Chapter 2
Mittag-Leffler Functions and Fractional Calculus

[*This chapter is based on the lectures of Professor R.K. Saxena of Jai Narain Vyas University, Jodhpur, Rajasthan, India.*]

2.0 Introduction

This section deals with Mittag-Leffler function and its generalizations. Its importance is realized during the last one and a half decades due to its direct involvement in the problems of physics, biology, engineering and applied sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differential equations and fractional order integral equations. Various properties of Mittag-Leffler functions are described in this section. Among the various results presented by various researchers, the important ones deal with Laplace transform and asymptotic expansions of these functions, which are directly applicable in the solution of differential equations and in the study of the behavior of the solution for small and large values of the argument. Hille and Tamarkin in 1920 have presented a solution of Abel-Volterra type integral equation

$$\phi(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)}{(x-t)^{1-\alpha}} \mathrm{d}t = f(x), \quad 0 < x < 1$$

in terms of Mittag-Leffler function. Dzherbashyan (1966) has shown that both the functions defined by (2.1.1) and (2.1.2) are entire functions of order $p = \frac{1}{\alpha}$ and type $\sigma = 1$. A detailed account of the basic properties of these functions is given in the third volume of Batemann Manuscript Project written by Erdélyi et al (1955) under the heading "Miscellaneous Functions".

2.1 Mittag-Leffler Function

Notation 2.1.1. $E_\alpha(x)$: **M**ittag-Leffler function

Notation 2.1.2. $E_{\alpha,\beta}(x)$: **G**eneralized Mittag-Leffler function

Note 2.1.1: According to Erdélyi, et al (1955) both $\boldsymbol{E_\alpha(\mathbf{x})}$ and $\boldsymbol{E_{\alpha,\beta}(\mathbf{x})}$ are called Mittag-Leffler functions.

Definition 2.1.1.

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)},\ \alpha \in C, \Re(\alpha) > 0. \tag{2.1.1}$$

Definition 2.1.2.

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha,\beta \in C, \Re(\alpha) > 0, \Re(\beta) > 0. \tag{2.1.2}$$

The function $E_\alpha(z)$ was defined and studied by Mittag-Leffler in the year 1903. It is a direct generalization of the exponential series. For $\alpha = 1$ we have the exponential series. The function defined by (2.1.2) gives a generalization of (2.1.1). This generalization was studied by Wiman in 1905, Agarwal in 1953, Humbert and Agarwal in 1953, and others.

Example 2.1.1. Prove that $E_{1,2}(z) = \frac{e^z-1}{z}$.

Solution 2.1.1: We have

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{1}{z}(e^z-1).$$

Definition 2.1.3. Hyperbolic function of order n.

$$h_r(z,n) = \sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk+r-1)!} = z^{r-1}E_{n,r}(z^n),\ \ r = 1,2,\ldots. \tag{2.1.3}$$

Definition 2.1.4. Trigonometric functions of order n.

$$K_r(z,n) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{kn+r-1}}{(kn+r-1)!} = z^{r-1}E_{n,r}(-z^n). \tag{2.1.4}$$

$$E_{\frac{1}{2},1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\frac{k}{2}+1\right)} = e^{z^2}\operatorname{erfc}(-z), \tag{2.1.5}$$

where erfc is complementary to the error function erf.

Definition 2.1.5. Error function.

$$\operatorname{erfc}(z) = \frac{2}{\pi^{\frac{1}{2}}}\int_z^{\infty} e^{-u^2}du = 1 - \operatorname{erf}(z),\ \ z \in C. \tag{2.1.6}$$

To derive (2.1.5), we see that Dzherbashyan (1966, P.297, Eq.7.1.) reads as

$$w(z) = \mathrm{e}^{-z^2}\operatorname{erfc}(-iz) \tag{2.1.7}$$

whereas Dzherbashyan (1966, P.297, Eq.7.1.8) is

$$w(z) = \sum_{n=0}^{\infty} \frac{(iz)^n}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{2.1.8}$$

From (2.1.7) and (2.1.8) we easily obtain (2.1.5). In passing, we note that $w(z)$ is also an error function (Dzherbashyan (1966)).

Definition 2.1.6. Mellin-Ross function.

$$E_t(\nu,a) = t^{\nu} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\nu+k+1)} = t^{\nu} E_{1,\nu+1}(at). \tag{2.1.9}$$

Definition 2.1.7. Robotov's function.

$$R_{\alpha}(\beta,t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma((1+\alpha)(k+1))} = t^{\alpha} E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1}). \tag{2.1.10}$$

Example 2.1.2. Prove that $E_{1,3}(z) = \frac{\mathrm{e}^z - z - 1}{z^2}$.

Solution 2.1.2: We have

$$\begin{aligned} E_{1,3}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} \\ &= \frac{1}{z^2}(\mathrm{e}^z - z - 1). \end{aligned}$$

Example 2.1.3. Prove that

$$E_{1,r}(z) = \frac{1}{z^{r-1}} \left\{ \mathrm{e}^z - \sum_{k=0}^{r-2} \frac{z^k}{k!} \right\},\ r = 1,2,\ldots.$$

The proof is similar to that in Example 2.1.2.

Revision Exercises 2.1.

2.1.1. Prove that

$$H_{1,2}^{1,1}\left[x\Big|_{(a,A),(0,1)}^{(a,A)}\right] = A^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k x^{(k+a)/A}}{\Gamma(1+(k+a)A)},$$

and write the right side in terms of a generalized Mittag-Leffler function.

2.1.2. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}H_{1,2}^{1,1}\left[x\Big|_{(a,A),(0,1)}^{(a,A)}\right]=H_{1,2}^{1,1}\left[x\Big|_{(a-A,A),(0,1)}^{(a-A,A)}\right].$$

2.1.3. Prove that

$$H_{2,1}^{1,1}\left[\frac{1}{x}\Big|_{(1-a,A)}^{(1-a,A),(1,1)}\right]=A^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{1}{x})^{\frac{k+1-a}{A}}}{\Gamma(1-(k+1-a)/A)}.$$

2.2 Basic Properties of Mittag-Leffler Function

As a consequence of the definitions (2.1.1) and (2.1.2) the following results hold:

Theorem 2.2.1. *There hold the following relations:*

$$\text{(i) } E_{\alpha,\beta}(z)=zE_{\alpha,\alpha+\beta}(z)+\frac{1}{\Gamma(\beta)} \tag{2.2.1}$$

$$\text{(ii) } E_{\alpha,\beta}(z)=\beta E_{\alpha,\beta+1}(z)+\alpha z\frac{\mathrm{d}}{\mathrm{d}z}E_{\alpha,\beta+1}(z) \tag{2.2.2}$$

$$\text{(iii) } \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\left[z^{\beta-1}E_{\alpha,\beta}(z^\alpha)\right]=z^{\beta-m-1}E_{\alpha,\beta-m}(z^\alpha), \tag{2.2.3}$$

$$\Re(\beta-m)>0,\ m=0,1,\ldots. \tag{2.2.4}$$

Solutions 2.2.1: (i) We have

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}=\sum_{k=-1}^{\infty}\frac{z^{k+1}}{\Gamma(\alpha+\beta+\alpha k)}$$
$$=zE_{\alpha,\alpha+\beta}(z)+\frac{1}{\Gamma(\beta)},\ \Re(\beta)>0.$$

(ii) We have

$$R.H.S.=\beta E_{\alpha,\beta+1}(z)+\alpha z\frac{\mathrm{d}}{\mathrm{d}z}\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta+1)}$$
$$=\sum_{k=0}^{\infty}\frac{(\alpha k+\beta)z^k}{\Gamma(\alpha k+\beta+1)}=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$$
$$=E_{\alpha,\beta}(z)=L.H.S.$$

(iii)

$$
\begin{aligned}
L.H.S. &= \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \sum_{k=0}^{\infty} \frac{z^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} \\
&= \sum_{k=0}^{\infty} \frac{z^{\alpha k+\beta-m-1}}{\Gamma(\alpha k+\beta-m)},\ \Re(\beta-m)>0,
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{k=0}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m (z^{\alpha k+\beta-1}) &= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta-m)} z^{\alpha k+\beta-m-1} \\
&= z^{\beta-m-1}E_{\alpha,\beta-m}(z^{\alpha}),\ m=0,1,2,\ldots \\
&= R.H.S.
\end{aligned}
$$

Following special cases of (2.2.3) are worth mentioning. If we set $\alpha=\frac{m}{n},\ m,n=1,2,\ldots$ then

$$
\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \left[z^{\beta-1}E_{\frac{m}{n},\beta}(z^{\frac{m}{n}})\right] &= z^{\beta-m-1}E_{\frac{m}{n},\beta-m}(z^{\frac{m}{n}}) \\
&= z^{\beta-m-1}\sum_{k=0}^{\infty}\frac{z^{\frac{mk}{n}}}{\Gamma\left(\frac{mk}{n}+\beta-m\right)}.
\end{aligned}
$$

for $\Re(\beta-m)>0$, (replacing k by $k+n$)

$$
\begin{aligned}
&= z^{\beta-m-1}\sum_{k=-n}^{\infty}\frac{z^{\frac{m(k+n)}{n}}}{\Gamma\left(\beta+\frac{mk}{n}\right)} \\
&= z^{\beta-1}E_{\frac{m}{n},\beta}(z^{\frac{m}{n}}) + z^{\beta-1}\sum_{k=1}^{n}\frac{z^{-\frac{mk}{n}}}{\Gamma\left(\beta-\frac{mk}{n}\right)},\ m,n=1,2,3.
\end{aligned}
\tag{2.2.5}
$$

$$
\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \left[z^{\beta-1}E_{m,\beta}(z^m)\right] = z^{\beta-1}E_{m,\beta}(z^m) + \frac{z^{-m}}{\Gamma(\beta-m)},\ \Re(\beta-m)>0. \tag{2.2.6}
$$

Putting $z=t^{\frac{n}{m}}$ in (2.2.3) it yields

$$
\begin{aligned}
\left(\frac{m}{n}t^{1-\frac{n}{m}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^m [t^{(\beta-1)\frac{n}{m}}E_{\frac{m}{n},\beta}(t)] &= t^{(\beta-1)\frac{n}{m}}E_{\frac{m}{n},\beta}(t) \\
&+ t^{(\beta-1)\frac{n}{m}}\sum_{k=1}^{n}\frac{t^{-k}}{\Gamma(\beta-\frac{mk}{n})},\ \Re(\beta-m)>0,\ m,n=1,2,\ldots.
\end{aligned}
\tag{2.2.7}
$$

When $m=1$, (2.2.7) reduces to

$$
\frac{t^{1-n}}{n}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{(\beta-1)n}E_{\frac{1}{n},\beta}(t)\right] = t^{(\beta-1)n}E_{\frac{1}{n},\beta}(t) + t^{(\beta-1)n}\sum_{k=1}^{n}\frac{t^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)},
$$

for $\Re(\beta) > 1$, which can be written as

$$\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{(\beta-1)n}E_{\frac{1}{n},\beta}(t)\right] = t^{\beta n-1}E_{\frac{1}{n},\beta}(t) + t^{\beta n-1}\sum_{k=1}^{n}\frac{t^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)},\ \Re(\beta) > 1. \quad (2.2.8)$$

2.2.1 Mittag-Leffler functions of rational order

Now we consider the Mittag-Leffler functions of rational order $\alpha = \frac{p}{q}$ with $p, q = 1, 2, ...$ relatively prime. The following relations readily follow from the definitions (2.1.1) and (2.1.2).

$$\text{(i)}\quad \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{p} E_p(z^p) = E_p(z^p) \quad (2.2.9)$$

$$\text{(ii)}\quad \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{p} E_{\frac{p}{q}}(z^{\frac{p}{q}}) = E_{\frac{p}{q}}(z^{\frac{p}{q}}) + \sum_{k=1}^{q-1}\frac{z^{k\frac{p}{q}-p}}{\Gamma(k\frac{p}{q}+1-p)}, \quad (2.2.10)$$

$q = 1, 2, 3, ...$ We now derive the relation

$$\text{(iii)}\quad E_{\frac{1}{q}}(z^{\frac{1}{q}}) = \mathrm{e}^{z}\left[1 + \sum_{k=1}^{q-1}\frac{\gamma(1-\frac{k}{q}, z)}{\Gamma(1-\frac{k}{q})}\right], \quad (2.2.11)$$

where $q = 2, 3, ...$ and $\gamma(\alpha, z)$ is the incomplete gamma function, defined by

$$\gamma(\alpha, z) = \int_0^z \mathrm{e}^{-u}u^{\alpha-1}\mathrm{d}u$$

To prove (2.2.11), set $p = 1$ in (2.2.10) and multiply both sides by e^{-z} and use the definition of $\gamma(\alpha, z)$. Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{e}^{-z}E_{\frac{1}{q}}(z^{\frac{1}{q}})\right] = \mathrm{e}^{-z}\sum_{k=1}^{q-1}\frac{z^{-\frac{k}{q}}}{\Gamma\left(1-\frac{k}{q}\right)}. \quad (2.2.12)$$

Integrating (2.2.12) with respect to z, we obtain (2.2.11).

2.2.2 Euler transform of Mittag-Leffler function

By virtue of beta function formula it is not difficult to show that

$$\int_0^1 z^{\rho-1}(1-z)^{\sigma-1}E_{\alpha,\beta}(xz^{\gamma})\mathrm{d}z = \Gamma(\sigma)\,{}_2\psi_2\left[x\Big|^{(\rho,\gamma),(1,1)}_{(\beta,\alpha),(\sigma+\rho,\gamma)}\right] \quad (2.2.13)$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\sigma) > 0, \gamma > 0$. Here ${}_2\psi_2$ is the generalized Wright function and $\alpha, \beta, \rho, \sigma \in C$.

Special cases of (2.2.13):

(i) When $\rho = \beta, \gamma = \alpha$, (2.2.13) yields

$$\int_0^1 z^{\beta-1}(1-z)^{\sigma-1} E_{\alpha,\beta}(xz^{\alpha})\mathrm{d}z = \Gamma(\sigma) E_{\alpha,\sigma+\beta}(x), \tag{2.2.14}$$

where $\alpha > 0; \beta, \sigma \in C, \Re(\beta) > 0, \Re(\sigma) > 0$ and,

(ii)

$$\int_0^1 z^{\sigma-1}(1-z)^{\beta-1} E_{\alpha,\beta}[x(1-z)^{\alpha}]\mathrm{d}z = \Gamma(\sigma) E_{\alpha,\beta+\sigma}(x), \tag{2.2.15}$$

where $\alpha > 0; \beta, \sigma \in C, \Re(\beta) > 0, \Re(\sigma) > 0$.

(iii) When $\alpha = \beta = 1$ we have

$$\begin{aligned}\int_0^1 z^{\rho-1}(1-z)^{\sigma-1} \exp(xz^{\gamma})\mathrm{d}z &= \Gamma(\sigma)\,{}_2\psi_2\left[x\Big|^{(\rho,\gamma),(1,1)}_{(1,1),(\sigma+\rho,\gamma)}\right] \\ &= \Gamma(\sigma)\,{}_1\psi_1\left[x\Big|^{(\rho,\gamma)}_{(\sigma+\rho,\gamma)}\right],\end{aligned} \tag{2.2.16}$$

where $\gamma > 0, \rho, \sigma \in C, \Re(\rho) > 0, \Re(\sigma) > 0$.

2.2.3 Laplace transform of Mittag-Leffler function

Notation 2.2.1. $F(s) = L\{f(t);s\} = (Lf)(s)$: Laplace transform of $f(t)$ with parameter s.

Notation 2.2.2. $L^{-1}\{F(s);t\}$: Inverse Laplace transform

Definition 2.2.1. The Laplace transform of a function $f(t)$, denoted by $F(s)$, is defined by the equation

$$F(s) = (Lf)(s) = L\{f(t);s\} = \int_0^{\infty} \mathrm{e}^{-st} f(t)\mathrm{d}t, \tag{2.2.17}$$

where $\Re(s) > 0$, which may be symbolically written as

$$F(s) = L\{f(t);s\} \text{ or } f(t) = L^{-1}\{F(s);t\},$$

provided that the function $f(t)$ is continuous for $t \geq 0$, it being tacitly assumed that the integral in (2.2.17) exists.

Example 2.2.1. Prove that

$$L^{-1}\{s^{-\rho}\} = \frac{t^{\rho-1}}{\Gamma(\rho)},\ \Re(s) > 0,\ \Re(\rho) > 0. \tag{2.2.18}$$

It follows from the Laplace integral

$$\int_0^\infty \mathrm{e}^{-st} t^{\rho-1} \mathrm{d}t = \frac{\Gamma(s)}{s^\rho},\ \Re(s) > 0,\ \Re(\rho) > 0. \tag{2.2.19}$$

Example 2.2.2. Find the inverse Laplace transform of $\frac{F(s)}{a+s^\alpha}$; $a, \alpha > 0$; where $\Re(s) > 0, F(s) = L\{f(t); s\}$.

Solution 2.2.1: Let

$$G(s) = \frac{1}{a+s^\alpha} = \sum_{r=0}^{\infty} (-a)^r s^{-\alpha-\alpha r},\ |\frac{a}{s^\alpha}| < 1.$$

Therefore,

$$\begin{aligned} L^{-1}\{G(s)\} = g(t) &= L^{-1}\left\{\sum_{r=0}^{\infty} (-a)^r s^{-\alpha-\alpha r}\right\} \\ &= t^{\alpha-1} E_{\alpha,\alpha}(-at^\alpha). \end{aligned} \tag{2.2.20}$$

Application of convolution theorem of Laplace transform yields the result

$$L^{-1}\left\{\frac{F(s)}{a+s^\alpha}; t\right\} = \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(-a(x-t)^\alpha)\ f(t)\mathrm{d}t \tag{2.2.21}$$

where $\Re(\alpha) > 0$.

By the application of Laplace integral, it follows that

$$\int_0^\infty z^{\rho-1} \mathrm{e}^{-az} E_{\alpha,\beta}(xz^\gamma)\mathrm{d}z = \frac{1}{a^\rho}\, {}_2\psi_1\left[\frac{x}{a^\gamma}\Big|^{(1,1),(\rho,\gamma)}_{(\beta,\alpha)}\right], \tag{2.2.22}$$

where $\rho, a, \alpha, \beta \in C, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(a) > 0, \Re(\rho) > 0$ and $|\frac{z}{a^\gamma}| < 1$. Special cases of (2.2.22) are worth mentioning.

(i) For $\rho = \beta, \gamma = \alpha, \Re(\alpha) > 0$, (2.2.22) gives

$$\int_0^\infty \mathrm{e}^{-az} z^{\beta-1} E_{\alpha,\beta}(xz^\alpha)\mathrm{d}z = \frac{a^{\alpha-\beta}}{a^\alpha - x}, \tag{2.2.23}$$

where $a, \alpha, \beta \in C, \Re(\alpha) > 0, \Re(\beta) > 0, |\frac{x}{a^\alpha}| < 1$.

When $a = 1$, (2.2.23) yields a known result.

$$\int_0^\infty e^{-z} z^{\beta-1} E_{\alpha,\beta}(xz^\alpha) dz = \frac{1}{1-x}, |x| < 1, \tag{2.2.24}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$. If we further take $\beta = 1$, (2.2.24) reduces to

$$\int_0^\infty e^{-z} E_\alpha(xz^\alpha) dz = \frac{1}{1-x}, |x| < 1.$$

(ii) When $\beta = 1$, (2.2.23) gives

$$\int_0^\infty e^{-az} E_\alpha(xz^\alpha) dz = \frac{a^{\alpha-1}}{a^\alpha - x}, \tag{2.2.25}$$

where $\Re(a) > 0, \Re(\alpha) > 0, |\frac{x}{a^\alpha}| < 1$.

2.2.4 Application of Lalace transform

From (2.2.23) we find that

$$L\{x^{\beta-1} E_{\alpha,\beta}(ax^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha - a} \tag{2.2.26}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$. We also have

$$L\{x^{\gamma-1} E_{\alpha,\gamma}(-ax^\alpha)\} = \frac{s^{\alpha-\gamma}}{s^\alpha + a}. \tag{2.2.27}$$

Now

$$\left[\frac{s^{\alpha-\beta}}{s^\alpha - a}\right]\left[\frac{s^{\alpha-\gamma}}{s^\alpha + a}\right] = \frac{s^{2\alpha-(\beta+\gamma)}}{s^{2\alpha} - a^2} \text{ for } \Re(s^2) > \Re(a). \tag{2.2.28}$$

By virtue of the convolution theorem of the Laplace transform, it readily follows that

$$\int_0^t u^{\beta-1} E_{\alpha,\beta}(au^\alpha)(t-u)^{\gamma-1} E_{\alpha,\gamma}(-a(t-u)^\alpha) du = t^{\beta+\gamma-1} E_{2\alpha,\beta+\gamma}(a^2 t^{2\alpha}), \tag{2.2.29}$$

where $\Re(\beta) > 0, \Re(\gamma) > 0$. Further, if we use the identity

$$\frac{1}{s^2} = \frac{s^{\alpha-\beta}}{s^\alpha - 1}\left[s^{\beta-2} - s^{\beta-\alpha-2}\right] \tag{2.2.30}$$

and the relation

$$L\{t^{\rho-1}; s\} = \Gamma(\rho) s^{-\rho}, \tag{2.2.31}$$

where $\Re(\rho) > 0, \Re(s) > 0$, we obtain

$$\int_0^t u^{\beta-1} E_{\alpha,\beta}(u^\alpha) \left[\frac{(t-u)^{1-\beta}}{\Gamma(2-\beta)} - \frac{(t-u)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \right] \mathrm{d}u = t, \tag{2.2.32}$$

where $0 < \beta < 2, \Re(\alpha) > 0$. Next we note that the following result (2.2.34) can be derived by the application of inverse Laplace transform to the identity

$$\left[\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1} \right] [s^{-\alpha}] = -\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1} + \frac{s^{\alpha-\beta}}{s^\alpha - 1}, \ \Re(s^\alpha) > 1. \tag{2.2.33}$$

We have

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} E_{2\alpha,\beta}(t^{2\alpha}) t^{\beta-1} \mathrm{d}t = -x^{\beta-1} E_{2\alpha,\beta}(x^{2\alpha}) + x^{\beta-1} E_{\alpha,\beta}(x^\alpha), \tag{2.2.34}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$. If we set $\beta = 1$ in (2.2.34), it reduces to

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} E_{2\alpha}(t^{2\alpha}) \mathrm{d}t = E_\alpha(x^\alpha) - E_{2\alpha}(x^{2\alpha}) \tag{2.2.35}$$

where $\Re(\alpha) > 0$.

2.2.5 Mittag-Leffler functions and the H-function

Both the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ belong to H-function family. We derive their relations with the H-function.

Lemma 2.2.1: Let $\alpha \in R_+ = (0, \infty)$. Then $E_\alpha(z)$ is represented by the Mellin-Barnes integral

$$E_\alpha(z) = \frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(1-\alpha s)} \mathrm{d}s, |\arg z| < \pi, \tag{2.2.36}$$

where the contour of integration L, beginning at $c - i\infty$ and ending at $c + i\infty, 0 < c < 1$, separates all poles $s = -k, k = 0, 1, 2, ...$ to the left and all poles $s = 1 + n, n = 0, 1, ...$ to the right.

Proof. We now evaluate the integral (2.2.36) as the sum of the residues at the points $s = 0, -1, -2, \ldots$. We find that

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(1-\alpha s)}\mathrm{d}s = \sum_{k=0}^{\infty}\lim_{s\to -k}\left[\frac{(s+k)\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(1-\alpha s)}\right] \quad (2.2.37)$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(1+k)}{k!\Gamma(1+\alpha k)}(-z)^k$$
$$= E_\alpha(z),$$

which yields (2.2.36) in accordance with the definition (2.1.1). It readily follows from the definition of the H-function and (2.2.36) that $E_\alpha(z)$ can be represented in the form

$$E_\alpha(z) = H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(0,\alpha)}^{(0,1)}\right], \quad (2.2.38)$$

where $H_{1,2}^{1,1}$ is the H-function, which is studied in Chapter 1.

Lemma 2.2.2: Let $\alpha \in R_+ = (0,\infty), \beta \in C$, then

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(\beta-\alpha s)}\mathrm{d}s. \quad (2.2.39)$$

The proof of (2.2.39) is similar to that of (2.2.36). Hence the proof is omitted. From (2.2.39) and the definition of the H-function we obtain the relation

$$E_{\alpha,\beta}(z) = H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(1-\beta,\alpha)}^{(0,1)}\right]. \quad (2.2.40)$$

In particular, $E_\alpha(z)$ can be expressed in terms of generalized Wright function in the form

$$E_\alpha(z) = {}_1\psi_1\left[z\Big|_{(1,\alpha)}^{(1,1)}\right]. \quad (2.2.41)$$

Similarly, we have

$$E_{\alpha,\beta}(z) = {}_1\psi_1\left[z\Big|_{(\beta,\alpha)}^{(1,1)}\right]. \quad (2.2.42)$$

Next, if we calculate the residues at the poles of the gamma function $\Gamma(1-s)$ at the points $s = 1+n, n = 0,1,2,...$ it gives

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s}\mathrm{d}s = \sum_{n=0}^{\infty}\lim_{s\to 1+n}\frac{(s-1-n)\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(1-\alpha s)}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(1+n)(-z)^{-n-1}}{n!\Gamma(1-\alpha(1+n))}$$
$$= -\sum_{n=1}^{\infty}\frac{z^{-n}}{\Gamma(1-\alpha n)}, \quad (2.2.43)$$

for $\alpha \neq 1,2,\cdots$. Similarly for $\alpha \neq 1,2,\cdots, E_{\alpha,\beta}(z)$, gives

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\mathrm{d}s = -\sum_{n=1}^{\infty}\frac{z^{-n}}{\Gamma(\beta-\alpha n)}. \quad (2.2.44)$$

Exercises 2.2.

2.2.1. Let

$$U_1(t) = t^{\beta-1}E_{\frac{m}{n},\beta}(t^{\frac{m}{n}})$$
$$U_2(t) = t^{\beta-1}E_{m,\beta}(t^m)$$
$$U_3(t) = t^{(\beta-1)\frac{n}{m}}E_{\frac{m}{n},\beta}(t)$$
$$U_4(t) = t^{(\beta-1)n}E_{\frac{1}{n},\beta}(t).$$

Then show that these functions respectively satisfy the following differential equations of Mittag-Leffler functions when m,n are relatively prime.

(i) $\dfrac{\mathrm{d}^m}{\mathrm{d}t^m}U_1(t) - U_1(t) = t^{\beta-1}\sum_{k=1}^{n}\dfrac{t^{-\frac{m}{n}k}}{\Gamma(\beta-\frac{mk}{n})}$
$\Re(\beta) > m, (m,n = 1,2,3,...)$;

(ii) $\dfrac{\mathrm{d}^m}{\mathrm{d}t^m}U_2(t) - U_2(t) = \dfrac{t^{-m+\beta-1}}{\Gamma(\beta-m)}$, $\Re(\beta) > m, m = 1,2,\ldots$;

(iii) $\left(\dfrac{m}{n}t^{1-\frac{n}{m}}\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^m U_3(t) - U_3(t) = t^{(\beta-1)\frac{n}{m}}\sum_{k=1}^{n}\dfrac{t^{-k}}{\Gamma(\beta-\frac{mk}{n})}$
$m,n = 1,2,3,...,\Re(\beta) > m$;

(iv) $\dfrac{1}{n}\left[\dfrac{\mathrm{d}}{\mathrm{d}t}U_4(t)\right] - t^{n-1}U_4(t) = t^{n\beta-1}\sum_{k=1}^{n}\dfrac{t^{-k}}{\Gamma(\beta-\frac{k}{n})}$
$n = 1,2,3,...,\Re(\beta) > 1$.

2.2.2. Prove that

$$\frac{\lambda}{\Gamma(\alpha)}\int_0^x \frac{E_\alpha(\lambda t^\alpha)}{(x-t)^{1-\alpha}}\mathrm{d}t = E_\alpha(\lambda x^\alpha) - 1, \Re(\alpha) > 0.$$

2.2.3. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{\gamma-1}E_{\alpha,\beta}(ax^\alpha)] = x^{\gamma-2}E_{\alpha,\beta-1}(ax^\alpha) + (\gamma-\beta)x^{\gamma-2}E_{\alpha,\beta}(ax^\alpha), \beta \neq \gamma.$$

2.2.4. Prove that

$$\frac{1}{\Gamma(\nu)}\int_0^z t^{\beta-1}(z-t)^{\nu-1}E_{\alpha,\beta}(\lambda t^\alpha)\mathrm{d}t = z^{\beta+\nu-1}E_{\alpha,\beta+\nu}(\lambda z^\alpha),$$
$$\Re(\beta) > 0, \Re(\nu) > 0, \Re(\alpha) > 0.$$

2.2.5. Prove that

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\cosh(\sqrt{\lambda}t)\mathrm{d}t = z^\alpha E_{2,\alpha+1}(\lambda z^2), \Re(\alpha) > 0.$$

2.2.6. Prove that

$$\frac{1}{\Gamma(\alpha)}\int_0^z \mathrm{e}^{\lambda t}(z-t)^{\alpha-1}\mathrm{d}t = z^{\alpha}E_{1,\alpha+1}(\lambda z), \Re(\alpha) > 0.$$

2.2.7. Prove that

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\frac{\sinh(\sqrt{\lambda}t)}{\sqrt{\lambda}}\mathrm{d}t = z^{\alpha+1}E_{2,\alpha+2}(\lambda z^2), \Re(\alpha) > 0.$$

2.2.8. Prove that

$$\int_0^{\infty} \mathrm{e}^{-sx}x^{\beta-1}E_{\alpha,\beta}(x^{\alpha})\mathrm{d}x = \frac{s^{\alpha-\beta}}{s^{\alpha}-1}, \Re(s) > 1.$$

2.2.9. Prove that

$$\int_0^{\infty} \mathrm{e}^{-st}E_{\alpha}(t^{\alpha})\mathrm{d}t = \frac{1}{s-s^{1-\alpha}}, \Re(s) > 1.$$

2.2.10. Prove that

$$\int_0^x u^{\gamma-1}E_{\alpha,\gamma}(yu^{\alpha})(x-u)^{\beta-1}E_{\alpha,\beta}[z(x-u)^{\alpha}]\mathrm{d}u$$
$$= \frac{yE_{\alpha,\beta+\gamma}(yx^{\alpha}) - zE_{\alpha,\beta+\gamma}(zx^{\alpha})}{y-z}x^{\beta+\gamma-1},$$

where $y,z \in C;\ y \neq z, \gamma > 0, \beta > 0$.

2.3 Generalized Mittag-Leffler Function

Notation 2.3.1. $E_{\beta,\gamma}^{\delta}(z)$: **Generalized Mittag-Leffler function**

Definition 2.3.1.

$$E_{\beta,\gamma}^{\delta}(z) = \sum_{n=0}^{\infty}\frac{(\delta)_n z^n}{\Gamma(\beta n+\gamma)n!}, \tag{2.3.1}$$

where $\beta,\gamma,\delta \in C$ with $\Re(\beta) > 0$. For $\delta = 1$, it reduces to Mittag-Leffler function (2.1.2). This function was introduced by T.R. Prabhakar in 1971. It is an entire function of order $\rho = [\Re(\beta)]^{-1}$.

2.3.1 Special cases of $E^{\delta}_{\beta,\gamma}(z)$

$$\text{(i)}\ E_\beta(z) = E^1_{\beta,1}(z) \tag{2.3.2}$$

$$\text{(ii)}\ E_{\beta,\gamma}(z) = E^1_{\beta,\gamma}(z) \tag{2.3.3}$$

$$\text{(iii)}\ \phi(\gamma,\delta;z) = {}_1F_1(\gamma;\delta;z) = \Gamma(\delta)E^{\gamma}_{1,\delta}(z), \tag{2.3.4}$$

where $\phi(\gamma,\delta;z)$ is Kummer's confluent hypergeometric function.

2.3.2 Mellin-Barnes integral representation

Lemma 2.3.1: Let $\beta \in R_+ = (0,\infty); \gamma,\delta \in C, \gamma \neq 0, \Re(\delta) > 0$. Then $E^{\delta}_{\beta,\gamma}(z)$ is represented by the Mellin-Barnes integral

$$E^{\delta}_{\beta,\gamma}(z) = \frac{1}{\Gamma(\delta)}\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(\delta-s)}{\Gamma(\gamma-\beta s)}(-z)^{-s}\mathrm{d}s, \tag{2.3.5}$$

where $|\arg(z)| < \pi$; the contour of integration beginning at $c - i\infty$ and ending at $c + i\infty, 0 < c < \Re(\delta)$, separates all the poles at $s = -k, k = 0,1,...$ to the left and all the poles at $s = n+\delta, n = 0,1,...$ to the right.

Proof. We will evaluate the integral on the R.H.S. of (2.3.5) as the sum of the residues at the poles $s = 0,-1,-2,...$. We have

$$\begin{aligned}\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(\delta-s)}{\Gamma(\gamma-\beta s)}(-z)^{-s}\mathrm{d}s &= \sum_{k=0}^{\infty}\lim_{s\to -k}\left[\frac{(s+k)\Gamma(s)\Gamma(\delta-s)(-z)^{-s}}{\Gamma(\gamma-\beta s)}\right]\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(\delta+k)}{\Gamma(\gamma+\beta k)}(-z)^k\\ &= \Gamma(\delta)\sum_{k=0}^{\infty}\frac{(\delta)_k}{\Gamma(\beta k+\gamma)}\frac{z^k}{k!} = \Gamma(\delta)E^{\delta}_{\beta,\gamma}(z)\end{aligned}$$

which proves (2.3.5).

2.3.3 Relations with the H-function and Wright function

It follows from (2.3.5) that $E^{\delta}_{\beta,\gamma}(z)$ can be represented in the form

$$E^{\delta}_{\beta,\gamma}(z) = \frac{1}{\Gamma(\delta)}H^{1,1}_{1,2}\left[-z\Big|^{(1-\delta,1)}_{(0,1),(1-\gamma,\beta)}\right] \tag{2.3.6}$$

where $H^{1,1}_{1,2}(z)$ is the H-function, the theory of which can be found in Chapter 1. This function can also be represented by

$$E^{\delta}_{\beta,\gamma}(z) = \frac{1}{\Gamma(\delta)}\,{}_1\psi_1\left[z\Big|^{(\delta,1)}_{(\gamma,\beta)}\right] \tag{2.3.7}$$

where ${}_1\psi_1$ is the Wright hypergeometric function ${}_p\psi_q(z)$.

2.3.4 Cases of reducibility

In this subsection we present some interesting cases of reducibility of the function $E^{\delta}_{\beta,\gamma}(z)$. The results are given in the form of five theorems. The results are useful in the investigation of the solutions of certain fractional order differential and integral equations.The proofs of the following theorems can be developed on similar lines to that of equation (2.2.1).

Theorem 2.3.1. *If $\beta,\gamma,\delta \in C$ with $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(\gamma-\beta) > 0$, then there holds the relation*

$$zE^{\delta}_{\beta,\gamma}(z) = E^{\delta}_{\beta,\gamma-\beta}(z) - E^{\delta-1}_{\beta,\gamma-\beta}(z). \tag{2.3.8}$$

Corollary 2.3.1: *If $\beta,\gamma \in C, \Re(\gamma) > \Re(\beta) > 0$, then we have*

$$zE^{1}_{\beta,\gamma}(z) = E_{\beta,\gamma-\beta}(z) - \frac{1}{\Gamma(\gamma-\beta)}. \tag{2.3.9}$$

Theorem 2.3.2. *If $\beta,\gamma,\delta \in C, \Re(\beta) > 0, \Re(\gamma) > 1$, then there holds the formula*

$$\beta E^{2}_{\beta,\gamma}(z) = E_{\beta,\gamma-1}(z) + (1+\beta-\gamma)E_{\beta,\gamma}(z). \tag{2.3.10}$$

Theorem 2.3.3. *If $\Re(\beta) > 0, \Re(\gamma) > 2+\Re(\beta)$, then there holds the formula*

$$\begin{aligned} zE^{3}_{\beta,\gamma}(z) = \frac{1}{2\beta^2}[&E_{\beta,\gamma-\beta-2}(z) - (2\gamma-3\beta-3)E_{\beta,\gamma-\beta-1}(z) \\ &+ (2\beta^2+\gamma^2-3\beta\gamma+3\beta-2\gamma+1)E_{\beta,\gamma-\beta}(z)]. \end{aligned} \tag{2.3.11}$$

Theorem 2.3.4. *If $\Re(\beta) > 0, \Re(\gamma) > 2$, then there holds the formula*

$$\begin{aligned} E^{3}_{\beta,\gamma}(z) = \frac{1}{2\beta^2}[&E_{\beta,\gamma-2}(z) + (3+3\beta-2\gamma)E_{\beta,\gamma-1}(z) \\ &+ (2\beta^2+\gamma^2+3\beta-3\beta\gamma-2\gamma+1)E_{\beta,\gamma}(z)]. \end{aligned} \tag{2.3.12}$$

2.3.5 Differentiation of generalized Mittag-Leffler function

Theorem 2.3.5. *Let $\beta,\gamma,\delta,\rho,w \in C$. Then for any $n = 1,2,\ldots$ there holds the formula, for $\Re(\gamma) > n$,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n [z^{\gamma-1}E^{\delta}_{\beta,\gamma}(wz^{\beta})] = z^{\gamma-n-1}E^{\delta}_{\beta,\gamma-n}(wz^{\beta}). \tag{2.3.13}$$

In particular, for $\Re(\gamma) > n$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n [z^{\gamma-1}E_{\beta,\gamma}(wz^{\beta})] = z^{\gamma-n-1}E_{\beta,\gamma-n}(wz^{\beta}) \tag{2.3.14}$$

and for $\Re(\gamma) > n$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n [z^{\gamma-1}\phi(\delta;\gamma;wz)] = \frac{\Gamma(\gamma)}{\Gamma(\gamma-n)}z^{\gamma-n-1}\phi(\delta;\gamma-n;wz). \tag{2.3.15}$$

Proof. Using (2.3.1) and taking term by term differentiation under the summation sign, which is possible in accordance with uniform convergence of the series in (2.3.1) in any compact set of C, we obtain

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n [z^{\gamma-1}E^{\delta}_{\beta,\gamma}(wz^{\beta})] &= \sum_{k=0}^{\infty}\frac{(\delta)_k}{\Gamma(\beta k+\gamma)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left[\frac{w^k z^{\beta x+\gamma-1}}{k!}\right]\\ &= z^{\gamma-n-1}E^{\delta}_{\beta,\gamma-n}(wz^{\beta}),\ \Re(\gamma) > n,\end{aligned}$$

which establishes (2.3.13). Note that (2.3.14) follows from (2.3.13) when $\delta = 1$ due to (2.3.3), and (2.3.15) follows from (2.3.13) when $\beta = 1$ on account of (2.3.4).

2.3.6 Integral property of generalized Mittag-Leffler function

Corollary 2.3.2: *Let $\beta,\gamma,\delta,w \in C, \Re(\gamma) > 0, \Re(\beta) > 0, \Re(\delta) > 0$. Then*

$$\int_0^z t^{\gamma-1}E^{\delta}_{\beta,\gamma}(wt^{\beta})\mathrm{d}t = z^{\gamma}E^{\delta}_{\beta,\gamma+1}(wz^{\beta}) \tag{2.3.16}$$

and (2.3.16) follows from (2.3.13). In particular,

$$\int_0^z t^{\gamma-1}E_{\beta,\gamma}(wt^{\beta})\mathrm{d}t = z^{\gamma}E_{\beta,\gamma+1}(wz^{\beta}) \tag{2.3.17}$$

and

$$\int_0^z t^{\delta-1}\phi(\gamma,\delta;wt)\mathrm{d}t = \frac{1}{\delta}z^{\delta}\phi(\gamma,\delta+1;wz) \tag{2.3.18}$$

Remark 2.3.1: The relations (2.3.15) and (2.3.18) are well known.

2.3.7 Integral transform of $E^{\delta}_{\beta,\gamma}(z)$

By appealing to the Mellin inversion formula, (2.3.5) yields the Mellin transform of the generalized Mittag-Leffler function.

$$\int_0^{\infty} t^{s-1} E^{\delta}_{\beta,\gamma}(-wt)\mathrm{d}t = \frac{\Gamma(s)\Gamma(\delta-s)w^{-s}}{\Gamma(\delta)\Gamma(\gamma-s\beta)}. \tag{2.3.19}$$

If we make use of the integral

$$\int_0^{\infty} t^{\nu-1}\mathrm{e}^{-\frac{t}{2}} W_{\lambda,\mu}(t)\mathrm{d}t = \frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}-\mu+\nu\right)}{\Gamma(1-\lambda+\nu)} \tag{2.3.20}$$

where $\Re(\nu \pm \mu) > -\frac{1}{2}$, we obtain the Whittaker transform of the Mittag-Leffler function

$$\int_0^{\infty} t^{\rho-1}\mathrm{e}^{-\frac{1}{2}pt} W_{\lambda,\mu}(pt) E^{\delta}_{\beta,\gamma}(wt^{\alpha})\mathrm{d}t = \frac{p^{-\rho}}{\Gamma(\delta)}\,{}_3\psi_2\left[\frac{w}{p^{\alpha}}\Big|^{(\delta,1),(\frac{1}{2}\pm\mu+\rho,\alpha)}_{(\gamma,\beta),(1-\lambda+\rho,\alpha)}\right] \tag{2.3.21}$$

where ${}_3\psi_2$ is the generalized Wright function, and $\Re(\rho) > |\Re(\mu)| - \frac{1}{2}, \Re(p) > 0, |\frac{w}{p^{\alpha}}| < 1$. When $\lambda = 0$ and $\mu = \frac{1}{2}$, then by virtue of the identity

$$W_{\pm\frac{1}{2},0}(t) = \exp\left(-\frac{t}{2}\right), \tag{2.3.22}$$

the Laplace transform of the generalized Mittag-Leffler function is obtained.

$$\int_0^{\infty} t^{\rho-1}\mathrm{e}^{-pt} E^{\delta}_{\beta,\gamma}(wt^{\alpha})\mathrm{d}t = \frac{p^{-\rho}}{\Gamma(\delta)}\,{}_2\psi_1\left[\frac{w}{p^{\alpha}}\Big|^{(\delta,1),(\rho,\alpha)}_{(\gamma,\beta)}\right] \tag{2.3.23}$$

where $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(\rho) > 0, \Re(p) > 0, p > |w|^{\frac{1}{\Re(\alpha)}}$. In particular, for $\rho = \gamma$ and $\alpha = \beta$ we obtain a result given by Prabhakar (1971, Eq.2.5).

$$\int_0^{\infty} t^{\gamma-1}\mathrm{e}^{-pt} E^{\delta}_{\beta,\gamma}(wt^{\beta})\mathrm{d}t = p^{-\gamma}(1-wp^{-\beta})^{-\delta} \tag{2.3.24}$$

where $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(p) > 0$ and $p > |w|^{\frac{1}{\Re(\beta)}}$.

The Euler transform of the generalized Mittag-Leffler function follows from the beta function.

$$\int_0^{1} t^{a-1}(1-t)^{b-1} E^{\delta}_{\beta,\gamma}(xt^{\alpha})\mathrm{d}t = \frac{\Gamma(b)}{\Gamma(\delta)}\,{}_2\psi_2\left[x\Big|^{(\delta,1),(a,\alpha)}_{(\gamma,\beta),(a+b,\alpha)}\right], \tag{2.3.25}$$

where $\Re(a) > 0, \Re(b) > 0, \Re(\delta) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\alpha) > 0$.

Theorem 2.3.6. *We have*

$$\int_0^\infty \mathrm{e}^{-pt} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(at^\alpha)\mathrm{d}t = \frac{k!p^{\alpha-\beta}}{(p^\alpha - a)^{k+1}}, \tag{2.3.26}$$

where $\Re(p) > |a|^{\frac{1}{\Re(\alpha)}}, \Re(\alpha) > 0, \Re(\beta) > 0$, *and* $E_{\alpha,\beta}^{(k)}(y) = \frac{\mathrm{d}^k}{\mathrm{d}y^k} E_{\alpha,\beta}(y)$.

Proof: We will use the following result:

$$\int_0^\infty \mathrm{e}^{-t} t^{\beta-1} E_{\alpha,\beta}(zt^\alpha)\mathrm{d}t = \frac{1}{1-z}, |z| < 1. \tag{2.3.27}$$

The given integral

$$\begin{aligned} &= \frac{\mathrm{d}^k}{\mathrm{d}a^k} \int_0^\infty \mathrm{e}^{-pt} t^{\beta-1} E_{\alpha,\beta}(\pm at^\alpha)\mathrm{d}t \\ &= \frac{\mathrm{d}^k}{\mathrm{d}a^k} \frac{p^{\alpha-\beta}}{(p^\alpha - a)} = \frac{k!p^{\alpha-\beta}}{(p^\alpha - a)^{k+1}}, \Re(\beta) > 0. \end{aligned}$$

Corollary 2.3.3:

$$\int_0^\infty \mathrm{e}^{-pt} t^{\frac{k-1}{2}} E_{\frac{1}{2},\frac{1}{2}}^{(k)}(a\sqrt{t})\mathrm{d}t = \frac{k!}{(\sqrt{p} - a)^{k+1}} \tag{2.3.28}$$

where $\Re(p) > a^2$.

Exercises 2.3.

2.3.1. Prove that

$$\frac{1}{\Gamma(\alpha)} \int_0^1 u^{\gamma-1}(1-u)^{\alpha-1} E_{\beta,\gamma}^\delta(zu^\beta)\mathrm{d}u = E_{\beta,\gamma+\alpha}^\delta(z), \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0.$$

2.3.2. Prove that

$$\frac{1}{\Gamma(\alpha)} \int_t^x (x-u)^{\alpha-1}(u-t)^{\gamma-1} E_{\beta,\gamma}^\delta[\lambda(u-t)^\beta]\mathrm{d}u = (x-t)^{\gamma+\alpha-1} E_{\beta,\gamma+\alpha}^\delta[\lambda(x-t)^\beta]$$

where $\Re(\alpha) > 0, \Re(\gamma) > 0, \Re(\beta) > 0$.

2.3.3. Prove that for $n = 1, 2, \ldots$

$$E_{n,\gamma}^\delta(z) = \frac{1}{\Gamma(\gamma)} {}_1F_n(\delta; \Delta(n;\gamma); n^{-n}z),$$

where $\Delta(n;\gamma)$ represents the sequence of n parameters $\frac{\gamma}{n}, \frac{\gamma+1}{n}, \ldots, \frac{\gamma+n-1}{n}$.

2.3.4. Show that for $\Re(\beta) > 0, \Re(\gamma) > 0$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m E^{\delta}_{\beta,\gamma}(z) = (\delta)_m E^{\delta+m}_{\beta,\gamma+m\beta}(z).$$

2.3.5. Prove that for $\Re(\beta) > 0, \Re(\gamma) > 0$,

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z} + \delta\right) E^{\delta}_{\beta,\gamma}(z) = \delta E^{\delta+1}_{\beta,\gamma}(z).$$

2.3.6. Prove that for $\Re(\gamma) > 1$,

$$(\gamma - \beta\delta - 1)E^{\delta}_{\beta,\gamma}(z) = E^{\delta}_{\beta,\gamma-1}(z) - \beta\delta E^{\delta+1}_{\beta,\gamma}(z).$$

2.3.7. Prove that

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1} E^{\gamma}_{\rho,\mu}[w(x-t)^{\rho}] E^{\sigma}_{\rho,\nu}(wt^{\rho})\mathrm{d}t = x^{\mu+\nu-1} E^{\gamma+\sigma}_{\rho,\mu+\nu}(wx^{\rho}),$$

where $\rho,\mu,\gamma,\nu,\sigma,w \in C; \Re(\rho),\Re(\mu),\Re(\nu) > 0$.

2.3.8. Find

$$L^{-1}\left[s^{-\lambda}\left(1-\frac{z}{s^{\rho}}\right)^{-\alpha}\right]$$

and give the conditions of validity.

2.3.9. Prove that

$$L^{-1}\left[s^{-\lambda}\left(1-\frac{z_1}{s}\right)^{-\alpha_1}\left(1-\frac{z_2}{s}\right)^{-\alpha_2}\right] = \frac{t^{\lambda-1}}{\Gamma(\lambda)}\Phi_2(\alpha_1,\alpha_2;\lambda;z_1 t, z_2 t),$$

where $\Re(\lambda) > 0, \Re(s) > \max[0,\Re(z_1),\Re(z_2)]$ and Φ_2 is the confluent hypergeometric function of two variables defined by

$$\Phi_2(b,b';c;u,z) = \sum_{k,j=0}^{\infty} \frac{(b)_k (b')_j u^k z^j}{(c)_{k+j} k! j!}. \tag{2.3.29}$$

2.3.10. From the above result deduce the formula

$$L^{-1}\left[s^{-\lambda}(1-\frac{z}{s})^{-\alpha}\right] = \frac{t^{\lambda-1}}{\Gamma(\lambda)}\phi(\alpha,\lambda;zt), \tag{2.3.30}$$

where $\Re(\lambda) > 0, \Re(s) > \max[0,|z|]$.

2.4 Fractional Integrals

This section deals with the definition and properties of various operators of fractional integration and fractional differentiation of arbitrary order. Among the various operators studied are the Riemann-Liouville fractional integral operators, Riemann-Liouville fractional differential operators, Weyl operators, Kober operators etc. Besides the basic properties of these operators, their behaviors under Laplace, Fourier and Mellin transforms are also presented. Application of Riemann-Liouville operators in the solution of fractional order differential and fractional order integral equations is demonstrated.

2.4.1 Riemann-Liouville fractional integrals of arbitrary order

Notation 2.4.1. ${}_aI_x^n, {}_aD_x^{-n}, n \in \mathbb{N} \cup 0$: **Fractional integral of integer order n**

Definition 2.4.1.

$$ {}_aI_x^n f(x) = {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) \mathrm{d}t \tag{2.4.1} $$

where $n \in \mathbb{N} \cup 0$.

We begin our study of fractional calculus by introducing a fractional integral of integer order n in the form of Cauchy formula.

$$ {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) \mathrm{d}t. \tag{2.4.2} $$

It will be shown that the above integral can be expressed in terms of n-fold integral, that is,

$$ {}_aD_x^{-n} f(x) = \int_0^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} \mathrm{d}x_3 ... \int_a^{x_{n-1}} f(t) \mathrm{d}t. \tag{2.4.3} $$

Proof. When $n = 2$, by using the well-known Dirichlet formula, namely

$$ \int_a^b \mathrm{d}x \int_a^x f(x,y) \mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(x,y) \mathrm{d}x \tag{2.4.4} $$

(2.4.3) becomes

$$ \begin{aligned} \int_a^x \mathrm{d}x_1 \int_a^{x_1} f(t) \mathrm{d}t &= \int_a^x \mathrm{d}t f(t) \int_t^x \mathrm{d}x_1 \\ &= \int_a^x (x-t) f(t) \mathrm{d}t. \end{aligned} \tag{2.4.5} $$

This shows that the two-fold integral can be reduced to a single integral with the help of Dirichlet formula. For $n = 3$, the integral in (2.4.3) gives

$$ \begin{aligned} {}_aD_x^{-3} f(x) &= \int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} f(t) \mathrm{d}t \\ &= \int_a^x \mathrm{d}x_1 \left[\int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} f(t) \mathrm{d}t \right]. \end{aligned} \tag{2.4.6} $$

By using the result in (2.4.5) the integrals within big brackets simplify to yield

$$ {}_aD_x^{-3} f(x) = \int_a^x \mathrm{d}x_1 \left[\int_a^{x_1} (x_1 - t) f(t) \mathrm{d}t \right]. \tag{2.4.7} $$

If we use (2.4.4), then the above expression reduces to

$$_aD_x^{-3}f(x) = \int_a^x \mathrm{d}t f(t) \int_x^t (x_1 - t)\mathrm{d}x_1 = \int_a^x \frac{(x-t)^2}{2!} f(t)\mathrm{d}t. \tag{2.4.8}$$

Continuing this process, we finally obtain

$$_aD_x^{-n}f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\mathrm{d}t. \tag{2.4.9}$$

It is evident that the integral in (2.4.9) is meaningful for any number n provided its real part is greater than zero.

2.4.2 Riemann-Liouville fractional integrals of order α

Notation 2.4.2. $_xI_b^\alpha, {}_xD_b^{-\alpha}, I_{b-}^\alpha$: **Riemann-Liouville right-sided fractional integral of order α.**

Definition 2.4.2. Let $f(x) \in L(a,b), \alpha \in C, \Re(\alpha) > 0$, then

$$_aI_x^\alpha f(x) = {}_aD_x^{-\alpha} f(x) = I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\mathrm{d}t, x > a \tag{2.4.10}$$

is called Riemann-Liouville left-sided fractional integral of order α.

Definition 2.4.3. Let $f(x) \in L(a,b), \alpha \in C, \Re(\alpha) > 0$, then

$$_xI_b^\alpha f(x) = {}_xD_b^{-\alpha} f(x) = I_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\mathrm{d}t, x < b \tag{2.4.11}$$

is called Riemann-Liouville right-sided fractional integral of order α.

Example 2.4.1. If $f(x) = (x-a)^{\beta-1}$, then find the value of $_aI_x^\alpha f(x)$.

Solution 2.4.1: We have

$$_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\mathrm{d}t.$$

If we substitute $t = a + y(x-a)$ in the above integral, it reduces to

$$\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}$$

where $\Re(\beta) > 0$. Thus

$$_aI_x^\alpha f(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}. \tag{2.4.12}$$

Example 2.4.2. It can be similarly shown that

$$ {}_xI_b^\alpha g(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1}, x<b \tag{2.4.13}$$

where $\Re(\beta)>0$ and $g(x)=(b-x)^{\beta-1}$.

Note 2.4.1: It may be noted that (2.4.12) and (2.4.13) give the Riemann-Liouville integrals of the power functions $f(x)=(x-a)^{\beta-1}$ and $g(x)=(b-x)^{\beta-1}, \Re(\beta)>0$.

2.4.3 Basic properties of fractional integrals

Property: Fractional integrals obey the following properties:

$$\begin{aligned} {}_aI_x^\alpha\ {}_aI_x^\beta\phi &= {}_aI_x^{\alpha+\beta}\phi = {}_aI_x^\beta\ {}_aI_x^\alpha\phi, \\ {}_xI_b^\alpha\ {}_xI_b^\beta\phi &= {}_xI_b^{\alpha+\beta}\phi = {}_xI_b^\beta\ {}_xI_b^\alpha\phi. \end{aligned} \tag{2.4.14}$$

Proof: By virtue of the definition(2.4.10), it follows that

$$\begin{aligned} {}_aI_x^\alpha\ {}_aI_x^\beta\phi &= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}}\frac{1}{\Gamma(\beta)}\int_a^t \frac{\phi(u)\mathrm{d}u}{(t-u)^{1-\beta}} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \mathrm{d}u\phi(u)\int_u^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}(t-u)^{1-\beta}}. \end{aligned} \tag{2.4.15}$$

If we use the substitution $y=\frac{t-u}{x-u}$, the value of the second integral is

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)(x-u)^{1-\alpha-\beta}}\int_0^1 y^{\beta-1}(1-y)^{\alpha-1}\mathrm{d}y = \frac{(x-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},$$

which, when substituted in (2.4.15) yields the first part of (2.4.14). The second part can be similarly established. In particular,

$$ {}_aI_x^{n+\alpha} f = {}_aI_x^n\ {}_aI_x^\alpha f, n\in\mathbb{N}, \Re(\alpha)>0 \tag{2.4.16}$$

which shows that the n-fold differentiation

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,{}_aI_x^{n+\alpha} f(x) = {}_aI_x^\alpha f(x), n\in\mathbb{N}, \Re(\alpha)>0 \tag{2.4.17}$$

for all x. When $\alpha=0$, we obtain

$$ {}_aI_x^0 f(x) = f(x);\ {}_aI_x^{-n} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = f^{(n)}(x). \tag{2.4.18}$$

Note 2.4.2: The property given in (2.4.14) is called semigroup property of fractional integration.

Notation 2.4.3. $L(a,b)$: **space of Lebesgue measurable real or complex valued functions.**

Definition 2.4.4. $L(a,b)$, **consists of Lebesgue measurable real or complex valued functions** $f(x)$ **on** $[a,b]$**:**

$$L(a,b) = \{f : ||f||_1 \sim \int_a^b |f(t)|\mathrm{d}t < \infty\}. \tag{2.4.19}$$

Note 2.4.3: The operators ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ are defined on the space $L(a,b)$.

Property: The following results hold:

$$\int_a^b f(x)({}_aI_x^\alpha g)\mathrm{d}x = \int_a^b g(x)({}_xI_b^\alpha f)\mathrm{d}x. \tag{2.4.20}$$

(2.4.20) can be established by interchanging the order of integration in the integral on the left-hand side of (2.4.20) and then using the Dirichlet formula (2.4.4).

The above property is called the property of "integration by parts" for fractional integrals.

2.4.4 A useful integral

We now evaluate the following integral given by Saxena and Nishimoto [Journal of Fractional Calculus, Vol. 6, 1994, 65-75].

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^\gamma \mathrm{d}t = (ac+d)^\gamma (b-a)^{\alpha+\beta-1} \times B(\alpha,\beta)\,{}_2F_1\left[\alpha,-\gamma;\alpha+\beta;\frac{(a-b)c}{(ac+d)}\right], \tag{2.4.21}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, |\arg\frac{(d+bc)}{(d+ac)}| < \pi, a, c$ and d are constants.

Solution Let

$$\begin{aligned} I &= \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^\gamma \mathrm{d}t \\ &= (ac+d)^\gamma \sum_{k=0}^\infty \frac{(-1)^k(-\gamma)_k c^k}{(ac+d)^k} \int_a^b (t-a)^{\alpha+k-1}(b-t)^{\beta-1}\mathrm{d}t \\ &= (ac+d)^\gamma (b-a)^{\alpha+\beta-1} B(\alpha,\beta)\,{}_2F_1\left(-\gamma,\alpha;\alpha+\beta;\frac{(a-b)c}{(ac+d)}\right). \end{aligned}$$

In evaluating the inner integral the modified form of the beta function, namely

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta), \tag{2.4.22}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$, is used.

Example 2.4.3. As a consequence of (2.4.21) it follows that

$$\begin{aligned} {}_aI_x^{\alpha}[(x-a)^{\beta-1}(cx+d)^{\gamma}] &= (ac+d)^{\gamma}(x-a)^{\alpha+\beta-1}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ &\quad \times {}_2F_1\left(\beta,-\gamma;\alpha+\beta;\frac{(a-x)c}{(ac+d)}\right), \end{aligned} \tag{2.4.23}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, |\arg\frac{(a-x)c}{(ac+d)}| < \pi, a, c$ and d being constants. In a similar manner we obtain the following result:

Example 2.4.4. We also have

$$\begin{aligned} {}_xI_b^{\alpha}[(b-x)^{\beta-1}(cx+d)^{\gamma} &= (cx+d)^{\gamma}](b-x)^{\alpha+\beta-1}\frac{\gamma(\beta)}{\Gamma(\alpha+\beta)} \\ &\quad \times {}_2F_1\left(\alpha,-\gamma;\alpha+\beta;\frac{(x-b)c}{(cx+d)}\right), \end{aligned} \tag{2.4.24}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, |\arg\frac{(x-b)c}{(cx+d)}| < \pi$.

Example 2.4.5. On the other hand if we set $\gamma = -\alpha-\beta$ in (2.4.21) it is found that

$${}_aD_x^{-\alpha}[(x-a)^{\beta-1}(cx+d)^{-\alpha-\beta}] = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(ac+d)^{-\alpha}(x-a)^{\alpha+\beta-1}(d+cx)^{-\beta}, \tag{2.4.25}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$.

Example 2.4.6. Similarly, we have

$${}_xI_b^{\alpha}[(b-x)^{\beta-1}(cx+d)^{-\alpha-\beta}] = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(cx+d)^{-\beta}(bc+d)^{-\alpha}(b-x)^{\alpha+\beta-1} \tag{2.4.26}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$.

2.4.5 The Weyl integral

Notation 2.4.4. ${}_xW_\infty^\alpha,\ {}_xI_\infty^\alpha$: **Weyl integral of order** α.

Definition 2.4.5. The Weyl fractional integral of $f(x)$ of order α, denoted by ${}_xW_\infty^\alpha$, is defined by

$$ {}_xW_\infty^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\mathrm{d}t,\ -\infty < x < \infty \tag{2.4.27}$$

where $\alpha \in C, \Re(\alpha) > 0$. (2.4.27) is also denoted by ${}_xI_\infty^\alpha f(x)$.

Example 2.4.7. Prove that

$$ {}_xW_\infty^\alpha \mathrm{e}^{-\lambda x} = \frac{\mathrm{e}^{-\lambda x}}{\lambda^\alpha} \text{ where } \Re(\alpha) > 0. \tag{2.4.28}$$

Solution: We have

$$\begin{aligned} {}_xW_\infty^\alpha \mathrm{e}^{-\lambda x} &= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} \mathrm{e}^{-\lambda t}\mathrm{d}t,\ \lambda > 0 \\ &= \frac{\mathrm{e}^{-\lambda x}}{\Gamma(\alpha)\lambda^\alpha} \int_0^\infty u^{\alpha-1}\mathrm{e}^{-u}\mathrm{d}u \\ &= \frac{\mathrm{e}^{-\lambda x}}{\lambda^\alpha},\ \Re(\alpha) > 0. \end{aligned}$$

Notation 2.4.5. ${}_xD_\infty^\alpha,\ D_-^\alpha$ **Weyl fractional derivative.**

Definition 2.4.6. The Weyl fractional derivative of order α, denoted by ${}_xD_\infty^\alpha$, is defined by

$$\begin{aligned} {}_xD_\infty^\alpha f(x) = D_-^\infty f(x) &= (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left({}_xW_\infty^{m-\alpha} f(x)\right) \\ &= (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \frac{1}{\Gamma(m-\alpha)} \int_x^\infty \frac{f(t)}{(t-x)^{1+\alpha-m}}\mathrm{d}t,\ -\infty < x < \infty \end{aligned} \tag{2.4.29}$$

where $m-1 \le \alpha < m,\ \alpha \in C, m = 0,1,2,\dots$.

Example 2.4.8. Find ${}_xD_\infty^\alpha \mathrm{e}^{-\lambda x},\ \lambda > 0$.

Solution: We have

$$\begin{aligned} {}_xD_\infty^\alpha \mathrm{e}^{-\lambda x} &= (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m {}_xW_\infty^{m-\alpha} \mathrm{e}^{-\lambda x} \\ &= (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \lambda^{-(m-\alpha)} \mathrm{e}^{-\lambda x} \\ &= \lambda^\alpha \mathrm{e}^{-\lambda x}. \end{aligned} \tag{2.4.30}$$

2.4.6 Basic properties of Weyl integral

Property: The following relation holds:

$$\int_0^\infty \phi(x)\,({}_0I_x^\alpha \psi(x))\,\mathrm{d}x = \int_0^\infty ({}_xW_\infty^\alpha \phi(x))\,\psi(x)\mathrm{d}x. \tag{2.4.31}$$

(2.4.31) is called the formula for fractional integration by parts. It is also called Parseval equality. (2.4.31) can be established by interchanging the order of integration.

Property: Weyl fractional integral obeys the semigroup property. That is,

$$\left({}_xW_\infty^\alpha\, {}_xW_\infty^\beta f\right) = \left({}_xW_\infty^{\alpha+\beta} f\right) = \left({}_xW_\infty^\beta\, {}_xW_\infty^\alpha f\right). \tag{2.4.32}$$

Proof: We have

$$\begin{aligned} {}_xW_\infty^\alpha\, {}_xW_\infty^\beta f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^\infty \mathrm{d}t(t-x)^{\alpha-1} \\ &\times \frac{1}{\Gamma(\beta)} \int_x^\infty (u-t)^{\beta-1} f(u)\mathrm{d}u. \end{aligned}$$

By using the modified form of the Dirichlet formula (2.4.4), namely

$$\int_x^a \mathrm{d}t(t-x)^{\alpha-1} \int_t^a (u-t)^{\beta-1} f(u)\mathrm{d}u = B(\alpha,\beta) \int_t^a (u-t)^{\alpha+\beta-1} f(u)\mathrm{d}u, \tag{2.4.33}$$

and letting $a \to \infty$, (2.4.33) yields the desired result:

$$\left({}_xW_\infty^\alpha\, {}_xW_\infty^\beta f\right) = \left({}_xW_\infty^{\alpha+\beta} f\right). \tag{2.4.34}$$

Notation 2.4.6. ${}_{-\infty}W_x^\alpha$, I_+^α : **Weyl integral with lower limit $-\infty$.**

Definition 2.4.7. Another companion to the operator (2.4.27) is the following:

$${}_{-\infty}W_x^\alpha f(x) = I_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} f(t)\mathrm{d}t,\ -\infty < x < \infty \tag{2.4.35}$$

where $\Re(\alpha) > 0$.

Note 2.4.4: The operator defined by (2.4.35) is useful in fractional diffusion problems in astrophysics and related areas.

Example 2.4.9. Prove that

$${}_{-\infty}W_x^\alpha \mathrm{e}^{ax} = \frac{\mathrm{e}^{ax}}{a^\alpha}. \tag{2.4.36}$$

Solution: We have the result by setting $x - t = u$.

Note 2.4.5: An alternative form of (2.4.35) in terms of convolution is given by

$$ {}_{-\infty}W_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} t_+^{\alpha-1} f(x-t)\mathrm{d}t \tag{2.4.37}$$

where

$$t_+^{\alpha-1} = \begin{cases} t^{\alpha-1}, \ t > 0 \\ 0, \ t < 0 \end{cases}$$

Example 2.4.10. Prove that

$$ {}_xW_{\infty}^{\nu}(\cos ax) = a^{-\nu} \cos\left(ax + \frac{1}{2}\pi\nu\right) \tag{2.4.38}$$

where $a > 0, 0 < \Re(\nu) < 1$.

Solution: The result follows from the known integral

$$\int_u^{\infty} (x-u)^{\nu-1} \cos ax \, \mathrm{d}x = \frac{\Gamma(\nu)}{a^{\nu}} \cos\left(au + \frac{\nu\pi}{2}\right). \tag{2.4.39}$$

Example 2.4.11. Prove that

$$ {}_xW_{\infty}^{\nu}(\sin ax) = a^{-\nu} \sin\left(ax + \frac{1}{2}\pi\nu\right). \tag{2.4.40}$$

Hint: Use the integral

$$\int_u^{\infty} (x-u)^{\nu-1} \sin ax \, \mathrm{d}x = \frac{\Gamma(\nu)}{a^{\nu}} \sin\left(au + \frac{1}{2}\pi\nu\right) \tag{2.4.41}$$

where $a > 0, 0 < \Re(\nu) < 1$.

Exercises 2.4.

2.4.1. Prove that

$$\left({}_aI_x^{\alpha}(x-a)^{\beta-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}, \ \Re(\beta) > 0.$$

2.4.2. Prove that

$$\left({}_aI_x^{\alpha}(x \pm c)^{\gamma-1}\right) = \frac{(a \pm c)^{\gamma-1}}{\Gamma(\alpha+1)}(x-a)^{\alpha} {}_2F_1\left(1, 1-\gamma; \alpha+1; \frac{a-x}{a \pm c}\right)$$

where $\Re(\beta) > 0, \gamma \in C, a \neq c, \ \left|\frac{a-x}{a \pm c}\right| < 1$.

2.4.3. Prove that

$$\left({}_aI_x^{\alpha}[(x-a)^{\beta-1}(b-x)^{\gamma-1}]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(b-a)^{1-\gamma}} \times {}_2F_1\left(\beta, 1-\gamma; \alpha+\beta; \frac{x-a}{b-a}\right)$$

where $\Re(\beta) > 0, \gamma \in C,\ a < x < b$.

2.4.4. Prove that

$$\left({}_aI_x^{\alpha}\left[\frac{(x-a)^{\beta-1}}{(b-x)^{\alpha+\beta}}\right]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(b-a)^{\alpha}(b-x)^{\beta}}$$

where $\Re(\beta) > 0,\ a < x < b$.

2.4.5. Prove that

$$\left({}_aI_x^{\alpha}\left[(x-a)^{\beta-1}(x\pm c)^{\gamma-1}\right]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(a\pm c)^{1-\gamma}} \times {}_2F_1\left(\beta, 1-\gamma; \alpha+\beta; \frac{(a-x)}{a\pm c}\right),$$

where $\Re(\beta) > 0,\ \gamma \in C, a \neq c, |\frac{a-x}{a\pm c}| < 1$.

2.4.6. Prove that for $\Re(\beta) > 0$,

$$\left({}_aI_x^{\alpha}\left[\frac{(x-a)^{\beta-1}}{(x\pm c)^{\alpha+\beta}}\right]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(a\pm c)^{\alpha}(x\pm c)^{\beta}}, \left|\frac{a-x}{a\pm c}\right| < 1.$$

2.4.7. Prove that

$$\left({}_aI_x^{\alpha}[\mathrm{e}^{\lambda x}]\right) = \mathrm{e}^{\lambda a}(x-a)^{\alpha}E_{1,\alpha+1}(\lambda x - \lambda a).$$

2.4.8. Prove that

$$\left({}_aI_x^{\alpha}[\mathrm{e}^{\lambda x}(x-a)^{\beta-1}]\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\mathrm{e}^{\lambda a}(x-a)^{\alpha+\beta-1}{}_1F_1(\beta;\alpha+\beta;\lambda x-\lambda a),$$

where $\Re(\beta) > 0, \Re(\alpha) > 0$.

2.4.9. Prove that

$$\left({}_aI_x^{\alpha}\left[(x-a)^{\frac{\nu}{2}}J_{\nu}(\lambda\sqrt{x-a})\right]\right) = \left(\frac{2}{\lambda}\right)^{\alpha}(x-a)^{\frac{\alpha+\nu}{2}}J_{\alpha+\nu}(\lambda\sqrt{x-a}),$$

where $\Re(\nu) > -1$.

2.4.10. Prove that

$$\left({}_aI_x^{\nu}\left[(x-a)^{\beta-1}{}_2F_1(\mu,\nu;\beta;\lambda(x-a))\right]\right)$$
$$=\frac{\Gamma(\beta)}{\Gamma(\nu+\beta)}(x-a)^{\nu+\beta-1}{}_2F_1(\mu,\nu;\nu+\beta;\lambda x-\lambda a),$$

where $\Re(\beta)>0$.

2.4.7 Laplace transform of the fractional integral

We have

$${}_0I_x^{\nu}f(x)=\frac{1}{\Gamma(\nu)}\int_0^x(x-t)^{\nu-1}f(t)dt, \tag{2.4.42}$$

where $\Re(\nu)>0$. Application of convolution theorem of the Laplace transform gives

$$\begin{aligned}L\{{}_0I_x^{\nu}f(x)\};s &= L\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}\right\}L\{f(t);s\}\\ &= s^{-\nu}F(s),\end{aligned} \tag{2.4.43}$$

where $\Re(s)>0$, $\Re(\nu)>0$.

2.4.8 Laplace transform of the fractional derivative

If $n\in\mathbb{N}$, then by the theory of the Laplace transform, we know that

$$L\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}f;s\right\}=s^nF(s)-\sum_{k=0}^{n-1}s^{n-k-1}f^{(k)}(0+) \tag{2.4.44}$$

$$=s^nF(s)-\sum_{k=0}^{n-1}s^kf^{(n-k-1)}(0+),\ (n-1\le\alpha<n) \tag{2.4.45}$$

where $\Re(s)>0$ and $F(s)$ is the Laplace transform of $f(t)$. By virtue of the definition of the derivative, we find that

$$\begin{aligned}L\{{}_0D_x^{\alpha}f;s\} &= L\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}{}_0I_x^{n-\alpha}f;s\right\}\\ &= s^nL\left\{{}_0I_x^{n-\alpha}f;s\right\}-\sum_{k=0}^{n-1}s^k\frac{\mathrm{d}^{n-k-1}}{\mathrm{d}x^{n-k-1}}\,{}_0I_x^{n-\alpha}f(0+)\end{aligned}$$

$$= s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} f(0+), \left(D = \frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{2.4.46}$$

$$= s^{\alpha}F(s) - \sum_{k=1}^{n} s^{k-1} D^{\alpha-k} f(0+) \tag{2.4.47}$$

where $\Re(s) > 0$.

2.4.9 Laplace transform of Caputo derivative

Notation 2.4.7. $\underset{0}{C}\,{}_aD_x^{\alpha}$

Definition 2.4.8. The Caputo derivative of a casual function $f(t)$ (that is $f(t) = 0$ for $t < 0$) with $\alpha > 0$ was defined by Caputo (1969) in the form

$$\underset{0}{C}\,{}_aD_x^{\alpha} f(x) = {}_aI_x^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = {}_aD_t^{-(n-\alpha)} f^{(n)}(t) \tag{2.4.48}$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t, (n-1 < \alpha < n) \tag{2.4.49}$$

where $n \in \mathbb{N}$.

From (2.4.43) and (2.4.49), it follows that

$$L\{\underset{0}{C}\,{}_0D_t^{\alpha} f(t); s\} = s^{-(n-\alpha)} L\{f^{(n)}(t)\}. \tag{2.4.50}$$

On using (2.4.44), we see that

$$L\{\underset{0}{C}\,{}_0D_t^{\alpha} f(t); s\} = s^{-(n-\alpha)} \left[s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0+) \right]$$

$$= s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0+), \;\; (n-1 < \alpha \leq n), \tag{2.4.51}$$

where $\Re(s) > 0$ and $\Re(\alpha) > 0$.

Note 2.4.6: From (2.4.48), it can be seen that

$$\underset{0}{C}\,{}_0D_t^{\alpha} A = 0, \text{ where } A \text{ is a constant,}$$

whereas the Riemann-Liouville derivative

$${}_0D_t^{\alpha} A = \frac{At^{-\alpha}}{\Gamma(1-\alpha)}, \;\; (\alpha \neq 1, 2, \cdots), \tag{2.4.52}$$

which is a surprising result.

Exercises 2.4.

2.4.11. Prove that

$$({}_0I_x^{\nu} f(x)) = L^{-1} s^{-\nu} L\{f(x); s\}, \tag{2.4.53}$$

where $\Re(\nu) > 0$.

2.4.12. Prove that the solution of Abel integral equation of the second kind

$$\phi(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)\mathrm{d}t}{(x-t)^{1-\alpha}} = f(x),\ 0 < x < 1$$

$\alpha > 0$, is given by

$$\phi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x E_\alpha[\lambda(x-t)^\alpha]\, f(t)\, \mathrm{d}t, \tag{2.4.54}$$

where $E_\alpha(x)$ is the Mittag-Leffler function defined by equation (2.1.1).

2.4.13. Show that

$$\frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{E_\alpha(\lambda t^\alpha)}{(x-t)^{1-\alpha}} \mathrm{d}t = E_\alpha(\lambda x^\alpha) - 1,\ \alpha > 0. \tag{2.4.55}$$

2.5 Mellin Transform of the Fractional Integrals and the Fractional Derivatives

2.5.1 Mellin transform

Notation 2.5.1. $m\{f(x);\ s\}, f^*(s)$: The Mellin transform

Notation 2.5.2. $m^{-1}\{f^*(s);\ x\}$: Inverse Mellin transform

Definition 2.5.1. The Mellin transform of a function $f(x)$, denoted by $f^*(s)$, is defined by

$$f^*(s) = m\{f(x);\ s\} = \int_0^\infty x^{s-1}\, f(x)\mathrm{d}x,\ x > 0. \tag{2.5.1}$$

The inverse Mellin transform is given by the contour integral

$$f(x) = m^{-1}\{f^*(s);\ x\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) x^{-s} \mathrm{d}s,\ i = \sqrt{-1} \tag{2.5.2}$$

where γ is real.

2.5.2 Mellin transform of the fractional integral

Theorem 2.5.1. *The following result holds true.*

$$m({}_0I_x^\alpha\, f)(s) = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-\alpha)} f^*(s+\alpha), \tag{2.5.3}$$

where $\Re(\alpha) > 0$ *and* $\Re(\alpha+s) < 1$.

Proof 2.5.1: We have

$$\begin{aligned} m({}_0I_x^\alpha\, f)(s) &= \int_0^\infty z^{s-1} \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} f(t)\, \mathrm{d}t\mathrm{d}z \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty f(t)\, \mathrm{d}t \int_t^\infty z^{s-1}(z-t)^{\alpha-1}\, \mathrm{d}z. \end{aligned} \tag{2.5.4}$$

On setting $z = \frac{t}{u}$, the z-integral becomes

$$t^{\alpha+s-1} \int_0^1 u^{-\alpha-s}(1-u)^{\alpha-1} \mathrm{d}u = t^{\alpha+s-1} B(\alpha, 1-\alpha-s), \tag{2.5.5}$$

where $\Re(\alpha) > 0$, $\Re(\alpha+s) < 1$. Putting the above value of z-integral, the result follows.

Similarly we can establish

Theorem 2.5.2. *The following result holds true.*

$$\begin{aligned} m({}_xI_\infty^\alpha\, f)(s) &= \frac{\Gamma(s)}{\Gamma(s+\alpha)}\ m\,\{t^\alpha f(t);\, s\} \\ &= \frac{\Gamma(s)}{\Gamma(s+\alpha)} f^*(s+\alpha), \end{aligned} \tag{2.5.6}$$

where $\Re(\alpha) > 0$, $\Re(s) > 0$.

Note 2.5.1: If we set $f(x) = x^{-\alpha}\phi(x)$, then using the property of the Mellin transform

$$x^\alpha \phi(x) \leftrightarrow \phi^*(s+\alpha), \tag{2.5.7}$$

the results (2.5.3) and (2.5.6) become

$$({}_0I_x^\alpha\, x^{-\alpha} f(x))(s) = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)}\ f^*(s), \tag{2.5.8}$$

where $\Re(\alpha) > 0$, $\Re(\alpha+s) < 1$ and

$$({}_xI_\infty^\alpha\, x^{-\alpha} f(x))(s) = \frac{\Gamma(s)}{\Gamma(s+\alpha)}\ f^*(s), \tag{2.5.9}$$

where $\Re(\alpha) > 0$, and $\Re(s) > 0$, respectively.

2.5.3 Mellin transform of the fractional derivative

Theorem 2.5.3. *If $n \in \mathbb{N}$ and $\lim_{t\to\infty} t^{s-1} f^{(\nu)}(t) = 0,\ \nu = 0, 1, \cdots, n$, then*

$$m\{f^{(n)}(t);\ (s)\} = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} m\{f(t);\ s-n\}, \tag{2.5.10}$$

where $\Re(s) > 0, \Re(s-n) > 0$.

Proof 2.5.2: Integrate by parts and using the definition of the Mellin transform, the result follows.

Example 2.5.1. Find the Mellin transform of the fractional derivative.

Solution 2.5.1: We have

$$ {}_0D_x^\alpha f = {}_0D_x^n\, {}_0D_x^{\alpha-n} f = {}_0D_x^n\, {}_0I_x^{n-\alpha} f.$$

Therefore,

$$m({}_0D_x^\alpha f)(s) = \frac{(-1)^n\,\Gamma(s)}{\Gamma(s-n)}\, m\left\{{}_0I_x^{n-\alpha} f\right\}(s-n), (n-1 \le \Re(\alpha) < n) \tag{2.5.11}$$

$$= \frac{(-1)^n\,\Gamma(s)\Gamma(1-(s-\alpha))}{\Gamma(s-n)\Gamma(1-s+n)}\, m\{f(t); s-\alpha\}, \tag{2.5.12}$$

where $\Re(s) > 0, \Re(s) < 1 + \Re(\alpha)$.

Remark 2.5.1: An alternative form of (2.5.12) is given in Exercise 2.5.2.

Exercises 2.5.

2.5.1. Prove Theorem 2.5.2.

2.5.2. Prove that the Mellin transform of fractional derivative is given by

$$m({}_0D_x^\alpha f)(s) = \frac{(-1)^n\,\Gamma(s)\sin[\pi(s-n)]}{\Gamma(s-\alpha)\sin[\pi(s-\alpha)]} m\{f(t); s-\alpha\}, \tag{2.5.13}$$

where $\Re(s) > 0, \Re(\alpha - s) > -1$.

2.5.3. Find the Mellin transform of $(1+x^a)^{-b}; a, b > 0$.

2.6 Kober Operators

Kober operators are the generalization of Riemann-Liouville and Weyl operators. These operators have been used by many authors in deriving the solution of single, dual and triple integral equations possessing special functions of mathematical physics, as their kernels.

Notation 2.6.1. Kober operator of the first kind

$$\mathbb{I}[f(x)],\ \mathbb{I}[\alpha,\eta : f(x)],\ \mathbb{I}(\alpha,\eta)f(x), E_{0,x}^{\alpha,\eta}\ f,\ \mathbb{I}_x^{n,\alpha}\ f.$$

Notation 2.6.2. Kober operator of the second kind

$$\mathbb{R}[f(x)],\ \mathbb{R}[\alpha,\zeta : f(x)],\ \mathbb{R}(\alpha,\zeta)f(x), K_{x,\infty}^{\alpha,\zeta}\ f, K_x^{\zeta,\alpha}\ f.$$

Definition 2.6.1.

$$\begin{aligned}\mathbb{I}[f(x)] &= \mathbb{I}[\alpha,\eta : f(x)] = \mathbb{I}(\alpha,\eta)f(x) = E_{0,x}^{\alpha,\eta} f \\ &= \mathbb{I}_x^{\eta,\alpha}\ f = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta} f(t)\mathrm{d}t,\end{aligned} \tag{2.6.1}$$

where $\Re(\alpha) > 0$.

Definition 2.6.2.

$$\begin{aligned}\mathbb{R}[f(x)] &= \mathbb{R}[\alpha,\zeta : f(x)] = \mathbb{R}(\alpha,\zeta)f(x) = K_{x,\infty}^{\alpha,\zeta} f \\ &= K_x^{\zeta,\alpha}\ f = \frac{x^{\zeta}}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1}t^{-\zeta-\alpha} f(t)\mathrm{d}t,\end{aligned} \tag{2.6.2}$$

where $\Re(\alpha) > 0$.

(2.6.1) and (2.6.2) hold true under the following conditions:

$$f \in L_p(0,\infty), \Re(\alpha) > 0, \Re(\eta) > -\frac{1}{q}, \Re(\zeta) > -\frac{1}{p}, \frac{1}{p} + \frac{1}{q} = 1, p \geq 1.$$

When $\eta = 0$, (2.6.1) reduces to Riemann-Liouville operator. That is,

$$I_x^{0,\alpha} f = x^{-\alpha}\ {}_0I_x^{\alpha} f. \tag{2.6.3}$$

For $\zeta = 0$, (2.6.2) yields the Weyl operator of $t^{-\alpha}f(t)$. That is,

$$K_x^{0,\alpha} f = {}_xW_{\infty}^{\alpha}\ t^{-\alpha} f(t). \tag{2.6.4}$$

Theorem 2.6.1. *[Kober (1940)].*

If $\Re(\alpha) > 0, \Re(\eta - s) > -1, f \in L_p(o,\infty), 1 \leq p \leq 2$ *(or* $f \in M_p(o,\infty)$, *a subspace of* $L_p(o,\infty)$ *and* $p > 2$ *),* $\Re(\eta) > -\frac{1}{q}, \frac{1}{p} + \frac{1}{q} = 1$, *then there holds the formula*

$$m\{\mathbb{I}(\alpha,\eta)f\}(s) = \frac{\Gamma(1+\eta-s)}{\Gamma(\alpha+\eta+1-s)} m\{f(x);s\}. \tag{2.6.5}$$

Proof 2.6.1: It is similar to the proof of Theorem 2.6.1.

In a similar manner, we can establish

Theorem 2.6.2. *[Kober (1940)].*

If $\Re(\alpha) > 0, \Re(s+\zeta) > 0, f \in L_p(o,\infty), 1 \leq p \leq 2$ *(or* $f \in M_p(o,\infty)$*, a subspace of* $L_p(o,\infty)$ *and* $p > 2$ *)*

$$\Re(\zeta) > -\frac{1}{p}, \frac{1}{p} + \frac{1}{q} = 1,$$

then,

$$m\{\Re(\alpha,\zeta)f\}(s) = \frac{\Gamma(\zeta+s)}{\Gamma(\alpha+\zeta+s)} m\{f(x);s\}. \tag{2.6.6}$$

Semigroup property of the Kober operators has been given in the form of

Theorem 2.6.3. *If* $f \in L_p(o,\infty), g \in L_q(o,\infty), \frac{1}{p} + \frac{1}{q} = 1, \Re(\eta) > -\frac{1}{q}, \Re(\zeta) > -\frac{1}{p}, 1 \leq p \leq 2,$ *(or* $f \in M_p(o,\infty)$*, a subspace of* $L_p(o,\infty)$ *and* $p > 2$ *), then*

$$\int_0^\infty g(x)(\mathbb{I}(\alpha,\eta : f))(x)\mathrm{d}x = \int_0^\infty f(x)(\mathbb{R}(\alpha,\eta : g))(x)\mathrm{d}x. \tag{2.6.7}$$

Proof 2.6.2: Interchange the order of integration.

Remark 2.6.1: Operators defined by (2.6.1.) and (2.6.2) are also called Erdélyi-Kober operators.

Exercises 2.6.

2.6.1. Prove Theorem 2.6.1.

2.6.2. For the modified Erdélyi-Kober operators, defined by the following equations for $m > 0$:

$$\begin{aligned} \mathbb{I}(\alpha,\eta : m)f(x) &= \mathbb{I}(f(x) : \alpha,\eta,m) \\ &= \frac{m}{\Gamma(\alpha)} x^{-\eta-m\alpha+m-1} \int_o^x t^\eta (x^m - t^m)^{\alpha-1} f(t)\mathrm{d}t, \end{aligned} \tag{2.6.8}$$

and

$$\begin{aligned} \mathbb{R}(\alpha,\zeta : m)f(x) &= \mathbb{R}(f(x) : \alpha,\zeta,m) \\ &= \frac{mx^\zeta}{\Gamma(\alpha)} \int_x^\infty t^{-\zeta-m\alpha+m-1} (t^m - x^m)^{\alpha-1} f(t)\mathrm{d}t, \end{aligned} \tag{2.6.9}$$

where $f \in L_p(0,\infty), \Re(\alpha) > 0, \Re(\eta) > -\frac{1}{q}, \Re(\zeta) > -\frac{1}{p}, \frac{1}{p} + \frac{1}{q} = 1$, find the Mellin transforms of (i) $\mathbb{I}(\alpha, \eta : m) f(x)$ and (ii) $\mathbb{R}(\alpha, \zeta : m) f(x)$, giving the conditions of validity.

2.6.3. For the operators defined by (2.6.8) and (2.6.9.), show that

$$\int_0^\infty \mathbb{R}(f(x) : \alpha, \eta, m) g(x) \mathrm{d}x = \int_0^\infty f(x) \mathbb{I}(g(x) : \alpha, \eta, m) \mathrm{d}x, \tag{2.6.10}$$

where the parameters α, η, m are the same in both the operators $\mathbb{I}$ and $\mathbb{R}$. Give conditions of validity of (2.6.10).

2.6.4. For the Erdélyi-Kober operator, defined by

$$I_{\eta,\alpha} f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x (x^2 - t^2)^{\alpha-1} t^{2\eta+1} f(t) \mathrm{d}t, \tag{2.6.11}$$

where $\Re(\alpha) > 0$, establish the following results (Sneddon (1975)):

$$\text{(i) } I_{\eta,\alpha} x^{2\beta} f(x) = x^{2\beta} I_{\eta+\beta,\alpha} f(x) \tag{2.6.12}$$

$$\text{(ii) } I_{\eta,\alpha} I_{\eta+\alpha,\beta} = I_{\eta,\alpha+\beta} = I_{\eta+\alpha,\beta}, I_{\eta,\alpha} \tag{2.6.13}$$

$$\text{(iii) } I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha}. \tag{2.6.14}$$

Remark 2.6.2: The results of Exercise 2.6.4 also hold for the operator, defined by

$$\mathbb{K}_{\eta,\alpha} f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty (t^2 - x^2)^{\alpha-1} t^{-2\alpha-2\eta+1} f(t) \mathrm{d}t, \tag{2.6.15}$$

where $\Re(\alpha) > 0$.

Remark 2.6.3: Operators more general than the operators defined by (2.6.11) and (2.6.15) are recently defined by Galué et al [Integral Transform & Spec. Funct. Vol. 9 (2000), No. 3, pp. 185-196] in the form

$${}_a I_x^{\eta,\alpha} f(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} t^\eta f(t) \mathrm{d}t, \tag{2.6.16}$$

where $\Re(\alpha) > 0$.

2.7 Generalized Kober Operators

Notation 2.7.1. $\mathbb{I}[\alpha, \beta, \gamma : m, \mu, \eta, a : f(x)], \mathbb{I}[f(x)]$

Notation 2.7.2. $\mathbb{I}[\alpha, \beta, \gamma : m, \mu, \delta, a : f(x)], \mathbb{I}[f(x)]$

Notation 2.7.3. $\mathbb{R}[f(x)], \mathbb{R}\left[\begin{smallmatrix}\alpha,\beta,\gamma; \\ \sigma,\rho,a;\end{smallmatrix} : f(x)\right]$

Notation 2.7.4. $\mathbb{K}[f(x)], \mathbb{K}\left[\begin{smallmatrix}\alpha,\beta,\gamma; \\ \delta,\rho,a;\end{smallmatrix} : f(x)\right]$

Notation 2.7.5. $I_{0,x}^{\alpha,\beta,\eta;} f(x)$ (Saigo, 1978)

Notation 2.7.6. $J_{x,\alpha}^{\alpha,\beta,\eta;} f(x)$ (Saigo, 1978)

Definition 2.7.1.

$$\begin{aligned}\mathbb{I}[f(x)] &= \mathbb{I}[\alpha,\beta,\gamma : m,\mu,\eta,a : f(x)] \\ &= \frac{\mu x^{-\eta-1}}{\Gamma(1-\alpha)} \int_0^x {}_2F_1\left(\alpha,\beta+m,\gamma;\frac{at^\mu}{x^\mu}\right) t^\eta f(t)\mathrm{d}t, \end{aligned} \tag{2.7.1}$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function.

Definition 2.7.2.

$$\begin{aligned}\mathbb{I}[f(x)] &= \mathbb{I}[\alpha,\beta,\gamma : m,\mu,\delta,a : f(x)] \\ &= \frac{\mu x^{\delta}}{\Gamma(1-\alpha)} \int_x^\infty {}_2F_1\left(\alpha,\beta+m;\gamma;\frac{ax^\mu}{t^\mu}\right) t^{-\delta-1} f(t)\mathrm{d}t. \end{aligned} \tag{2.7.2}$$

Operators defined by (2.7.1) and (2.7.2) exist under the following conditions:

(i) $1 \le p,\ q < \infty,\ p^{-1}+q^{-1} = 1,\ |\arg(1-a)| < \pi$
(ii) $\Re(1-\alpha) > m, \Re(\eta) > -\frac{1}{q}, \Re(\delta) > -\frac{1}{p}, \Re(\gamma-\alpha-\beta-m) > -1, m \in \mathbb{N}_0;$ $\gamma \neq 0,-1,-2,\cdots$
(iii) $f \in L_p(0,\infty)$

Equations (2.7.1) and (2.7.2) are introduced by Kalla and Saxena (1969).

For $\gamma = \beta$, (2.7.1) and (2.7.2) reduce to generalized Kober operators, given by Saxena (1967).

Definition 2.7.3.

$$\begin{aligned}\mathbb{R}[f(x)] &= \mathbb{R}\left[\begin{smallmatrix}\alpha,\beta,\gamma; \\ \sigma,\rho,a;\end{smallmatrix} f(x)\right] \\ &= \frac{x^{-\sigma-\rho}}{\Gamma(\rho)} \int_0^x t^\sigma (x-t)^{\rho-1}\, {}_2F_1\left[\alpha,\beta;\gamma;a\left(1-\frac{t}{x}\right)\right] f(t)\mathrm{d}t. \end{aligned} \tag{2.7.3}$$

Definition 2.7.4.

$$\begin{aligned}\mathbb{K}[f(x)] &= \mathbb{K}\left[\begin{smallmatrix}\alpha,\beta,\gamma; \\ \delta,\rho,a;\end{smallmatrix} f(x)\right] \\ &= \frac{x^{\delta}}{\Gamma(\rho)} \int_x^\infty t^{-\delta-\rho} (t-x)^{\rho-1}\, {}_2F_1\left[\alpha,\beta;\gamma;a\left(1-\frac{x}{t}\right)\right] f(t)\mathrm{d}t. \end{aligned} \tag{2.7.4}$$

The conditions of validity of the operators (2.7.3) and (2.7.4) are given below:

(i) $p \geq 1,\ q < \infty,\ p^{-1} + q^{-1} = 1,\ |\arg(1-a)| < \pi.$
(ii) $\Re(\sigma) > -\frac{1}{q}, \Re(\delta) > -\frac{1}{p}, \Re(\rho) > 0.$
(iii) $\gamma \neq 0, -1, -2, \cdots; \Re(\gamma - \alpha - \beta) > 0.$
(iv) $f \in L_p(0, \infty).$

The operators defined by (2.7.3) and (2.7.4) are given by Saxena and Kumbhat (1973). When a is replaced by $\frac{a}{\alpha}$ and α tends to infinity, the operators defined by (2.7.3) and (2.7.4) reduce to the following operators associated with confluent hypergeometric functions.

Definition 2.7.5.

$$\mathbb{R}\left[{}^{\beta,\gamma;}_{\sigma,\rho,a;} f(x)\right] = \lim_{\alpha \to \infty} \mathbb{R}\left[{}^{\alpha,\beta,\gamma;}_{\sigma,\rho,\frac{a}{\alpha};} f(x)\right]$$
$$= \frac{x^{-\sigma-\rho}}{\Gamma(\rho)} \int_0^x \Phi\left[\beta, \gamma; a\left(1 - \frac{t}{x}\right)\right] t^{\sigma}(x-t)^{\rho-1} f(t)\mathrm{d}t. \tag{2.7.5}$$

Definition 2.7.6.

$$\mathbb{K}\left[{}^{\beta,\gamma;}_{\sigma,\rho,a;} f(x)\right] = \lim_{\alpha \to \infty} \mathbb{K}\left[{}^{\alpha,\beta,\gamma;}_{\delta,\rho,\frac{a}{\alpha};} f(x)\right]$$
$$= \frac{x^{\delta}}{\Gamma(\rho)} \int_x^{\infty} \Phi\left[\beta, \gamma; a\left(1 - \frac{x}{t}\right)\right] t^{-\delta-\rho}(t-x)^{\rho-1} f(t)\mathrm{d}t, \tag{2.7.6}$$

where $\Re(\rho) > 0, \Re(\delta) > 0.$

Remark 2.7.1: Many interesting and useful properties of the operators defined by (2.7.3) and (2.7.4) are investigated by Saxena and Kumbhat (1975), which deal with relations of these operators with well-known integral transforms, such as Laplace, Mellin and Hankel transforms. Equation (2.7.3) was first considered by Love (1967).

Remark 2.7.2: In the special case, when α is replaced by $\alpha + \beta, \gamma$ by α, σ by zero, ρ by α and β by $-\eta$, then (2.7.3) reduces to the operator (2.7.7) considered by Saigo (1978). Similarly, (2.7.4) reduces to another operator (2.7.9) introduced by Saigo (1978).

Definition 2.7.7. Let $\alpha, \beta, \eta \in \mathbb{C}$, and let $x \in \mathbb{R}_+$ the fractional integral ($\Re(\alpha) > 0$) and the fractional derivative ($\Re(\alpha) < 0$) of the first kind of a function $f(x)$ on $\mathbb{R}_+$ are defined by Saigo (1978) in the form

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}$$
$$\times\ {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x}\right) f(t)\mathrm{d}t,\ \Re(\alpha) > 0 \tag{2.7.7}$$
$$= \frac{\mathrm{d}^n}{\mathrm{d}x^n} I_{0,x}^{\alpha+n,\beta-n,\eta-n} f(x),\ 0 < \Re(\alpha) + n \leq 1,\ (n \in \mathbb{N}_0). \tag{2.7.8}$$

Definition 2.7.8. The fractional integral ($\Re(\alpha) > 0$) and fractional derivative ($\Re(\alpha) < 0$) of the second kind of a function $f(x)$ on $\mathbb{R}_+$ are given by Saigo (1978) in the form

$$J_{x,\infty}^{\alpha,\beta,\eta}\, f(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta} \times {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) f(t)\mathrm{d}t,\ \Re(\alpha)>0 \tag{2.7.9}$$

$$= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} J_{x,\infty}^{\alpha+n,\beta-n,\eta}\, f(x),\ 0<\Re(\alpha)+n\le 1,\ (n\in\mathbb{N}_0). \tag{2.7.10}$$

Example 2.7.1. Find the value of

$$I_{0,x}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\,{}_2F_1(a,b;c;-a'x)\right\}.$$

Solution 2.7.1: We have

$$\begin{aligned} K &= I_{0,x}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\,{}_2F_1(a,b;c;-a'x)\right\} \\ &= \sum_{r=0}^\infty \frac{(a)_r(b)_r(-1)^r(a')^r}{(c)_r r!} I_{0,x}^{\alpha,\beta,\eta} x^{r+\sigma-1}. \end{aligned}$$

Applying the result of Exercise 2.7.1, we obtain

$$\begin{aligned} K &= x^{\sigma-\beta-1}\sum_{r=0}^\infty (-1)^r \frac{(a)_r(b)_r\ \Gamma(\sigma+r)\Gamma(\sigma-\beta+\eta+r)(a')^r}{(c)_r r!\ \Gamma(\sigma-\beta+r)\Gamma(\alpha+\eta+\sigma+r)}\, x^r \\ &= x^{\sigma-\beta-1}\,\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} \\ &\quad\times {}_4F_3(a,b,\sigma,\sigma+\eta-\beta;c,\sigma-\beta,\sigma+\alpha+\eta;-a'x), \end{aligned}$$

where $\Re(\alpha)>0, \Re(\sigma)>0, \Re(\sigma+\eta-\beta)>0, c\ne 0,-1,-2,\cdots;\ |a'x|<1$.

Example 2.7.2. Find the value of

$$J_{x,\infty}^{\alpha,\beta,\eta}\left(x^\lambda\,{}_2F_1\left(a,b;c;\ \frac{a'}{x}\right)\right).$$

Solution 2.7.2: Following a similar procedure and using the result of Exercise 2.7.3, it gives

$$\begin{aligned} J_{x,\infty}^{\alpha,\beta,\eta}\left(x^\lambda\,{}_2F_1\left(a,b;c;\ \frac{a'}{x}\right)\right) &= \frac{\Gamma(\beta-\lambda)\Gamma(\eta-\lambda)}{\Gamma(-\lambda)\Gamma(\alpha+\beta+\eta-\lambda)} x^{\lambda-\beta} \\ &\quad\times {}_4F_3\left(a,b,\beta-\lambda,\eta-\lambda;c,-\lambda,\alpha+\beta+\eta-\lambda;\frac{a'}{x}\right), \end{aligned}$$

where $\Re(\alpha)>0, \Re(\beta-\lambda)>0, \Re(\eta-\lambda)>0, x>0, c\ne 0,-1,-2,\cdots;\ |x|>|a'|$.

Remark 2.7.3: Special cases of the operators $I_{0,x}^{\alpha,\beta,\eta}$ and $J_{x,\infty}^{\alpha,\beta,\eta}$ are the operators of Riemann -Liouville:

$$I_{0,x}^{\alpha,-\alpha,\eta}\ f(x) = {}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t,\ (\Re(\alpha) > 0) \tag{2.7.11}$$

the Weyl:

$$J_{x,\infty}^{\alpha,-\alpha,\eta} f(x) = {}_xW_\infty^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\mathrm{d}t,\ (\Re(\alpha) > 0) \tag{2.7.12}$$

and the Erdélyi-Kober operators:

$$I_{0,x}^{\alpha,0,\eta}\ f(x) = E_{0,x}^{\alpha,\eta}\ f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\mathrm{d}t,\ (\Re(\alpha) > 0) \tag{2.7.13}$$

and

$$J_{x,\infty}^{\alpha,0,\eta}\ f(x) = K_{x,\infty}^{\alpha,\eta}\ f(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\mathrm{d}t,\ (\Re(\alpha) > 0) \tag{2.7.14}$$

Example 2.7.3. Prove the following theorem.

If $\Re(\alpha) > 0$ and $\Re(s) < 1 + \min[0, \Re(\eta-\beta)]$, then the following formula holds for $f(x) \in L_p(0,\infty)$ with $1 \le p \le 2$ or $f(x) \in M_p(0,\infty)$ with $p > 2$:

$$m\left\{x^{\beta} I_{0,x}^{\alpha,\beta,\eta} f\right\} = \frac{\Gamma(1-s)\Gamma(\eta-\beta+1-s)}{\Gamma(1-s-\beta)\Gamma(\alpha+\eta+1-s)} m\{f(x)\}. \tag{2.7.15}$$

Solution 2.7.3: Use the integral

$$\int_x^\infty u^{-\sigma-\gamma}(u-x)^{\gamma-1}\ {}_2F_1\ (\alpha,\beta;\gamma;1-\frac{x}{u})\mathrm{d}u = \frac{\Gamma(\gamma)\Gamma(\sigma)\Gamma(\gamma+\sigma-\alpha-\beta)}{\Gamma(\gamma+\sigma-\alpha)\Gamma(\gamma+\sigma-\beta)}, \tag{2.7.16}$$

where $\Re(\gamma) > 0$, $\Re(\sigma) > 0$, $\Re(\gamma+\sigma-\alpha-\beta) > 0$.

Exercises 2.7.

2.7.1. Prove that

$$I_{0,x}^{\alpha,\beta,\eta}\ x^{\lambda} = \frac{\Gamma(1+\lambda)\Gamma(1+\lambda+\eta-\beta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)} x^{\lambda-\beta}, \tag{2.7.17}$$

and give the conditions of validity.

2.7.2. Find the Mellin transform of $x^{\beta}\ J_{x,\infty}^{\alpha,\beta,\eta}\ f(x)$, giving conditions of its validity.

2.7.3. Prove that

$$J_{x,\infty}^{\alpha,\beta,\eta}\ x^{\lambda} = \frac{\Gamma(\beta-\lambda)\Gamma(\eta-\lambda)}{\Gamma(-\lambda)\Gamma(\alpha+\beta+\eta-\lambda)} x^{\lambda-\beta} \tag{2.7.18}$$

and give the conditions of validity.

2.7.4. Prove that

$$\begin{aligned} I_{0,x}^{\alpha,\beta,\eta}\ (x^{k}\mathrm{e}^{-\lambda x}) &= \frac{\Gamma(k+1)\Gamma(\eta+k-\beta+1)}{\Gamma(k-\beta+1)\Gamma(\alpha+\eta+k+1)} x^{k-\beta} \\ &\times\ {}_2F_2(k+1,\eta+k-\beta+1;\ k-\beta+1,\alpha+\eta+k+1;-\lambda x), \end{aligned} \tag{2.7.19}$$

and give the conditions of validity.

2.7.5. Prove that

$$\begin{aligned} J_{x,\infty}^{\alpha,\beta,\eta}\ \mathrm{e}^{-sx} &= s^{\eta}x^{\eta-\beta}\frac{\Gamma(\beta-\eta)}{\Gamma(\alpha+\beta)}\Phi(1-\alpha-\beta,1+\eta-\beta;-sx) \\ &+ s^{\beta}\frac{\Gamma(\eta-\beta)}{\Gamma(\alpha+\eta)}\Phi(1-\alpha-\eta,1+\beta-\eta;-sx), \end{aligned} \tag{2.7.20}$$

and give the conditions of its validity. Deduce the results for $L[\ {}_xW_{\infty}^{\alpha}\ f](s)$ and $L[\ K_{x,\infty}^{\alpha,\eta}\ f](s)$.

2.7.6. Prove that [Saxena and Nishimoto (2002)]

$$\begin{aligned} I_{0,x}^{\alpha,\beta,\eta}\ [x^{\sigma-1}(a+bx)^{c}] &= a^{c}\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} x^{\sigma-\beta-1} \\ &\times\ {}_3F_2\left(\sigma,\sigma+\eta-\beta,-c;\ \sigma-\beta,\sigma+\alpha+\eta;-\frac{bx}{a}\right), \end{aligned} \tag{2.7.21}$$

where $\Re(\sigma) > \max[0,\Re(\beta-\eta)], |\frac{bx}{a}| < 1$.

2.7.7. Evaluate

$$I_{0,x}^{\alpha,\beta,\eta}\ \left\{x^{\sigma-1}H_{p,q}^{m,n}\left[ax^{\lambda}\ \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\},\ \lambda > 0, \tag{2.7.22}$$

and give the conditions of its validity.

2.7.8. Evaluate

$$J_{x,\infty}^{\alpha,\beta,\eta}\ \left\{x^{\sigma-1}H_{p,q}^{m,n}\left[ax^{-\lambda}\ \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\},\ \lambda > 0, \tag{2.7.23}$$

and give the conditions of its validity.

2.7.9. Establish the following property of Saigo operators called "Integration by parts".

$$\int_0^\infty f(x)\left(I_{0,x}^{\alpha,\beta,\eta} g\right)(x)\mathrm{d}x = \int_0^\infty g(x)\left(J_{x,\infty}^{\alpha,\beta,\eta} f\right)(x)\mathrm{d}x.$$

2.7.10. From Exercise 2.7.6, deduce the formula for

$$I_{0,x}^{\alpha,-\alpha,\eta}(a+bx)^c, \tag{2.7.24}$$

given by B. Ross (1993).

2.7.11. Prove that

$${}_0I_x^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}x^{k+\alpha}, \tag{2.7.25}$$

where $\Re(\alpha) > 0$, $\Re(k) > -1$,

2.7.12. Prove that

$$W_{x,\infty}^\alpha x^k = \frac{\Gamma(-\alpha-k)}{\Gamma(-k)}x^{k+\alpha}, \tag{2.7.26}$$

where $\Re(\alpha) > 0$, $\Re(k) < -\Re(\alpha)$.

2.7.13. Show that

$$J_{x,\infty}^{\alpha,\beta,\eta}\ (x^\lambda \mathrm{e}^{-px}) = x^{\lambda-\beta} G_{2,3}^{3,0}\left[px|_{0,\beta-\lambda,\eta-\lambda}^{-\lambda,\alpha+\beta+\eta-\lambda}\right], \tag{2.7.27}$$

where $G_{2,3}^{3,0}(\cdot)$ is the Meijer's G-function, $\Re(px) > 0$, $\Re(\alpha) > 0$.

Hint: Use the integral

$$\mathrm{e}^{-px} = \frac{1}{2\pi i}\int_L \Gamma(-s)(px)^s \mathrm{d}s. \tag{2.7.28}$$

2.7.14. Evaluate

$$I_{0,x}^{\alpha,\beta,\eta} x^{\sigma-1} H_{p,q}^{m,n}\left[ax^{-\lambda}|_{(b_q,B_q)}^{(a_p,A_p)}\right],\ \lambda > 0, \tag{2.7.29}$$

giving the conditions of its validity.

2.7.15. Evaluate

$$J_{x,\infty}^{\alpha,\beta,\eta} x^{\sigma-1} H_{p,q}^{m,n}\left[ax^{\lambda}|_{(b_q,B_q)}^{(a_p,A_p)}\right],\ \lambda > 0 \tag{2.7.30}$$

and give the conditions of validity of the result.

2.7.16. With the help of the following chain rules for Saigo operators (Saigo, 1985)

$$I_{0,x}^{\alpha,\beta,\eta} I_{0,x}^{\gamma,\delta,\alpha+\eta} f = I_{0,x}^{\alpha+\gamma,\beta+\delta,\eta} f, \tag{2.7.31}$$

and

$$J_{x,\infty}^{\alpha,\beta,\eta} J_{x,\infty}^{\gamma,\delta,\alpha+\eta} f = J_{x,\infty}^{\alpha+\gamma,\beta+\delta,\eta} f, \tag{2.7.32}$$

derive the inverses

$$(I_{0,x}^{\alpha,\beta,\eta})^{-1} = I_{0,x}^{-\alpha,-\beta,\alpha+\eta}. \tag{2.7.33}$$

and

$$(J_{x,\infty}^{\alpha,\beta,\eta})^{-1} = J_{x,\infty}^{-\alpha,-\beta,\alpha+\eta}. \tag{2.7.34}$$

2.8 Compositions of Riemann-Liouville Fractional Calculus Operators and Generalized Mittag-Leffler Functions

In this section, composition relations between Riemann-Liouville fractional calculus operators and generalized Mittag-Leffler functions are derived. These relations may be useful in the solution of fractional differintegral equations. For details, one can refer to the work of Saxena and Saigo (2005). For ready reference some of the definitions are repeated here.

2.8.1 Composition Relations Between R-L Operators and $E_{\beta},\gamma^{\delta}(z)$

Notation 2.8.1. $E_\alpha(x)$: Mittag-Leffler function.

Notation 2.8.2. $E_{\alpha,\beta}(x)$: Generalized Mittag-Leffler function.

Notation 2.8.3. $I_{0+}^{\alpha} f$: Riemann-Liouville left-sided integral.

Notation 2.8.4. $I_{-}^{\alpha} f$: Riemann-Liouville right-sided integral.

Notation 2.8.5. $D_{0+}^{\alpha} f$: Riemann-Liouville left-sided derivative.

Notation 2.8.6. $D_{-}^{\alpha} f$: Riemann-Liouville right-sided derivative.

Notation 2.8.7. $E_{\beta,\gamma}^{\delta}(z)$: Generalized Mittag-Leffler function (Prabhakar, 1971).

Definition 2.8.1.

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \ (\alpha\in C,\Re(\alpha)>0). \tag{2.8.1}$$

Definition 2.8.2.

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \ (\alpha,\beta\in C,\Re(\alpha)>0,\Re(\beta)>0). \tag{2.8.2}$$

Definition 2.8.3.

$$(I_{0+}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}\mathrm{d}t,\ \ \Re(\alpha)>0. \tag{2.8.3}$$

Definition 2.8.4.

$$(I_{-}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty\frac{f(t)}{(t-x)^{1-\alpha}}\mathrm{d}t,\ \ \Re(\alpha)>0. \tag{2.8.4}$$

Definition 2.8.5.

$$(D_{0+}^{\alpha}f)(x)=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}\right)(x);\ \ \Re(\alpha)>0 \tag{2.8.5}$$

$$=\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\int_0^x\frac{f(t)}{(x-t)^{\{\alpha\}}}\mathrm{d}t,\ \ \Re(\alpha)>0. \tag{2.8.6}$$

Definition 2.8.6.

$$(D_{-}^{\alpha}f)(x)=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}(I_{-}^{1-\{\alpha\}}f)(x),\ \ \Re(\alpha)>0 \tag{2.8.7}$$

$$=\frac{1}{\Gamma(1-\{\alpha\})}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\int_x^\infty\frac{f(t)}{(t-x)^{\{\alpha\}}}\mathrm{d}t,\ \ \Re(\alpha)>0. \tag{2.8.8}$$

Remark 2.8.1: Here $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α. Note that $\Gamma(1-\{\alpha\})=\Gamma(m-\alpha), [\alpha]+1=m, \{\alpha\}=1+\alpha-m$.

Definition 2.8.7.

$$E_{\beta,\gamma}^{\delta}(z)=\sum_{k=0}^{\infty}\frac{(\delta)_k z^k}{\Gamma(\beta k+\gamma)k!},\ \ (\beta,\gamma,\delta\in C;\Re(\gamma)>0,\Re(\beta)>0). \tag{2.8.9}$$

For $\delta=1$, (2.8.9) reduces to (2.8.2).

Theorem 2.8.1. *Let $\alpha > 0,\ \beta > 0,\ \gamma > 0$ and $\alpha \in \mathbb{R}$. Let I_{0+}^{α} be the left-sided operator of Riemann-Liouville fractional integral (2.8.3). Then there holds the formula*

$$(I_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})])(x) = x^{\alpha+\gamma-1}E_{\beta,\alpha+\gamma}^{\delta}(ax^{\beta}). \tag{2.8.10}$$

Proof 2.8.1: By virtue of (2.8.3) and (2.8.9), we have

$$K \equiv (I_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})])(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\sum_{n=0}^{\infty}\frac{(\delta)_n a^n t^{n\beta+\gamma-1}}{\Gamma(\beta n+\gamma)n!}\mathrm{d}t.$$

Interchanging the order of integration and summation and evaluating the inner integral by means of beta-function formula, it gives

$$K \equiv x^{\alpha+\gamma-1}\sum_{n=0}^{\infty}\frac{(\delta)_n(ax^{\beta})^n}{\Gamma(\alpha+\beta n+\gamma)(n)!} = x^{\alpha+\gamma-1}E_{\beta,\alpha+\gamma}^{\delta}(ax^{\beta}).$$

This completes the proof of Theorem 2.8.1.

Corollary 2.8.1: *For $\alpha > 0, \beta > 0, \gamma > 0$ and $\alpha \in \mathbb{R}$, there holds the formula*

$$(I_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}(at^{\beta})])(x) = x^{\alpha+\gamma-1}E_{\beta,\alpha+\gamma}(ax^{\beta}). \tag{2.8.11}$$

Remark 2.8.2: For $\beta = \alpha$, (2.8.11) reduces to

$$(I_{0+}^{\alpha}[t^{\gamma-1}E_{\alpha,\gamma}(at^{\alpha})])(x) = \frac{x^{\gamma-1}}{a}\left[E_{\alpha,\gamma}(ax^{\alpha}) - \frac{1}{\Gamma(\gamma)}\right], (a \neq 0) \tag{2.8.12}$$

by virtue of the identity

$$E_{\alpha,\gamma}(x) = \frac{1}{\Gamma(\gamma)} + xE_{\alpha,\alpha+\gamma}(x), (a \neq 0). \tag{2.8.13}$$

Theorem 2.8.2. *Let $\alpha > 0, \beta > 0, \gamma > 0$ and $\alpha \in \mathbb{R}$, $(a \neq 0)$ and let I_{0+}^{α} be the left-sided operator of Riemann-Liouville fractional integral (2.8.3). Then there holds the formula*

$$(I_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})])(x) = \frac{1}{a}x^{\alpha+\gamma-\beta-1}[E_{\beta,\alpha+\gamma-\beta}^{\delta}(ax^{\beta}) - E_{\beta,\alpha+\gamma-\beta}^{\delta-1}(ax^{\beta})]. \tag{2.8.14}$$

Proof. Use Theorem 2.8.1.

The following two theorems can be established in the same way.

Theorem 2.8.3. *Let $\alpha > 0, \beta > 0, \gamma > 0$ and $\alpha \in \mathbb{R}$ and let I_{-}^{α} be the right-sided operator of Riemann-Liouville fractional integral (2.8.4). Then we arrive at the following result:*

$$(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\beta,\gamma}^{\delta}(at^{-\beta})])(x) = x^{-\gamma}[E_{\beta,\alpha+\gamma}^{\delta}(ax^{-\beta})] \tag{2.8.15}$$

Corollary 2.8.2: *For $\alpha > 0, \beta > 0, \gamma > 0$ and $\alpha \in \mathbb{R}$, there holds the formulas:*

$$(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\beta,\gamma}(at^{-\beta})])(x) = x^{-\gamma}[E_{\beta,\alpha+\gamma}(ax^{-\beta})] \tag{2.8.16}$$

and

$$(I_{-}^{\alpha}t^{-\alpha-1}E_{\beta}(at^{-\beta}))(x) = x^{-1}[E_{\beta,\alpha+1}(ax^{-\beta})]. \tag{2.8.17}$$

Theorem 2.8.4. *Let $\alpha > 0, \beta > 0, \gamma > 0, \alpha \in \mathbb{R},\ (a \neq 0), \alpha+\gamma > \beta$ and let I_{-}^{α} be the right-sided operator of Riemann-Liouville fractional integral (2.8.4). Then there holds the formula*

$$(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\beta,\gamma}^{\delta}(at^{-\beta})])(x) = \frac{1}{a}x^{\beta-\gamma}[E_{\beta,\alpha+\gamma-\beta}^{\delta}(ax^{-\beta}) - E_{\beta,\alpha+\gamma-\beta}^{\delta-1}(ax^{-\beta})]. \tag{2.8.18}$$

Corollary 2.8.3: *For $\alpha > 0, \beta > 0, \gamma > 0$ with $\alpha+\gamma > \beta$ and for $\alpha \in \mathbb{R},\ (a \neq 0)$, there holds the formula*

$$(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\beta,\gamma}(at^{-\beta})])(x) = \frac{1}{a}x^{\beta-\gamma}\left[E_{\beta,\alpha+\gamma-\beta}(ax^{-\beta}) - \frac{1}{\Gamma(\alpha+\gamma-\beta)}\right]. \tag{2.8.19}$$

Remark 2.8.3: (Kilbas and Saigo, (1998))

$$(I_{-}^{\alpha}[t^{-\alpha-\gamma}E_{\alpha,\gamma}(at^{-\alpha})])(x) = \frac{x^{\alpha-\gamma}}{a}\left[E_{\alpha,\gamma}(ax^{-\alpha}) - \frac{1}{\Gamma(\gamma)}\right],\ (a \neq 0) \tag{2.8.20}$$

$$(I_{-}^{\alpha}[t^{-\alpha-1}E_{\alpha}(at^{-\alpha})])(x) = \frac{x^{\alpha-1}}{a}\left[E_{\alpha}(ax^{-\alpha}) - 1\right],\ (a \neq 0). \tag{2.8.21}$$

Theorem 2.8.5. *Let $\alpha > 0, \beta > 0, \gamma > 0, \gamma > \alpha, \alpha \in \mathbb{R}$ and let D_{0+}^{α} be the left-sided operator of Riemann -Liouville fractional derivative (2.8.6). Then there holds the formula.*

$$(D_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})])(x) = x^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}^{\delta}(ax^{\beta}). \tag{2.8.22}$$

Proof 2.8.2: By virtue of (2.8.9) and (2.8.6), we have

$$K \equiv (D_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})])(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}\left[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})\right]\right)(x)$$

$$= \sum_{n=0}^{\infty}\frac{a^n(\delta)_n}{\Gamma(\gamma+n\beta)\Gamma(1-\{a\})n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}\int_0^x t^{n\beta+\gamma-1}(x-t)^{-\{\alpha\}}\mathrm{d}t$$

$$= \sum_{n=0}^{\infty}\frac{a^n(\delta)_n}{\Gamma(\gamma+n\beta+1-\{\alpha\})n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1}x^{n\beta+\gamma-\{\alpha\}}$$

$$= \sum_{n=0}^{\infty}\frac{a^n(\delta)_n x^{\gamma+n\beta-\alpha-1}}{\Gamma(n\beta+\gamma-\alpha)n!} = x^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}^{\delta}(ax^{\beta}),$$

which proves the theorem.

By using a similar procedure, we arrive at the following theorem.

Theorem 2.8.6. *Let $\alpha > 0, \gamma > \beta > 0, \alpha \in \mathbb{R}$, $(a \neq 0)$, $\gamma > \alpha+\beta$ and let D_{0+}^{α} be the left-sided operator of Riemann-Liouville fractional derivative (2.8.6). Then there holds the formula*

$$\left(D_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}^{\delta}(at^{\beta})]\right)(x) = \frac{1}{a}x^{\gamma-\alpha-\beta-1}\left[E_{\beta,\gamma-\alpha-\beta}^{\delta}(ax^{\beta}) - E_{\beta,\gamma-\alpha-\beta}^{\delta-1}(ax^{\beta})\right]. \tag{2.8.23}$$

Corollary 2.8.4: *Let $\alpha > 0, \gamma > \beta > 0, \alpha \in \mathbb{R}$, $(a \neq 0)$, $\gamma > \alpha+\beta$, then there holds the formula.*

$$\left(D_{0+}^{\alpha}[t^{\gamma-1}E_{\beta,\gamma}(at^{\beta})]\right)(x) = \frac{1}{a}x^{\gamma-\alpha-\beta-1}\left[E_{\beta,\gamma-\alpha-\beta}(ax^{\beta}) - \frac{1}{\Gamma(\gamma-\alpha-\beta)}\right]. \tag{2.8.24}$$

Theorem 2.8.7. *Let $\alpha > 0, \gamma > 0, \gamma-\alpha > 0$ with $\gamma-\alpha+\{\alpha\} > 1, \alpha \in \mathbb{R}$, and let D_{-}^{α} be the right-sided operator of Riemann-Liouville fractional derivative (2.8.8). Then there holds the formula.*

$$\left(D_{-}^{\alpha}[t^{\alpha-\gamma}E_{\beta,\gamma}^{\delta}(at^{-\beta})]\right)(x) = x^{-\gamma}E_{\beta,\gamma-\alpha}^{\delta}(ax^{-\beta}). \tag{2.8.25}$$

Theorem 2.8.8. *Let $\alpha > 0, \beta > 0$ with $\gamma-\{\alpha\} > 1$, $\alpha \in \mathbb{R}$, $\gamma > \alpha+\beta$, $(a \neq 0)$ and let D_{-}^{α} be the right-sided operator of Riemann-Liouville fractional derivative (2.8.8). Then there holds the formula*

$$\left(D_{-}^{\alpha}[t^{\alpha-\gamma}E_{\beta,\gamma}^{\delta}(at^{-\beta})]\right)(x) = \frac{x^{\beta-\gamma}}{a}\left[E_{\beta,\gamma-\alpha-\beta}^{\delta}(ax^{-\beta}) - E_{\beta,\gamma-\alpha-\beta}^{\delta-1}(ax^{-\beta})\right]. \tag{2.8.26}$$

Exercises 2.8.

2.8.1. Show that

$$ax^{\beta}E^{\delta}_{\beta,\gamma}(ax^{\beta}) = E^{\delta}_{\beta,\gamma-\beta}(ax^{\beta}) - E^{\delta-1}_{\beta,\gamma-\beta}(ax^{\beta}), (a \neq 0) \tag{2.8.27}$$

2.8.2. Show that

$$\left(I^{\alpha}_{0+}[t^{\gamma-1}E_{\alpha,\gamma}(at^{\alpha})]\right)(x) = \frac{x^{\gamma-1}}{a}\left[E_{\alpha,\gamma}(ax^{\alpha}) - \frac{1}{\Gamma(\gamma)}\right], \ (a \neq 0). \tag{2.8.28}$$

2.8.3. Prove Theorem 2.8.3.

2.8.4. Prove Theorem 2.8.4.

2.8.5. Prove Theorem 2.8.6.

2.8.6. Prove Theorem 2.8.7.

2.8.7. Prove Theorem 2.8.8.

2.8.8. Prove that

$$\left(I^{\alpha}_{0+}t^{\omega}H^{m,n}_{p,q}\left[t^{\sigma}\Big|^{(a_p,A_p)}_{(bq,Bq)}\right]\right)(x) = x^{\omega+\alpha}H^{m,n+1}_{p+1,q+1}\left[x^{\sigma}\Big|^{(-\omega,\sigma),(a_p,A_p)}_{(bq,Bq),(-\omega-\alpha,\sigma)}\right], \tag{2.8.29}$$

giving conditions of validity.

2.8.9. Evaluate

$$\left(I^{\alpha}_{-}t^{\omega}H^{m,n}_{p,q}\left[t^{\sigma}\Big|^{(a_p,A_p)}_{(bq,Bq)}\right]\right)(x), \tag{2.8.30}$$

and give the conditions of validity.

2.9 Fractional Differential Equations

Differential equations contain integer order derivatives, whereas fractional differential equations involve fractional derivatives, like $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}$, which are defined for $\alpha > 0$. Here α is not necessarily an integer and can be rational, irrational or even complex-valued. Today, fractional calculus models find applications in physical, biological, engineering, biomedical and earth sciences. Most of the problems discussed involve relaxation and diffusion models in the so called complex or disordered systems. Thus, it gives rise to the generalization of initial value problems involving ordinary differential equations to generalized fractional-order differential equations and Cauchy problems involving partial differential equations to fractional reaction, fractional diffusion and fractional reaction-diffusion equations. Fractional calculus plays a dominant role in the solution of all these physical problems.

2.9.1 Fractional relaxation

In order to formulate a relaxation process, we require a physical law, say the relaxation equation

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t)+\frac{1}{c}f(t)=0, t>0, c>0, \tag{2.9.1}$$

to be solved for the initial value $f(t=0)=f_0$. The unique solution of (2.9.1) is given by

$$f(t)=f_0\ \mathrm{e}^{-\frac{t}{c}}, t\geq 0, c>0. \tag{2.9.2}$$

Now the problem is as to how we can generalize the initial-value problem (2.9.1) into a fractional value problem with physical motivation. If we incorporate the initial value f_0 into the integrated relaxation equation (2.9.1), we find that

$$f(t)-f_0=-\frac{1}{c}\ {}_0D_t^{-1}\ f(t), \tag{2.9.3}$$

where ${}_0D_t^{-1}$ is the standard Riemann integral of $f(t)$. On replacing $\frac{1}{c}\ {}_0D_t^{-1}\ f(t)$ by $\frac{1}{c^\alpha}\ {}_0D_t^{-\alpha}\ f(t)$, it yields the fractional integral equation

$$f(t)-f_0=-\left(\frac{1}{c^\alpha}\right)\ {}_0D_t^{-\alpha}\ f(t), \alpha>0 \tag{2.9.4}$$

with initial value

$$f_0=f(t=0).$$

Applying the Riemann-Liouville differential operator ${}_0D_t^\alpha$ from the left and making use of the formula (2.4.16), we arrive at

$${}_0D_t^\alpha[f(x)-f_0]=-c^{-\alpha}f(t),\ \alpha>0, c>0, \tag{2.9.5}$$

with initial condition $f_0=f(t=0)$.

Theorem 2.9.1. *The solution of the fractional differential equation (2.9.4) is given by*

$$f(t)=f_0H_{1,2}^{1,1}\left[\left(\frac{t}{c}\right)^\alpha\Big|_{(0,1),(0,\alpha)}^{(0,1)}\right], \tag{2.9.6}$$

where $\alpha>0, c>0$.

Proof 2.9.1: If we apply the Laplace transform to equation (2.9.4), it gives

$$F(s)-f_0s^{-1}=-\frac{1}{c^\alpha}s^{-\alpha}F(s), \tag{2.9.7}$$

where we have used the result (2.4.7) and $F(s)$ is the Laplace transform of $f(t)$. Solving for $F(s)$, we have

$$F(s) = L\{f(t)\} = f_0 \left[\frac{s^{-1}}{1+(cs)^{-\alpha}}\right]. \tag{2.9.8}$$

Taking inverse Laplace transform, (2.9.8) gives

$$\begin{aligned}
f(t) = L^{-1}\{F(s)\} &= f_0 L^{-1}\left[\frac{s^{-1}}{1+(cs)^{-\alpha}}\right] \\
&= f_0 L^{-1}\left[\sum_{k=0}^{\infty}(-1)^k c^{-\alpha k} s^{-\alpha k-1}\right] \\
&= f_0 \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{t}{c})^{\alpha k}}{\Gamma(\alpha k+1)} \\
&= f_0 E_\alpha\left[-\left(\frac{t}{c}\right)^{\alpha}\right],
\end{aligned} \tag{2.9.9}$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function. (2.9.9) can be written in terms of the H-function as

$$f(t) = f_0 H_{1,2}^{1,1}\left[\left(\frac{t}{c}\right)^{\alpha}\Big|_{(0,1),(0,\alpha)}^{(0,1)}\right], \tag{2.9.10}$$

where $c > 0, \alpha > 0$. This completes the proof of the Theorem 2.9.1.

Alternative form of the solution. By virtue of the identity

$$H_{p,q}^{m,n}\left[x^{\mu}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{1}{\mu} H_{p,q}^{m,n}\left[x\Big|_{(b_q,\frac{B_q}{\mu})}^{(a_p,\frac{A_p}{\mu})}\right],\ (\mu > 0) \tag{2.9.11}$$

the solution (2.9.10) can be written as

$$f(t) = \frac{f_0}{\alpha} H_{1,2}^{1,1}\left[\frac{t}{c}\Big|_{(0,\frac{1}{\alpha}),(0,1)}^{(0,\frac{1}{\alpha})}\right], \tag{2.9.12}$$

where $\alpha > 0, c > 0$.

Remark 2.9.1: In the limit as $\alpha \to 1$, one recovers the result (2.9.2)

$$f(t) = f_0 \exp\left(-\frac{t}{c}\right) = f_0 E_1\left(\frac{t}{c}\right). \tag{2.9.13}$$

Remark 2.9.2: In terms of Wright's function, the solution (2.9.10) can be expressed in the form

$$f(t) = f_0\ {}_1\psi_1\ \left[{}^{(1,1)}_{(1,\alpha)}\Big|; (\frac{t}{c})^{\alpha}\right], \tag{2.9.14}$$

where $\alpha > 0, c > 0$.

In a similar manner, we can establish Theorems 2.9.2 and 2.9.3 given below.

Theorem 2.9.2. *The solution of the fractional integral equation*

$$N(t) - N_0 t^{\mu-1} = -c^{\nu}\ {}_0D_t^{-\nu}\, N(t), \tag{2.9.15}$$

is given by

$$N(t) = N_0 \Gamma(\mu) t^{\mu-1} E_{\nu,\mu}(-c^{\nu} t^{\mu}), \tag{2.9.16}$$

where $E_{\nu,\mu}(\cdot)$ is the generalized Mittag-Leffler function (2.1.2), $\nu > 0, \mu > 0$.

Remark 2.9.3: When $\mu = 1$, we obtain the result given by Haubold and Mathai (2000).

Theorem 2.9.3. *If $c > 0, \nu > 0, \mu > 0$, then for the solution of the integral equation*

$$N(t) - N_0\, t^{\mu-1} E^{\gamma}_{\nu,\mu}[-(ct)^{\nu}] = -c^{\nu}{}_0D_t^{-\nu}\, N(t), \tag{2.9.17}$$

there holds the formula

$$N(t) = N_0\, t^{\mu-1} E^{\gamma+1}_{\nu,\mu}[-(ct)^{\nu}]. \tag{2.9.18}$$

Hint: Use the formula

$$L^{-1}\left\{ s^{-\beta}(1 - a s^{-\alpha})^{-\gamma} \right\} = t^{\beta-1} E^{\gamma}_{\alpha,\beta}(a t^{\alpha}), \tag{2.9.19}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(s) > |a|^{\frac{1}{\Re(\alpha)}}, \Re(s) > 0.$

Corollary 2.9.1: *If $c > 0, \mu > 0, \nu > 0$, then for the solution of*

$$N(t) - N_0 t^{\mu-1} E_{\nu,\mu}[-c^{\nu} t^{\nu}] = -c^{\nu}\ {}_0D_t^{-\nu}\, N(t), \tag{2.9.20}$$

there holds the relation

$$N(t) = \frac{N_0}{\nu} t^{\mu-1} \left[E_{\nu,\mu-1}(-c^{\nu} t^{\nu}) + (1 + \nu - \mu) E_{\nu,\mu}(-c^{\nu} t^{\nu}) \right]. \tag{2.9.21}$$

Theorem 2.9.4. *The Cauchy problem for the integro-differential equation*

$${}_0D_x^{\mu}\, f(x) + \lambda\ {}_0D_x^{-\nu}\, f(x) = h(x),\ (\lambda, \mu, \nu \in \mathbb{C}) \tag{2.9.22}$$

with the initial condition

$$D_x^{\mu-k-1}\, f(0) = a_k, k = 0, 1, \cdots, [\mu], \tag{2.9.23}$$

where $\Re(\nu) > 0, \Re(\mu) > 0$ and $h(x)$ is any integrable function on the finite interval [0,b] has the unique solution, given by

$$\begin{aligned} f(x) = &\int_0^x (x-t)^{\mu-1} E_{\mu+\nu,\mu}[-\lambda (x-t)^{\mu+\nu}] h(t) \mathrm{d}t \\ &+ \sum_{k=0}^{n-1} a_k x^{\mu-k-1} E_{\mu+\nu,\mu-k}(-\lambda x^{\mu+\nu}) \end{aligned} \tag{2.9.24}$$

Proof 2.9.2: Exercise.

Theorem 2.9.5. *The solution of the equation*

$$ {}_0D_t^{\frac{1}{2}} f(t) + bf(t) = 0; \left[{}_0D_t^{-\frac{1}{2}} f(t) \right]_{t=0} = C, \tag{2.9.25}$$

where C is a constant is given by

$$ f(t) = C\, t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}\left(-bt^{\frac{1}{2}}\right), \tag{2.9.26}$$

where $E_{\frac{1}{2},\frac{1}{2}}(\cdot)$ is the Mittag-Leffler function.

Proof 2.9.3: Exercise see (2.4.47).

Remark 2.9.4: Theorem 2.9.5 gives the generalized form of the equation solved by Oldham and Spanier (1974).

Exercises 2.9.

2.9.1. Prove that if $c > 0, \nu > 0, \mu > 0$, then the solution of

$$ N(t) - N_0 t^{\mu-1} E^2_{\nu,\mu}(c^\nu t^\nu) = -c^\nu {}_0D_t^{-\nu} N(t), \tag{2.9.27}$$

is given by

$$\begin{aligned} N(t) = N_0 t^{\mu-1} E^3_{\nu,\mu}(-c^\nu t^\nu) = \frac{N_0 t^{\mu-1}}{2\nu^2} \Big[& E_{\nu,\mu-2}(-c^\nu t^\nu) \\ & + \{3(\nu+1) - 2\mu\} E_{\nu,\mu-1}(-c^\nu t^\nu) \\ & + \left\{2\nu^2 + \mu^2 + 3\nu - 2\mu - 3\nu\mu + 1\right\} E_{\nu,\mu}(-c^\nu t^\nu) \Big], \end{aligned} \tag{2.9.28}$$

where $\Re(\nu) > 0,\ \Re(\mu) > 2$.

2.9.2. Prove that if $\nu > 0, c > 0, d > 0, \mu > 0, c \neq d$, then for the solution of the equation

$$ N(t) - N_0 t^{\mu-1} E_{\nu,\mu}(-d^\nu t^\nu) = -c^\nu {}_0D_t^{-\nu} N(t), \tag{2.9.29}$$

there holds the formula.

$$ N(t) = N_0 \frac{t^{\mu-\nu-1}}{c^\nu - d^\nu} \left[E_{\nu,\mu-\nu}(-d^\nu t^\nu) - E_{\nu,\mu-\nu}(-c^\nu t^\nu) \right]. \tag{2.9.30}$$

2.9.3. Prove that if $c > 0, \nu > 0, \mu > 0$, then for the solution of the equation

$$N(t) - N_0 t^{\mu-1} E_{\nu,\mu}(-c^\nu t^\nu) = -c^\nu {}_0D_t^{-\nu}\, N(t), \tag{2.9.31}$$

the following result holds:

$$N(t) = \frac{N_0}{\nu} t^{\mu-1} \left[E_{\nu,\mu-1}(-c^\nu t^\nu) + (1+\nu-\mu) E_{\nu,\mu}(-c^\nu t^\nu) \right]. \tag{2.9.32}$$

2.9.4. Solve the equation

$${}_0D_t^Q\, f(t) + {}_0D_t^q\, f(t) = g(t),$$

where $q - Q$ is not an integer or a half integer and the initial condition is

$$\left[{}_0D_t^{q-1}\, f(t) + {}_0D_t^{Q-1}\, f(t) \right]_{t=0} = C \tag{2.9.33}$$

where C is a constant.
2.9.5. Solve the equation

$${}_0D_t^\alpha\, x(t) - \lambda x(t) = h(t), \;\; (t > 0), \tag{2.9.34}$$

subject to the initial conditions

$$\left[{}_0D_t^{\alpha-k}\, h(t) \right]_{t=0} = b_k, \quad (k = 1, \cdots, n) \tag{2.9.35}$$

where $n - 1 < \alpha < n$.

2.9.6. Prove Theorem 2.9.4.

2.9.7. Prove Theorem 2.9.5.

2.9.2 Fractional diffusion

Theorem 2.9.6. *The solution of the following initial value problem for the fractional diffusion equation in one dimension*

$${}_0D_t^\alpha\, U(x,t) = \lambda^2 \frac{\partial^2 U(x,t)}{\partial x^2}, \;\; (t > 0, -\infty < x < \infty) \tag{2.9.36}$$

with initial conditions :

$$\lim_{x\to\pm\infty} U(x,t) = 0; \left[{}_0D_t^{\alpha-1}\, U(x,t) \right]_{t=0} = \phi(x) \tag{2.9.37}$$

is given by

$$U(x,t)=\int_{-\infty}^{\infty}G(x-\zeta,t)\phi(\zeta)\mathrm{d}\zeta, \tag{2.9.38}$$

where

$$G(x,t)=\frac{1}{\pi}\int_{0}^{\infty}t^{\alpha-1}E_{\alpha,\alpha}(-k^2\lambda^2t^{\alpha})\cos kx\,\mathrm{d}k. \tag{2.9.39}$$

Solution 2.9.1: Let $0<\alpha<1$. Using the boundary conditions (2.9.37), the Fourier transform of (2.9.36) with respect to variable x gives

$$_0D_x^{\alpha}\,\bar{U}(k,t)+\lambda^2k^2\bar{U}(k,t)=0 \tag{2.9.40}$$

$$\left[{}_0D_t^{\alpha-1}\,\bar{U}(k,t)\right]_{t=0}=\bar{\phi}(k), \tag{2.9.41}$$

where k is a Fourier transform parameter and ‘ $-$ ’ indicates Fourier transform. Applying the Laplace transform to (2.9.40) and using (2.9.41), it gives

$$\tilde{\bar{U}}(k,s)=\frac{\bar{\phi}(k)}{s^{\alpha}+k^2\lambda^2}, \tag{2.9.42}$$

where ‘ $\sim$ ’ indicates Laplace transform. The inverse Laplace transform of (2.9.42) yields

$$\bar{U}(k,t)=t^{\alpha-1}\bar{\phi}(k)E_{\alpha,\alpha}(-\lambda^2k^2t^2), \tag{2.9.43}$$

and then the solution is obtained by taking inverse Fourier transform. By taking inverse Fourier transform of (2.9.43) and using the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-ikx}f(k)\mathrm{d}k=\frac{1}{\pi}\int_{0}^{\infty}f(k)\cos(kx)\mathrm{d}k \tag{2.9.44}$$

we have

$$U(x,t)=\int_{-\infty}^{\infty}G(x-\zeta,t)\phi(\zeta)\mathrm{d}\zeta, \tag{2.9.45}$$

where

$$G(x,t)=\frac{1}{\pi}\int_{0}^{\infty}t^{\alpha-1}E_{\alpha,\alpha}(-k^2\lambda^2t^{\alpha})\cos(kx)\mathrm{d}k \tag{2.9.46}$$

with $\Re(\alpha)>0, k>0$.

Exercises 2.9.

2.9.8. Evaluate the integral in (2.9.46).

2.9.9. Find the solution of the Fick’s diffusion equation

$$\frac{\partial}{\partial t}P(x,t)=\lambda\frac{\partial^2}{\partial x^2}P(x,t),$$

with the initial condition $P(x,t=0)=\delta(x)$, where $\delta(x)$ is the Dirac delta function.

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http://www.springer.com/978-0-387-75893-0

Special Functions for Applied Scientists
Mathai, A.M.; Haubold, H.J.
2008, XXVI, 470 p. 10 illus., Hardcover
ISBN: 978-0-387-75893-0