *be two square* $n \times n$ *matrices with coefficients in a commutative pre-semiring* $(E, \oplus, \otimes)$.

Then:

$$\det{}^+(A \otimes B) = \det{}^+(A) \otimes \det{}^+(B) \oplus \det{}^-(A) \otimes \det{}^-(B) \oplus \gamma$$

and:

$$\det{}^-(A \otimes g) = \det{}^+(A) \otimes \det{}^-(B) \oplus \det{}^-(A) \otimes \det{}^+(B) \oplus \gamma$$

where:

$$\gamma = \sum_{f \in F'(n)} \sum_{\pi \in Per^{+}(n)} \left(\prod_{i=1}^{n} a_{i,f(i)} \otimes b_{f(i),\pi(i)} \right)$$
$$= \sum_{f \in F'(n)} \sum_{\pi \in Per^{-}(n)} \left(\prod_{i=1}^{n} a_{i,f(i)} \otimes b_{f(i),\pi(i)} \right)$$

$F'(n)$*, in the above expressions, denoting the set of the mappings* $f: \{1, \ldots n\} \to \{1, \ldots n\}$ *which are not permutations.* □

As an immediate consequence of the above, we find again the well-known result:

Corollary 5.2. *If* $(E, \oplus)$ *is a group, then:*

$$\det(A \otimes B) = \det(A) \otimes \det(B)$$

As already pointed out in the introduction, Theorem 1 above clearly *does not* directly follow from the classical result (on the real field). Indeed a different proof is needed for the case of pre-semirings to get the exact expression of the additional term γ arising in both expressions of $\det^{+}(A \otimes B)$ and $\det^{-}(A \otimes B)$.

6. Cayley–Hamilton Theorem in Pre-Semirings

The Cayley–Hamilton theorem is a classical result of linear algebra (on the field of real numbers) according to which a matrix satisfies its own characteristic equation.

Combinatorial proofs of this theorem have been provided by Straubing (1983) and previously by Rutherford (1964). Rutherford's result constituted, moreover, a generalization of the theorem to the case of *semirings*.

Below we give a combinatorial proof inspired from Straubing (1983) and Zeilberger (1985), but which further generalizes the theorem to the case of *commutative pre-semirings* (indeed, it does not need to assume that ε, the neutral element of $\oplus$, is absorbing for $\otimes$).

Theorem 2. *Let* $(E, \oplus, \otimes)$ *be a commutative pre-semiring with neutral elements* ε *and* e.

Let A *be a square* $n \times n$ *matrix with coefficients in* $(E, \oplus, \otimes)$*, and let* $(P_A^{+}(\lambda), P_A^{-}(\lambda))$ *be the characteristic bipolynomial of* A.

$$\text{Then we have: } P_A^{+}(A) = P_A^{-}(A) \tag{23}$$

where:

$P_A^{+}(A)$ *and* $P_A^{-}(A)$ *are matrices obtained by replacing* λ^{n-q} *by the matrix* A^{n-q} *in the expression of* $P_A^{+}(\lambda)$ *and* $P_A^{-}(\lambda)$*, and where the following conventional notation is used:* A^{o} *denotes the matrix with diagonal terms equal to e and nondiagonal terms equal to* ε*; for every* $\alpha \in E$, $\alpha \otimes A^{o}$ *denotes the matrix with diagonal terms equal to* α *and nondiagonal terms equal to* ε.

Proof. We show that each entry (i, j) of the matrix $P_A^+(A)$ is equal to the entry (i, j) of the matrix $P_A^-(A)$.

Let us therefore consider i and j as fixed.

For $q = 0, 1, \dots, n-1$, the value of term (i, j) of the matrix A^{n-q} is:

$$(A^{n-q})_{ij} = \sum_{\substack{p \in P_{ij} \\ |p| = n-q}} \left(\prod_{(k,l) \in p} a_{k,l} \right)$$

where P_{ij} is the set of (nonnecessarily elementary) paths joining i to j in the complete directed graph on the set of vertices $\{1, \dots, n\}$, and where $|p|$ denotes the cardinality (number of arcs) of the path $p \in P_{ij}$.

For $q = n$, consistently with the adopted notational convention, $(A^{n-q})_{i,j} = (A^0)_{ij}$ is equal to ε for $i \neq j$, and to e for $i = j$.

Furthermore, the coefficient of A^{n-q} in $P_A^+(A)$ is:

$$\sum_{\substack{\sigma \in \mathrm{Part}^+(n) \\ |\sigma| = q}} \left(\prod_{i \in \mathrm{dom}(\sigma)} a_{i,\sigma(i)} \right)$$

and, consequently, the term (i, j) of the matrix $P_A^+(A)$ (by using the distributivity of $\otimes$ with respect to $\oplus$) is given by the following formulae. For $i \neq j$:

$$\sum_{q=1}^{n-1} \left[\left(\sum_{\substack{p \in P_{ij} \\ |p| = n-q}} \prod_{(k,l) \in p} a_{k,l} \right) \otimes \left(\sum_{\substack{\sigma \in \mathrm{Part}^+(n) \\ |\sigma| = q}} \prod_{i \in \mathrm{dom}(\sigma)} a_{i,\sigma(i)} \right) \right] \oplus \left[\sum_{\substack{p \in P_{ij} \\ |p| = n}} \prod_{(k,l) \in p} a_{k,l} \right] \tag{24}$$

For $i = j$, we must add to expression (24) the extra term:

$$\sum_{\substack{\sigma \in \mathrm{Part}^+(n) \\ |\sigma| = n}} \left(\prod_{i \in \mathrm{dom}(\sigma)} a_{i,\sigma(i)} \right)$$

(which may be viewed as corresponding to the value $q = n$).

Let us denote $\mathcal{F}_{ij}^+$ (resp. $\mathcal{F}_{ij}^-$) the family of graphs having $X = \{1, 2, \dots, n\}$ as vertex set and whose set of arcs U decomposes into: $U = P \cup C$ where:

- P is a set of arcs forming a path from i to j;
- C is a set of arcs such that the graph $G = [X, C]$ is the graph associated with a partial permutation σ of X with $\sigma \in \mathrm{Part}^+(n)$ (resp. $\sigma \in \mathrm{Part}^-(n)$).
  In other words, [X, C] is a union of an even (resp. odd) number of disjoint circuits (loops are allowed) not necessarily covering all the vertices.
- $|U| = |P| + |C| = n$.

The *weight* w(G) of a graph G = [X, U] belonging to $\mathcal{F}_{ij}^{+}$ or to $\mathcal{F}_{ij}^{-}$ is defined as:

$$w(G) = \prod_{(k,1)\in U} a_{k,1}$$

In the case where $i \neq j$, by expanding (24) (distributivity) we then observe that entry (i, j) of $P_A^{+}(A)$ is:

$$\sum_{G\in\mathcal{F}_{ij}^{+}} w(G) \tag{25}$$

In the case where $i = j$, by considering that the path P can be empty in the decomposition $U = P \cup C$, the additional term corresponding to $q = n$ is clearly taken into account in expression (25).

Similarly, it is easy to see that the entry (i, j) of $P_A^{-}(A)$ is, in all cases, ($i = j$ and $i \neq j$), equal to:

$$\sum_{G\in\mathcal{F}_{ij}^{-}} w(G) \tag{26}$$

It therefore remains to show that the two expressions (25) and (26) are equal. To do so, let us show that, with any graph G of $\mathcal{F}_{ij}^{+}$, we can associate a graph G′ of $\mathcal{F}_{ij}^{-}$ of the same weight, $w(G') = w(G)$, the correspondence thus exhibited between $\mathcal{F}_{ij}^{+}$ and $\mathcal{F}_{ij}^{-}$ being one-to-one.

Let us therefore consider $G = [X, P \cup C] \in \mathcal{F}_{ij}^{+}$. [X, C] is a union of an even number (possibly zero) of vertex-disjoint circuits (Fig. 3 shows an example where $n = 8$, $i = 1$, $j = 4$).

Since $|P| + |C| = n$, we observe that the sets of vertices covered by P and C necessarily have at least one common element. Furthermore, the path P not necessarily being elementary, P can contain one (or several) circuit(s).

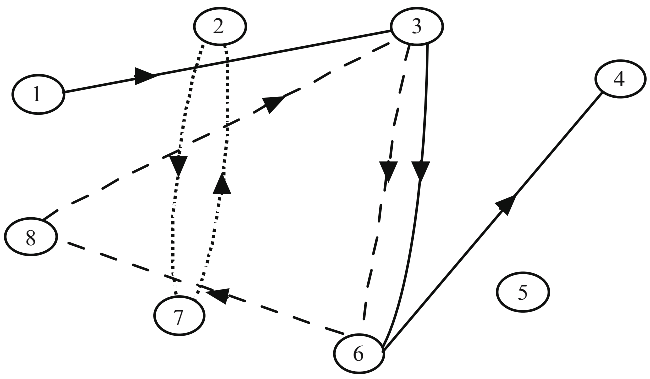

Fig. 3 Example illustrating the proof of the Cayley–Hamilton theorem. A graph $G \in \mathcal{F}_{ij}^{+}$ for $n = 8$, with $i = 1$ and $j = 4$. The path P is indicated in full lines and the partial permutation σ of characteristic +1 (as it contains two vertex-disjoint circuits) is indicated with dotted lines

Let us follow the path P starting from i until one of the following two situations occurs:

Case 1. We arrive at a vertex of P already traversed without meeting a vertex covered by C;
Case 2. We arrive at a vertex k covered by C.

In *case 1* we have identified a circuit Γ of P which does not contain any vertex covered by C. In this case, we construct $G' = [X, P' \cup C']$ where:

- P' is deduced from P by eliminating the circuit Γ;
- C' is deduced from C by adding the circuit Γ.

We observe that C' now contains an odd number of disjoint circuits, therefore $G' \in \mathcal{F}^{-}_{ij}$.

In *case 2*, let Γ be the circuit of C containing the vertex k. We construct $G' = [X, P' \cup C']$ where:

- P' is deduced from P by adding the circuit Γ;
- C' is deduced from C by eliminating the circuit Γ.

Here again, C' contains an odd number of disjoint circuits, therefore $G' \in \mathcal{F}^{-}_{ij}$.

Furthermore, we observe that in the two cases, G and G' have the same set of arcs, therefore $w(G') = w(G)$.

Finally, it is easy to see that, the same construction by which G is transformed into G' can be used to transform G' back into G: there is therefore a one-to-one correspondence between $\mathcal{F}^{+}_{ij}$ and $\mathcal{F}^{-}_{ij}$. (see illustration in Fig. 4)

From the above we deduce:

$$\sum_{G \in \mathcal{F}^{+}_{ij}} w(G) = \sum_{G \in \mathcal{F}^{-}_{ij}} w(G)$$

which completes the proof of Theorem 2. $\square$

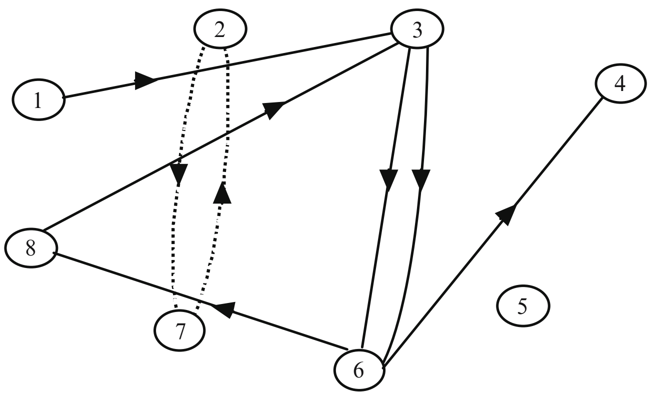

Fig. 4 The graph G' obtained by including the circuit (3, 6, 8) in P is an element of $\mathcal{F}^{-}_{ij}$ and it has the same weight as G

7. Semirings, Bideterminants and Arborescences

In the present section we consider a square $n \times n$ matrix, $A = (a_{ij})$ with elements in a commutative semiring $(E, \oplus, \otimes)$. We assume therefore:

- That $\oplus$ has a neutral element ε
- That $\otimes$ has a neutral element e.
- That ε is *absorbing* for $\otimes$ that is to say,

$$\forall x \in E: \qquad \varepsilon \otimes x = x \otimes \varepsilon = \varepsilon$$

For $r \in [1, n]$ we denote $\bar{A}$ the $(n-1) \times (n-1)$ matrix deduced from A by deleting row r and column r.

We denote I the $(n-1) \times (n-1)$ identity matrix of $M_{n-1}(E)$ with all diagonal terms equal to e and all other terms equal to ε.

7.1. An Extension to Semirings of the Matrix-Tree Theorem

Let us begin by stating below the result which will be proved in Sect. 7.2, and which may be viewed as a generalization, to semirings, of the classical "Matrix-Tree-Theorem" by Borchardt (1860) and Tutte (1948).

Theorem 3. *(Minoux, 1997)*

Let A *be a square* $n \times n$ *matrix with coefficients in a commutative semiring* $(E, \oplus, \otimes)$. *Let* $\bar{A}$ *be the matrix deduced from* A *by deleting row* r *and column* $r (r \in [1, n])$ *and let* B *be the* $(2n-2) \times (2n-2)$ *matrix of the form:*

$$B = \left[\begin{array}{c:c} \overline{D} & \bar{A} \\ \hdashline I & I \end{array}\right]$$

where I is the identity matrix of $M_{n-1}(E)$ *and* $\overline{D}$ *the diagonal matrix whose diagonal terms are:*

$$d_{ii} = \sum_{j=1}^{n} a_{ij} \quad \forall i \in \{1, \ldots, n\} \backslash \{r\}$$

(sum in the sense of $\oplus$*).*

Let us denote by $\mathcal{G}$ *the complete directed 1-graph on the vertex set* $X = \{1, 2, \ldots, n\}$ *and by* $\mathcal{T}_r$ *the set of the arborescences rooted at* r *in* $\mathcal{G}$*. For an arbitrary partial graph* G *of* $\mathcal{G}$*, the weight of* G*, denoted* w (G)*, is the product (in the sense of* $\otimes$*) of the values* a_{ij} *for all the arcs* (i, j) *of* G.

Then we have the identity:

$$\det{}^{+}(B) = \det{}^{-}(B) \oplus \sum_{\mathcal{G} \in \mathcal{T}_r} w(G) \qquad \square$$

7.2. Proof of Extended Theorem

To prove Theorem 3, let us consider the following $(2n-2) \times (2n-2)$ square matrix:

$$B' = \left[\begin{array}{c:c} \bar{A} & \overline{D} \\ \hdashline I & I \end{array}\right]$$

We observe that the permutation applied to the columns of B to obtain B' is even if $n - 1$ is even, and odd if $n - 1$ is odd. Consequently, if $n - 1$ is even we have $\det^+(B) = \det^+(B')$ and $\det^-(B) = \det^-(B')$. If $n - 1$ is odd, we have: $\det^+(B) = \det^-(B')$ and $\det^-(B) = \det^+(B')$.

Let us begin by studying the properties of the bideterminant of $B' = (b'_{ij})$. We have:

$$\det^+(B') = \sum_{\pi \in Per^+(2n-2)} \left(\prod_{i=1}^{2n-2} b'_{i,\pi(i)} \right) \tag{27}$$

In the above expression, all the terms corresponding to permutations π of $\{1, \ldots, 2n - 2\}$ such that $b'_{i,\pi(i)} = \varepsilon$ for some $i \in [1, 2n - 2]$ disappear because of the absorption property.

Consequently, in (27), we only have to take into account the permutations π of $Per^+(2n - 2)$ such that, for $1 \le i \le n - 1$:

$$\pi(i + n - 1) = i \quad \text{or} \quad \pi(i + n - 1) = i + n - 1$$

Each admissible permutation π can therefore be associated with a partition of $\overline{X} = \{1, \ldots, n - 1\}$ in two subsets U and V where:

$$U = \{i/i \in \overline{X}; \quad \pi(i + n - 1) = i\}$$
$$V = \{i/i \in \overline{X}; \quad \pi(i + n - 1) = i + n - 1\}$$

Furthermore, we observe that the columns of B' indexed $i + n - 1$ with $i \in U$ can only be covered by rows with index $i \in U$. Given that $\overline{D}$ is diagonal, we must therefore have:

$$\forall i \in U: \quad \pi(i) = i + n - 1$$

Each admissible permutation π can therefore be considered as derived from a permutation σ of V (a partial permutation of $X = \{1, \ldots, n\}$) as follows:

$$\begin{cases} \forall i \in V: \begin{cases} \pi(i) = \sigma(i) \\ \pi(i + n - 1) = i + n - 1 \end{cases} \\ \forall i \in U: \begin{cases} \pi(i) = i + n - 1 \\ \pi(i + n - 1) = i \end{cases} \end{cases}$$

The graph representing π on the set of vertices $\{1, \ldots, 2n-2\}$ therefore consists of:

- Elementary circuits representing the partial permutation σ;
- $|V|$ loops on the vertices $i+n-1$ ($i \in V$);
- $|U|$ circuits of length 2 (therefore even) of the form $(i, i+n-1)$, $i \in U$.

The signature of π is therefore equal to

$$\mathrm{sign}(\pi) = \mathrm{sign}(\sigma) \times (-1)^{|U|}$$

hence:

$$\begin{aligned}\mathrm{sign}(\pi) &= \mathrm{sign}(\pi) \times (-1)^{2\times|V|}\\ &= \mathrm{sign}(\sigma) \times (-1)^{|V|} \times (-1)^{|U|+|V|}\\ &= \mathrm{char}(\sigma) \times (-1)^{n-1}\end{aligned}$$

(since $V = \mathrm{dom}(\sigma)$).

Let us first assume that $n-1$ is even. In this case, $\mathrm{sign}(\pi)$ is none other than the characteristic of σ as a partial permutation of $\overline{X}$, and $\pi \in \mathrm{Per}^+(2n-2)$ if and only if $\sigma \in \mathrm{Part}^+(n-1)$. Then, (27) can be rewritten:

$$\begin{aligned}\det{}^+(B') &= \sum_{\sigma\in \mathrm{Part}^+(n-1)} \left(\prod_{i\in V} a_{i,\sigma(i)}\right) \otimes \left(\prod_{i\in U} d_{ii}\right)\\ &= \det{}^+(B)\end{aligned} \tag{28}$$

We would obtain a similar expression for $\det^-(B') = \det^-(B)$ simply by replacing $\sigma \in \mathrm{Part}^+(n-1)$ in (28) with $\sigma \in \mathrm{Part}^-(n-1)$. (Fig. 5)

Let us now consider the case where $n-1$ is odd. We then have $\mathrm{sign}(\pi) = -\mathrm{char}(\sigma)$, and, consequently, we have:

$$\begin{aligned}\det{}^+(B') &= \sum_{\sigma\in \mathrm{Part}^-(n-1)} \left(\prod_{i\in V} a_{i,\sigma(i)}\right) \otimes \left(\prod_{i\in U} d_{ii}\right)\\ &= \det{}^-(B)\end{aligned} \tag{29}$$

(we obtain the expression of $\det^-(B') = \det^+(B)$ by replacing $\sigma \in \mathrm{Part}^-(n-1)$ in (29) with $\sigma \in \mathrm{Part}^+(n-1)$).

Thus it is seen that, in both cases ($n-1$ even or odd), the expression giving $\det^+(B)$ is:

$$\det{}^+(B) = \sum_{\sigma\in \mathrm{Part}^+(n-1)} \left(\prod_{i\in V} a_{i,\sigma(i)}\right) \otimes \left(\prod_{i\in U} d_{ii}\right) \tag{30}$$

(where $V = \mathrm{dom}(\sigma)$ and $U = \overline{X}\backslash V$). The expression giving $\det^-(B)$ is simply deduced from the above by replacing $\sigma \in \mathrm{Part}^+(n-1)$ with $\sigma \in \mathrm{Part}^-(n-1)$.

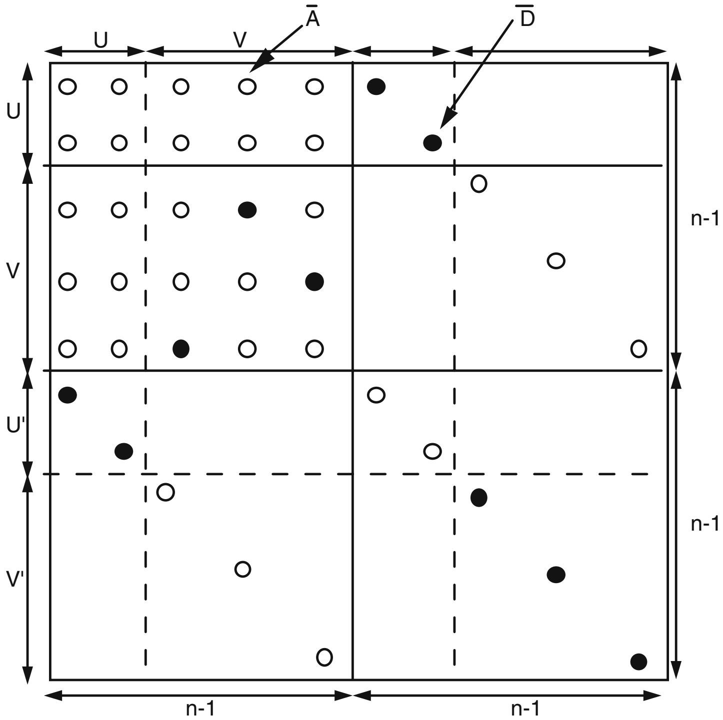

Fig. 5 The matrix B' and a partition of $\overline{X} = \{1, \ldots, n-1\}$ into two subsets U and V corresponding to an admissible permutation π of $\{1, \ldots, 2n-2\}$. Only the terms distinct from ε (neutral element of $\oplus$) are represented (by circles). The terms indicated in black are those corresponding to the permutation π. The partial permutation σ is the one induced by π on the sub-matrix of $\bar{A}$ restricted to the rows and columns of V

Let us denote $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) the family of all directed graphs constructed on the vertex set $X = \{1, 2, \ldots, n\}$, of the form $G = [X, C \cup Y]$ where:

- C is a set of arcs constituting vertex-disjoint circuits and containing an even (resp. odd) number of circuits;
- Y is a set of arcs such that, for every $i \in X\backslash\{r\}$ not covered by C, Y contains a single arc of the form (i, j) (the possibility $j = i$ being authorized, as well as the possibility $j = r$).

By expanding expression (30), that is to say by replacing each term d_{ii} by $\sum_{j=1}^{n} a_{ij}$ and by using distributivity, we then observe that $\det^+(B)$ can be expressed in the form:

$$\det^+(B) = \sum_{G \in \mathcal{F}^+} w(G) \tag{31}$$

where the "weight" w(G) of the graph $G = [X, C \cup Y]$ is:

$$w(G) = \prod_{(k,1)\in C\cup Y} a_{k,1}$$

We would prove similarly that:

$$\det{}^{-}(B) = \sum_{G\in\mathcal{F}^{-}} w(G) \tag{32}$$

Among the graphs of $\mathcal{F}^{+}\cup\mathcal{F}^{-}$, those *which do not contain a cycle* play a special role. Indeed, in this case, $C = \emptyset$, and the set Y does not contain a cycle and is composed of $n - 1$ arcs (an arc originating at each vertex $i \in X\backslash\{r\}$). Y therefore forms an *arborescence rooted at* r.

Since $C = \emptyset$, the subclass $\mathcal{T}_r$ (the set of arborescences rooted at r) is necessarily included in $\mathcal{F}^{+}$.

If we denote $\mathcal{F}^{+} = \mathcal{T}_r \cup \mathcal{F}_c^{+}$

we can therefore write:

$$\det{}^{+}(B) = \sum_{G\in\mathcal{T}_r} w(G) \oplus \sum_{G\in\mathcal{F}_c^{+}} w(G) \tag{33}$$

The end of the proof uses the following result (Zeilberger, 1985):

Lemma 7.2.1.

$$\sum_{G\in\mathcal{F}_c^{+}} w(G) = \sum_{G\in\mathcal{F}^{-}} w(G) \tag{34}$$

Proof. It proceeds by showing that, with each graph $G \in \mathcal{F}_c^{+}$ we can associate a graph G' of $\mathcal{F}^{-}$ with $w(G') = w(G)$, and that the correspondence is one-to-one.

Let us therefore consider a graph G of $\mathcal{F}_c^{+}$ of the form $G = [X, C \cup Y]$.

This graph contains at least one circuit and [X, C] contains an even number (possibly zero) of circuits. Among all the circuits of G, let us consider the one which meets the vertex with the smallest index number and let Γ be the set of its arcs.

If $\Gamma \subset Y$ then let us define $G' = [X, C' \cup Y']$ with

$$C' = C \cup \Gamma$$
$$Y' = Y\backslash\Gamma$$

If $\Gamma \subset C$ then let us define C' and Y' as:

$$C' = C\backslash\Gamma$$
$$Y' = Y \cup \Gamma$$

In both cases, C' contains an odd number of circuits, therefore $G' \in \mathcal{F}^-$, and as G and G' have the same sets of arcs:

$$w(G') = w(G).$$

Furthermore, we observe that the same construction which transforms G to G' enables one to transform G' back to G.

We would prove in the same way that, with every $G \in \mathcal{F}^-$ we can associate $G' \in \mathcal{F}_c^+$ such that $w(G') = w(G)$.

This completes the proof of Lemma 7.2.1. □

By using Lemma 7.2.1, (33) is then rewritten:

$$\det{}^+(B) = \sum_{G \in \mathcal{T}_r} w(G) \oplus \det{}^-(B), \text{ which establishes Theorem 3.} \quad \square$$

7.3. *The Classical Matrix-Tree Theorem as a Special Case*

In the special case where A is a real matrix on the field of real numbers, we see that

$$\sum_{G \in \mathcal{T}_r} w(G) = \det{}^+(B) - \det{}^-(B) = \det(B)$$

where det(B) is the determinant of B in the usual sense and:

$$\begin{aligned}
\det(B) &= \det \begin{bmatrix} \overline{D} & \vdots & \bar{A} \\ \cdots & \vdots & \cdots \\ I & \vdots & I \end{bmatrix} \\
&= \det \begin{bmatrix} \overline{D} - \bar{A} & \vdots & \bar{A} \\ \cdots\cdots & \vdots & \cdots \\ 0 & \vdots & I \end{bmatrix} \\
&= \det(\overline{D} - \bar{A})
\end{aligned}$$

From the above, we deduce the following corollary, known as the "Matrix Tree Theorem", due independently to Borchardt (1860) and Tutte (1948):

Corollary 7.3.1. *Let* $A = (a_{ij})$ *be a square* $n \times n$ *matrix with real coefficients;* D *the diagonal matrix whose ith diagonal term is* $d_{ii} = \sum_{j=1}^{n} a_{ij}$; $\bar{A}$ *and* $\overline{D}$ *the matrices deduced from* A *and* D *by eliminating the rth row and the rth column (for any fixed* r, $1 \le r \le n$*). Then det* $(\overline{D} - \bar{A})$ *is equal to the sum of the weights of the arborescences rooted at* r *in the graph associated with matrix* A.

Theorem 3 can thus be considered as an extension to semirings of the "Matrix-Tree Theorem".

7.4. A Still More General Version of the Theorem

A more general version of the "Matrix Tree Theorem", known as the "All Minors Matrix Tree Theorem" (see Chen (1976), Chaiken (1982)) can also be extended to semirings. We present this extension below (Theorem 4).

Let $A = (a_{ij})$ be a square $n \times n$ matrix with coefficients in a commutative semi-ring $(E, \oplus, \otimes)$, such that $\forall i = 1, \ldots, n$: $a_{ii} = \varepsilon$ (the neutral element of $\oplus$ in E).

For every $i \in X = \{1, 2, \ldots, n\}$ set:

$$d_{ii} = \sum_{\substack{k=1 \\ k \neq i}}^{n} a_{ik}$$

Let $L \subset X$ be a subset of rows of A and $K \subset X$ a subset of columns of A with $|L| = |K|$.

Let $\bar{A}$ be the sub-matrix of A obtained by eliminating the rows of L and the columns of K. The rows and the columns of $\bar{A}$ are therefore indexed by $\overline{L} = X \backslash L$ and $\overline{K} = X \backslash K$.

By setting $m = |\overline{L}| = |\overline{K}|$ and $p = |\overline{L} \cap \overline{K}|$ let us consider the $(m + p) \times (m + p)$ square matrix B having the block structure:

$$B = \begin{bmatrix} \bar{A} & \vdots & \overline{Q} \\ \cdots & \vdots & \cdots \\ R & \vdots & I_p \end{bmatrix}$$

where:

I_p is the $p \times p$ identity matrix of the semiring $(E, \oplus, \otimes)$.

Q is a $m \times p$ matrix whose rows are indexed by $\overline{L}$ and whose columns are indexed by $\overline{L} \cap \overline{K}$; all its terms are equal to ε except those indexed (i, i) with $i \in \overline{L} \cap \overline{K}$ which are equal to d_{ii}.

R is a $p \times m$ matrix whose lines are indexed by $\overline{L} \cap \overline{K}$ and whose columns are indexed by $\overline{K}$; all its terms are equal to ε except those indexed (i, i) with $i \in \overline{L} \cap \overline{K}$ which are equal to e (the neutral element of $\otimes$ in E).

For every subset $Y \subset X = \{1, 2, \ldots, n\}$ let us denote sign $(Y, X) = (-1)^{\nu(Y,X)}$ where:

$$\nu(Y, X) = |\{(i, j)/i \in X \backslash Y, j \in Y, i < j\}|$$

and $s(L, K) = \text{sign}(L, X) \times \text{sign}(K, X) \times (-1)^m$.

Let us also consider the set $\mathcal{T} = \mathcal{T}^+ \cup \mathcal{T}^-$ of all the directed forests H on the vertex set X satisfying the following three properties:

(i) H contains exactly $|L| = |K|$ trees;
(ii) Each tree of H contains exactly a vertex of L and a vertex of K;
(iii) Each tree of H is an arborescence, the root of which is the unique vertex of K which it contains.

The subsets $\mathcal{T}^+$ and $\mathcal{T}^-$ are then defined as follows.

With each $H \in \mathcal{T}$ we can associate a one-to-one correspondence $\pi^*: L \to K$ defined as: $\pi^*(j) = i$ if and only if $i \in K$ and $j \in L$ belong to the same tree of H.

Then $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) is the set of the directed forests of $\mathcal{T}$ such that sign $(\pi^*) = +1$ (resp. sign $(\pi^*) = -1$).
We can then state:

Theorem 4. *(Minoux, 1998a)*
If $s(L, K) = +1$ *then there exists* $\Delta \in E$
such that:

$$\begin{cases} \det^+(B) = \sum\limits_{H \in \mathcal{T}^+} w(H) \oplus \Delta \\ \det^-(B) = \sum\limits_{H \in \mathcal{T}^-} w(H) \oplus \Delta \end{cases}$$

If $s(L, K) = -1$ *then there exists* $\Delta \in E$ *such that:*

$$\begin{cases} \det^+(B) = \sum\limits_{H \in \mathcal{T}^-} w(H) \oplus \Delta \\ \det^-(B) = \sum\limits_{H \in \mathcal{T}^+} w(H) \oplus \Delta \end{cases}$$

Proof. Refer to Exercise 1 at the end of the chapter where the exact expression of Δ is specified. □

The above result suggests, once again, an essential remark concerning the general approach followed in the present chapter. In fact, suppose that we apply the simple trick which consists in formally deducing the generalized result from the classical result. The reader will easily be convinced that we can reformulate the classical "All-Minors Matrix-Tree Theorem" as:

$$\det(B) = \sum_{H \in \mathcal{T}^+} w(H) - \sum_{H \in \mathcal{T}^-} w(H)$$

If one thinks that it then suffices to rewrite the classical result by switching each term appearing negatively to the other side of the equation, one is led to propose a generalized version of the form:

$$\det^+(B) \oplus \sum_{H \in \mathcal{T}^-} w(H) = \det^-(B) \oplus \sum_{H \in \mathcal{T}^+} w(H)$$

which is not correct. Indeed, the above formula does not take into account the additional term Δ which cancels itself in the classical result.

Only a direct proof, specialized to the semiring structure, can exhibit this term and provide the exact expression (see Exercise 1 at the end of the chapter).

8. A Generalization of the Mac Mahon Identity to Commutative Pre-Semirings

Let us consider a square $n \times n$ matrix, $A = (a_{ij})$ with coefficients in a commutative pre-semiring $(E, \oplus, \otimes)$.

$x_1, x_2, \ldots, x_n$ being indeterminates and $m_1, m_2, \ldots, m_n$ natural integers, we consider the expression:

$$\begin{gathered}(a_{11} \otimes x_1 \oplus a_{12} \otimes x_2 \oplus \cdots a_{1n} \otimes x_n)^{m_1} \\ \otimes (a_{21} \otimes x_1 \oplus \cdots \oplus a_{2n} \otimes x_n)^{m_2} \\ \cdot \\ \cdot \\ \cdot \\ \otimes (a_{n1} \otimes x_1 \oplus \cdots \oplus a_{nn} \otimes x_n)^{m_n}\end{gathered} \tag{35}$$

and we denote $K(m_1, m_2, \ldots, m_n)$ the coefficient of the term involving $x_1^{m_1} \otimes x_2^{m_2} \otimes \cdots \otimes x_n^{m_n}$ in the expansion of expression (35).

The Mac Mahon identity (1915) (recalled in Sect. 8.2 below) establishes a link between the formal series S in $x_1, x_2, \ldots, x_n$, with coefficients $K(m_1, m_2, \ldots, m_n)$, and the expansion of the inverse of the determinant of the matrix $I - A\,D_x$, where D_x is the diagonal matrix whose diagonal terms are the indeterminates $x_1, x_2, \ldots, x_n$.

In Sect. 8.1, we establish a more general version of this result for commutative pre-semirings by giving a combinatorial proof generalizing that of Foata (1965), Cartier and Foata (1969) (see also Zeilberger, 1985). In Sect. 8.2 we show that the classical identity can be found again as a special case.

8.1. The Generalized Mac Mahon Identity

Theorem 5. *(Minoux 1998b, 2001)*

Let $(E, \oplus, \otimes)$ *be a commutative pre-semiring and* $A = (a_{ij}) \in M_n(E)$.

Let S denote the formal series:

$$S = \sum_{(m_1,\ldots m_n)} K(m_1, \ldots, m_n) \otimes x_1^{m_1} \otimes x_2^{m_2} \otimes \cdots \otimes x_n^{m_n} \tag{36}$$

where the sum extends to all distinct n*-tuples of natural integers.*

Then we have the following generalized Mac Mahon identity:

$$\begin{aligned} S \otimes &\left(\sum_{\sigma \in \mathrm{Part}^+(n)} \prod_{i \in \mathrm{dom}(\sigma)} a_{i,\sigma(i)} \otimes x_{\sigma(i)} \right) \\ &= e \oplus S \otimes \left(\sum_{\sigma \in \mathrm{Part}^-(n)} \prod_{i \in \mathrm{dom}(\sigma)} a_{i,\sigma(i)} \otimes x_{\sigma(i)} \right) \end{aligned} \tag{37}$$

Proof. Let us consider the family $\mathcal{G}(m_1, \ldots, m_n)$ of all the directed multigraphs of the form $G = [X, Y]$ where $X = \{1, 2, \ldots, n\}$ is the vertex set and where the set of arcs Y satisfies the two conditions:

(1) $\forall i \in X$, Y contains exactly m_i arcs origining at i
(2) $\forall i \in X$, Y contains exactly m_i arcs terminating at i

(observe that the graphs of the family $\mathcal{G}(m_1, \ldots, m_n)$ can obviously contain loops).

The weight of $G = [X, Y]$ is defined as the formal expression:

$$w(G) = \prod_{(k,l)\in Y} (a_{kl} \otimes x_l)$$

(product in the sense of $\otimes$) with the convention $w(G) = e$ if $Y = \emptyset$.

We then verify that:

$$K(m_1, \ldots, m_n) x_1^{m_1} \otimes x_2^{m_2} \otimes \cdots \otimes x_n^{m_n} = \sum_{G\in\mathcal{G}(m_1,\ldots,m_n)} w(G)$$

Consequently, the expression S given by (36) can be rewritten:

$$S = \sum_{(m_1,\ldots,m_n)} \sum_{G\in\mathcal{G}(m_1,\ldots,m_n)} w(G) = \sum_{G\in\mathcal{G}} w(G)$$

$$\text{with} \quad \mathcal{G} = \bigcup_{(m_1,\ldots,m_n)} \mathcal{G}(m_1, \ldots, m_n)$$

(union extended to all distinct n-tuples of natural integers).

Let us now consider the family $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) of all the graphs of the form $G = [X, Y \cup C]$ where:

- $[X, Y] \in \mathcal{G}$
- $[X, C]$ is the graph representative of a partial permutation $\sigma \in \text{Part}^+(n)$ (resp. $\sigma \in \text{Part}^-(n)$). It is therefore a set of arcs forming an even number (resp. odd number) of elementary vertex-disjoint circuits (some of these circuits may be loops).

We then observe that the left-hand side of (37) is equal to: $\sum_{G\in\mathcal{F}^+} w(G)$ and the right-hand side of (37) is equal to: $e \oplus \sum_{G\in\mathcal{F}^-} w(G)$.

Among all the graphs of the family $\mathcal{F}^+ \cup \mathcal{F}^-$, let us consider $G_0 = [X, Y \cup C]$ with $Y = \emptyset$ and $C = \emptyset$. In this case, the graph $[X, Y]$ corresponds to $m_1 = 0, m_2 = 0, \ldots m_n = 0$, it is therefore the unique element of the family $\mathcal{G}(0, 0, \ldots 0)$. Furthermore, $G_0 \in \mathcal{F}^+$ since $C = \emptyset$ corresponds to an even number of circuits, and $w(G_0) = e$.

Consequently, it suffices to establish that:

$$\sum_{G\in\mathcal{F}^+\backslash G_0} w(G) = \sum_{G\in\mathcal{F}^-} w(G) \tag{38}$$

To do so, we are going to exhibit a one-to-one correspondence between $\mathcal{F}^+\backslash G_0$ and $\mathcal{F}^-$ such that, if $G \in \mathcal{F}^+\backslash G_0$ and $G' \in \mathcal{F}^-$ are images through this one-to-one correspondence, then $w(G') = w(G)$.

All the graphs of the form $[X, Y \cup C]$ in $(\mathcal{F}^+\backslash G_o) \cup \mathcal{F}^-$ are assumed to be represented by adjacency lists with the following convention: for every $i \in X$, if i belongs to a circuit in $[X, C]$, then the arc of origin i in C is placed in the first position of the list of the arcs of origin i.

Now, let us consider $G = [X, Y \cup C] \in \mathcal{F}^+ \backslash G_0$. Since $G \neq G_0$, there exists at least one vertex of nonzero degree in G. Among these, let i_0 be the vertex having minimum index number.

Observe that C consists of an even number of vertex-disjoint circuits (this number may possibly be zero).

Let us traverse the partial graph [X, Y] starting from vertex i_0 by using the arcs of Y as follows: from every intermediate vertex i encountered that is not covered by C, we take the arc (i, j) which appears first in the adjacency list of vertex i. The traversal stops when one of the two following situations arises:

Case 1. We arrive at a vertex already encountered in the pathway before having encountered a vertex covered by C;
Case 2. We arrive at a vertex k covered by C.

In the first case, we have exhibited a circuit of the partial graph [X, Y], which does not have a common vertex with C. Let $\Gamma \subset Y$ be the set of its arcs.

$$\begin{aligned} \text{We then form} \quad & G' = [X, Y' \cup C'] \\ \text{with} \quad & Y' = Y \backslash \Gamma \\ & C' = C \cup \Gamma \end{aligned}$$

In the second case, C contains a circuit passing through k and let Γ be the set of its arcs. Then we form $G' = [X, Y' \cup C']$ with:

$$\begin{aligned} Y' &= Y \cup \Gamma \\ C' &= C \backslash \Gamma \end{aligned}$$

Moreover, the adjacency list of each node i covered by the circuit Γ is modified in such a way that the arc of Γ which originates at i becomes the first in the adjacency list for i.

In both cases, C' contains an odd number of vertex-disjoint circuits. Furthermore, the sets of arcs of G and G' being the same, we have $w(G') = w(G)$.

Finally, we observe that, thanks to the convention established concerning the order of arcs in the adjacency lists, the same construction which transforms G into G' enables one to transform G into G'. This is therefore a one-to-one correspondence between $\mathcal{F}^+ \backslash G_0$ and $\mathcal{F}^-$, which completes the proof of Theorem 5. □

8.2. The Classical Mac Mahon Identity as a Special Case

It is interesting to verify that the generalized form (37) of the Mac Mahon identity includes, as a special case, the usual form on the field of real numbers, which is expressed by the following corollary:

Corollary 8.2.1. S *being defined as in expression (*36*), and* B *denoting the matrix* $B = (b_{ij})_{\substack{i=1,\dots n \\ j=1,\dots n}} = (a_{ij} x_j)_{\substack{i=1,\dots n \\ j=1,\dots n}}$, *we have:*

$$S \times \det(I - B) = 1 \tag{39}$$

Proof. See Exercise 2 at the end of the chapter and Minoux (1998b, 2001). □

Exercises

Exercise 1. We consider the real matrix:

$$A = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 0 & 3 & -1 & 2 \\ 2 & 0 & 5 & -3 \\ -2 & 1 & 6 & 0 \end{bmatrix}$$

on a dioid $(\overline{\mathbb{R}}, \oplus, \otimes)$.

(1) Give the formal expression of the bideterminant of A, by formally stating $\det^+(A)$ and $\det^-(A)$.
(2) Compute the value of the bideterminant when the dioid under consideration is $(\overline{\mathbb{R}}, \mathrm{Max}, \mathrm{Min})$. Check that $\mathrm{Max}\{\det^+(A);\ \det^-(A)\}$ is indeed equal to the optimal value of the « bottleneck » (Max-Min) assignment problem.
(3) Compute the value of the bideterminant when the dioid under consideration is $(\overline{\mathbb{R}}, \mathrm{Max}, +)$. Check that $\mathrm{Max}\{\det^+(A);\ \det^-(A)\}$ is indeed equal to the optimal value of the assignment problem (where the objective is to maximize the sum of the selected entries).
(4) Check the Cayley–Hamilton theorem for A in both cases ($(\overline{\mathbb{R}}, \mathrm{Max}, \mathrm{Min})$ and $(\overline{\mathbb{R}}, \mathrm{Max}, +)$).

Exercise 2. We consider the real 4×4 matrix with entries in the dioid $(\overline{\mathbb{R}}, \mathrm{Min}, +)$:

$$A = \begin{bmatrix} \infty & 4 & 0 & 1 \\ 0 & \infty & -1 & 2 \\ 3 & 5 & \infty & -3 \\ -2 & 1 & 6 & \infty \end{bmatrix}$$

which is a generalized adjacency matrix corresponding to the complete oriented graph.

(1) Set up the list of all arborescences with root $r = 1$ in the above graph, and calculate the sum S (in the sense of $\oplus = \mathrm{Min}$) of the weights of these arborescences. We recall that, in the Matrix-Tree Theorem (see Theorem 3, Sect. 7.1), the arborescences involved are those having arcs oriented from the pending vertices to the root. The vertex $r = 1$ has thus zero out-degree.
(2) Check the generalized version of the « matrix tree theorem » on this example, in other words that $\det^+(B) = \mathrm{Min}\,\{\det^-(B);\ S\}$

where B is the 6×6 matrix: $\begin{bmatrix} \overline{D} & \vdots & \bar{A} \\ \cdots & \cdot & \cdots \\ I & \vdots & I \end{bmatrix}$

where:

$\bar{A}$ is deduced from A by deleting the first row and the first column of A; $\overline{D}$ is the diagonal matrix with diagonal entries:

$$d_{ii} = \underset{j=1,\dots,n}{\mathrm{Min}} \left\{a_{ij}\right\} \quad \forall i = 2, 3, 4.$$

[*Answers*:

(1) There are 16 distinct arborescences rooted at $r = 1$ in this example. For instance the arborescence composed of the arcs (2,1) (3,1) (4,1) with weight 1 ($= 0 + 3 - 2$); the arborescence composed of the arcs (2, 1) (3, 1) (4, 2) with weight 4, etc. The minimum of the weights of these 16 arborescences is $S = -6$, and corresponds to the arborescence (4, 1)(2, 3)(3, 4).

(2) We have $\bar{A} = \begin{bmatrix} \infty & -1 & 2 \\ 5 & \infty & -3 \\ 1 & 6 & \infty \end{bmatrix}$ and $\overline{D} = \begin{bmatrix} -1 & \infty & \infty \\ \infty & -3 & \infty \\ \infty & \infty & -2 \end{bmatrix}$

and it can be checked that:

$$\det{}^{+}(B) = -6, \quad \det{}^{-}(B) = -3$$

and that the extended Matrix-Tree Theorem holds since:

$$\det{}^{+}(B) = \mathrm{Min}\{\det{}^{-}(B), S\} = \mathrm{Min}\{-3, -6\}.]$$

Exercise 3. (Proof of Theorem 4: generalized "All Minors Matrix Tree Theorem")

In this exercise, we refer to the concepts and notation used in Sect. 7.4.

Given two subsets U and V of X of equal cardinality ($|U| = |V|$), we refer to as *matching* every one-to-one correspondence $\pi: U \to V$. The *signature* of a matching $\pi: U \to V$, denoted $\mathrm{sign}(\pi)$, is defined as follows. A pair (i, j) of elements of U is said to be in *inversion* relatively to π if $i < j$ and $\pi(i) > \pi(j)$. By denoting $\nu(\pi)$ the number of pairs (i, j) $i \in U$, $j \in U$, which are in inversion relatively to π, then sign $(\pi) = (-1)^{\nu(\pi)}$. We observe that, in the special case where $U = V = X$, a matching is none other than a permutation of X, and we verify that in this case the definition of the matching signature is consistent with that of the permutation signature.

The *characteristic* of a matching $\pi: U \to V$ is defined as:

$$\mathrm{char}(\pi) = \mathrm{sign}(\pi) \times (-1)^{|W|}$$
$$\text{where} \quad W = \{i / i \in U, \pi(i) = i\}$$

We now denote by $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) the set of all the directed graphs on X having as set of arcs $S \cup T$ where:

- S is the set of arcs of the form $(i, \pi(i))$ for every $i \in L$ such that $i \neq \pi(i)$, where $\pi: \overline{L} \to \overline{K}$ is a matching of characteristic $+1$ (resp. of characteristic -1).
- T is a set of arcs such that, for every $i \in \overline{L}$ satisfying $\pi(i) = i$, there is exactly one arc in T of the form (k, i) with $k \in X$, $k \neq i$ (note that $\pi(i) = i$ implies $i \in \overline{L} \cap \overline{K}$).

Among the graphs H of the family $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) those which are circuitless are exactly those of $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) (see Sect. 7.4). We can therefore write:

$\mathcal{F}^+ = \mathcal{T}^+ \cup \mathcal{F}_c^+$ and $\mathcal{F}^- = \mathcal{T}^- \cup \mathcal{F}_c^-$ where $\mathcal{F}_c^+$ (resp. $\mathcal{F}_c^-$) denotes the family of sub-graphs $H \in \mathcal{F}^+$ (resp. $H \in \mathcal{F}^-$) which contain nontrivial circuits (i.e. circuits which are not loops).

(1) Prove that we have:

$$\det{}^{+}(B) = \sum_{H\in\mathcal{F}^{+}} w(H) \quad \text{and} \quad \det{}^{-}(B) = \sum_{H\in\mathcal{F}^{-}} w(H)$$

where B is the matrix $\begin{bmatrix} \bar{A} & \vdots & Q \\ \cdots & \vdots & \cdots \\ R & \vdots & I_p \end{bmatrix}$ defined in Sect. 7.4.

(2) Show, by using an argument similar to the one used by Chaiken (1982), that

$$\sum_{H\in\mathcal{F}_c^{+}} w(H) = \sum_{H\in\mathcal{F}_c^{-}} w(H)$$

(3) Then show that Theorem 4 is deduced from the above by taking:

$$\Delta = \sum_{H\in\mathcal{F}_c^{+}} w(H) = \sum_{H\in\mathcal{F}_c^{-}} w(H)$$

[*Answers*: refer to Minoux (1998a)].

Exercise 4. Where we recover the classical Mac Mahon identity

Here we take the field of real numbers as the basic algebraic structure.

(1) Let B be a $n \times n$ matrix with coefficients in $\mathbb{R}$, and I the identity matrix of $M_n(\mathbb{R})$. Prove that:

$$\det(I - B) = \sum_{\sigma\in\text{Part}^{+}(n)} \left(\prod_{j\in\text{dom}(\sigma)} b_{i,\sigma(i)} \right) - \sum_{\sigma\in\text{Part}^{-}(n)} \left(\prod_{i\in\text{dom}(\sigma)} b_{i,\sigma(i)} \right).$$

(2) By using the above relation, deduce from Theorem 5 (see Sect. 8.1) the classical Mac Mahon identity:

$$\begin{aligned} & S \times \det(I - B) = 1 \\ \text{with} \quad & B = (b_{ij})_{\substack{i=1\cdots n \\ j=1\cdots n}} = (a_{ij}x_j)_{\substack{i=1\cdots n \\ j=1\cdots n}}. \end{aligned}$$

[*Answers*: refer to Minoux (1998b, 2001)]

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