# Chapter 2
# Black-Scholes

> *Through my parents and relatives I became interested in economics and, in particular, finance. My mother loved business and wanted me to work with her brother in his book publishing and promotion business. During my teenage years, I was always treasurer of my various clubs; I traded extensively among my friends; I gambled to understand probabilities and risks; and worked with my uncles to understand their business activities. I invested in the stock market while in high school and university. I was fascinated with the determinants of the level of stock prices. I spent long hours reading reports and books to glean the secrets of successful investing, but, alas, to no avail.*
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> —Myron Scholes, Nobel autobiography, 1997

At this point let us interrupt the narrative in order to explain the Black-Scholes-Merton revolution. Otherwise you will not be able to make much sense out of what follows. Derman gives a good qualitative discussion of this, but, as the great nineteenth-century Scottish physicist James Clerk Maxwell used to say, "I didn't see the 'go' of it. " If you put "Black-Scholes" into Google you will find something like 1.46 million entries. Most of them are technical, proposing solutions to the equations or trying to generalize or derive them. Some of these sites have clearly been posted by ex-physicists, who note, for example, that the Black-Scholes equation can be morphed into the equation that describes the flow of heat. (We will explain this, too.) There are offers to tutor you for a considerable fee. While wandering through this jungle I came across the perfect site for my purposes. It is called "Black-Scholes the Easy Way. " You can find it at http://homepage.mac.com/j.norstad. The person who put it up, John Norstad, is a computer scientist who was learning this stuff as a hobby. His posted notes represent his own learning process and are very clear. We will use his examples in case someone wants to consult the site for more details. But what is the basic problem?

At this time, as mentioned, financial institutions were doing a substantial business in the sale of derivatives. A typical example is a stock option. This is a contract between two parties that allows the buyer of the option to purchase a particular stock at a future time from the seller of the option at a contractually specified price called the "strike price, " which is often but not always the price of the stock when you buy the option. Until that future time you do not own the stock. You own an option *to buy* the stock at the fixed price. If the stock has gone up by the time you buy it in the future you are, using the term of choice, "in the money. " If the stock goes down, you don't buy it but are out the cost of the option— "out of the money. " The question is, what should the price of the option be when you buy it? This is what the Black-Scholes equation purports to allow you to compute.

To see what is involved we will, following Norstad, consider a "toy" model. This illustrates many of the general features of the problem without the mathematical complexity.

In the toy model there is a stock whose current value is \$100, which, in this example, will be the strike price. What makes the model a toy is that at the time the option is to be exercised there are only two possible prices; \$120 and \$80. In the real world there will be a continuum of prices, which is what Black-Scholes must deal with. The kind of option we are considering here is called a "European call option. " It can only be exercised at one definite time in the future. An "American call option" can be exercised at any time. We will further assume that the probability of the stock's rising to \$120 is three quarters, while the probability of its falling to \$80 is one quarter. What then should you be willing to pay for the option? At first sight this seems like a simple question to answer. With these probabilities the expected outcome is $\frac{3}{4} \times \$20 + \frac{1}{4} \times \$0$, which is \$15. Thus one might assume that the option should be worth \$15 to you and that you can then, with a high probability, expect to earn \$5 if you buy the option. This would be true if we were not able to engage in financial engineering—an activity that goes under the rubric of "arbitrage. " With arbitrage one can gain a certain profit with no risk at all. The cost of this arbitrage is what determines the cost of the option. This changes everything and explains why the financial institutions were hiring quants by the carload. Here is how the arbitrage works in this case.

Let's assume there is a friendly bank that is willing to lend money interest free. This is a simplification in the analysis that can readily be corrected. If you want to see how including interest modifies the results in the toy model, you can look at Norstad's website. Only a little high school algebra is required. We do not lose any matters of principle if we make this assumption. Likewise we assume that we can buy fractional shares of this stock from a friendly broker commission-free. To use another physics/economics term, we make the problem "frictionless. " In physics a friction force generates heat without doing any useful work. Here, the friction generates money loss without helping us with the bottom line.

Now suppose you have made the expectation calculation given above and are willing to give \$15 to buy the option. We will now see how we can, using arbitrage, always pocket \$5 no matter what the outcome is. To achieve this we take the \$15 from you and put \$5 in our pocket. You will never see the fiver again. We then borrow \$40 from the friendly bank-"leverage. " We take your \$10 and the borrowed \$40 and buy a half share of the stock. This is called the "hedge. " In this case it has cost us \$10 to replicate the option. This will turn out to be its true value.

Now we have the two cases. If the final price is a \$120, you will exercise your option to buy the stock at a \$100. We are obliged to deliver the stock to you at that price. What we will do is to sell our half-share for $60, repay the bank its $40, and add the remaining $20 to the $100 you gave us to buy the share so we can give it to you. We have, of course, pocketed the $5. If, on the other hand, the final price is $80 you will not exercise your option. If you really like the stock you can simply buy it outright for $80. We will sell our half-share for $40, which we will then return to the bank, still pocketing the $5. This means that if you have given us $15 for the option you have overpaid by $5. If you think about it, you will see that the $10 price is a kind of tipping point. If we can sell the option for more than $10 we will make money, and if someone wants to sell us the option for less than $10 we will buy it and again make money. The $10 is a kind of equilibrium price at which it is not profitable to either buy or sell. Black and Scholes approached the problem of option evaluation using arbitrage as an equilibrium problem—you need to find the price at which there is equilibrium between buying and selling. Merton had a different approach, and this is the one that is now more generally used.

To understand it, note that in finding the correct option price in the presence of the possibility of arbitrage, the probabilities of three-quarters and one quarter played no role. These probabilities only entered when, in the absence of arbitrage, we used them to compute the expected gain. We never had to use them when we found the cost of the hedge—which is the correct option price. This is important because in real life there is little likelihood that we would ever be given these probabilities in any reliable way. In fact, in an important sense, the presence of a buyer of the option is irrelevant. Suppose we just construct a portfolio that consists of $10, and a $40 loan from the bank, which we then invest in half a share of the stock which is selling for $100 a share. This is called a "synthetic option. " You can easily persuade yourself that if we sell this stock when its value is either $120 or $80, the amount that we gain or lose is the same as what the buyer of the option gains or loses in the preceding example if he or she pays $10 for the option. The essence of Merton's approach is to show that one can, in general, construct synthetic options that have the same outcome as the real options. The cost of the synthetic option is the same as the cost of the real option and, by what is called the Law of One Price, this *is* the cost of the option. This is what these brokerage firms do—they construct synthetic options.

In discussing the equivalence of these two methods of pricing options, Derman was reminded of what happened in quantum electrodynamics in the late 1940s. There were two approaches. Julian Schwinger started from first principles and by carrying out a series of horrendous calculations—which, as Oppenheimer said, only Schwinger could have done—produced numbers for physical quantities that could be compared to experiment. Richard Feynman, on the other hand, arrived at these numbers by using pictorial methods and intuitive arguments. A reasonably competent graduate student could learn them in a few days. I remember the disquiet I felt when I first studied Feynman's papers. Why did these tricks work? Indeed, did they always work? The matter was laid to rest when Freeman Dyson in a mathematical tour de force showed that Schwinger and Feynman had found two equivalent representations of the same theory. You could use Feynman's Mozartean calculus knowing that Schwinger's Bach-like logic made it legitimate.

What Black, Scholes, and Merton had to confront was the fact that in the real world we do not have a situation in which there are just two future prices but in fact a continuum. This gets one into the question of how you can predict the future of a stock price. To deal with this Black and Scholes adopted a model that supposed that stock prices follow what is known as a "random walk." We will explain this, but first remarkably the same model was used by a French mathematician named Louis Bachelier to derive the price of what is known as a "barrier option, " an option that is extinguished if the stock price crosses a certain barrier. (One feature of Bachelier's model that was not realistic was his use of negative as well as positive stock prices.)[3] This work was contained in his Ph.D. thesis, *Théorie de la Spéculation*, which he presented at the Sorbonne in 1900! Although Bachelier wrote both books and papers on this kind of probability theory, his work was not much appreciated, even in France. A great deal of it was rediscovered and made more rigorous by people such as Norbert Weiner. It is now embedded in what is known as the "stochastic calculus."

In 1905, Einstein, who had not heard of Bachelier, used these ideas to analyze what is known as "Brownian motion. " In 1827, the Scottish botanist Robert Brown made a very odd discovery. He noticed that if pollen grains, which could only be seen though a microscope, were suspended in water they executed a curious dancing motion. Brown made the natural assumption that these pollens were alive, but then he tried other microscopic particles made of things that were clearly not alive and found the same effect. By the end of the century the correct explanation had been guessed at; namely, that these very small particles were being bombarded by the still smaller—indeed invisible—water molecules and that the "dance" was in response to these collisions. In 1905, Einstein showed that this assumption had precise and measurable consequences. The same work was done independently at about the same time by the Polish physicist Marian Smoluchowski. The basic idea can be illustrated by what is known as the "drunkard's walk. "

A drunkard begins his walk at, say, a lamp post. At each step he can go, say, 2 feet, but in a totally random direction. The question then is how far on the average the drunkard will go, as the crow flies, away from the lamp post after a given number of these steps, say $N$. When first confronted with this question many people are tempted to say the drunk won't get anywhere, since he will simply retrace each step. But a moment's reflection convinces one that this is essentially impossible. After each step the drunkard's new direction is totally random.[4] The path will appear jagged, but the distance from the lamp post continues to increase. Indeed, the fundamental result of this analysis is that the average distance increases as the square root of the number of steps N.[5] This square root feature shows up in the

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[3] Bachelier's random-walk model predicted that the spread in stock prices would increase as the square root of the time. But there was no limit in his model as to how low a stock could go. It could become negative, which means the company would be paying you to buy the stock. This flaw in Bachelier was first noted by the economist Paul Samuelson.

[4] A special case is a one-dimensional random walk. After each step the drunkard is equally likely to go forward or backward. Nonetheless he will move inexorably away from the lamp post. This example makes it clear why it takes longer to reach a certain average distance than it would if one just walked there.

[5] To be precise we are talking about the root mean square distance. This is the square root of the average of the square of the distance.

Brownian motion. Here the "drunkard" is a grain pollen that is being driven hither and yon by its collisions with the invisible water molecules. The distance that the pollen goes is, as Einstein showed, proportional to the square root of the time during which you observe it. In his papers Einstein put in a few numbers for various substances at room temperature. He predicted that typically such a suspended object might go some thousandths of a centimeter in a second. Experiments were done and, from the results, one could deduce the constants that went into his formula—constants that included the number of molecules in a cubic centimeter. These experiments convinced most skeptics that atoms really existed. They did not convince the physicist-philosopher Ernst Mach, who liked asking atomists, "Have you seen one? "

Assuming that stock prices follow a continuous random walk, Black and Scholes could make a prediction for the future distribution of the price of a stock. In Derman's book he has a typical graph that plots this. Prices on the graph deviate from the initial price, making a kind of wedge with the pointed end at the initial price. The wedge continually widens as time goes on, so that the price becomes more uncertain. Knowing the probable future prices of the stock, Black and Scholes were able to derive an equation for the value of the option at any given time. It is a differential equation—one involving the sort of derivatives that I mistakenly thought that Scholes was first learning about when I met him. Since the value of the stock is constantly changing—unlike our toy example, where it changed only once—the hedge must also be constantly adjusted. In the general case, the price of the option is the total price of this constantly adjusted hedge. It is this hedge price that is used by the people who sell these options.

The transport of heat is also such a diffusion process in which the faster moving "hot" molecules transfer their momentum to the cooler particles until the two reach an equilibrium. Therefore it is not entirely surprising that one can mathematically transform the Black-Scholes equation into the equation that describes heat diffusion. This equation has been studied for well over a century so that there are a lot of mathematical tools available. Indeed, in their paper, Black and Scholes transform their equation into the heat equation, which they then solve.

So why would I, as a physicist, find the Black-Scholes model quite odd? All physical theories are models. For example, quantum electrodynamics, which is the most precise theory ever created, operates in a model universe that contains only electrons and quanta of light-photons. The rest of the universe, with its neutrons, protons and mesons, quarks, and the like, are ignored. But the object of this model, like all the other models in physics, is to predict the future. If the model is correct, then the numbers and curves one calculates with it should be confirmed by experiment. But the Black-Scholes model is quite different. It uses a model of the future to describe the *present*.[6] In the absence of this, or some equivalent model, present

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[6] This is reflected in the conditions under which the equation is solved. What is specified is the *future* value of the option at the time it is exercised. This is either zero or the strike price less the amount you have borrowed depending on how the stock's price has evolved. When you solve a physics equation you generally use data from the present to find your solution, which then allows you to predict the future.

stock options have no reasonable assigned value. Presumably the test of the model is that if one uses it as a guide to buy these options and, as a result, goes broke, one would be inclined to re-examine the assumptions. But we have digressed. Let us continue the discussion of quants in the next essay.

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