2
Structural Stability

2.1 Structural stability, Darcy model

Structural stability is the study of stability of the model itself. The classical definition of stability involves continuous dependence of the solution on changes in the initial data, cf. section 1.1.2. However, it is increasingly being realised that continuous dependence on changes in the coefficients, in the model, in boundary data, or even in the partial differential equations themselves, is very important. This aspect of continuous dependence, or stability, is what we refer to as structural stability. (Hirsch and Smale, 1974) were prominent in introducing the ideas of structural stability. In chapter 16 of their book (Hirsch and Smale, 1974) ask, ... "What effect does changing the differential equation itself have on the solution? ... This is the problem of structural stability." The book of (Hirsch and Smale, 1974) gives an authoritative account of structural stability in an ordinary differential equation context. Structural stability is also emphasized in the books by (Bellomo and Preziosi, 1995), (Doering and Gibbon, 1995), (Drazin and Reid, 1981), and (Flavin and Rionero, 1995), although the topic of porous media is not specifically addressed in the context of structural stability in these works. In this chapter we focus on examples of structural stability in the context of the equations of porous media. It is extremely important, because if a small change in the equations, or a coefficient in an equation, causes a major change in the solution it may well say something about how accurate the model is as a vehicle to describe flow in porous media.

B. Straughan, *Stability and Wave Motion in Porous Media*,
DOI: 10.1007/978-0-387-76543-3_2,

Early articles dealing with structural stability questions in porous flows are those of (Ames and Payne, 1994), (Franchi and Straughan, 1993a; Franchi and Straughan, 1996), and (Payne and Straughan, 1996) investigates in some detail the continuous dependence of the solution on changes in the initial-time geometry. We do not describe the work of (Payne and Straughan, 1999a), but this paper establishes continuous dependence on the coefficients of Forchheimer and of Brinkman, and also investigates how the solution to the Brinkman equations converges to that of the Darcy equations as the Brinkman coefficient tends to zero. We focus on examples which illustrate various different effects, and the sections on continuous dependence on the Dufour, Krishnamurti, and Vadasz coefficients are new.

We commence with a result of (Payne and Straughan, 1998b) which establishes continuous dependence on the cooling coefficient for Newton's law of cooling in a Darcy porous material. (Franchi and Straughan, 1996) proved a similar result for a Brinkman porous material, but their method is inadequate to deal with the less dissipative Darcy system. (Payne and Straughan, 1998b) were able to prove *a priori* continuous dependence in three space dimensional problems without having to restrict the size of the time interval or the size of the initial data. In contrast, when one considers the Navier-Stokes equations, such a restriction is evidently necessary, (Ames and Payne, 1997)

We do not consider in this chapter structural stability questions for the porous medium equation model based on a distribution of voids in an elastic body, see section 7.2. However, this topic is investigated in (Chirita et al., 2006). (Chirita and Ciarletta, 2008) develop the structural stability analysis further by including temperature effects in the model.

A class of nonlinear models which possess properties not dissimilar to those of the model in section 2.1.1 are those studied by (Payne and Straughan, 1999c). These writers investigated continuous dependence on the spatial geometry for a Stokes' flow system when the nonlinearity in the temperature equation was regarded as important. This class of Stokes' flow is called a nonlinear Stokes' problem by (Duka et al., 2007). The paper by (Duka et al., 2007) derives interesting bounds for a solution to a nonlinear Stokes' system for thermal convection in a horizontal annulus.

2.1.1 Newton's law of cooling

The Darcy equations for non-isothermal flow in a porous medium are as in chapter 1, sections 1.2, 1.6.1, namely,

$$v_i = -\frac{\partial p}{\partial x_i} + g_i T, \tag{2.1}$$

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{2.2}$$

$$\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = \Delta T, \tag{2.3}$$

where v_i, T, p and g_i are the velocity, temperature, pressure and the gravity vector. The density ρ in equation (1.15) has been assumed linear in T with the body force $f_i = g_i$, the constant part of the body force being absorbed in the pressure term. In this section equations (2.1) – (2.3) hold on a bounded spatial domain Ω with boundary Γ, for positive time. On the boundary Γ we suppose v_i and T satisfy the conditions

$$v_i n_i = 0, \qquad \text{and} \qquad \frac{\partial T}{\partial n} = -\kappa\big(T - T_a(\mathbf{x},t)\big)\,, \tag{2.4}$$

where $\kappa(> 0)$ is the cooling coefficient, $T_a(\mathbf{x}, t)$ is the temperature outside of the porous body at the boundary, n_i is the outward unit normal to Γ, and $\partial/\partial n$ denotes the outward normal derivative. The initial condition is

$$T(\mathbf{x}, t) = T_0(\mathbf{x}), \tag{2.5}$$

for T_0 given.

To investigate continuous dependence on κ we let (v_i, T, p) be a solution to (2.1) – (2.5) with a cooling coefficient κ_2, and we let (u_i, S, q) be another soultion to (2.1) – (2.5) for the same T_a and initial data, but for a different cooling coefficient κ_1. We wish to derive an *a priori* estimate for a measure of $T - S$ and $v_i - u_i$ in terms of the difference $\kappa_2 - \kappa_1$. To this end let w_i, θ, π and κ be the difference variables

$$w_i = v_i - u_i, \quad \theta = T - S, \quad \pi = p - q, \quad \kappa = \kappa_2 - \kappa_1, \tag{2.6}$$

and then from (2.1) – (2.5) we see that (w_i, θ, π) satisfies the partial differential equations

$$w_i = -\frac{\partial \pi}{\partial x_i} + g_i\theta, \tag{2.7}$$

$$\frac{\partial w_i}{\partial x_i} = 0, \tag{2.8}$$

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} + w_i \frac{\partial T}{\partial x_i} = \Delta\theta. \tag{2.9}$$

The boundary and initial conditions are

$$n_i w_i = 0, \quad \frac{\partial \theta}{\partial n} = -\kappa_1\theta - \kappa(T - T_a), \qquad \text{on } \Gamma \times [0, \mathcal{T}], \tag{2.10}$$

$$\theta(\mathbf{x}, 0) = 0, \quad x \in \Omega, \tag{2.11}$$

where $\mathcal{T} < \infty$ is an arbitrary (but preassigned) time.

We assume, without loss of generality, that

$$|\mathbf{g}| \leq 1. \tag{2.12}$$

To establish continuous dependence we multiply (2.7) by w_i and integrate over Ω, and using the Cauchy - Schwarz inequality, one finds

$$\|\mathbf{w}\|^2 = g_i(\theta, w_i) \le \|\theta\|\|\mathbf{w}\|,$$

and so

$$\|\mathbf{w}\| \le \|\theta\|. \tag{2.13}$$

Next, multiply (2.9) by θ and integrate over Ω to derive

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = \int_\Omega w_i T\theta_{,i}dx - \|\nabla\theta\|^2 - \kappa_1\oint_\Gamma \theta^2 dA - \kappa\oint_\Gamma \theta(T - T_a)dA. \tag{2.14}$$

Next, employ the arithmetic-geometric mean inequality to see that

$$-\kappa\oint_\Gamma \theta(T_1 - T_a)dA \le \frac{\kappa}{2\alpha}\oint_\Gamma \theta^2 dA + \frac{\kappa\alpha}{2}\oint_\Gamma (T_1 - T_a)^2 dA, \tag{2.15}$$

for $\alpha > 0$ arbitrary. We select $\alpha = \kappa/2\kappa_1$, and then use (2.15) in (2.14). In this manner we derive

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 \le \int_\Omega w_i T\theta_{,i}dx - \|\nabla\theta\|^2 + \frac{\kappa^2}{4\kappa_1}\oint_\Gamma (T - T_a)^2 dA. \tag{2.16}$$

2.1.2 A priori bound for T

To proceed we require an *a priori* bound for $|T|$. We establish such a bound for a function T satisfying (2.2) and (2.3), following (Payne and Straughan, 1998b). We simply use T, v_i and κ, rather than T, v_i and κ_2. Multiply (2.3) by T^{p-1} for $p > 1$ (we assume the temperature is scaled to be non-negative). Thus,

$$\frac{d}{dt}\int_\Omega T^p dx = -p(p-1)\int_\Omega T^{p-2}|\nabla T|^2 dx - \kappa p\oint_\Gamma T^{p-1}(T - T_a)dA.$$

With the aid of Young's inequality we have

$$\kappa p T^{p-1}T_a \le \kappa p T^p + \kappa T_a^p\Big(\frac{p-1}{p}\Big)^{p-1}.$$

Employing this in the previous inequality allows us to show that

$$\frac{d}{dt}\int_\Omega T^p dx \le -p(p-1)\int_\Omega T^{p-2}|\nabla T|^2 dx + \kappa\Big(\frac{p-1}{p}\Big)^{p-1}\oint_\Gamma T_a^p dA.$$

This inequality is integrated after discarding the first term on the right, to deduce

$$\left[\int_\Omega T^p dx\right]^{1/p} \le \left[\int_\Omega T_0^p dx + \kappa\Big(\frac{p-1}{p}\Big)^{p-1}\int_0^t ds\oint_\Gamma T_a^p dA\right]^{1/p}. \tag{2.17}$$

Now, let $p \to \infty$ in (2.17) to see that

$$\sup_\Omega |T| \le T_m, \tag{2.18}$$

where the constant T_m is given by

$$T_m = \max \left\{ \sup_{\Omega} |T_0| \, , \sup_{\Gamma \times [0, \mathcal{T}]} |T_a| \right\} .$$

Equipped with the estimate (2.18) for T (maximum principle), we bound the first term on the right of (2.16),

$$\begin{aligned} \int_{\Omega} w_i T \theta_{,i} dx &\leq T_m \|\mathbf{w}\| \, \|\nabla \theta\|, \\ &\leq T_m \|\theta\| \, \|\nabla \theta\|, \end{aligned}$$

where (2.13) has been used, and then after further use of the arithmetic-geometric mean inequality,

$$\int_{\Omega} w_i T \theta_{,i} dx \leq \frac{T_m^2}{4} \|\theta\|^2 + \|\nabla \theta\|^2. \tag{2.19}$$

Upon utilizing (2.19) in (2.16) we find

$$\frac{d}{dt} \|\theta\|^2 \leq \frac{T_m^2}{2} \|\theta\|^2 + A\kappa^2, \tag{2.20}$$

where the function A is defined by

$$A(t) = \frac{1}{2\kappa_1} \oint_{\Gamma} (T_m - T_a)^2 dA.$$

In deriving (2.20), bound (2.18) has been extended to the boundary by continuity. Inequality (2.20) may be integrated by an integrating factor method to see that

$$\|\theta(t)\|^2 \leq R(t)\kappa^2, \tag{2.21}$$

where R is defined as

$$R(t) = \int_0^t A(s) \exp \left[\frac{1}{2} T_m^2 (t - s) \right] ds.$$

The bound (2.21) is our continuous dependence estimate for θ. Now, from (2.13) we also find

$$\|\mathbf{w}(t)\|^2 \leq R(t)\kappa^2, \tag{2.22}$$

which establishes continuous dependence of v_i on the cooling coefficient. Continuous dependence on the cooling coefficient κ is established, since $R(t)$ is *a priori* because it only depends on data and the geometry of Ω.

2.2 Structural stability, Forchheimer model

In this section we describe work of (Franchi and Straughan, 2003) who consider the isothermal Forchheimer equations with quadratic and cubic terms, namely

$$\frac{\partial u_i}{\partial t} = -au_i - b|\mathbf{u}|u_i - c|\mathbf{u}|^2 u_i - p_{,i}, \qquad \frac{\partial u_i}{\partial x_i} = 0, \tag{2.23}$$

where u_i is the average fluid velocity in the porous medium, a is the Darcy coefficient (viscosity divided by permeability), b and c are the Forchheimer coefficients, and p is the pressure.

2.2.1 Continuous dependence on b

We commence with a study of continuous dependence on the coefficient b. Therefore let u_i and v_i solve the following boundary initial value problems for different Forchheimer coefficients b_1 and b_2, but for the same second Forchheimer coefficient c,

$$\begin{aligned} &\frac{\partial u_i}{\partial t} = -au_i - b_1|\mathbf{u}|u_i - c|\mathbf{u}|^2 u_i - p_{,i}, \quad \frac{\partial u_i}{\partial x_i} = 0, \quad \text{in } \Omega \times \{t > 0\}, \\ &n_i u_i = 0, \qquad \text{on } \Gamma \times \{t > 0\}, \\ &u_i(x, 0) = f_i(x), \qquad x \in \Omega, \end{aligned} \tag{2.24}$$

$$\begin{aligned} &\frac{\partial v_i}{\partial t} = -av_i - b_2|\mathbf{v}|v_i - c|\mathbf{v}|^2 v_i - q_{,i}, \quad \frac{\partial v_i}{\partial x_i} = 0, \quad \text{in } \Omega \times \{t > 0\}, \\ &n_i v_i = 0, \qquad \text{on } \Gamma \times \{t > 0\}, \\ &v_i(x, 0) = f_i(x), \qquad x \in \Omega. \end{aligned} \tag{2.25}$$

In these problems Ω is a bounded domain in $\mathbb{R}^3$ with boundary Γ, n_i is the unit outward normal to Γ, and f_i is the given initial data.

The difference variables w_i, π, b are defined by

$$w_i = u_i - v_i, \quad \pi = p - q, \quad b = b_1 - b_2. \tag{2.26}$$

By subtraction we see that w_i satisfies the boundary initial value problem

$$\begin{aligned} &\frac{\partial w_i}{\partial t} = -aw_i - \big(b_1|\mathbf{u}|u_i - b_2|\mathbf{v}|v_i\big) - c\big(|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i\big) - \pi_{,i}, \\ &\frac{\partial w_i}{\partial x_i} = 0, \quad \text{in } \Omega \times \{t > 0\}, \\ &n_i w_i = 0, \qquad \text{on } \Gamma \times \{t > 0\}, \\ &w_i(x, 0) = 0, \qquad x \in \Omega. \end{aligned} \tag{2.27}$$

The first step involves rearranging the b_1 and b_2 terms as

$$b_1|\mathbf{u}|u_i - b_2|\mathbf{v}|v_i = \frac{b}{2}(|\mathbf{u}|u_i + |\mathbf{v}|v_i) + \tilde{b}(|\mathbf{u}|u_i - |\mathbf{v}|v_i), \tag{2.28}$$

where $\tilde{b} = (b_1 + b_2)/2$, and observing (Payne and Straughan, 1999a) show that

$$(|\mathbf{u}|u_i - |\mathbf{v}|v_i)w_i = \frac{1}{2}(|\mathbf{u}| + |\mathbf{v}|)w_i w_i + \frac{1}{2}(|\mathbf{u}| - |\mathbf{v}|)^2(|\mathbf{u}| + |\mathbf{v}|). \tag{2.29}$$

Next, multiply $(2.27)_1$ by w_i and integrate over Ω, to find with the aid of (2.28) and (2.29),

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\mathbf{w}\|^2 =& -a\|\mathbf{w}\|^2 - \frac{b}{2}\int_\Omega (|\mathbf{u}|u_iw_i + |\mathbf{v}|v_iw_i)dx \\ & - \frac{\tilde{b}}{2}\int_\Omega (|\mathbf{u}| + |\mathbf{v}|)w_iw_i dx \\ & - \frac{\tilde{b}}{2}\int_\Omega (|\mathbf{u}| - |\mathbf{v}|)^2(|\mathbf{u}| + |\mathbf{v}|)dx \\ & - c\int_\Omega (|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i)w_i dx. \end{aligned} \tag{2.30}$$

(Franchi and Straughan, 2003) show that

$$\begin{aligned}(|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i)w_i =& \frac{1}{2}|\mathbf{u}|^2(u_i - v_i + v_i)w_i - \frac{1}{2}|\mathbf{v}|^2v_iw_i \\ & + \frac{1}{2}|\mathbf{u}|^2u_iw_i + \frac{1}{2}|\mathbf{v}|^2w_i(u_i - v_i - u_i) \\ =& \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{v}|^2)w_iw_i + \frac{1}{2}(u_i + v_i)w_i(|\mathbf{u}|^2 - |\mathbf{v}|^2) \\ =& \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{v}|^2)w_iw_i + \frac{1}{2}(|\mathbf{u}|^2 - |\mathbf{v}|^2)^2. \end{aligned} \tag{2.31}$$

This expression is employed in (2.30) to obtain

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\mathbf{w}\|^2 \le& -a\|\mathbf{w}\|^2 - \frac{b}{2}\int_\Omega (|\mathbf{u}|u_iw_i + |\mathbf{v}|v_iw_i)dx \\ & - \frac{\tilde{b}}{2}\int_\Omega (|\mathbf{u}| + |\mathbf{v}|)w_iw_i dx - \frac{c}{2}\int_\Omega (|\mathbf{u}|^2 + |\mathbf{v}|^2)w_iw_i dx. \end{aligned} \tag{2.32}$$

We suppose $c > 0$. The case where $c = 0$ is covered in (Franchi and Straughan, 2003). We use the Cauchy-Schwarz and arithmetic-geometric mean inequalities to see that

$$\begin{aligned}-\frac{b}{2}\left|\int_\Omega (|\mathbf{u}|u_iw_i + |\mathbf{v}|v_iw_i)\right|dx \le& \frac{b^2}{8c}\int_\Omega (u_iu_i + v_iv_i)dx \\ & + \frac{c}{2}\int_\Omega (|\mathbf{u}|^2 + |\mathbf{v}|^2)w_iw_i dx. \end{aligned} \tag{2.33}$$

Now use this inequality in (2.32) and discard the $\tilde{b}$ term to derive

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{w}\|^2 \le -a\|\mathbf{w}\|^2 + \frac{b^2}{8c}\int_\Omega (u_iu_i + v_iv_i)dx. \tag{2.34}$$

From equations (2.24) and (2.25) one shows

$$\|\mathbf{u}\|^2 \le \|\mathbf{f}\|^2 \exp(-2at) \quad \text{and} \quad \|\mathbf{v}\|^2 \le \|\mathbf{f}\|^2 \exp(-2at). \tag{2.35}$$

These bounds are now used in (2.34) to arrive at

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{w}\|^2 + a\|\mathbf{w}\|^2 \le \frac{b^2}{4c}\exp(-2at)\|\mathbf{f}\|^2.$$

With the aid of an integrating factor and integration one sees that

$$\|\mathbf{w}(t)\|^2 \le b^2\,\frac{\|\mathbf{f}\|^2}{2c}\,t\exp(-2at). \tag{2.36}$$

Inequality (2.36) establishes continuous dependence on b when $c > 0$.

2.2.2 Continuous dependence on c

In this subsection we establish continuous dependence on the coefficient c. Let now (u_i, p) and (v_i, q) solve the boundary initial value problems (2.24) and (2.25) for the same b but with c_1 and c_2 different.

Define in this case

$$w_i = u_i - v_i, \quad \pi = p - q, \quad c = c_1 - c_2.$$

Then (w_i, π) satisfies the boundary initial value problem

$$\begin{aligned}
&\frac{\partial w_i}{\partial t} = -aw_i - b(|\mathbf{u}|u_i - |\mathbf{v}|v_i) - c_1|\mathbf{u}|^2u_i + c_2|\mathbf{v}|^2v_i - \pi_{,i},\\
&\frac{\partial w_i}{\partial x_i} = 0, \quad \text{in } \Omega\times\{t>0\},\\
&n_iw_i = 0, \qquad \text{on } \Gamma\times\{t>0\},\\
&w_i(x,0) = 0, \qquad x\in\Omega.
\end{aligned} \tag{2.37}$$

(Franchi and Straughan, 2003) use the rearrangement

$$c_1|\mathbf{u}|^2u_i - c_2|\mathbf{v}|^2v_i = \frac{c}{2}(|\mathbf{u}|^2u_i + |\mathbf{v}|^2v_i) + \tilde{c}(|\mathbf{u}|^2u_i - |\mathbf{v}|^2v_i), \tag{2.38}$$

where $\tilde{c} = (c_1 + c_2)/2$.

Now multiply $(2.37)_1$ by w_i and integrate over Ω. We employ the rearrangements (2.29), (2.38) and (2.31) and then show

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 = {}& -a\|\mathbf{w}\|^2 - \frac{b}{2}\int_\Omega(|\mathbf{u}| + |\mathbf{v}|)w_iw_i\,dx\\
&- \frac{b}{2}\int_\Omega(|\mathbf{u}| - |\mathbf{v}|)^2(|\mathbf{u}| + |\mathbf{v}|)dx - \frac{c}{2}\int_\Omega(|\mathbf{u}|^2u_iw_i + |\mathbf{v}|^2v_iw_i)dx\\
&- \frac{\tilde{c}}{2}\int_\Omega(|\mathbf{u}|^2 + |\mathbf{v}|^2)w_iw_i\,dx - \frac{\tilde{c}}{2}\int_\Omega(|\mathbf{u}|^2 - |\mathbf{v}|^2)^2dx.
\end{aligned}$$

The two b terms and the $\tilde{c}$ term involving $(|\mathbf{u}|^2 - |\mathbf{v}|^2)^2$ are discarded to derive

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + a\|\mathbf{w}\|^2 \le &- \frac{c}{2}\int_\Omega (|\mathbf{u}|^2 u_i w_i + |\mathbf{v}|^2 v_i w_i)dx \\ &- \frac{\tilde{c}}{2}\int_\Omega (|\mathbf{u}|^2 + |\mathbf{v}|^2) w_i w_i dx. \end{aligned} \tag{2.39}$$

Next, the Cauchy-Schwarz and arithmetic-geometric mean inequalities are employed to see that

$$\begin{aligned} \frac{c}{2}\int_\Omega (|\mathbf{u}|^2 u_i w_i + |\mathbf{v}|^2 v_i w_i)dx \le &\frac{c^2}{8\tilde{c}}\int_\Omega (|\mathbf{u}|^4 + |\mathbf{v}|^4)dx \\ &+ \frac{\tilde{c}}{2}\int_\Omega (|\mathbf{u}|^2 + |\mathbf{v}|^2) w_i w_i dx. \end{aligned} \tag{2.40}$$

Upon use of (2.40) in (2.39) we see after integration,

$$\|\mathbf{w}\|^2 + 2a\int_0^t \|\mathbf{w}\|^2 ds \le \frac{c^2}{4\tilde{c}}\int_0^t \int_\Omega (|\mathbf{u}|^4 + |\mathbf{v}|^4)dx\, ds. \tag{2.41}$$

The right hand side of (2.41) is estimated by multiplying (2.24) by u_i, (2.25) by v_i, and integrating over $\Omega \times (0,t)$ to show that

$$\int_0^t \int_\Omega (|\mathbf{u}|^4 + |\mathbf{v}|^4)dx\, ds \le \left(\frac{c_1 + c_2}{2c_1 c_2}\right)\|\mathbf{f}\|^2.$$

Upon using this inequality in (2.41) one finds

$$\|\mathbf{w}\|^2 + 2a\int_0^t \|\mathbf{w}\|^2 ds \le \frac{\|\mathbf{f}\|^2}{4c_1 c_2}\, c^2. \tag{2.42}$$

Inequality (2.42) establishes continuous dependence on c. A further bound for w_i may be obtained from (2.42) with the use of an integrating factor, this is

$$\int_0^t \|\mathbf{w}\|^2 ds \le \frac{\|\mathbf{f}\|^2}{8ac_1 c_2}(1 - e^{-2at})\, c^2.$$

2.2.3 Energy bounds

Interesting upper and lower bounds for $\|\mathbf{u}\|$ are obtained by (Franchi and Straughan, 2003) who follow the method of (Payne and Straughan, 1999a). To derive these estimates we suppose u_i is a solution to (2.24) with b_1 replaced by b, so u_i satisfies the boundary initial value problem

$$\begin{aligned} &\frac{\partial u_i}{\partial t} = -au_i - b|\mathbf{u}|u_i - c|\mathbf{u}|^2 u_i - \pi_{,i}, \quad \frac{\partial u_i}{\partial x_i} = 0, \quad \text{in } \Omega \times \{t > 0\}, \\ &n_i u_i = 0, \qquad \text{on } \Gamma \times \{t > 0\}, \\ &u_i(x,0) = f(x), \qquad x \in \Omega. \end{aligned} \tag{2.43}$$

Multiply (2.43) by u_i and integrate over Ω to find

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 = -a\|\mathbf{u}\|^2 - b\int_\Omega |\mathbf{u}|^3 dx - c\int_\Omega |\mathbf{u}|^4 dx. \tag{2.44}$$

We first derive a lower bound for $\|\mathbf{u}\|$, and set $\Phi(t) = \|\mathbf{u}(t)\|^2$. From (2.44)

$$\frac{d\Phi}{dt} = -2a\|\mathbf{u}\|^2 - 2b\int_\Omega |\mathbf{u}|^3 dx - 2c\int_\Omega |\mathbf{u}|^4 dx. \tag{2.45}$$

Define the function χ by

$$\chi(t) = -2a\|\mathbf{u}\|^2 - \frac{4}{3}b\int_\Omega |\mathbf{u}|^3 dx - c\int_\Omega |\mathbf{u}|^4 dx, \tag{2.46}$$

and observe that $\chi \leq 0$. From (2.46) and (2.45) $d\Phi/dt \leq \chi$, and then

$$\Phi\frac{d\chi}{dt} = 4\|\mathbf{u}\|^2(u_{i,t}, u_{i,t}) \geq \Big(\frac{d\Phi}{dt}\Big)^2 \geq \Big(-\frac{d\Phi}{dt}\Big)(-\chi). \tag{2.47}$$

Hence, $(d\chi/dt)/\chi \leq (d\Phi/dt)/\Phi$, which after integration and rearrangement yields

$$-\chi(t) \leq \Phi(t)\,\frac{\{-\chi(0)\}}{\|\mathbf{f}\|^2}\,. \tag{2.48}$$

We may now show $2\chi \leq d\Phi/dt \leq \chi$, and so with the aid of (2.48) we deduce

$$\frac{1}{2}\frac{d\Phi}{dt} \geq \chi(t) \geq \Phi(t)\,\frac{\chi(0)}{\|\mathbf{f}\|^2}\,.$$

After integration we obtain

$$\|\mathbf{u}(t)\|^2 \geq \|\mathbf{f}\|^2 \exp\left[\frac{-2\{-\chi(0)\}t}{\|\mathbf{f}\|^2}\right]. \tag{2.49}$$

From inequality (2.49) one sees that u_i cannot vanish identically in a finite time.

We may use the Cauchy-Schwarz inequality to show

$$-\int_\Omega |\mathbf{u}|^4 dx \leq -\frac{\|\mathbf{u}\|^4}{m}\,,$$

where $m = m(\Omega)$ is the measure of Ω. If this inequality is utilized in (2.45) one may show

$$\frac{d}{dt}\|\mathbf{u}\|^2 + 2a\|\mathbf{u}\|^2 + \frac{2c}{m}\|\mathbf{u}\|^4 \leq 0.$$

Now since u_i cannot vanish in a finite time we divide by $\|\mathbf{u}\|^4$ and solve the resulting inequality for $\|\mathbf{u}\|^{-2}$. This leads to the upper bound

$$\|\mathbf{u}(t)\|^2 \leq \frac{\|\mathbf{f}\|^2}{e^{2at} + c\|\mathbf{f}\|^2(e^{2at}-1)/am}\,. \tag{2.50}$$

If we combine (2.50) and (2.49) we find the estimates for $\|\mathbf{u}(t)\|$,

$$\frac{\|\mathbf{f}\|^2}{\exp\left[\left(4a + 8b\int_\Omega |\mathbf{f}|^3 dx/3\|\mathbf{f}\|^2 + 2c\int_\Omega |\mathbf{f}|^4 dx/\|\mathbf{f}\|^2\right)t\right]}$$
$$\leq \|\mathbf{u}(t)\|^2$$
$$\leq \frac{\|\mathbf{f}\|^2}{e^{2at} + c\|\mathbf{f}\|^2(e^{2at}-1)/am}\,. \tag{2.51}$$

2.2.4 Brinkman-Forchheimer model

(Celebi et al., 2006) study structural stability for a version of the Brinkman-Forchheimer equations, namely, they study the boundary - initial value problem,

$$\begin{aligned}
&\frac{\partial u_i}{\partial t} = \gamma\Delta u_i - au_i - b|\mathbf{u}|^\alpha u_i - \pi_{,i},\\
&\frac{\partial u_i}{\partial x_i} = 0, \ \text{ in } \Omega\times\{t>0\},\\
&u_i = 0, \qquad \text{on } \Gamma\times\{t>0\},\\
&u_i(x,0) = f(x), \qquad x\in\Omega,
\end{aligned} \tag{2.52}$$

where γ is a Brinkman coefficient and $\alpha \in [1,2]$ is a constant.

(Celebi et al., 2006) establish existence and uniqueness of a solution to (2.52), and show that there is a constant D, depending on f and the coefficients in (2.52), such that

$$\sup_{0\leq t\leq T}\|\nabla\mathbf{u}(t)\| \leq D, \qquad \int_0^T \left\|\frac{\partial \mathbf{u}}{\partial t}(t)\right\|^2 dt \leq D,$$

for any $T>0$. They also show that the solution u_i depends continuously on the Forchheimer coefficient b, and on the Brinkman coefficient γ. This is an interesting paper and the proofs employ the Sobolev inequality in a non-trivial manner.

2.3 Forchheimer model, non-zero boundary conditions

(Payne et al., 1999) studied continuous dependence on changes in the viscosity for a Forchheimer and a Brinkman model. The motivation of (Payne et al., 1999) was to analyse mathematically a model for the process of salinization, whereby salts are transported upwards in soils in dry regions. A model for this was developed by (Gilman and Bear, 1996) and this model has a strong viscosity - concentration dependence. The work of (Gilman and

Bear, 1996) involves a nonlinear set of equations, and similar models are studied in (Wooding et al., 1997a; Wooding et al., 1997b) and in (van Duijn et al., 2002). (Payne et al., 1999) analyses the manner in which the velocity and concentration depend on changes in the viscosity. The reason for the need to study continuous dependence on the viscosity is that (Gilman and Bear, 1996) point out that the viscosity dependence on concentration is 1.5 to 3 times greater than that of pure water. By comparison the variation in density is only of order 0.15 to 0.30 times greater. Certainly such a strong variation indicates that convective motion of salt in a porous medium ought to take into account viscosity dependence on salt concentration.

The model based on Darcy's law studied by (Payne et al., 1999) is now presented. If we let u_i, c and p denote the fields of velocity, concentration and pressure, the Forchheimer equations for flow in a porous medium studied by (Payne et al., 1999) are

$$\begin{aligned} &bu_i|\mathbf{u}| + (1+\gamma_1 c)u_i = -p_{,i} + g_i c, \\ &\frac{\partial u_i}{\partial x_i} = 0, \\ &\frac{\partial c}{\partial t} + u_i \frac{\partial c}{\partial x_i} = \Delta c, \end{aligned} \tag{2.53}$$

where γ_1 and b are positive constants, $g_i(\mathbf{x})$ is a gravity field which we again assume satisfies

$$|\mathbf{g}| \le 1. \tag{2.54}$$

Equations (2.53) hold on the region $\Omega \times (0, \mathcal{T})$ for Ω a bounded domain in $\mathbb{R}^3$ and for some time $\mathcal{T}$, $0 < \mathcal{T} < \infty$. The viscosity variation is represented by the term $1+\gamma_1 c$, i.e. we allow a linear variation in c so that the viscosity μ has form $\mu = \mu_1(1+\gamma_1 c)$. The $g_i c$ term represents a linear variation in c for the density, i.e. a Boussinesq like approximation. Since c is a concentration it is reasonable to assume that it is non-negative, although if we knew *a priori* that u_i is bounded then $c \ge 0$ would follow from the maximum principle.

On the boundary Γ (of Ω) the conditions imposed are

$$u_i n_i = f(\mathbf{x}, t), \quad c = h(\mathbf{x}, t), \qquad \mathbf{x} \in \Gamma, \tag{2.55}$$

for known functions f and h. The initial condition is that concentration is prescribed at $t = 0$, i.e.

$$c(\mathbf{x}, 0) = c_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{2.56}$$

c_0 given.

We note in passing that existence and uniqueness questions of solutions to systems like that studied here may be answered by the methods of (Ly and Titi, 1999) or those of (Rodrigues, 1986; Rodrigues, 1992).

The work of (Payne et al., 1999) relies on establishing an upper bound for c. We now give very brief details of how this is achieved.

2.3.1 A maximum principle for c

To derive a maximum principle for c (Payne et al., 1999) use the method of (Payne and Straughan, 1998a).

They introduce a function H by

$$\begin{aligned} &\Delta H(\mathbf{x},t) = 0 \qquad \text{in} \quad \Omega \times (0,\mathcal{T}),\\ &H(\mathbf{x},t) = h^{2p-1}(\mathbf{x},t) \qquad \text{on} \quad \Gamma \times (0,\mathcal{T}). \end{aligned}$$

The analysis commences with the identity

$$\int_0^t ds \int_\Omega (H - c^{2p-1})\big\{c_{,t} + u_i c_{,i} - \Delta c\big\}\, dx = 0.$$

An integration by parts and rearrangement leads to

$$\begin{aligned} \int_\Omega c^{2p} dx + \frac{2(2p-1)}{p} \int_0^t ds \int_\Omega c^p_{,i} c^p_{,i} dx &= \int_\Omega c_0^{2p} dx \\ &+ 2p(H,c) - 2p(H_0,c_0) - 2p \int_0^t ds \int_\Omega H_{,s} c\, dx \\ &+ 2p \int_0^t ds \int_\Omega H u_i c_{,i} dx + 2p \int_0^t ds \oint_\Gamma \frac{\partial H}{\partial n}\, h\, dA \\ &- \int_0^t ds \oint_\Gamma f c^{2p} dA. \end{aligned} \tag{2.57}$$

The remainder of the proof of the maximum principle for c is from this point very technical. The purpose of this section is to describe continuous dependence on γ_1 and so we refer to (Payne et al., 1999) or (Payne and Straughan, 1998a) for full details. After many steps the proof arrives at an inequality of form

$$\|c\|_{2p} \le \left[\|c_0\|_{2p}^{2p} + \Big(\sum_{i=1}^5 r_i\Big) h_m^{2p} \right]^{1/2p}, \tag{2.58}$$

where $\|\cdot\|_{2p}$ is the norm on $L^{2p}(\Omega)$, r_i involve h or c_0, and $h_m = \max_{\Gamma\times[0,\mathcal{T}]} |h|$. Taking the limit $2p \to \infty$ leads to the *a priori* bound

$$\sup_{\Omega\times[0,\mathcal{T}]} |c| \le \max\Big\{ |c_0|_m\,,\ \sup_{[0,\mathcal{T}]} h_m \Big\} = c_m \tag{2.59}$$

where $|c_0|_m = \max_\Omega |c_0|$, and c_m is defined as indicated.

2.3.2 Continuous dependence on the viscosity

To investigate continuous dependence on the viscosity coefficient γ_1 in (2.53) suppose (u_i, c_1, p) and (v_i, c_2, q) are solutions to (2.53) – (2.56) for the same data functions f, h and c_0, but for different viscosity coefficients,

γ_1 and γ_2, respectively. The difference solution (w_i, ϕ, π) is introduced as

$$w_i = u_i - v_i, \quad \phi = c_1 - c_2, \quad \pi = p - q, \quad \gamma = \gamma_1 - \gamma_2. \tag{2.60}$$

By calculation (w_i, ϕ, π) is seen to satisfy the boundary-initial value problem

$$\begin{aligned} &b[u_i|\mathbf{u}| - v_i|\mathbf{v}|] + w_i + \gamma c_1 u_i + \gamma_2 \phi u_i + \gamma_2 c_2 w_i = -\pi_{,i} + g_i \phi, \\ &w_{i,i} = 0, \\ &\phi_{,t} + w_i c_{1,i} + v_i \phi_{,i} = \Delta\phi, \end{aligned} \tag{2.61}$$

in $\Omega \times (0, \mathcal{T})$, with the boundary and initial conditions

$$w_i = \phi = 0 \quad \text{on } \Gamma, \qquad \phi(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega. \tag{2.62}$$

It is convenient to also rearrange $(2.61)_1$ in the form

$$b[u_i|\mathbf{u}| - v_i|\mathbf{v}|] + w_i + \gamma_1 c_1 w_i + \gamma c_1 v_i + \gamma_2 \phi v_i = -\pi_{,i} + g_i \phi. \tag{2.63}$$

The proof starts by multiplying $(2.61)_1$ by w_i and integrating to find

$$\begin{aligned} b\int_\Omega (u_i|\mathbf{u}| - v_i|\mathbf{v}|) w_i dx &+ \int_\Omega (1 + \gamma_2 c_2) w_i w_i dx \\ &= g_i(\phi, w_i) - \gamma \int_\Omega c_1 u_i w_i dx - \gamma_2 \int_\Omega \phi u_i w_i dx. \end{aligned} \tag{2.64}$$

The right hand side is estimated using the maximum principle and Hölder's inequality. Identity (2.29) is used on the first term on the left and we drop a term to derive

$$\begin{aligned} &\frac{b}{2}\int_\Omega (|\mathbf{u}| + |\mathbf{v}|) w_i w_i dx + \int_\Omega (1 + \gamma_2 c_2) w_i w_i dx \\ &\le \|\phi\| \, \|\mathbf{w}\| + \gamma c_m \|\mathbf{u}\| \, \|\mathbf{w}\| + \gamma_2 \Big(\int_\Omega |\mathbf{u}| w_i w_i dx\Big)^{1/2} \Big(\int_\Omega |\mathbf{u}| \phi^2 dx\Big)^{1/2} \\ &\le \frac{1}{2\alpha}\|\phi\|^2 + \Big(\frac{\alpha}{2} + \frac{\beta}{2}\Big)\|\mathbf{w}\|^2 + \frac{\gamma^2 c_m^2}{2\beta}\|\mathbf{u}\|^2 \\ &\quad + b\int_\Omega |\mathbf{u}| w_i w_i dx + \frac{\gamma_2^2}{4b}\Big(\int_\Omega |\mathbf{u}|^3 dx\Big)^{1/3} \Big(\int_\Omega |\phi|^3 dx\Big)^{2/3}, \end{aligned} \tag{2.65}$$

where $\alpha, \beta > 0$ are constants to be chosen. We next use the Sobolev inequality

$$\int_\Omega \phi^4 dx \le k^2 \Big(\int_\Omega \phi^2 dx\Big)^{1/2} \Big(\int_\Omega |\nabla\phi|^2 dx\Big)^{3/2},$$

for $k > 0$ constant, together with the Cauchy-Schwarz inequality in (2.65) to obtain

$$\begin{aligned}&\frac{b}{2}\int_\Omega (|\mathbf{u}|+|\mathbf{v}|)w_iw_i dx + \int_\Omega (1+\gamma_2 c_2)w_iw_i dx\\ \leq&\frac{1}{2\alpha}\|\phi\|^2 + \frac{1}{2}(\alpha+\beta)\|\mathbf{w}\|^2 + \gamma^2\frac{c_m^2}{2\beta}\|\mathbf{u}\|^2\\ &+ b\int_\Omega |\mathbf{u}|w_iw_i dx + \frac{\gamma_2^2 k^{2/3}}{4b}\|\mathbf{u}\|_3\|\phi\|\,\|\nabla\phi\|. \end{aligned}\tag{2.66}$$

An analogous procedure starting from (2.64) leads to

$$\begin{aligned}&\frac{b}{2}\int_\Omega (|\mathbf{u}|+|\mathbf{v}|)w_iw_i dx + \int_\Omega (1+\gamma_1 c_1)w_iw_i dx\\ \leq&\frac{1}{2\alpha}\|\phi\|^2 + \frac{1}{2}(\alpha+\beta)\|\mathbf{w}\|^2 + \gamma^2\frac{c_m^2}{2\beta}\|\mathbf{v}\|^2\\ &+ b\int_\Omega |\mathbf{v}|w_iw_i dx + \frac{\gamma_2^2 k^{2/3}}{4b}\|\mathbf{v}\|_3\|\phi\|\,\|\nabla\phi\|. \end{aligned}\tag{2.67}$$

Upon addition of (2.66) and (2.67) we see that

$$\begin{aligned}\int_\Omega (2+&\gamma_1 c_1 + \gamma_2 c_2)w_iw_i dx \leq \frac{1}{\alpha}\|\phi\|^2 + (\alpha+\beta)\|\mathbf{w}\|^2\\ &+ \gamma^2\frac{c_m^2}{2\beta}(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2) + \frac{\gamma_2^2 k^{2/3}}{4b}(\|\mathbf{u}\|_3+\|\mathbf{v}\|_3)\|\phi\|\,\|\nabla\phi\|. \end{aligned}\tag{2.68}$$

A further use of the arithmetic-geometric mean inequality shows that, for a constant $\epsilon > 0$ to be chosen

$$\begin{aligned}\big[2-(\alpha+\beta)\big]\|\mathbf{w}\|^2 \leq &\left[\frac{1}{\alpha} + \frac{\gamma_2^2 k^{2/3}}{64 b^2\epsilon}(\|\mathbf{u}\|_3+\|\mathbf{v}\|_3)^2\right]\|\phi\|^2\\ &+ \gamma^2\frac{c_m^2}{2\beta}(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2) + \epsilon\|\nabla\phi\|^2. \end{aligned}\tag{2.69}$$

Directly from (2.53) we may deduce for a constant d involving data

$$\begin{aligned}\|\mathbf{u}\|^2 &\leq 4\|c_1\|^2 + d, \qquad \|\mathbf{u}\|_3 \leq \frac{1}{b^{1/3}}(4\|c_1\|^2+d)^{1/3},\\ \|\mathbf{v}\|^2 &\leq 4\|c_2\|^2 + d, \qquad \|\mathbf{v}\|_3 \leq \frac{1}{b^{1/3}}(4\|c_2\|^2+d)^{1/3}. \end{aligned}\tag{2.70}$$

Employing (2.70) in (2.69) yields for computable constants $\beta_1,\dots,\beta_3$, dependent only on data, choosing $\alpha=\beta=1/2$, an inequality of form

$$\|\mathbf{w}\|^2 \leq \beta_1\|\phi\|^2 + \beta_2 + \beta_3\|\nabla\phi\|^2. \tag{2.71}$$

To estimate the $\|\phi\|$ and $\|\nabla\phi\|$ terms we multiply $(2.61)_3$ by ϕ and integrate to find

$$\frac{1}{2}\|\phi\|^2 + \int_0^t \|\nabla\phi\|^2 ds = \int_0^t ds\int_\Omega w_i c_1 \phi_{,i} dx.$$

Bounding c_1 and using the Cauchy-Schwarz inequality yields

$$\|\phi\|^2 + \int_0^t \|\nabla\phi\|^2 ds \leq c_m^2 \int_0^t \|\mathbf{w}\|^2 ds. \tag{2.72}$$

Use of (2.72) in (2.71) shows that after integration

$$\int_0^t \|\mathbf{w}\|^2 ds \leq k_1 \int_0^t (t-s)\|\mathbf{w}\|^2 ds + k_2(t)\gamma^2, \tag{2.73}$$

where k_1, k_2 depend only on data. From this inequality we may establish the estimates

$$\int_0^t (t-s)\|\mathbf{w}\|^2 ds \leq k_3(t)\gamma^2, \qquad \text{and} \qquad \int_0^t \|\mathbf{w}\|^2 ds \leq k_4\gamma^2, \tag{2.74}$$

for k_3 and k_4 computable data bounds. These are continuous dependence estimates for w_i. An analogous estimate for ϕ follows from (2.72), of the form

$$\|\phi(t)\|^2 + \int_0^t \|\nabla\phi\|^2 ds \leq k_4 c_m^2 \gamma^2. \tag{2.75}$$

The inequalities (2.74) and (2.75) demonstrate continuous dependence on the viscosity coefficient γ_1. They are truly *a priori* since the coefficients of γ^2 depend only on boundary and initial data, and on the geometry of Ω.

2.4 Brinkman model, non-zero boundary conditions

In this section we review work of (Payne et al., 1999) which establishes continuous dependence on the viscosity coefficient γ_1 for the following Brinkman system,

$$\begin{aligned} &-\Delta u_i + (1+\gamma_1 c)u_i = -p_{,i} + g_i c, \\ &\frac{\partial u_i}{\partial x_i} = 0, \\ &\frac{\partial c}{\partial t} + u_i \frac{\partial c}{\partial x_i} = \Delta c, \end{aligned} \tag{2.76}$$

on $\Omega \times (0, \mathcal{T})$. The boundary and initial conditions in this case are

$$u_i = f_i(\mathbf{x}, t), \quad c = h(\mathbf{x}, t), \qquad \mathbf{x} \in \Gamma \times \{t > 0\}, \tag{2.77}$$

$$c(\mathbf{x}, 0) = c_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \tag{2.78}$$

(Payne et al., 1999) first compare the solution u_i to (2.76) with a solution a_i which solves the Stokes' flow problem in Ω, namely

$$\begin{aligned} &\Delta a_i = \rho_{,i}, \qquad \frac{\partial a_i}{\partial x_i} = 0 \qquad \text{in } \Omega, \\ &a_i = f_i \qquad \text{on } \Gamma \end{aligned} \tag{2.79}$$

where ρ is a pressure term. For a data term d_0 they go via a_i to show that

$$\|\mathbf{u}\|^2 \leq 5\|c\|^2 + d_0. \tag{2.80}$$

Continuous dependence on γ_1 proceeds via letting (u_i, c_1, p) and (v_i, c_2, q) solve (2.76) – (2.78) for the same data functions f_i, h and c_0, but for different viscosity coefficients γ_1 and γ_2, respectively. The difference variables (w_i, ϕ, π) and γ are defined as in equations (2.60). The boundary-initial value problem is

$$\begin{aligned} &-\Delta w_i + (1 + \gamma_2 c_2) w_i + \gamma c_1 u_i + \gamma_2 \phi u_i = -\pi_{,i} + g_i \phi, \\ &\frac{\partial w_i}{\partial x_i} = 0, \\ &\frac{\partial \phi}{\partial t} + w_i \frac{\partial c_1}{\partial x_i} + v_i \frac{\partial \phi}{\partial x_i} = \Delta \phi, \\ &w_i = \phi = 0 \text{ on } \Gamma, \quad \phi(\mathbf{x}, 0) = 0, \ \mathbf{x} \in \Omega. \end{aligned} \tag{2.81}$$

By using inequality estimates (Payne et al., 1999) show that one may compute data constants α_1 and α_2 such that

$$\|\mathbf{w}(t)\|^2 + \|\nabla \mathbf{w}(t)\|^2 \leq \alpha_1 \gamma^2, \qquad \|\phi\|^2 \leq \alpha_2 \gamma^2. \tag{2.82}$$

Inequalities (2.82) are *a priori* bounds which demonstrate continuous dependence of the solution on the viscosity coefficient γ_1. Note that the stronger dissipation in the Brinkman model allows continuous dependence to be proven in the $\|\mathbf{w}\|$ and $\|\nabla \mathbf{w}\|$ measures.

Further novel structural stability results for the Brinkman equations may be found in (Lin and Payne, 2007a; Lin and Payne, 2007b). Also, interesting structural stability results for the Brinkman-Forchheimer equations are established by (Celebi et al., 2006).

2.5 Convergence, non-zero boundary conditions

(Payne et al., 1999) also consider the question of convergence of the solution to an equivalent Darcy system to (2.53) to the case where $\gamma_1 = 0$. That is, (Payne et al., 1999) also consider the viscosity variation in (2.53), but they neglect the b (Forchheimer) term. Their goal is to investigate the behaviour as $\gamma_1 \to 0$. To state this result let (u_i, c_1, p) satisfy the following boundary-initial value problem, where γ_1 has been replaced by γ,

$$\begin{aligned} &(1+\gamma c_1)u_i = -p_{,i} + g_i c_1, \qquad \frac{\partial u_i}{\partial x_i} = 0, \\ &\frac{\partial c_1}{\partial t} + u_i \frac{\partial c_1}{\partial x_i} = \Delta c_1, \end{aligned} \tag{2.83}$$

in $\Omega \times (0, \mathcal{T})$, with

$$\begin{aligned} &u_i n_i = f, \qquad c_1 = h \qquad \text{on } \Gamma \times (0, \mathcal{T}), \\ &c_1(\mathbf{x}, 0) = c_0(\mathbf{x}), \ \ \mathbf{x} \in \Omega, \end{aligned} \tag{2.84}$$

i.e. the equivalent Darcy system to (2.53). We let (v_i, c_2, q) satisfy the analogous Darcy system when $\gamma = 0$, i.e.

$$\begin{aligned} &v_i = -q_{,i} + g_i c_2, \qquad \frac{\partial v_i}{\partial x_i} = 0, \\ &\frac{\partial c_2}{\partial t} + v_i \frac{\partial c_2}{\partial x_i} = \Delta c_2, \end{aligned} \tag{2.85}$$

in $\Omega \times (0, \mathcal{T})$, with

$$\begin{aligned} &v_i n_i = f, \qquad c_2 = h \qquad \text{on } \Gamma \times (0, \mathcal{T}), \\ &c_2(\mathbf{x}, 0) = c_0(\mathbf{x}), \ \ \mathbf{x} \in \Omega. \end{aligned} \tag{2.86}$$

By defining $w_i = u_i - v_i$ (Payne et al., 1999) show that

$$\int_0^t \|\mathbf{w}\|^2 ds \le \alpha_3 \gamma^2, \tag{2.87}$$

for a data term α_3.

Inequality (2.87) demonstrates convergence of u_i to v_i as $\gamma \to 0$ in the measure indicated. (Payne et al., 1999) also obtain convergence of w_i in $L^2(\Omega)$ norm and convergence of $\phi = c_1 - c_2$ in $L^2(\Omega)$ and $H^1(\Omega)$ norms.

2.6 Continuous dependence, Vadasz coefficient

(Vadasz, 1995; Vadasz, 1996; Vadasz, 1997; Vadasz, 1998a; Vadasz, 1998b) has made an extensive investigation of convection in a porous medium when the layer of saturated porous medium is rotating about a fixed axis. (Vadasz, 1998a) is a very interesting contribution. In this paper he employs linear instability and weakly nonlinear analysis to investigate the instability mechanisms governing convection in a rotating porous layer. Of particular interest is the fact that he discovers that if the inertia term is left in the momentum equation, then convection may commence by oscillatory convection. This is a striking result which implies that the inertia term plays a predominant role in determining the character of convection. In view of this we now examine how the solution to the equations for convection in a saturated porous material depends on the coefficient of the inertia term. The coefficient of the inertia term is denoted by $1/Va$, where Va is the Vadasz number. The usual Darcy law is recovered by letting $Va \to \infty$.

If we let u_i, T and p be the velocity, temperature and pressure, then the equations for non-isothermal flow in a saturated porous medium, taking inertia into account may be taken to be, cf. (Vadasz, 1998a), (Straughan, 2001b),

$$\frac{1}{Va}\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - u_i + g_i T, \tag{2.88}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.89}$$

$$\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \Delta T. \tag{2.90}$$

These equations hold on $\Omega \times (0, \mathcal{T})$, $\Omega \subset \mathbb{R}^3$ bounded, and g_i, $|\mathbf{g}| \leq 1$, is the gravity vector. The boundary conditions we consider are

$$u_i n_i = 0 \qquad \text{and} \qquad T = h(\mathbf{x}, t), \tag{2.91}$$

where $\mathbf{n}$ is the unit outward normal to Γ, the boundary of Ω. The initial conditions are that

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \qquad T(\mathbf{x}, 0) = T_0(\mathbf{x}). \tag{2.92}$$

It is convenient to employ $\alpha = 1/Va$ in (2.88), so this equation is rewritten as

$$\alpha \frac{\partial u_i}{\partial t} = -p_{,i} - u_i + g_i T. \tag{2.93}$$

In this section we study the continuous dependence of the solution on the coefficient α. To achieve this we need a maximum principle for T.

2.6.1 A maximum principle for T

A weak maximum principle for T is established by (Payne et al., 2001) (see also (Temam, 1988)) and we outline their proof. For a test function ϕ which vanishes on Γ, T satisfies the equation

$$\int_\Omega (T_{,t}\phi - u_i T \phi_{,i} + T_{,i}\phi_{,i})dx = 0. \tag{2.94}$$

Note that equation (2.94) may be obtained from (2.90) by multiplying that equation by ϕ and integrating over Ω. Define the number T_m by

$$T_m = \max\Big\{\sup_\Omega |T_0|, \sup_{\Omega\times[0,\mathcal{T}]} |h|\Big\}. \tag{2.95}$$

The function ϕ is chosen as

$$\phi = [T - T_m]^+ = \sup(T - T_m, 0).$$

Since $\phi_{,i} = T_{,i}$ when $T > T_m$, $\phi_{,i} = 0$ for $T \leq T_m$, (2.94) reduces to, after integration

$$\frac{1}{2}\int_0^t ds \int_\Omega |[T - T_m]^+|_{,s}^2 dx + \int_0^t ds \int_\Omega |\nabla[T - T_m]^+|^2 dx = 0.$$

(Note that $\int_\Omega u_i T\phi_{,i} dx = 0$.) From the last inequality we deduce that $[T - T_m]^+ = 0$, or $T \leq T_m$.

Next, select $\phi = [-T - T_m]^+$ in (2.94). A similar calculation to the above shows $T \geq -T_m$. Thus,

$$|T| \leq T_m, \quad (\mathbf{x}, t) \in \Omega \times [0, \mathcal{T}]. \tag{2.96}$$

2.6.2 Continuous dependence on α.

Let (u_i, T, p) be a solution to (2.89) – (2.93) with coefficient α_1 and let (v_i, S, q) be a solution to (2.89) – (2.93) for the same boundary and initial functions h, u_i^0, T_0 in (2.91), (2.92), but for a different Vadasz coefficient α_2. Define the difference variables w_i, θ and π, and the difference of the Vadasz coefficients α by

$$w_i = u_i - v_i, \quad \theta = T - S, \quad \pi = p - q, \quad \alpha = \alpha_1 - \alpha_2. \tag{2.97}$$

From equations (2.89) – (2.93) we find (w_i, θ, π) satisfy the boundary-initial value problem

$$\begin{aligned}
&\alpha_1 \frac{\partial w_i}{\partial t} + \alpha \frac{\partial v_i}{\partial t} = -\frac{\partial \pi}{\partial x_i} + g_i\theta - w_i, \\
&\frac{\partial w_i}{\partial x_i} = 0, \\
&\frac{\partial \theta}{\partial t} + w_i \frac{\partial T}{\partial x_i} + v_i \frac{\partial \theta}{\partial x_i} = \Delta\theta,
\end{aligned} \tag{2.98}$$

these equations holding on $\Omega \times (0, \mathcal{T})$, with

$$w_i n_i = 0, \quad \theta = 0, \qquad \text{on } \Gamma \times [0, \mathcal{T}], \tag{2.99}$$

$$w_i(\mathbf{x}, 0) = 0, \quad \theta(\mathbf{x}, 0) = 0, \qquad \mathbf{x} \in \Omega. \tag{2.100}$$

The analysis begins by multiplying $(2.98)_1$ by w_i and integrating over Ω to find, with the aid of $(2.98)_2$ and (2.99),

$$\begin{aligned}
\|\mathbf{w}\|^2 + \frac{\alpha_1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 =& (w_i, g_i\theta) - \alpha(v_{i,t}, w_i), \\
\leq& \frac{1}{2\zeta}\|\mathbf{w}\|^2 + \frac{\zeta}{2}\|\theta\|^2 + \frac{\alpha^2}{2\beta}(v_{i,t}, v_{i,t}) + \frac{\beta}{2}\|\mathbf{w}\|^2,
\end{aligned} \tag{2.101}$$

where the arithmetic-geometric mean inequality has been employed and $\beta, \zeta > 0$ are to be chosen. Next, multiply $(2.98)_3$ by θ and integrate over Ω

to obtain with the aid of $(2.98)_2$ and (2.99),

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\theta\|^2 =&(w_iT,\theta_{,i})-\|\nabla\theta\|^2,\\ \le&T_m\|\mathbf{w}\|\,\|\nabla\theta\|-\|\nabla\theta\|^2,\\ \le&\frac{T_m^2}{4}\|\mathbf{w}\|^2,\end{aligned}\tag{2.102}$$

where T has been bounded using (2.96), and the Cauchy-Schwarz and arithmetic-geometric mean inequalities have been employed. Integrate (2.102) over $(0,t)$ and use (2.100) to find

$$\|\theta(t)\|^2\le\frac{T_m^2}{2}\int_0^t\|\mathbf{w}\|^2ds.\tag{2.103}$$

We next integrate (2.101) over $(0,t)$ and pick $\beta/2+1/2\zeta=1$, e.g. $\beta=\zeta=1$. This yields

$$\alpha_1\|\mathbf{w}\|^2\le\int_0^t\|\theta\|^2ds+\alpha^2\int_0^t\|v_{i,s}\|^2ds.\tag{2.104}$$

To bound the first term on the right we integrate (2.103) to obtain

$$\int_0^t\|\theta\|^2ds\le\frac{T_m^2\mathcal{T}}{2}\int_0^t\|\mathbf{w}\|^2ds.$$

Thus, from (2.104) we may derive,

$$\alpha_1\|\mathbf{w}(t)\|^2\le\frac{T_m^2\mathcal{T}}{2}\int_0^t\|\mathbf{w}\|^2ds+\alpha^2\int_0^t\|v_{i,s}\|^2ds.\tag{2.105}$$

To estimate the $v_{i,t}$ term we multiply the equivalent v_i equation from (2.93) by $v_{i,t}$ and integrate over Ω then $(0,t)$ to find

$$\alpha_2\|v_{i,t}\|^2+\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2=(g_iS,v_{i,t}),$$

$$\begin{aligned}&\alpha_2\int_0^t\|v_{i,s}\|^2ds+\frac{1}{2}\|\mathbf{v}\|^2\\ &\le\frac{1}{2}\|\mathbf{v}_0\|^2+\frac{\alpha_2}{2}\int_0^t\|v_{i,s}\|^2ds+\frac{1}{2\alpha_2}\int_0^t\|S\|^2ds,\end{aligned}\tag{2.106}$$

where the arithmetic-geometric mean inequality has been employed. From (2.106) we see that

$$\alpha_2\int_0^t\|v_{i,s}\|^2ds\le\|\mathbf{v}_0\|^2+\frac{1}{\alpha_2}\int_0^t\|S\|^2ds\le\|\mathbf{v}_0\|^2+\frac{T_m^2mt}{\alpha_2}\,,\tag{2.107}$$

where (2.96) has been used.

Now, employ (2.107) in (2.105) and we may show that

$$\|\mathbf{w}\|^2-\frac{T_m^2\mathcal{T}}{2\alpha_1}\int_0^t\|\mathbf{w}\|^2ds\le\alpha^2\left[\frac{{\mathbf{v}_0}^2}{\alpha_2}+\frac{T_m^2m\mathcal{T}}{\alpha_2^2}\right].$$

This inequality is integrated by an integrating factor method and we derive

$$\int_0^t \|\mathbf{w}\|^2 ds \le K\alpha^2, \tag{2.108}$$

where

$$K = \frac{2\alpha_1\|\mathbf{v}_0\|^2}{\alpha_2 T_m^2 \mathcal{T}} + \frac{2\alpha_1 m}{\alpha_2^2}\,.$$

Inequality (2.108) establishes continuous dependence on α in the measure $\int_0^t \|\mathbf{w}\|^2 ds$. We may determine continuous dependence estimates in the measures $\|\theta(t)\|^2$ and $\|\mathbf{w}(\mathbf{t})\|^2$ from (2.103) and (2.105) and (2.107) and these are

$$\|\theta(t)\|^2 \le \frac{KT_m^2}{2}\,\alpha^2, \tag{2.109}$$

$$\|\mathbf{w}(t)\|^2 \le K_2\alpha^2, \tag{2.110}$$

where

$$K_2 = \frac{KT_m^2\mathcal{T}}{2\alpha_1} + \frac{\|\mathbf{v}_0\|^2}{\alpha_1\alpha_2} + \frac{mT_m^2\mathcal{T}}{\alpha_1\alpha_2^2}\,.$$

2.7 Continuous dependence, Krishnamurti coefficient

A very interesting model to describe a situation of penetrative convection in a viscous fluid was developed by (Krishnamurti, 1997). She also produced an experiment which captured the phenomenon and motivated her model. Linear instability and nonlinear energy stability bounds for a solution to the Krishnamurti model were derived by (Straughan, 2002b). The theoretical model of (Krishnamurti, 1997) relies on a pH indicator called thymol blue being dissolved in water. This gives rise to a double diffusive model with an equation for the temperature of the fluid coupled to an equation for the concentration of thymol blue. The penetrative effect is provided by the heat source depending on the thymol blue concentration. In this section we consider continuous dependence for a Krishnamurti model in a Darcy porous medium. Linear instability and nonlinear energy stability analyses for this model are given by (Hill, 2005a). In his work (Hill, 2005a) also develops stability analyses for a Brinkman theory, a theory where the heat source is nonlinear, and for a theory in which the density in the buoyancy force depends on temperature and concentration.

The partial differential equations governing the Krishnamurti model in a Darcy porous medium are

$$
\begin{aligned}
& v_i = -p_{,i} + g_i T, \\
& \frac{\partial v_i}{\partial x_i} = 0, \\
& \frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = \Delta T + \alpha C, \\
& \frac{\partial C}{\partial t} + v_i \frac{\partial C}{\partial x_i} = \Delta C.
\end{aligned}
\tag{2.111}
$$

In these equations v_i, p, T, C are the velocity, pressure, temperature and concentration, g_i is the gravity vector ($|\mathbf{g}| \leq 1$), and the Krishnamurti effect is introduced via the αC term in $(2.111)_3$. The Krishnamurti term arises because (Krishnamurti, 1997) takes the heat supply to depend (linearly) on concentration and this gives rise to equations $(2.111)_3$. We here assume (2.111) hold on $\Omega \times (0, \mathcal{T})$ with the boundary conditions

$$
v_i n_i = 0, \quad T = h(\mathbf{x}, t), \quad C = r(\mathbf{x}, t), \qquad \text{on } \Gamma \times (0, \mathcal{T}]. \tag{2.112}
$$

The initial conditions are

$$
T(\mathbf{x}, 0) = T_0(\mathbf{x}), \qquad C(\mathbf{x}, 0) = C_0(\mathbf{x}). \tag{2.113}
$$

The goal of this section is to show that the solution (v_i, p, T, C) depends continuously on changes in the Krishnamurti coefficient α. It is important in analysing a model to know that the addition of a term like the αC Krishnamurti term still retains the well posedness of the original system.

To establish continuous dependence we find it necessary to have an *a priori* bound for the temperature T. We may invoke the analysis of section 2.6 to see that C is bounded by its initial and boundary values, precisely,

$$
|C| \leq C_m = \max \left\{ \sup_{\Omega} |C_0|, \sup_{\Omega \times [0, \mathcal{T}]} |r| \right\}.
$$

The presence of the αC term in (2.111) prevents us from immediately deducing a maximum principle for T.

2.7.1 An a priori bound for T

We introduce the function H which solves

$$
\begin{aligned}
\Delta H &= 0 \qquad \text{in } \Omega, \\
H &= h^{2p-1} \qquad \text{on } \Gamma,
\end{aligned}
\tag{2.114}
$$

where $H = H(\mathbf{x}, t)$ since $h = h(\mathbf{x}, t)$, and p is an integer.

Because of equation $(2.111)_3$ we may write

$$
\int_0^t ds \int_{\Omega} (T^{2p-1} - H)(T_{,t} + v_i T_{,i} - \Delta T - \alpha C) dx = 0. \tag{2.115}
$$

After several integrations by parts we deduce from (2.115)

$$\begin{aligned}\int_\Omega T^{2p}dx+\frac{2(2p-1)}{p}\int_0^t ds\int_\Omega T^p_{,i}T^p_{,i}dx=&\int_\Omega T_0^{2p}dx+2p(H,T)\\&-2p(H_0,T_0)-2p\int_0^t(H_{,s},T)ds+2p\int_0^t ds\int_\Omega Hv_iT_{,i}dx\\&+2p\int_0^t ds\int_\Gamma\frac{\partial H}{\partial n}h\,dA-\alpha\int_0^t(H,C)ds\\&+\alpha\int_0^t ds\int_\Omega T^{2p-1}Cdx.\end{aligned}\tag{2.116}$$

The second - sixth terms on the right of (2.116) are handled as in (Payne and Straughan, 1998a) and the new terms are the seventh and eighth. The arithmetic-geometric mean inequality is used to see that

$$-\alpha\int_0^t(H,C)ds\le\frac{\alpha}{2}\int_0^t\|H\|^2ds+\frac{mC_m^2\mathcal{T}}{2}\alpha,\tag{2.117}$$

where m is the measure of Ω. To handle the last term in (2.116) we employ Young's inequality as follows,

$$\begin{aligned}\int_0^t ds\int_\Omega T^{2p-1}Cdx\le&\Big(\frac{2p-1}{p}\Big)\int_0^t ds\int_\Omega T^{2p}dx\\&+\frac{1}{2p}\int_0^t ds\int_\Omega C^{2p}dx.\end{aligned}\tag{2.118}$$

From the maximum principle, (Protter and Weinberger, 1967), we know $H\le h_m^{2p-1}$, $h_m=\max_\Gamma|h|$, and then since from $(2.111)_1$ we find $\|\mathbf{v}\|\le\|T\|$, we use (2.117) and (2.118) and follow the analysis of (Payne and Straughan, 1998a) to derive

$$\begin{aligned}\int_\Omega T^{2p}dx\le&\int_\Omega T_0^{2p}dx+2p(\|H\|\,\|T\|+\|H_0\|\,\|T_0\|)\\&+2p\sqrt{\int_0^t\|H_{,s}\|^2ds\int_0^t\|T\|^2ds}\\&+2ph_m^{2p-1}\sqrt{\int_0^t\|\nabla T\|^2ds\int_0^t\|T\|^2ds}\\&+2p\sqrt{\int_0^t ds\int_\Gamma h^2dA\int_0^t ds\int_\Gamma\Big(\frac{\partial H}{\partial n}\Big)^2dA}\\&+\frac{\alpha}{2}\int_0^t\|H\|^2ds+\frac{mC_m^2\mathcal{T}}{2}\alpha+\alpha\Big(\frac{2p-1}{p}\Big)\int_0^t ds\int_\Omega T^{2p}dx\\&+\frac{mC_m^{2p}\mathcal{T}}{2p}\alpha.\end{aligned}\tag{2.119}$$

The next step is to bound the $\|T\|$ and $\|\nabla T\|$ terms and their integrals. To this end we introduce the function G which satisfies

$$\Delta G = 0 \quad \text{in } \Omega, \qquad G = h(\mathbf{x}, t) \quad \text{on } \Gamma. \tag{2.120}$$

Now form the combination

$$\int_0^t ds \int_\Omega (T - G)(T_{,t} + v_i T_{,i} - \Delta T - \alpha C) dx = 0.$$

After integrations by parts we may derive from this

$$\begin{aligned}
\frac{1}{2}\|T\|^2 + \int_0^t \|\nabla T\|^2 ds = {} & \frac{1}{2}\|T_0\|^2 + \int_0^t ds \int_\Omega T_{,i} G_{,i} dx \\
& + (G, T) - (G_0, T_0) - \int_0^t ds \int_\Omega T G_{,s} dx \\
& + \int_0^t ds \int_\Omega G v_i T_{,i} dx + \alpha \int_0^t ds \int_\Omega C T dx \\
& - \alpha \int_0^t ds \int_\Omega C G dx.
\end{aligned}$$

We modify the argument of (Payne and Straughan, 1998a), p. 328, to find

$$\begin{aligned}
\int_0^t ds \int_\Omega G v_i T_{,i} dx \leq & G_m \int_0^t ds \int_\Omega |\mathbf{v}|\, |\nabla T|\, dx \\
\leq & \frac{h_m^2}{2} \int_0^t \|T\|^2 ds + \frac{1}{2} \int_0^t \|\nabla T\|^2 ds.
\end{aligned}$$

Thus, use of this and the arithmetic-geometric mean inequality in the above allows us to deduce

$$\begin{aligned}
\frac{1}{4}\|T\|^2 + \frac{1}{2}\int_0^t \|\nabla T\|^2 ds \leq {} & \|T_0\|^2 + \int_0^t ds \int_\Gamma h \frac{\partial G}{\partial n} dA \\
& + \|G\|^2 + \frac{1}{2}\|G_0\|^2 + \frac{1}{2}\int_0^t \|G_{,s}\|^2 ds + \frac{\alpha}{2}\int_0^t \|G\|^2 ds \\
& + \alpha \mathcal{T}_m C_m^2 + \left(\frac{1}{2} + \frac{\alpha}{2} + \frac{h_m^2}{4}\right) \int_0^t \|T\|^2 ds.
\end{aligned} \tag{2.121}$$

(Payne and Straughan, 1998a) show how to use a Rellich identity to bound the G terms in (2.121). The new term here is the $\alpha \int_0^t \|G\|^2 ds/2$ one but this also responds to the (Payne and Straughan, 1998a) treatment. We define the data term $D_1(t)$, for computable constants $h_1, \ldots, h_6$

dependent only on data, by

$$\begin{aligned}\frac{1}{4}D_1(t) = & h_1 \int_\Gamma h^2 dA + h_2 \int_\Gamma |\nabla_s h|^2 dA \\ & + h_3 \sqrt{\int_0^t ds \int_\Gamma h^2 dA \int_0^t d\eta \int_\Gamma |\nabla_s h|^2 dA} \\ & + h_4 \int_0^t ds \int_\Gamma h_{,s}^2 dA + h_5 \int_0^t ds \int_\Gamma h^2 dA \\ & + h_6 \int_0^t d\eta \int_\Omega |\nabla_s h_{,\eta}|^2 dA,\end{aligned}$$

where ∇_s is the tangential derivative on Γ. We may show $D_1/4$ is a data bound for all five terms on the right of (2.121) which involve G.

Thus, put $a = 2 + 2\alpha + h_m^2$, then (2.121) leads to

$$\frac{1}{4}\|T\|^2 + \frac{1}{2}\int_0^t \|\nabla T\|^2 ds \le \|T_0\|^2 + \frac{1}{4}D_1 + \alpha \mathcal{T} C_m^2 m + \frac{a}{4}\int_0^t \|T\|^2 ds.$$

This inequality may be integrated to find

$$\|T(t)\|^2 \le D_2(t) + a \int_0^t \|T\|^2 ds, \tag{2.122}$$

where

$$D_2(t) = 4D_1 + 4\|T_0\|^2 + 4m\alpha \mathcal{T} C_m^2.$$

Inequality (2.122) may be integrated to obtain the following three bounds,

$$\begin{aligned}\|T(t)\|^2 &\le D_2 + a \int_0^t e^{a(t-s)} D_2(s) ds = D_3(t), \\ \int_0^t \|T\|^2 ds &\le \int_0^t e^{a(t-s)} D_2(s) ds = D_4(t), \\ \int_0^t \|\nabla T\|^2 ds &\le \frac{1}{2}D_2 + \frac{a}{2}D_4 = D_5(t).\end{aligned}$$

We now return to (2.119). (Payne and Straughan, 1998a) show there are constants $\psi_1, c_1 > 0$ such that

$$\begin{aligned}\|H\|^2 &\le \psi_1 \int_\Gamma h^{4p-2} dA, \\ \|H_{,t}\|^2 &\le \psi_1 \int_\Gamma |(h^{2p-1})_{,t}|^2 dA, \\ \int_\Gamma \left(\frac{\partial H}{\partial n}\right)^2 dA &\le c_1 \int_\Gamma |\nabla_s h^{2p-1}|^2 dA.\end{aligned}$$

Using these inequalities and the bounds for $\|T\|$ and $\|\nabla T\|$ in (2.119) we may derive

$$\begin{aligned}\int_\Omega T^{2p}dx \le \int_\Omega T_0^{2p}dx + 2p\Big(D_{3\,max}^{1/2} + \|T_0\|\Big)\psi_1^{1/2}\sqrt{\int_\Gamma h^{4p-2}dA}\\
+ 2pD_4^{1/2}\sqrt{\int_0^t \psi_1 d\eta \int_\Gamma h_{,\eta}^2 h^{4p-4}dA}\\
+ 2ph_m^{2p-1}\sqrt{\int_0^t D_3(s)ds\int_0^t D_5(s)ds}\\
+ 2pc_1^{1/2}\sqrt{\int_0^t ds\int_\Gamma h^2dA\int_0^t d\eta\int_\Gamma |\nabla_s h^{2p-1}|^2dA}\\
+ m\alpha\mathcal{T}\Big(\frac{C_m^{2p}}{2p} + \frac{C_m^2}{2}\Big) + \frac{\alpha}{2}\psi_1\int_0^t ds\int_\Gamma h^{4p-2}dA\\
+ \alpha\Big(\frac{2p-1}{p}\Big)\int_0^t ds\int_\Omega T^{2p}dx.\end{aligned}\tag{2.123}$$

The first seven terms on the right of (2.123) are data and we denote these by $F(h)$. With $Q = \int_0^t ds \int_\Omega T^{2p}dx$, (2.123) is

$$Q' - \mu Q \le F,$$

where $\mu = \alpha(2p-1)/p$. This inequality integrates to yield

$$\int_\Omega T^{2p}dx \le \mu\int_0^t F(s)e^{\mu(t-s)}ds + F.$$

We raise both sides of this inequality to the power $1/2p$ to see that

$$\Big(\int_\Omega T^{2p}dx\Big)^{1/2p} \le \Big[F + \mu\int_0^t F(s)e^{\mu(t-s)}ds\Big]^{1/2p}.\tag{2.124}$$

Let $p\to\infty$ and since the right hand side of (2.124) is composed of $\int_\Omega T_0^{2p}dx$, h_m^{2p}, C_m^{2p} raised to the power $1/2p$ we arrive at

$$\sup_{\Omega\times[0,\mathcal{T}]}|T| \le \max\Big\{|T_0|_m\,, \sup_{[0,\mathcal{T}]} h_m, C_m\Big\} = T_B.\tag{2.125}$$

This is the *a priori* bound we sought to achieve.

2.7.2 Continuous dependence

We now let (u_i, T, C_1, p) be a solution to (2.111) – (2.113) for Krishnamurti coefficient α_1 and we let (v_i, S, C_2, q) be another solution for a different Krishnamurti coefficient α_2, but for the same data functions h, r, T_0 and

C_0. Thus (u_i, T, C_1, p) and (v_i, S, C_2, q) satisfy the boundary-initial value problems,

$$\begin{aligned} &u_i = -p_{,i} + g_i T, \\ &u_{i,i} = 0, \\ &T_{,t} + u_i T_{,i} = \Delta T + \alpha_1 C_1, \\ &C_{1,t} + u_i C_{1,i} = \Delta C_1, \end{aligned} \tag{2.126}$$

in $\Omega \times (0, \mathcal{T})$,

$$u_i n_i = 0, \quad T = h, \quad C_1 = r \qquad \text{on } \Gamma \times (0, \mathcal{T}], \tag{2.127}$$

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}), \quad C_1(\mathbf{x}, 0) = C_0(\mathbf{x}), \tag{2.128}$$

and

$$\begin{aligned} &v_i = -q_{,i} + g_i S, \\ &v_{i,i} = 0, \\ &S_{,t} + v_i S_{,i} = \Delta S + \alpha_2 C_2, \\ &C_{2,t} + v_i C_{2,i} = \Delta C_2, \end{aligned} \tag{2.129}$$

in $\Omega \times (0, \mathcal{T})$,

$$v_i n_i = 0, \quad S = h, \quad C_2 = r \qquad \text{on } \Gamma \times (0, \mathcal{T}], \tag{2.130}$$

$$S(\mathbf{x}, 0) = S_0(\mathbf{x}), \quad C_2(\mathbf{x}, 0) = C_0(\mathbf{x}). \tag{2.131}$$

The difference variables w_i, θ, ϕ, π and α are defined by

$$w_i = u_i - v_i,\ \theta = T - S,\ \phi = C_1 - C_2,\ \pi = p - q,\ \alpha = \alpha_1 - \alpha_2. \tag{2.132}$$

By direct calculation we see that (w_i, θ, ϕ, π) satisfies the boundary-initial value problem

$$\begin{aligned} &w_i = -\pi_{,i} + g_i \theta, \\ &w_{i,i} = 0, \\ &\theta_{,t} + w_i T_{,i} + v_i \theta_{,i} = \Delta\theta + \alpha_1 \phi + \alpha C_2, \\ &\phi_{,t} + w_i C_{1,i} + v_i \phi_{,i} = \Delta\phi, \end{aligned} \tag{2.133}$$

in $\Omega \times (0, \mathcal{T})$,

$$w_i n_i = 0, \quad \theta = 0, \quad \phi = 0, \qquad \text{on } \Gamma \times (0, \mathcal{T}], \tag{2.134}$$

$$\theta(\mathbf{x}, 0) = 0, \quad \phi(\mathbf{x}, 0) = 0. \tag{2.135}$$

First, observe that multiplying $(2.133)_1$ by w_i, integrating over Ω and using the Cauchy-Schwarz inequality we find

$$\|\mathbf{w}\| \leq \|\theta\|. \tag{2.136}$$

By multiplying $(2.133)_3$ by θ and integrating over Ω,

$$\frac{d}{dt}\|\theta\|^2 = 2\int_\Omega w_i T\theta_{,i} dx - 2\|\nabla\theta\|^2 + 2\alpha(C_2, \theta) + 2\alpha_1(\phi, \theta).$$

Now use the bound for T and the arithmetic-geometric mean inequality to find

$$\frac{d}{dt}\|\theta\|^2 \leq a\|\theta\|^2 + \alpha_1\|\phi\|^2 + k\alpha^2, \tag{2.137}$$

where we have set

$$a = \frac{T_B^2}{2} + 1 + \alpha_1, \qquad k = mC_m^2.$$

Next, multiply $(2.133)_4$ by ϕ and integrate over Ω to find

$$\begin{aligned}\frac{d}{dt}\|\phi\|^2 =& 2\int_\Omega w_i C_1 \phi_{,i} dx - 2\|\nabla\phi\|^2,\\ \leq& \frac{C_m^2}{2}\|\mathbf{w}\|^2,\\ \leq& \frac{C_m^2}{2}\|\theta\|^2,\end{aligned} \tag{2.138}$$

where (2.136) has also been employed.

We put $\beta = a + C_m^2/2$ and add (2.137) and (2.138) to deduce

$$\frac{d}{dt}(\|\theta\|^2 + \|\phi\|^2) \leq \beta(\|\theta\|^2 + \|\phi\|^2) + k\alpha^2.$$

This inequality is integrated to arrive at

$$\|\theta(t)\|^2 + \|\phi(t)\|^2 \leq \zeta(t)\alpha^2, \tag{2.139}$$

where $\zeta(t) = ke^{\beta t}/\beta$.

Inequality (2.139) is an *a priori* bound and establishes continuous dependence on the Krishnamurti coefficient α for equations (2.111).

2.8 Continuous dependence, Dufour coefficient

This section is devoted to studying the influence the Dufour effect has on double diffusive convective motion in a porous medium of Brinkman type. We focus on the Brinkman equations rather than the Darcy equations. As pointed out in chapter 1, the Brinkman equations of flow in porous media (Brinkman, 1947) have been the subject of intense recent attention. Among recent papers dealing with Brinkman models we cite (Franchi and Straughan, 1996), (Givler and Altobelli, 1994), (Guo and Kaloni, 1995c; Guo and Kaloni, 1995a), (Kladias and Prasad, 1991), (Kwok and Chen, 1987), (Lombardo and Mulone, 2002a; Lombardo and Mulone, 2002b; Lombardo and Mulone, 2003), (Nield and Bejan, 2006), (Qin and Chadam, 1996), (Qin et al., 1995), (Qin and Kaloni, 1992; Qin and Kaloni, 1994), (Payne and Song, 1997; Payne and Song, 2000), (Payne and Straughan, 1996; Payne and Straughan, 1999a), and the references therein. Double diffusive convective motion is the phenomenon involving the diffusion and convection of two independent fields, such as temperature and a salt field. In section 2.7 we analysed another double diffusive problem. Stability analyses of double diffusive phenomena, in a variety of practical contexts, have occupied much recent attention, cf. (Avramenko and Kuznetsov, 2004), (Bardan et al., 2000; Bardan et al., 2001), (Bardan and Mojtabi, 1998), (Bresch and Sy, 2003), (Budu, 2002), (Carr, 2003a; Carr, 2003b), (Chang, 2004), (Charrier-Mojtabi et al., 1998), (Clark et al., 2002), (Guo and Kaloni, 1995c; Guo and Kaloni, 1995a; Guo and Kaloni, 1995b), (Guo et al., 1994), (Hill, 2005a; Hill, 2003; Hill, 2004b; Hill, 2004a; Hill, 2004c; Hill, 2005b), (Hurle and Jakeman, 1971), (Karimi-Fard et al., 1999), (Knutti and Stocker, 2000), (Lombardo and Mulone, 2002b), (Lombardo et al., 2001), (Malashetty et al., 2006), (Song, 2002), (Stocker, 2001), (Stocker and Schmittner, 1997), (Straughan and Tracey, 1999) and (Ybarra and Velarde, 1979). (Straughan, 2004a), chapter 14 discusses double diffusive and even multi-diffusive convection in detail in a variety of contexts. Further practical studies of double diffusive convection to energy conversion and management via a solar pond occupy the papers by (Rothmeyer, 1980), (Tabor, 1980), and (Zangrando, 1991), the one by Rothmeyer investigating in particular the Soret effect, which is in some sense the mathematical adjoint to the Dufour effect.

To describe the Dufour effect, the equations for convective - diffusive motion in an incompresssible fluid in a Brinkman porous medium may be written as, employing a Boussinesq approximation in the body force term in the momentum equation,

$$\begin{aligned} &v_i - \lambda\Delta v_i = -p_{,i} + g_i T + h_i C, \qquad v_{i,i} = 0, \\ &T_{,t} + v_i T_{,i} = -J_{i,i}\,, \\ &C_{,t} + v_i C_{,i} = -K_{i,i}\,, \end{aligned} \tag{2.140}$$

where v_i, T, C and p represent velocity, temperature, salt concentration and pressure fields, respectively, g_i and h_i are the gravity vector terms arising in the density equation of state, and $\mathbf{J}$ and $\mathbf{K}$ are fluxes of heat and solute, respectively. In equations (2.140) λ is the Brinkman coefficient. The Brinkman equations are discussed at length in (Nield and Bejan, 2006) and in chapter 1, section 1.4 of this book. We observe that in $(2.140)_1$ the T, C terms arise from the body force in a Boussinesq approximation. The v_i term is essentially an interaction force between the fluid and porous matrix. The $\lambda\Delta v_i$ term is an effective viscosity contribution and is believed appropriate when the porosity is not too small. In the Brinkman equations the nonlinear convective terms of Navier-Stokes theory are omitted as is the acceleration, $\partial v_i/\partial t$, term; this is consistent with flow through a porous matrix where the convection and acceleration terms are likely to be negligible. (Hurle and Jakeman, 1971) argue that the general forms for the fluxes **J** and **K** should be

$$J_i = -\kappa T_{,i} - \rho T C\Big(\frac{\partial\mu}{\partial C}\Big) D' C_{,i}\,, \;\; K_i = -\rho D\big[S_T C(1-C)T_{,i} + C_{,i}\big]\,, \quad (2.141)$$

where κ, D, D', S_T, ρ and μ are, respectively, thermal conductivity, diffusion constant, Dufour coefficient, Soret coefficient, density and chemical potential of the solute. Continuous dependence of the solution on the Soret coefficient is treated in (Straughan and Hutter, 1999). In this section we set the Soret coefficient $S_T = 0$ and concentrate on a Dufour effect. As a first step we treat a linear Dufour effect. This means we treat the $\rho TCD'(\partial\mu/\partial C)$ term in (2.141) as constant. This is in keeping with the approach of (Ybarra and Velarde, 1979). From a mathematical viewpoint we may then, without loss of generality, reduce system (2.140), incorporating the reduced version of (2.141), to the form

$$\begin{aligned} & v_i - \lambda\Delta v_i = -p_{,i} + g_i T + h_i C, \qquad v_{i,i} = 0, \\ & T_{,t} + v_i T_{,i} = \Delta T + \gamma\Delta C\,, \\ & C_{,t} + v_i C_{,i} = \Delta C\,, \end{aligned} \qquad (2.142)$$

where $\gamma > 0$ is a constant and $\gamma\Delta C$ represents the Dufour effect. We now develop *a priori* bounds to enable us to establish continuous dependence of the solution on changes in the Dufour coefficient (constant) γ.

2.8.1 Continuous dependence on γ.

The continuous dependence result we now establish is truly *a priori* in that the coefficients appearing in the stability estimate are dependent only on initial and boundary data, and on the geometry of the domain. The proof given here is not identical to that of (Straughan and Hutter, 1999). However, it can be adapted very quickly since the Soret system studied in (Straughan and Hutter, 1999) is obtained by exchanging T and C in (2.142). On the boundary Γ we consider the given data

$$v_i = 0, \quad T = h, \quad C = g, \qquad x \in \Gamma, \qquad (2.143)$$

for prescribed functions h and g. Note that since we are dealing with the Brinkman equations all components of the velocity are prescribed on Γ. The initial data are

$$T(\mathbf{x},0) = T_0(\mathbf{x}), \quad C(\mathbf{x},0) = C_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \qquad (2.144)$$

To study continuous dependence on γ we let (u_i, T, C_1, p) and (v_i, S, C_2, q) be solutions to (2.142) – (2.144) for the same boundary and initial data, but for different Dufour coefficients γ_1 and γ_2. Thus, let (u_i, T, C_1, p) and

(v_i, S, C_2, q) solve the boundary-initial value problems

$$\begin{aligned} &u_i - \lambda\Delta u_i = -p_{,i} + g_i T + h_i C_1, \\ &u_{i,i} = 0, \\ &T_{,t} + u_i T_{,i} = \Delta T + \gamma_1 \Delta C_1, \\ &C_{1,t} + u_i C_{1,i} = \Delta C_1, \end{aligned} \tag{2.145}$$

in $\Omega \times (0, \mathcal{T})$,

$$u_i = 0, \quad T = h, \quad C_1 = g, \qquad \text{on } \Gamma \times (0, \mathcal{T}), \tag{2.146}$$

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}), \quad C_1(\mathbf{x}, 0) = C_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2.147}$$

and

$$\begin{aligned} &v_i - \lambda\Delta v_i = -q_{,i} + g_i S + h_i C_2, \\ &v_{i,i} = 0, \\ &S_{,t} + v_i S_{,i} = \Delta S + \gamma_2 \Delta C_2, \\ &C_{2,t} + v_i C_{2,i} = \Delta C_2, \end{aligned} \tag{2.148}$$

in $\Omega \times (0, \mathcal{T})$,

$$v_i = 0, \quad S = h, \quad C_2 = g, \qquad \text{on } \Gamma \times (0, \mathcal{T}), \tag{2.149}$$

$$S(\mathbf{x}, 0) = T_0(\mathbf{x}), \quad C_2(\mathbf{x}, 0) = C_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{2.150}$$

Define the difference solution (w_i, θ, ϕ, π) and the gamma-difference, γ, by

$$w_i = u_i - v_i, \quad \theta = T - S, \quad \phi = C_1 - C_2, \quad \pi = p - q, \quad \gamma = \gamma_1 - \gamma_2.$$

The solution (w_i, θ, ϕ, π) satisfies the partial differential equations

$$\begin{aligned} &w_i - \lambda\Delta w_i = -\pi_{,i} + g_i\theta + h_i\phi, \qquad w_{i,i} = 0, \\ &\theta_{,t} + w_i T_{,i} + v_i \theta_{,i} = \Delta\theta + \gamma\Delta C_1 + \gamma_2 \Delta\phi\,, \\ &\phi_{,t} + v_i \phi_{,i} + w_i C_{1,i} = \Delta\phi, \end{aligned} \tag{2.151}$$

in $\Omega \times (0, \mathcal{T})$, together with the boundary and initial conditions,

$$w_i = 0, \quad \theta = 0, \quad \phi = 0, \qquad \text{on } \Gamma \times (0, \mathcal{T}), \tag{2.152}$$

$$\theta(\mathbf{x}, 0) = 0, \quad \phi(\mathbf{x}, 0) = 0. \tag{2.153}$$

Our analysis commences by multiplying $(2.151)_1$ by w_i and integrating over Ω to derive

$$\|\mathbf{w}\|^2 + \lambda\|\nabla\mathbf{w}\|^2 = g_i(\theta, w_i) + h_i(\phi, w_i). \tag{2.154}$$

Again we suppose, $|\mathbf{g}| \le 1, \quad |\mathbf{h}| \le 1$.

Multiply $(2.151)_3$ by θ and integrate over Ω. Multiply $(2.151)_4$ by ϕ and likewise integrate over Ω. In this way one derives

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 = -(w_iT_{,i},\theta) - \|\nabla\theta\|^2 - \gamma(\nabla C_1,\nabla\theta) - \gamma_2(\nabla\theta,\nabla\phi), \tag{2.155}$$

and

$$\frac{d}{dt}\frac{1}{2}\|\phi\|^2 = -(w_iC_{1,i},\phi) - \|\nabla\phi\|^2. \tag{2.156}$$

We form the combination (2.155)+Γ(2.156) for a constant $\Gamma(>0)$ to be chosen. In this way we obtain

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\big(\Gamma\|\phi\|^2+\|\theta\|^2\big) &= -\Gamma(w_iC_{1,i},\phi) - (w_iT_{,i},\theta) - \Gamma\|\nabla\phi\|^2\\ &\quad - \gamma_2(\nabla\theta,\nabla\phi) - \|\nabla\theta\|^2 - \gamma(\nabla C_1,\nabla\theta). \end{aligned}\tag{2.157}$$

The first two terms on the right of this expression are cubic. We wish to make a positive - definite form from the next three. So, the idea now is to require Γ so large that

$$\Gamma\|\nabla\phi\|^2 + \gamma_2(\nabla\theta,\nabla\phi) + \|\nabla\theta\|^2 \geq \xi_1\|\nabla\phi\|^2 + \xi_2\|\nabla\theta\|^2\,,$$

for positive numbers ξ_1,ξ_2. For example by using the arithmetic-geometric mean inequality on the γ_2 term we may deduce

$$\Gamma\|\nabla\phi\|^2 + \gamma_2(\nabla\theta,\nabla\phi) + \|\nabla\theta\|^2 \geq \Big(\Gamma - \frac{\gamma_2}{2\alpha}\Big)\|\nabla\phi\|^2 + \Big(1 - \frac{\alpha\gamma_2}{2}\Big)\|\nabla\theta\|^2\,,$$

for $\alpha > 0$ at our disposal. Let us now choose $\alpha = 1/\gamma_2$ and then select $\Gamma = \gamma_2^2$. Thus, the inequality above becomes

$$\Gamma\|\nabla\phi\|^2 + \gamma_2(\nabla\theta,\nabla\phi) + \|\nabla\theta\|^2 \geq \frac{\gamma_2^2}{2}\|\nabla\phi\|^2 + \frac{1}{2}\|\nabla\theta\|^2. \tag{2.158}$$

Now use the arithmetic-geometric mean inequality on the last term of (2.157). We balance the $\|\nabla\theta\|^2$ term which arises with a piece of the same term from (2.158). Thus, (2.157) together with (2.158) allows us to derive

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\big(\Gamma\|\phi\|^2+\|\theta\|^2\big) &\leq -\Gamma(w_iC_{1,i},\phi) - (w_iT_{,i},\theta) - \frac{\gamma_2^2}{2}\|\nabla\phi\|^2\\ &\quad - \frac{1}{4}\|\nabla\theta\|^2 + \gamma^2\|\nabla C_1\|^2. \end{aligned}\tag{2.159}$$

Since we have extra dissipation provided by the Brinkman term (as opposed to the Darcy term of section 2.7) we can bound the cubic terms in (2.159) in a different manner. We begin with the following Sobolev inequality

$$\|\mathbf{w}\|_4 \leq c_1\|\nabla\mathbf{w}\|, \tag{2.160}$$

where $\|\cdot\|_4$ is the norm on $L^4(\Omega)$ and $c_1 = c_1(\Omega)$. We also utilise the Poincaré inequality $\lambda_1\|\mathbf{w}\|^2 \leq \|\nabla\mathbf{w}\|^2$. Next, use the Cauchy - Schwarz, Sobolev and Poincaré inequalities together with the arithmetic-geometric mean inequality to find

$$\begin{aligned}|(w_iC_{1,i},\phi)| &\leq \|\nabla C_1\|\,\|\mathbf{w}\|_4\,\|\phi\|_4\\ &\leq c_1^2\,\|\nabla C_1\|\,\|\nabla\mathbf{w}\|\,\|\nabla\phi\|\\ &\leq \frac{c_1^4}{2}\,\|\nabla C_1\|^2\,\|\nabla\mathbf{w}\|^2+\frac{1}{2}\|\nabla\phi\|^2.\end{aligned}\tag{2.161}$$

A similar procedure leads to

$$|(w_iT_{,i},\theta)| \leq c_1^4\|\nabla T\|^2\,\|\nabla\mathbf{w}\|^2+\frac{1}{4}\|\nabla\theta\|^2.\tag{2.162}$$

Now, combine (2.161) and (2.162) in inequality (2.159) to arrive at

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\big(\gamma_2^2\|\phi\|^2+\|\theta\|^2\big) \leq&\frac{c_1^4\gamma_2^2}{2}\,\|\nabla C_1\|^2\,\|\nabla\mathbf{w}\|^2\\ &+c_1^4\|\nabla T\|^2\,\|\nabla\mathbf{w}\|^2+\gamma^2\|\nabla C_1\|^2.\end{aligned}\tag{2.163}$$

We need to estimate $\|\nabla\mathbf{w}\|^2$ and then from (2.154) we may find

$$\begin{aligned}\|\mathbf{w}\|^2+\lambda\|\nabla\mathbf{w}\|^2 =&(g_i\theta,w_i)+(h_i\phi,w_i)\\ \leq&\|\theta\|\|\mathbf{w}\|+\|\phi\|\|\mathbf{w}\|\end{aligned}$$

and then we use Poincaré's inequality on a part of $\|\nabla\mathbf{w}\|^2$ to find

$$\|\mathbf{w}\|^2+\lambda\sqrt{\lambda_1}\|\mathbf{w}\|\|\nabla\mathbf{w}\| \leq \|\theta\|\|\mathbf{w}\|+\|\phi\|\|\mathbf{w}\|.$$

From this inequality we derive the estimate

$$\|\mathbf{w}\|+\lambda\sqrt{\lambda_1}\|\nabla\mathbf{w}\| \leq \|\theta\|+\|\phi\|.\tag{2.164}$$

What we require in (2.159) is an upper bound for $\|\nabla\mathbf{w}\|^2$ and we may derive this from (2.164), since this inequality shows

$$\|\nabla\mathbf{w}\| \leq \frac{\|\theta\|+\|\phi\|}{\lambda\sqrt{\lambda_1}}\,,$$

and squaring

$$\|\nabla\mathbf{w}\|^2 \leq \frac{(\|\theta\|+\|\phi\|)^2}{\lambda^2\lambda_1} \leq \frac{2}{\lambda^2\lambda_1}\big(\|\theta\|^2+\|\phi\|^2\big)\,.\tag{2.165}$$

Thus, we employ estimate (2.165) in inequality (2.163) to find

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\Big(\gamma_2^2\|\phi\|^2+\|\theta\|^2\Big) \leq&\frac{c_1^4}{\lambda^2\lambda_1}\Big(\gamma_2^2\,\|\nabla C_1\|^2+2\|\nabla T\|^2\Big)(\|\theta\|^2+\|\phi\|^2)\\ &+\gamma^2\|\nabla C_1\|^2.\end{aligned}\tag{2.166}$$

We now need *a priori* bounds for $\|\nabla C_1\|$ and $\|\nabla T\|$. To this end we follow analogous steps to section 2.7 and we introduce the harmonic function, H, which adopts the same boundary values as C_1. Thus, define

$$\Delta H=0,\ \text{in}\ \Omega\times(0,\mathcal{T}),\quad H(\mathbf{x},t)=g(\mathbf{x},t),\ \text{on}\ \Gamma\times(0,\mathcal{T}).\tag{2.167}$$

Form the identity

$$\int_0^t \int_\Omega (C_1 - H)(C_{1,t} + u_i C_{1,i} - \Delta C_1) dx\, d\eta = 0. \tag{2.168}$$

Next perform several integrations in (2.168) and use the boundary values and properties of H to see that

$$\frac{1}{2}\|C_1(t)\|^2 - \frac{1}{2}\|C_0\|^2 - (H, C_1) + (H_0, C_0) + \int_0^t \int_\Omega H_{,\eta} C_1 dx\, d\eta$$
$$- \int_0^t \int_\Omega H u_i C_{1,i} dx\, d\eta + \int_0^t \|\nabla C_1\|^2 d\eta - \int_0^t \oint_\Gamma g \frac{\partial H}{\partial n}\, dA\, d\eta = 0. \tag{2.169}$$

The point of introducing such an H is that we cannot work directly with T or C_1 to form energy-like estimates since they have non-zero boundary values. Instead we work with identities for $T - H$ or $C_1 - H$, functions which are zero on Γ. We may derive *a priori* bounds for H in a straightforward manner. To handle the cubic term in (2.169) we let g_m be the maximum value of g on $\Gamma \times [0, \mathcal{T})$ ($g_m$is taken positive) and then since H is harmonic we know by the maximum principle that $H \leq g_m$. Upon employing the Cauchy-Schwarz and arithmetic-geometric mean inequalities we derive

$$\begin{aligned}\int_0^t \int_\Omega H u_i C_{1,i} dx\, d\eta \leq & g_m \sqrt{\int_0^t \|\mathbf{u}\|^2 d\eta} \sqrt{\int_0^t \|\nabla C_1\|^2 d\eta} \\ \leq & \frac{1}{2} \int_0^t \|\nabla C_1\|^2 d\eta + \frac{1}{2} g_m^2 \int_0^t \|\mathbf{u}\|^2 d\eta,\end{aligned} \tag{2.170}$$

where the coefficient of $\int_0^t \|\nabla C_1\|^2 d\eta$ has been deliberately chosen less than 1 so we may dominate it by the equivalent term in (2.169).

From equation $(2.145)_1$ we may show that

$$\|\mathbf{u}\|^2 + \lambda \|\nabla \mathbf{u}\|^2 = g_i(T, u_i) + h_i(C_1, u_i).$$

We use this equation to derive a bound for $\int_0^t \|\mathbf{u}\|^2 d\eta$ to employ in (2.170). We now use the Cauchy - Schwarz inequality and Poincaré's inequality to derive

$$\|\mathbf{u}\|^2 + \lambda \|\nabla \mathbf{u}\|^2 \leq \|T\| \|\mathbf{u}\| + \|C_1\| \|\mathbf{u}\|,$$

then

$$\|\mathbf{u}\|^2 + \lambda \lambda_1 \|\mathbf{u}\|^2 \leq \|T\| \|\mathbf{u}\| + \|C_1\| \|\mathbf{u}\|.$$

Thus,

$$\|\mathbf{u}\| \leq \frac{\|T\| + \|C_1\|}{(1 + \lambda \lambda_1)}\,,$$

from whence,

$$\|\mathbf{u}\|^2 \leq \frac{2(\|T\|^2 + \|C_1\|^2)}{(1 + \lambda \lambda_1)}\,. \tag{2.171}$$

Therefore, from (2.170),

$$\int_0^t \int_\Omega H u_i C_{1,i} dx\, d\eta \leq \frac{1}{2} \int_0^t \|\nabla C_1\|^2 d\eta + g_m^2 \Big(\int_0^t \|T\|^2 d\eta + \int_0^t \|C_1\|^2 d\eta \Big). \tag{2.172}$$

By using the arithmetic-geometric mean inequality we may now show that

$$(H, C_1) \leq \|H\|^2 + \frac{1}{4}\|C_1\|^2, \qquad -(H_0, C_0) \leq \frac{1}{2}\|H_0\|^2 + \frac{1}{2}\|C_0\|^2, \tag{2.173}$$

and

$$\int_0^t \int_\Omega H_{,\eta} C_1\, dx\, d\eta \leq \frac{1}{2a} \int_0^t \int_\Omega H_{,\eta}^2 dx\, d\eta + \frac{a}{2} \int_0^t \int_\Omega C_1^2 dx\, d\eta, \tag{2.174}$$

for $a > 0$ to be selected.

We now use the Poincaré inequality on the C_1^2 term on the right, but since $C_1 = g$ on Γ the Poincaré inequality now takes form

$$\lambda_1 \int_\Omega C_1^2 dx \leq \int_\Omega |\nabla C_1|^2 dx + k_P \int_\Gamma g^2 dA,$$

where λ_1 and k_P are positive constants depending on Ω. We integrate this inequality over $(0, t)$ to find

$$\int_0^t ds \int_\Omega C_1^2 dx \leq \frac{1}{\lambda_1} \int_0^t ds \int_\Omega |\nabla C_1|^2 dx + \frac{k_P}{\lambda_1} \int_0^t ds \int_\Gamma g^2 dA. \tag{2.175}$$

Now use estimate (2.175) on the right of (2.174) to find

$$\int_0^t \int_\Omega H_{,\eta} C_1\, dx\, d\eta \leq \frac{1}{2a} \int_0^t \int_\Omega H_{,\eta}^2\, dx\, d\eta + \frac{a}{2\lambda_1} \int_0^t ds \int_\Omega |\nabla C_1|^2 dx + \frac{ak_P}{2\lambda_1} \int_0^t ds \int_\Gamma g^2 dA.$$

We choose $a/2\lambda_1 = 1/4$, i.e. $a = \lambda_1/2$, to balance the $\int_0^t ds \int_\Omega |\nabla C_1|^2 dx$ piece with an equivalent piece of the analogous term in (2.169). Thus, the necessary inequality is

$$\int_0^t \int_\Omega H_{,\eta} C_1\, dx\, d\eta \leq \frac{1}{\lambda_1} \int_0^t \int_\Omega H_{,\eta}^2\, dx\, d\eta + \frac{1}{4} \int_0^t ds \int_\Omega |\nabla C_1|^2 dx + \frac{k_P}{4} \int_0^t ds \int_\Gamma g^2 dA. \tag{2.176}$$

By use of the Cauchy-Schwarz inequality one finds

$$\int_0^t \int_\Gamma g \frac{\partial H}{\partial n}\, dA\, d\eta \leq \sqrt{\int_0^t \int_\Gamma g^2 dA\, d\eta} \sqrt{\int_0^t \int_\Gamma \Big(\frac{\partial H}{\partial n}\Big)^2 dA\, d\eta}\,. \tag{2.177}$$

We next employ (2.173), (2.176) and (2.177) together with (2.170) in equation (2.169) to arrive at

$$\begin{aligned}\frac{1}{4}\|C_1(t)\|^2+\frac{1}{4}\int_0^t\|\nabla C_1\|^2d\eta\leq\|C_0\|^2+\frac{k_P}{4}\int_0^t ds\int_\Gamma g^2dA+\|H\|^2\\+\frac{1}{2}\|H_0\|^2+\frac{1}{2}\int_0^t\|H_{,\eta}\|^2d\eta\\+\sqrt{\int_0^t\int_\Gamma g^2dA\,d\eta}\sqrt{\int_0^t\int_\Gamma\Big(\frac{\partial H}{\partial n}\Big)^2dA\,d\eta}\\+g_m^2\int_0^t\|T\|^2d\eta+g_m^2\int_0^t\|C_1\|^2d\eta.\end{aligned}\qquad(2.178)$$

The next stage involves use of a Rellich identity, cf. (Payne and Weinberger, 1958), to estimate the H terms on the right of (2.178). Details appropriate to the function H are similar to those in (Franchi and Straughan, 1994), p. 449. We now give details.

Recall how the function H is defined in (2.167). Thus we may write

$$\begin{aligned}0&=\int_\Omega x^iH_{,i}\Delta H\,dx\\&=\int_\Omega(x^iH_{,i}H_{,j})_{,j}dx-\int_\Omega x^i_{,j}H_{,i}H_{,j}dx-\int_\Omega x^iH_{,ij}H_{,j}dx\\&=\int_\Gamma x^iH_{,i}n_jH_{,j}dA-\int_\Omega\delta^i_jH_{,i}H_{,j}dx-\int_\Omega\frac{x^i}{2}(H_{,j}H_{,j})_{,i}dx\\&=\int_\Gamma x_iH_{,i}\frac{\partial H}{\partial n}\,dA-\int_\Omega H_{,i}H_{,i}dx\\&\quad-\frac{1}{2}\int_\Omega(x^iH_{,j}H_{,j})_{,i}dx+\frac{1}{2}\int_\Omega x^i_{,i}H_{,j}H_{,j}dx\\&=\int_\Gamma x_iH_{,i}\frac{\partial H}{\partial n}\,dA-\frac{1}{2}\int_\Gamma x_in_iH_{,j}H_{,j}dA\\&\quad-\int_\Omega H_{,i}H_{,i}dx+\frac{3}{2}\int_\Omega H_{,i}H_{,i}dx,\end{aligned}$$

where several integrations by parts and use of the divergence theorem have been performed. Thus, we see that

$$\frac{1}{2}\|\nabla H\|^2=\frac{1}{2}\int_\Gamma x_in_iH_{,j}H_{,j}dA-\int_\Gamma x_iH_{,i}\frac{\partial H}{\partial n}\,dA.\qquad(2.179)$$

On Γ we write ∇H as a normal and tangential part, thus

$$H_{,i}=\frac{\partial H}{\partial n}\,n_i+\nabla_sH\,s_i,$$

where $\nabla_sH\,s_i$ is the tangential derivative, $s^i\nabla_sH=x^i_{;\alpha}a^{\alpha\beta}H_{;\beta}$ where $a^{\alpha\beta}$ is the first fundamental form on Γ and $_{;\alpha}$ denotes surface differentiation.

From this decomposition it follows that $H_{,j}H_{,j} = (\partial H/\partial n)^2 + |\nabla_s H|^2$. Hence, we write the right hand side (RHS) of (2.179) as

$$
\begin{aligned}
RHS =& \frac{1}{2}\int_\Gamma x_i n_i \Big(\frac{\partial H}{\partial n}\Big)^2 dA + \frac{1}{2}\int_\Gamma x_i n_i |\nabla_s H|^2 dA \\
& - \int_\Gamma x_i n_i \Big(\frac{\partial H}{\partial n}\Big)^2 dA - \int_\Gamma x_i s^i \nabla_s H\, \frac{\partial H}{\partial n}\, dA \\
=& -\frac{1}{2}\int_\Gamma x_i n_i \Big(\frac{\partial H}{\partial n}\Big)^2 dA - \frac{1}{2}\int_\Gamma x_i n_i |\nabla_s H|^2 dA - \int_\Gamma x_i s^i \nabla_s H\, \frac{\partial H}{\partial n}\, dA
\end{aligned}
$$

So (2.179) becomes

$$
\begin{aligned}
\frac{1}{2}\|\nabla H\|^2 + \frac{1}{2}\int_\Gamma x_i n_i \Big(\frac{\partial H}{\partial n}\Big)^2 dA =& \frac{1}{2}\int_\Gamma x_i n_i |\nabla_s H|^2 dA \\
& - \int_\Gamma x_i s_i \frac{\partial H}{\partial n} \nabla_s H\, dA. \qquad (2.180)
\end{aligned}
$$

We suppose now Ω is star shaped and put $m_1 = \min_\Gamma x_i n_i > 0$. Thus, from (2.180) we may determine positive constants c_1 and c_2 depending on Γ such that

$$
\begin{aligned}
\|\nabla H\|^2 + c_1 \int_\Gamma \Big(\frac{\partial H}{\partial n}\Big)^2 dA \le& c_2 \int_\Gamma |\nabla_s H|^2 dA \\
=& c_2 \int_\Gamma |\nabla_s g|^2 dA. \qquad (2.181)
\end{aligned}
$$

The Poincaré inequality for H has form, since $H \neq 0$ on Γ,

$$
\lambda_1 \|H\|^2 \le \|\nabla H\|^2 + k_P \int_\Gamma H^2 dA,
$$

where $k_P = k_P(\Omega) > 0$ and so

$$
\|H\|^2 \le \frac{c_2}{\lambda_1}\int_\Gamma |\nabla_s g|^2 dA + \frac{k_P}{\lambda_1}\int_\Gamma g^2 dA. \qquad (2.182)
$$

Furthermore, $\Delta H_{,t} = 0$ in $\Omega \times [0, \mathcal{T}]$, with $H_{,t} = g_{,t}$ on Γ. We may apply the above analysis to $\phi = H_{,t}$ to derive an inequality analogous to (2.181) and from this we find

$$
\|\nabla H_{,t}\|^2 \le c_2 \int_\Gamma |\nabla_s g_{,t}|^2 dA. \qquad (2.183)
$$

Thus, inequalities (2.180) – (2.183) allow us to obtain estimates for the H terms on the right of (2.178). Clearly, we may determine constants c_α

dependent on Γ such that

$$\|H\|^2+\frac{1}{2}\|H_0\|^2\le\frac{3}{2}\,c_3\int_\Gamma g^2dA+\frac{3}{2}\,c_4\int_\Gamma|\nabla_s g|^2dA, \tag{2.184}$$

$$\int_0^t\|H_{,\eta}\|^2d\eta\le c_5\int_0^t\int_\Gamma g_{,\tau}^2dA\,d\tau+c_6\int_0^t\int_\Gamma|\nabla_s g_{,\tau}|^2dA\,d\tau, \tag{2.185}$$

$$\int_0^t\int_\Gamma\Big(\frac{\partial H}{\partial n}\Big)^2dA\,d\eta\le c_2\int_0^t\int_\Gamma|\nabla_s g|^2dA\,d\eta. \tag{2.186}$$

If we now denote by D_1 a data term of form

$$\begin{aligned}D_1(t)=4\|T_0\|^2&+k_1\int_\Gamma g^2dA+k_2\int_\Gamma|\nabla_s g|^2dA+k_3\int_0^t\int_\Gamma g_{,\tau}^2dA\,d\tau\\&+k_4\int_0^t\int_\Gamma|\nabla_s g_{,\tau}|^2dA\,d\tau+k_5\sqrt{\int_0^t\int_\Gamma g^2dA\,d\eta}\sqrt{\int_0^t\int_\Gamma|\nabla_s g|^2dA\,d\eta}\,,\end{aligned}$$

where k_α may be computed from (2.184) – (2.186), then from (2.178) we may arrive at the inequality

$$\|C_1(t)\|^2+\int_0^t\|\nabla C_1\|^2d\eta\le D_1(t)+4g_m^2\int_0^t(\|T\|^2+\|C_1\|^2)d\eta. \tag{2.187}$$

We must now carry out a similar procedure for bounding $\|T\|$ and $\|\nabla T\|$ and so we introduce the harmonic function G which assumes the same boundary values as T, i.e. define G to solve

$$\Delta G=0,\quad\text{in }\Omega\times(0,\mathcal{T}),\qquad G(\mathbf{x},t)=h(\mathbf{x},t),\quad\text{on }\Gamma\times(0,\mathcal{T}). \tag{2.188}$$

Since T satisfies (2.145) we may construct the identity

$$\int_0^t\int_\Omega(T-G)(T_{,t}+u_iT_{,i}-\Delta T-\gamma_1\Delta C_1)dx\,d\eta=0. \tag{2.189}$$

We now carry out several integrations in (2.189) to arrive at

$$\begin{aligned}&\frac{1}{2}\|T(t)\|^2-\frac{1}{2}\|T_0\|^2-(G,T)+(G_0,T_0)+\int_0^t(G_{,\eta},T)d\eta\\&-\int_0^t\int_\Omega Gu_iT_{,i}dx\,d\eta+\int_0^t\|\nabla T\|^2d\eta+\gamma_1\int_0^t(\nabla C_1,\nabla T)d\eta\\&-\int_0^t\int_\Gamma h\frac{\partial G}{\partial n}\,dA\,d\eta-\gamma_1\int_0^t\int_\Gamma g\frac{\partial G}{\partial n}\,dA\,d\eta=0.\end{aligned} \tag{2.190}$$

Let h_m denote the maximum value of h on Γ. Then following the procedure leading to (2.170) we estimate the cubic term in (2.190). The arithmetic-geometric mean inequality is used on the γ_1 term and these procedures furnish the bound

$$
\begin{aligned}
\int_0^t \int_\Omega G u_i T_{,i} dx\, d\eta - \gamma_1 \int_0^t (\nabla C_1, \nabla T) d\eta \leq\ & h_m^2 \int_0^t \|\mathbf{u}\|^2 d\eta \\
& + \frac{1}{2} \int_0^t \|\nabla T\|^2 d\eta + \gamma_1^2 \int_0^t \|\nabla C_1\|^2 d\eta \\
\leq\ & \frac{2h_m^2}{(1+\lambda\lambda_1)} \left(\int_0^t \|T\|^2 d\eta + \int_0^t \|C_1\|^2 d\eta \right) \\
& + \frac{1}{2} \int_0^t \|\nabla T\|^2 d\eta + \gamma_1^2 \int_0^t \|\nabla C_1\|^2 d\eta, \qquad (2.191)
\end{aligned}
$$

where in the last step (2.171) has been employed.

We estimate the $G_{,\eta}$ term as

$$
\begin{aligned}
\int_0^t (G_{,\eta}, T) d\eta \leq & \frac{1}{2a} \int_0^t \|G_{,\eta}\|^2 d\eta + \frac{a}{2} \int_0^t \|T\|^2 d\eta \\
\leq & \frac{1}{2a} \int_0^t \|G_{,\eta}\|^2 d\eta + \frac{a}{2\lambda_1} \int_0^t \|\nabla T\|^2 d\eta \\
& + \frac{a k_P}{2\lambda_1} \int_0^t d\eta \int_\Gamma h^2 dA
\end{aligned}
$$

where we have also used the Poincaré inequality for T. Now pick $a/2\lambda_1 = 1/4$, and then

$$
\begin{aligned}
\int_0^t (G_{,\eta}, T) d\eta \leq & \frac{1}{\lambda_1} \int_0^t \|G_{,\eta}\|^2 d\eta + \frac{1}{4} \int_0^t \|\nabla T\|^2 d\eta \\
& + \frac{k_P}{4} \int_0^t d\eta \int_\Gamma h^2 dA. \qquad (2.192)
\end{aligned}
$$

Upon employing (2.191) and (2.192) in (2.190) we may further use the arithmetic-geometric mean inequality to obtain

$$
\begin{aligned}
\frac{1}{4}\|T(t)\|^2 + \frac{1}{4} \int_0^t \|\nabla T\|^2 d\eta \leq & \|T_0\|^2 + \frac{1}{2}\|G_0\|^2 + \|G\|^2 + \frac{1}{\lambda_1} \int_0^t \|G_{,\eta}\|^2 d\eta \\
& + \int_0^t \int_\Gamma h \frac{\partial G}{\partial n} dA\, d\eta + \gamma_1 \int_0^t \int_\Gamma g \frac{\partial G}{\partial n} dA\, d\eta + \gamma_1^2 \int_0^t \|\nabla C_1\|^2 d\eta \\
& + 2g_m^2 \int_0^t \|C_1\|^2 d\eta + 2g_m^2 \int_0^t \|T\|^2 d\eta. \qquad (2.193)
\end{aligned}
$$

Next use the Cauchy-Schwarz inequality on the boundary terms,

$$\begin{aligned}\int_0^t\int_\Gamma h\frac{\partial G}{\partial n}\,dA\,d\eta+\gamma_1\int_0^t\int_\Gamma g\frac{\partial G}{\partial n}\,dA\,d\eta&\\ \leq\sqrt{\int_0^t\int_\Gamma h^2dA\,d\eta}\sqrt{\int_0^t\int_\Gamma\Big(\frac{\partial G}{\partial n}\Big)^2dA\,d\eta}&\\ +\gamma_1\sqrt{\int_0^t\int_\Gamma g^2dA\,d\eta}\sqrt{\int_0^t\int_\Gamma\Big(\frac{\partial G}{\partial n}\Big)^2dA\,d\eta}&\qquad(2.194)\end{aligned}$$

By using a Rellich identity argument one may show that analogous inequalities to (2.184) – (2.186) hold for G. We then define the data term D_2 for computable constants $\ell_1,\ldots,\ell_5$ as

$$\begin{aligned}D_2(t)=&4\|T_0\|^2+\ell_1\int_\Gamma h^2dA+\ell_2\int_\Gamma|\nabla_sh|^2dA\\ &+\ell_3\int_0^t\int_\Gamma h_{,\tau}^2dA\,d\tau+\ell_4\int_0^t\int_\Gamma|\nabla_sh_{,\tau}|^2dA\,d\tau\\ &+\ell_5\sqrt{\int_0^t\int_\Gamma h^2dA\,d\eta}\sqrt{\int_0^t\int_\Gamma|\nabla_sh|^2dA\,d\eta}\\ &+\ell_5\gamma_1\sqrt{\int_0^t\int_\Gamma g^2dA\,d\eta}\sqrt{\int_0^t\int_\Gamma|\nabla_sh|^2dA\,d\eta}.\qquad(2.195)\end{aligned}$$

Upon using (2.194) and (2.195) in (2.193) one may produce the inequality

$$\begin{aligned}\|T(t)\|^2+\int_0^t\|\nabla T\|^2d\eta\leq\ &D_2(t)+8g_m^2\int_0^t\|C_1\|^2d\eta\\ &+8g_m^2\int_0^t\|T\|^2d\eta+4\gamma_1^2\int_0^t\|\nabla C_1\|^2d\eta.\qquad(2.196)\end{aligned}$$

We now let α be a constant such that $\alpha>4\gamma_1^2$ and then form $\alpha(2.187)+(2.196)$. In this manner we obtain the bound

$$\begin{aligned}&\alpha\|C_1(t)\|^2+(\alpha-4\gamma_1^2)\int_0^t\|\nabla C_1\|^2d\eta+\|T(t)\|^2+\int_0^t\|\nabla T\|^2d\eta\\ &\leq\alpha D_1+D_2+\big[4\alpha g_m^2+8h_m^2\big]\int_0^t\|C_1\|^2d\eta\\ &\quad+(4\alpha g_m^2+8h_m^2)\int_0^t\|T\|^2d\eta.\qquad(2.197)\end{aligned}$$

Define now $K_1=4\alpha g_m^2+8h_m^2$, $D(t)=\alpha D_1+D_2$, and $K=K_1$ if $\alpha>1$ or $K=K_1/\alpha$ if $\alpha<1$. Then from (2.197) one may discard the $\|\nabla C_1\|^2$ and $\|\nabla T\|^2$ terms to derive

$$\alpha\|C_1(t)\|^2+\|T(t)\|^2\leq D+K\bigg[\alpha\int_0^t\|C_1\|^2d\eta+\int_0^t\|T\|^2d\eta\bigg].$$

Thus upon integration we see that

$$\alpha\int_0^t \|C_1\|^2 d\eta + \int_0^t \|T\|^2 d\eta \le P(t), \tag{2.198}$$

where P is the data term

$$P(t) = \int_0^t e^{K(t-s)} D(s)ds. \tag{2.199}$$

We still need *a priori* estimates for $\int_0^t \|\nabla T\|^2 d\eta$ and $\int_0^t \|\nabla C_1\|^2 d\eta$ and these follow by using (2.198) in (2.197) to find

$$\int_0^t \|\nabla T\|^2 d\eta \le P_2(t), \qquad \int_0^t \|\nabla C_1\|^2 d\eta \le P_1(t), \tag{2.200}$$

where P_1 and P_2 are data terms given by

$$P_1(t) = \frac{1}{(\alpha - 4\gamma_1^2)}\big[D(t) + KP(t)\big], \qquad P_2(t) = D(t) + KP(t).$$

We are now in a position to complete the continuous dependence estimate on γ. An integration of (2.166) yields

$$\begin{aligned}\gamma_2^2\|\phi(t)\|^2 + \|\theta(t)\|^2 \le{}& \frac{2c_1^4}{\lambda^2\lambda_1}\int_0^t \Big[\gamma_2^2\|\nabla C_1\|^2 + 2\|\nabla T\|^2\Big](\|\phi\|^2 + \|\theta\|^2)d\eta\\ &+ \gamma^2\int_0^t \|\nabla C_1\|^2 d\eta,\\ \le{}& \frac{2K_1c_1^4}{\lambda^2\lambda_1}\int_0^t \Big[\gamma_2^2\|\nabla C_1\|^2 + 2\|\nabla T\|^2\Big](\gamma_2^2\|\phi\|^2 + \|\theta\|^2)d\eta\\ &+ \gamma^2 P_1(t),\end{aligned} \tag{2.201}$$

where $K_2 = \max\{1, \gamma_2^{-2}\}$. Now define $f(t) = 2K_2c_1^4[\gamma_2^2\|\nabla C_1\|^2 + 2\|\nabla T\|^2]/\lambda^2\lambda_1$. Then an application of Gronwall's inequality to (2.201) furnishes the estimate

$$\begin{aligned}\gamma_2^2\|\phi(t)\|^2 + \|\theta(t)\|^2 \le{}& \gamma^2 P_1(t) + \gamma^2\int_0^t P_1(s)f(s)\Big[\exp\int_s^t f(u)du\Big]ds,\\ \le{}& \gamma^2 P_1(t) + \gamma^2\Big[\exp\int_0^t f(s)ds\Big]\bar{P}_1(t)\int_0^t f(s)ds,\end{aligned} \tag{2.202}$$

where $\bar{P}_1(t) = \max_{s\in[0,t]} P_1(s)$. Thanks to (2.200) we have $\int_0^t f(s)ds \le P_3(t)$, where the data term P_3 is given by $P_3(t) = 2K_2c_1^4\big[\gamma_2^2 P_2(t) + 2P_1(t)\big]/\lambda^2\lambda_1$. Therefore, from inequality (2.202) we may deduce

$$\gamma_2^2\|\phi(t)\|^2 + \|\theta(t)\|^2 \le R(t)\gamma^2, \tag{2.203}$$

where $R(t)$ is the data term given by $R(t) = P_1(t) + \bar{P}_1(t)P_3(t)\exp\big[P_3(t)\big]$.

Inequality (2.203) demonstrates continuous dependence on the Dufour coefficient γ, for the salt concentration C and temperature T.

We may also derive a continuous dependence inequality for the velocity $\mathbf{u}$ by employing (2.154) in combination with (2.203). From (2.154) one easily derives the estimates

$$\|\mathbf{w}\| \leq \frac{\|\theta\| + \|\phi\|}{(1+\lambda\lambda_1)}, \qquad \text{and} \qquad \|\nabla\mathbf{w}\| \leq \frac{1}{\lambda\sqrt{\lambda_1}}\,(\|\theta\| + \|\phi\|).$$

These inequalities together with (2.203) yield

$$\|\mathbf{w}(t)\|^2 \leq \frac{2K_2R(t)}{(1+\lambda\lambda_1)^2}\,\gamma^2, \qquad \text{and} \qquad \|\nabla\mathbf{w}(t)\|^2 \leq \frac{2K_2}{\lambda^2\lambda_1}\,R(t)\,\gamma^2. \tag{2.204}$$

Inequalities (2.204) establish continuous dependence on the Dufour coefficient γ in the L^2 and H^1 measures of $\mathbf{w}$ as indicated.

Very interesting *a priori* bounds and continuous dependence on the Soret coefficient for the system of equations (2.140) are established by (Lin and Payne, 2007a). These writers study equations (2.140) with zero *flux* boundary conditions. The methods they use are very interesting and of necessity different from those described in this section.

2.9 Initial - final value problems

Recently a new class of problem has been shown to be relevant to many applied mathematical situations. This is where the data are not given at time $t = 0$, but instead are prescribed as a linear combination at times $t = 0$ and $t = T$. We shall refer to such situations as initial - final value problems. Specific applications of these ideas are in (Payne and Schaefer, 2002), (Payne et al., 2004), (Ames et al., 2004a; Ames et al., 2004b), (Quintanilla and Straughan, 2005b; Quintanilla and Straughan, 2005a) and the references therein. This class of problem was originally introduced in order to stabilize solutions to the improperly posed problem when the data is given at $t = T$ and one wishes to compute the solution backward in time, see (Ames et al., 1998), (Ames and Payne, 1999) and the references therein. (Ames et al., 2004a) study an initial - final value problem for the first order abstract equation $u_t + Au = f$. (Ames et al., 2004b) investigate an initial - final value problem for the diffusion equation with the spatial domain being an infinite cylinder. (Payne and Schaefer, 2002) study an initial - final value problem for the second order in time abstract equation $u_{tt} + Au = F$. They also investigate a similar initial - final value problem for the equation $u_{tt} + au_t + Au = 0$, for $a > 0$ a constant. (Payne et al., 2004) study an initial - final value problem for some fluid mechanics problems, especially in connection with Stokes flow. Further analyses of initial - final value problems are by (Quintanilla and Straughan, 2005b) who investigate thermoelasticity according to the new developments of (Green and Naghdi, 1991; Green and Naghdi, 1992; Green and Naghdi, 1993).

Further analysis of these theories may be found in (Quintanilla and Racke, 2003), (Quintanilla and Straughan, 2000; Quintanilla and Straughan, 2002; Quintanilla and Straughan, 2004), (Zhang and Zuazua, 2003), (Puri and Jordan, 2004). Another article dealing with initial - final value problems is that of (Quintanilla and Straughan, 2005a) who concentrate on dipolar fluids, see also (Bleustein and Green, 1967), (Green and Naghdi, 1968; Green and Naghdi, 1970), (Green et al., 1965), (Green and Rivlin, 1967), (Akyildiz and Bellout, 2004), (Jordan and Puri, 1999; Jordan and Puri, 2002), (Puri and Jordan, 1999b; Puri and Jordan, 1999a), on the (Green and Naghdi, 1996) extended theory of viscous fluids, and on the Brinkman-Forchheimer model of flow in porous media. The last topic is of interest in this book.

The article of (Quintanilla and Straughan, 2005a) analyses the Brinkman-Forchheimer equations, as used by (Qin and Kaloni, 1998), namely

$$\begin{aligned} Au_{i,t} =& -p_{,i} - u_i + \lambda\Delta u_i - \beta|\mathbf{u}|u_i, \\ u_{i,i} =& 0. \end{aligned} \tag{2.205}$$

In these equations u_i, p represent the velocity and pressure, and A, λ, β are positive constants.

We take equations (2.205) to be defined on a bounded domain $\Omega \subset \mathbb{R}^3$ on the time interval $(0, T)$ for some $T < \infty$, with the boundary conditions being

$$u_i = 0 \qquad \text{on} \quad \Gamma. \tag{2.206}$$

The study of (Quintanilla and Straughan, 2005a) uses the initial - final condition

$$u_i(T) + \alpha u_i(0) = f_i, \tag{2.207}$$

where α is a constant, and $f_i(\mathbf{x})$ is a prescribed function. (The standard initial boundary value problem for (2.205) would replace (2.207) by $u_i(0) = f_i$. The standard final boundary value problem for (2.205) would employ $u_i(T) = f_i$ instead of (2.207).) Here, the objective is to obtain a bound on u_i in terms of f_i and α, employing the relation (2.207).

(Quintanilla and Straughan, 2005a) note that for the final value problem for (2.205), (2.206), i.e. with $\alpha = 0$, a global solution does not exist. By transforming $t \to T - t$ one may show (cf. for example, the arguments in (Straughan, 1998))

$$\|\mathbf{u}(t)\| \geq \frac{\|\mathbf{u}(0)\|}{e^{-\gamma t} - k_2\|\mathbf{u}(0)\|(1 - e^{-\gamma t})/2\gamma}\,. \tag{2.208}$$

In this inequality $\gamma = (\lambda\lambda_1 + 1)/A$, $k_2 = 2\beta/Am^{1/2}$, with λ_1 being the first eigenvalue in the membrane problem for Ω and where m is the volume of Ω. The right hand side of (2.208) blows-up at time $\mathcal{T} = [A/(\lambda\lambda_1 + 1)] \log\{1 + [(\lambda\lambda_1 + 1)m^{1/2}/\beta\|\mathbf{u}(0)\|]\}$, and so u_i cannot exist classically beyond this time. (Quintanilla and Straughan, 2005a) then argue that care must be taken with the initial - final value problem defined by (2.205) – (2.207).

(Quintanilla and Straughan, 2005a) derive a bound for u_i by commencing with multiplication of (2.205) by u_i and integration over Ω using the boundary conditions to find

$$\frac{d}{dt}\frac{A}{2}\|\mathbf{u}\|^2 = -\|\mathbf{u}\|^2 - \lambda\|\nabla\mathbf{u}\|^2 - \beta\int_\Omega |\mathbf{u}|^3 dx. \tag{2.209}$$

We employ the Poincaré inequality $-\|\nabla\mathbf{u}\|^2 \leq -\lambda_1\|\mathbf{u}\|^2$ and the Cauchy-Schwarz inequality to find $-\int_\Omega |\mathbf{u}|^3 dx \leq -\|\mathbf{u}\|^{3/2}/m^{1/2}$. Then from (2.209) with $\Phi(t) = \|\mathbf{u}(t)\|^2$ one may show

$$\frac{d\Phi}{dt} \leq -c_1\Phi - c_2\Phi^{3/2}, \tag{2.210}$$

where the constants c_1 and c_2 are given by

$$c_1 = \frac{2(1+\lambda\lambda_1)}{A}, \qquad c_2 = \frac{2\beta}{Am^{1/2}}.$$

Inequality (2.210) is integrated to obtain

$$\|\mathbf{u}(t)\| \leq \frac{\|\mathbf{u}(0)\|e^{-c_1 t/2}}{1 + c_2\|\mathbf{u}(0)\|(1-e^{-\gamma t})/c_1}, \tag{2.211}$$

for t in the interval $0 \leq t \leq T$.

This is a bound for $u_i(t)$ in terms of $u_i(0)$. However, $u_i(0)$ is unknown. We need to remove $\|\mathbf{u}(0)\|$ in (2.211) and convert it to an estimate involving f_i and α. The key is also to retain the c_2 term since this contains the Forchheimer effect (the β term). It is necessary to bound $\|\mathbf{u}(0)\|$ from both above and below.

(Quintanilla and Straughan, 2005a) show that one may demonstrate

$$\|\mathbf{u}(0)\| \geq \frac{\|\mathbf{f}\|}{\sqrt{2(\alpha^2 + e^{-c_1 T})}}, \tag{2.212}$$

and provided $|\alpha| > e^{-c_1 T/2}$,

$$\|\mathbf{u}(0)\| \leq \frac{1}{(|\alpha| - e^{-c_1 T/2})}\|\mathbf{f}\|. \tag{2.213}$$

The lower and upper bounds (2.212) and (2.213) used in (2.211) lead to the estimate

$$\|\mathbf{u}(t)\| \leq e^{-c_1 t/2}\frac{\|\mathbf{f}\|}{(|\alpha| - e^{-c_1 T/2})}\left[1 + \frac{c_2(1-e^{-c_1 t/2})\|\mathbf{f}\|}{c_1\sqrt{2(\alpha^2 + e^{-c_1 T})}}\right]^{-1}, \tag{2.214}$$

provided $|\alpha| > e^{-c_1 T/2}$, for t in the interval $0 \leq t \leq T$.

(Quintanilla and Straughan, 2005a) observe that while the bound in (2.214) is not optimal, the system of equations (2.205) is nonlinear, and so an optimal bound would be hard to achieve. If instead one were to consider the equivalent problem for the Brinkman equations, i.e. take $\beta = 0$ in (2.205), we may derive an optimal estimate. We do not include details since they follow very closely the arguments of (Payne et al., 2004) for the Stokes equations. The difference is the addition of the $-u_i$ term in (2.205). The Lagrange identity and non-uniqueness proofs of (Payne et al., 2004) apply here, *mutatis mutandis*.

2.10 The interface problem

In this section we study the problem where a viscous fluid adjoins a porous medium saturated with the same fluid. In thermal convection this was addressed in the fundamental papers by (Nield, 1977) and by (Chen and Chen, 1988). One of the fundamental problems in modelling flow of a fluid over a porous medium is that the conditions at the interface between the fluid and the porous medium are a contentious matter, see e.g. (Beavers and Joseph, 1967), (Caviglia et al., 1992b),(Ciesjko and Kubik, 1999), (Jäger and Mikelic, 1998), (Jäger et al., 1999), (Jones, 1973), (McKay, 2001), (Murdoch and Soliman, 1999), (Nield and Bejan, 2006), pp. 17 – 19, (Ochoa-Tapia and Whitaker, 1995a; Ochoa-Tapia and Whitaker, 1995b; Ochoa-Tapia and Whitaker, 1997), (Saffman, 1971), (Taylor, 1971). Very good agreement with experiment is often achieved by employing the experimentally suggested condition proposed by (Beavers and Joseph, 1967), or its generalization by (Jones, 1973). (Straughan, 2001c; Straughan, 2002a), (Carr, 2004) and (Carr and Straughan, 2003) have investigated various aspects and generalisations of the Nield and Chen-Chen problems. They find that the Beavers-Joseph and Jones boundary conditions give good results over a wide range of parameters. The Beavers-Joseph condition has been successful in the slow flow of a fluid past a porous sphere (Qin and Kaloni, 1993). If one is employing a method based on linearized instability and so is using Stokes' flow, use of a Beavers-Joseph or a Jones condition is probably justified. Numerical schemes are developed for the coupled fluid flow and porous flow problems by (Discacciati et al., 2002), by (Miglio et al., 2003), by (Hoppe et al., 2007), and by (Mu and Xu, 2007). Several computational simulations are reported in these papers. Another interesting numerical contribution to porous/fluid flow is by (Das et al., 2002). This paper presents a finite volume method in three-dimensions. The porous part of the domain is allowed to be anisotropic. It is shown that flow circulation may occur inside the porous medium and the direction of flow may reverse at the interface between the porous medium and fluid. (Layton et al., 2003) prove existence for weak solutions to the problem of Darcy porous media flow coupled to the Stokes equations in a fluid with the Beavers - Joseph interface boundary condition. They also analyse in detail a finite element scheme which formulates the coupled problem as uncoupled steps in the porous and fluid regions thereby allowing a user to employ some of the many existing numerical codes for the separate flow regions. (Das and Lewis, 2007) is another recent very interesting contribution. These writers are interested in the three-dimensional flow pattern and how heterogeneities in the porous medium will affect this. To achieve their aim they interestingly employ two porous layers with different permeabilities.

The purpose of this section is to review work of (Payne and Straughan, 1998a) which studies the manner in which a solution to flow in a fluid which borders a porous medium depends on a coefficient in the Jones boundary conditions. We adopt the notation of (Payne and Straughan, 1998a) and thus, let an appropriate part of the plane $z = x_3 = 0$ denote the boundary between a porous medium occupying a bounded region Ω_2 in $\mathbb{R}^3$, and a linear viscous fluid occupying a bounded region Ω_1 in $\mathbb{R}^3$. The porous region is in $z \geq 0$ while the fluid domain is in $z < 0$, although both Ω_1 and Ω_2 are bounded. The interface between Ω_1 and Ω_2 is denoted by L while the remaining parts of the boundaries of Ω_1 and Ω_2 are denoted, respectively, by Γ_1 and Γ_2. In Ω_1 the fluid velocity is slow such that the governing equations may be taken to be those of Stokes flow. The question of Navier-Stokes flow is addressed in (Payne and Straughan, 1998a). In the porous region Ω_2 the flow is assumed to satisfy the Darcy (1856) equations.

Let (u_i, T, p) denote the velocity, temperature and pressure in Ω_1 while (u_i^m, T^m, p^m) denotes the velocity, temperature and pressure in Ω_2. The Stokes flow equations which hold in the fluid region are

$$\begin{aligned} &\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \mu\Delta u_i + g_i T, \qquad \frac{\partial u_i}{\partial x_i} = 0\,, \\ &\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \kappa\Delta T, \end{aligned} \tag{2.215}$$

in $\Omega_1 \times (0, \mathcal{T})$, where μ is the dynamic viscosity, κ is the thermal diffusivity and g_i is the gravity vector which is scaled such that $|\mathbf{g}| \leq 1$.

The relevant Darcy equations which hold in the porous region are,

$$\begin{aligned} &\frac{\mu}{k} u_i^m = -\frac{\partial p^m}{\partial x_i} + g_i T^m, \qquad \frac{\partial u_i^m}{\partial x_i} = 0\,, \\ &\frac{\partial T^m}{\partial t} + u_i^m \frac{\partial T^m}{\partial x_i} = \kappa^m \Delta T^m, \end{aligned} \tag{2.216}$$

in $\Omega_2 \times (0, \mathcal{T})$. The constant k is the permeability and κ^m is the thermal diffusivity of the porous medium.

The functions u_i, T and T^m satisfy the initial data

$$\begin{aligned} &u_i(x,0) = f_i(x), \qquad T(x,0) = T_0(x), \quad x \in \Omega_1, \\ &T^m(x,0) = T_0^m(x), \quad x \in \Omega_2. \end{aligned} \tag{2.217}$$

On the outer boundary $\Gamma_1 \cup \Gamma_2$ we consider

$$\begin{aligned} &u_i = 0, \quad T = T_U(x,t), \qquad \text{on} \quad \Gamma_1 \times (0,\mathcal{T}),\\ &u_i^m n_i = 0, \quad T^m = T_L(x,t), \qquad \text{on} \quad \Gamma_2 \times (0,\mathcal{T}), \end{aligned} \tag{2.218}$$

for prescribed functions T_U and T_L, with n_i being the unit outward normal. The conditions on the interface L chosen by (Payne and Straughan, 1998a) are

$$\begin{aligned} &u_3 = u_3^m, \qquad T = T^m, \qquad \kappa T_{,3} = \kappa^m T_{,3}^m,\\ &p^m = p - 2\mu u_{3,3}, \qquad u_{\beta,3} + u_{3,\beta} = \frac{\alpha_1}{\sqrt{k}}\, u_\beta. \end{aligned} \tag{2.219}$$

The coefficient α_1 is determined by experiment for a given fluid and a given porous solid. These boundary conditions are discussed at length in (Nield and Bejan, 2006), see also chapter 6. The condition $u_{\beta,3} + u_{3,\beta} = u_\beta \alpha_1/\sqrt{k}$ essentially derives from the work of (Jones, 1973). The motivation for this arose from (Beavers and Joseph, 1967) who argued on the basis of experimental results that

$$u_{\beta,3} = \frac{\alpha_1}{\sqrt{k}}\,(u_\beta - u_\beta^m), \qquad \text{on } L \tag{2.220}$$

and (Jones, 1973) generalised this to include the shear stress at the interface, i.e.

$$u_{\beta,3} + u_{3,\beta} = \frac{\alpha_1}{\sqrt{k}}\,(u_\beta - u_\beta^m). \tag{2.221}$$

(Nield and Bejan, 2006) write that (Saffman, 1971) argues that the last term may essentially be dropped in equation (2.220). This is the justification for $(2.219)_5$.

The object of this section is to describe an *a priori* estimate showing how (u_i, T) and (u_i^m, T^m) depend continuously on the interface coefficient α_1. To do this, let (u_i, p, T) and (u_i^m, p^m, T^m) satisfy (2.215) – (2.219) and let (v_i, q, S) and (v_i^m, q^m, S^m) solve the same boundary initial value problem with identical data functions f_i, T_0, T_0^m, T_U and T_L, but with the Jones coefficient α_1 replaced by a different value α_2. The difference variables (w_i, π, θ) and σ are defined by

$$w_i = u_i - v_i, \quad \pi = p - q, \quad \theta = T - S, \quad \sigma = \alpha_1 - \alpha_2. \tag{2.222}$$

By direct calculation one finds that (w_i, π, θ) satisfy the partial differential equations

$$\begin{aligned} &\frac{\partial w_i}{\partial t} = -\frac{\partial \pi}{\partial x_i} + \mu \Delta w_i + g_i \theta,\\ &\frac{\partial w_i}{\partial x_i} = 0\,,\\ &\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} + w_i \frac{\partial S}{\partial x_i} = \kappa \Delta \theta, \end{aligned} \tag{2.223}$$

in $\Omega_1 \times (0, \mathcal{T})$,

$$\begin{aligned} &\frac{\mu}{k} w_i^m = -\frac{\partial \pi^m}{\partial x_i} + g_i \theta^m, \\ &\frac{\partial w_i^m}{\partial x_i} = 0, \\ &\frac{\partial \theta^m}{\partial t} + u_i^m \frac{\partial \theta^m}{\partial x_i} + w_i^m \frac{\partial S^m}{\partial x_i} = \kappa^m \Delta \theta^m, \end{aligned} \tag{2.224}$$

in $\Omega_2 \times (0, \mathcal{T})$.

The initial conditions become

$$w_i(x,0) = 0, \quad \theta(x,0) = 0, \quad x \in \Omega_1, \qquad \theta^m(x,0) = 0, \quad x \in \Omega_2. \tag{2.225}$$

On the outer boundary the relevant conditions become

$$\begin{aligned} &w_i = 0, \quad \theta = 0, \qquad \text{on} \quad \Gamma_1 \times (0, \mathcal{T}), \\ &w_i^m n_i = 0, \quad \theta^m = 0, \qquad \text{on} \quad \Gamma_2 \times (0, \mathcal{T}). \end{aligned} \tag{2.226}$$

The interface boundary conditions may be written

$$\begin{aligned} &w_3 = w_3^m, \qquad \theta = \theta^m, \qquad \kappa \theta_{,3} = \kappa^m \theta_{,3}^m, \\ &\pi^m = \pi - 2\mu w_{3,3}, \qquad w_{\beta,3} + w_{3,\beta} = \frac{\alpha_1}{\sqrt{k}} w_\beta + \frac{\sigma}{\sqrt{k}} v_\beta, \end{aligned} \tag{2.227}$$

these holding on $L \times (0, \mathcal{T})$.

(Payne and Straughan, 1998a) establish the following theorem which demonstrates continuous dependence of a solution on the interface coefficient α_1.

Theorem 2.10.1 *Suppose $\partial T/\partial n \in L^1(\Gamma_1 \times (0, \mathcal{T}))$ and $\partial T^m/\partial n \in L^1(\Gamma_2 \times (0, \mathcal{T}))$. Then there exist constants $\gamma (< 2\mu/k)$, B, C and $\hat{A}$, determined in (Payne and Straughan, 1998a) such that*

$$\begin{aligned} \int_{\Omega_1} w_i w_i \, dx + B \int_0^t \int_{\Omega_1} w_i w_i \, dx \, d\eta + \gamma \int_{\Omega_2} w_i^m w_i^m \, dx \\ \leq \frac{C e^{Bt}}{\alpha_1 \alpha_2} \left(\int_{\Omega_1} f_i f_i \, dx + \hat{A} t T_m^2 \right) \sigma^2. \end{aligned} \tag{2.228}$$

Furthermore, there is a constant M, depending on t, such that

$$\int_{\Omega_1} \theta^2 dx + \int_{\Omega_2} (\theta^m)^2 dx \leq \frac{M}{\alpha_1 \alpha_2} \sigma^2. \tag{2.229}$$

The proof of this theorem is technical, care must be taken with the interface terms, and we refer to (Payne and Straughan, 1998a) for full details. Nevertheless we note that the proof is interesting and is based on a combination function $\Phi(t)$ of the form

$$\Phi(t) = \int_{\Omega_1} w_i w_i dx + \gamma \int_0^t \int_{\Omega_2} w_i^m w_i^m \, dx \, d\eta \, .$$

2.11 Lower bounds on the blow-up time

(Payne and Schaefer, 2006; Payne and Schaefer, 2007) and (Payne and Song, 2007a) produce a clever argument to show that one can derive *lower bounds* for the blow-up time for a nonlinear differential equation and for the Navier-Stokes equations with nonlinear forcing terms. Prior to this work there had been many analyses of blow-up which had derived upper bounds on the blow-up time. However, the work of (Payne and Schaefer, 2006; Payne and Schaefer, 2007) and (Payne and Song, 2007a) is novel in that it produces a lower bound for the blow-up time. (Suzuki, 2006) shows how to derive a universal bound, independent of the initial data, which is useful in calculating the initial blow-up rate of a solution, whereas (Hirota and Ozawa, 2006) consider numerical techniques for estimating the blow-up time and the rate of solution increase. (Kirane et al., 2005) investigate critical exponents of Fujita type when fractional derivatives are present. (Fila and Winkler, 2008) demonstrate a solution which blows up in a finite time at a point with the solution remaining bounded elsewhere. Other interesting blow-up results and analysis showing prevention of blow-up are due to (Bhandar et al., 2004), (Boutat et al., 2004), (Tersenov, 2004).

We now consider an analogue of the (Payne and Song, 2007a) problem but for a Brinkman porous medium. The equations for the Brinkman problem with a non-zero inertia and nonlinear forces depending on temperature are, cf. equations (2.76)

$$\begin{aligned} &\alpha\frac{\partial u_i}{\partial t}=-u_i+\lambda\Delta u_i-\frac{\partial p}{\partial x_i}+h_i(T),\\ &\frac{\partial u_i}{\partial x_i}=0,\\ &\frac{\partial T}{\partial t}+u_i\frac{\partial T}{\partial x_i}=\Delta T+f(T). \end{aligned} \tag{2.230}$$

In these equations u_i, T, p are velocity, temperature and pressure, α, λ are the inertia and Brinkman coefficients and $h_i(T)$ and $f(T)$ are nonlinear functions of temperature. Equations (2.230) are defined on a bounded spatial region Ω over a time interval $(0,\mathcal{T})$. The boundary conditions considered are

$$u_i=0,\quad T=0 \qquad \text{on } \Gamma\times(0,\mathcal{T}), \tag{2.231}$$

while the initial conditions are

$$u_i(\mathbf{x},0)=u_i^0(\mathbf{x}),\quad T(\mathbf{x},0)=T_0(\mathbf{x})\,\geq\,0. \tag{2.232}$$

We here only consider the Brinkman model, but one could consider a Darcy model. Also, we only consider Dirichlet conditions on the boundary whereas one could alternatively employ Neumann boundary conditions following (Payne and Schaefer, 2006), (Payne and Song, 2007a). We also note that we could employ $T =$ constant in (2.231) although care would then need to be taken with the function f.

Since both equation $(2.230)_1$ and equation $(2.230)_3$ are forced by nonlinear functions of temperature, one may ask if blow-up occurs, will this be in the first instance via the velocity or the temperature field? We follow (Payne and Song, 2007a) to show this must be via the temperature.

Let t_1 be the blow-up time of the temperature T and t_2 be the blow-up time of the velocity u_i. We wish to show that $t_1 < t_2$. Suppose, therefore, this is false so that $t_2 < t_1$. Then, for $t < t_2$, we multiply equation $(2.230)_1$ by u_i and integrate over Ω to find after integrations by parts and use of the boundary conditions and $(2.230)_2$,

$$\frac{d}{dt}\frac{\alpha}{2}\|\mathbf{u}\|^2 = -\|\mathbf{u}\|^2 - \lambda\|\nabla\mathbf{u}\|^2 + \int_\Omega h_i u_i dx.$$

We employ the Poincaré inequality $\lambda_1\|\mathbf{u}\|^2 \leq \|\nabla\mathbf{u}\|^2$ and the arithmetic-geometric mean inequality for $\gamma > 0$ to now see that

$$\frac{d}{dt}\frac{\alpha}{2}\|\mathbf{u}\|^2 \leq -\Big(1 + \lambda\lambda_1 - \frac{\gamma}{2}\Big)\|\mathbf{u}\|^2 + \frac{\|\mathbf{h}\|^2}{2\gamma}. \tag{2.233}$$

Pick $\gamma = (1 + \lambda\lambda_1)$ and then from (2.233) one sees that

$$\frac{d}{dt}\|\mathbf{u}\|^2 \leq -\frac{\gamma}{\alpha}\|\mathbf{u}\|^2 + \frac{\|\mathbf{h}\|^2}{\gamma\alpha}. \tag{2.234}$$

Since $t < t_2 < t_1$, $h_i(T)$ is bounded and so $\|\mathbf{h}\|^2 \leq M^2$, for some constant M. Employ this bound in (2.234), and integrate with an integrating factor to obtain

$$\begin{aligned}\|\mathbf{u(t)}\|^2 \leq \|\mathbf{u_0}\|^2 \exp\Big[-\Big(\frac{1+\lambda\lambda_1}{\alpha}\Big)t\Big] \\ + \frac{M^2}{(1+\lambda\lambda_1)^2}\Big\{1 - \exp\Big[-\Big(\frac{1+\lambda\lambda_1}{\alpha}\Big)t\Big]\Big\},\end{aligned} \tag{2.235}$$

where $t \leq t_2$. Now let $t \to t_2$. By assumption $\|\mathbf{u(t)}\|^2$ blows up at $t = t_2$, but inequality (2.235) contradicts this. Thus, $t_1 \leq t_2$, and so t_1 is a lower bound for the blow-up time.

The conditions we now impose on the nonlinear function $f(T)$ are the same as those of (Payne and Schaefer, 2007), namely

$$f(0) = 0, \quad f(s) > 0, \qquad \text{for } s > 0, \tag{2.236}$$

$$\int_T^\infty \frac{ds}{f(s)} \quad \text{is bounded for } T > 0, \tag{2.237}$$

and there are constants $n > 2$ and $\beta > 0$ such that

$$f(T)\left(\int_T^\infty \frac{ds}{f(s)}\right)^{n+1} \to \infty \quad \text{as} \quad T \to 0^+, \tag{2.238}$$

$$f'(T)\int_T^\infty \frac{ds}{f(s)} \le n+1-\beta. \tag{2.239}$$

As (Payne and Schaefer, 2007) remark, from the work of (Ball, 1977) and (Kielhöfer, 1975), when the solution does cease to exist globally then the behaviour is that of blow-up.

To now derive a lower bound for the blow-up time t_1 we follow (Payne and Schaefer, 2007), (Payne and Song, 2007a). Put $R = \int_T^\infty ds/f(s)$, $v = 1/R$, and define the function $\phi(t)$ by

$$\phi(t) = \int_\Omega v^n dx.$$

By differentiation

$$\begin{aligned}\frac{d\phi}{dt} =& n\int_\Omega v^{n-1}v_t dx\\ =& n\int_\Omega \frac{v^{n+1}}{f(T)}\,T_t\,dx\\ =& n\int_\Omega \frac{v^{n+1}}{f(T)}\,\big[\Delta T - u_iT_{,i} + f(T)\big]\,dx.\end{aligned} \tag{2.240}$$

Using the chain rule one shows

$$\begin{aligned}\int_\Omega v^{n+1}\frac{u_iT_{,i}}{f(T)}\,dx &= \frac{1}{n}\int_\Omega (v^n)_{,i}u_i dx\\ &= \frac{1}{n}\left[\int_\Omega (v^n u_i)_{,i}dx - \int_\Omega v^n u_{i,i}dx\right] = 0\,.\end{aligned}$$

Thus, equation (2.240) reduces to

$$\frac{d\phi}{dt} = n\int_\Omega \frac{v^{n+1}}{f(T)}\,\big[\Delta T + f(T)\big]\,dx. \tag{2.241}$$

From this point, the estimate for t_1 effectively follows from the arguments of (Payne and Schaefer, 2007). Integrate the first term on the right of (2.241) by parts to find

$$n\int_\Omega \frac{v^{n+1}\Delta T}{f(T)}dx = -n\int_\Omega \left(\frac{v^{n+1}}{f}\right)_{,i}T_{,i}dx + n\int_\Gamma \frac{v^{n+1}}{f(T)}\,\frac{\partial T}{\partial\nu}\,dS, \tag{2.242}$$

where $\partial/\partial\nu$ denotes the unit outward normal derivative. Thanks to condition (2.238) the last term in (2.242) is zero. The first term in (2.242) is expanded and then (2.239) is employed to find

$$\begin{aligned} n\int_\Omega \frac{v^{n+1}\Delta T}{f(T)}dx &= n\int_\Omega \frac{v^{n+1}f'(T)}{f^2}\,T_{,i}T_{,i}dx \\ &\qquad - n(n+1)\int_\Omega \frac{T_{,i}}{f^2}\,v^{n+2}T_{,i}dx, \\ &\leq n\int_\Omega \frac{v^{n+2}}{f^2}\,T_{,i}T_{,i}[n+1-\beta]dx - n(n+1)\int_\Omega \frac{v^{n+2}}{f^2}\,T_{,i}T_{,i}dx \\ &= -\beta n\int_\Omega \frac{v^{n+2}}{f^2}\,T_{,i}T_{,i}dx\,. \end{aligned} \tag{2.243}$$

Inequality (2.243) is now employed in equation (2.241) to find

$$\frac{d\phi}{dt} \leq -\beta n\int_\Omega \frac{v^{n+2}}{f^2}\,T_{,i}T_{,i}dx + n\int_\Omega v^{n+1}dx.$$

Noting that $v^{(n/2+1)}T_{,i}/f = 2(v^{n/2})_{,i}/n$ this inequality is rearranged as

$$\frac{d\phi}{dt} \leq -\frac{4\beta}{n}\int_\Omega (v^{n/2})_{,i}(v^{n/2})_{,i}dx + n\int_\Omega v^{n+1}dx. \tag{2.244}$$

If m denotes the measure of Ω then from Hölder's inequality and the Cauchy-Schwarz inequality one sees

$$\begin{aligned} \int_\Omega v^{n+1}dx \leq& m^{(n-2)/3n}\Big(\int_\Omega v^{3n/2}dx\Big)^{2(n+1)/3n} \\ \leq& m^{(n-2)/3n}\Big(\int_\Omega v^{2n}dx\int_\Omega v^n dx\Big)^{(n+1)/3n}. \end{aligned} \tag{2.245}$$

We next use the Sobolev inequality

$$\int_\Omega \psi^4 dx \leq C\Big(\int_\Omega \psi^2 dx\Big)^{1/2}\Big(\int_\Omega \psi_{,i}\psi_{,i}dx\Big)^{3/2},$$

where a value for C is calculated in (Payne, 1964), taking $\psi = v^{n/2}$ to find

$$\int_\Omega v^{2n}dx \leq C\Big(\int_\Omega v^n dx\Big)^{1/2}\Big[\int_\Omega (v^{n/2})_{,i}(v^{n/2})_{,i}dx\Big]^{3/2}.$$

This estimate is now used in (2.245) to obtain

$$\begin{aligned} \int_\Omega v^{n+1}dx \leq\, & m^{(n-2)/3n}C^{(n+1)/3n}\Big(\alpha_1\int_\Omega |\nabla v^{n/2}|^2 dx\Big)^{(n+1)/2n} \\ & \times\Big(\frac{1}{\alpha_1}\int_\Omega v^n dx\Big)^{(n+1)/2n}, \end{aligned} \tag{2.246}$$

where the constant $\alpha_1 > 0$ has been added to allow removal of the $|\nabla v^{n/2}|^2$ term. Next, employ Young's inequality

$$XY \leq \frac{X^p}{p} + \frac{Y^s}{s}, \qquad \frac{1}{p} + \frac{1}{s} = 1,$$

with $X = \alpha_1 \int_\Omega |\nabla v^{n/2}|^2 dx$, $Y = \int_\Omega v^n dx/\alpha_1$, $p = 2n/(n+1) > 1$, and $s = 2n/(n-1)$. Then, from (2.246) we derive

$$\begin{aligned}\int_\Omega v^{n+1} dx \leq & m^{(n-2)/3n} C^{(n+1)/3n} \Big(\frac{n+1}{2n}\Big) \alpha_1^{2n/(n+1)} \int_\Omega |\nabla v^{n/2}|^2 dx \\ & + m^{(n-2)/3n} C^{(n+1)/3n} \Big(\frac{n-1}{2n}\Big) \frac{1}{\alpha_1^{2n/(n-1)}} \\ & \times \Big(\int_\Omega v^n dx\Big)^{(n+1)/(n-1)}. \end{aligned} \tag{2.247}$$

Inequality (2.247) is next employed in inequality (2.244) to find

$$\begin{aligned}\frac{d\phi}{dt} \leq & \Big[nm^{(n-2)/3n} C^{(n+1)/3n} \Big(\frac{n+1}{2n}\Big) \alpha_1^{2n/(n+1)} - \frac{4\beta}{n}\Big] \int_\Omega |\nabla v^{n/2}|^2 dx \\ & + nm^{(n-2)/3n} C^{(n+1)/3n} \Big(\frac{n-1}{2n}\Big) \frac{1}{\alpha_1^{2n/(n-1)}} \phi^{(n+1)/(n-1)}. \end{aligned} \tag{2.248}$$

The constant α_1 is now selected to make the first term on the right of (2.248) zero. Thus, for K computable, from (2.248) we derive

$$\frac{d\phi}{dt} \leq K\phi^{(n+1)/(n-1)}.$$

This inequality is integrated to obtain

$$\frac{1}{[\phi(0)]^{2/(n-1)}} - \frac{1}{[\phi(t)]^{2/(n-1)}} \leq \frac{2Kt}{(n-1)}. \tag{2.249}$$

When $t \to t_1$ (the blow-up time), then (2.249) yields the lower bound $\hat{t}$ for t_1, where

$$\begin{aligned} t_1 \geq \hat{t} = & \Big(\frac{n-1}{2K}\Big) \frac{1}{[\phi(0)]^{2/(n-1)}} \\ = & \Big(\frac{n-1}{2K}\Big) \Big(\int_\Omega \Big[\int_{T_0(x)}^\infty \frac{ds}{f(s)} \Big]^{-n} dx \Big)^{-2/(n-1)}. \end{aligned} \tag{2.250}$$

The above derivation simply adapts the clever analyses of (Payne and Schaefer, 2007) and (Payne and Song, 2007a) to a Brinkman model.

A lower bound with a more direct derivation may be found by adapting the method of (Payne, 1975), pp. 49, 50. To do this we work with equation $(2.230)_3$. The assumption on the nonlinearity is now inequality (8.31) of (Payne, 1975), namely

$$\int_\Omega T^{2p-1} f(T) dx \leq \int_\Omega |T|^{2p+\gamma}, \tag{2.251}$$

where γ is a positive constant, and (2.251) holds for any positive integer p.

Introduce the function

$$\Phi_p(t) = \int_\Omega T^{2p} dx.$$

Then,

$$\begin{aligned}\frac{d\Phi_p}{dt} =& 2p \int_\Omega T^{2p-1} \frac{\partial T}{\partial t}\, dx \\ =& 2p \int_\Omega T^{2p-1} \Delta T\, dx + 2p \int_\Omega T^{2p-1} f(T) dx \\ &- \int_\Omega T^{2p-1} u_i \frac{\partial T}{\partial x_i}\, dx.\end{aligned} \tag{2.252}$$

Integrating by parts and using the boundary conditions,

$$\begin{aligned}\int_\Omega T^{2p-1} u_i \frac{\partial T}{\partial x_i}\, dx =& \frac{1}{2p} \int_\Omega u_i \frac{\partial}{\partial x_i} T^{2p}\, dx \\ =& \frac{1}{2p} \int_\Gamma u_i n_i T^{2p}\, dS - \frac{1}{2p} \int_\Omega u_{i,i} T^{2p}\, dx \\ =& 0.\end{aligned} \tag{2.253}$$

Further integration by parts and use of the boundary conditions yield

$$\begin{aligned}2p \int_\Omega T^{2p-1} \Delta T\, dx =& 2p \int_\Gamma T^{2p-1} \frac{\partial T}{\partial n}\, dS - 2p(2p-1) \int_\Omega T^{2p-2} T_{,i} T_{,i} dx \\ =& -2p(2p-1) \int_\Omega T^{2p-2} T_{,i} T_{,i} dx\,.\end{aligned} \tag{2.254}$$

Now, use (2.253), (2.254) and inequality (2.251) in equation (2.252) to see that

$$\begin{aligned}\frac{d\Phi_p}{dt} \leq& -\frac{2(2p-1)}{p} \int_\Omega T^p_{,i} T^p_{,i}\, dx + 2p \int_\Omega |T|^{2p+\gamma}\, dx \\ \leq& 2p \int_\Omega T^{2p+\gamma} dx\,.\end{aligned} \tag{2.255}$$

Next, put

$$T_*(t) = \sup_{x \in \Omega} |T(\mathbf{x}, t)|\,.$$

Then from (2.255) we may derive

$$\frac{d\Phi_p}{dt} \leq 2p T_*^\gamma \Phi_p\,.$$

An integration of this inequality yields

$$\Phi_p(t) \leq \Phi_p(0) \exp\left[2p \int_0^t T_*^\gamma(s) ds\right].$$

Raise both sides of this inequality to the power $1/2p$ and then let $p \to \infty$. In this manner we obtain

$$T_*(t) \le T_*(0) \exp\left[\int_0^t T_*^\gamma(s)ds\right]. \tag{2.256}$$

Since t_1 is the blow-up time for T we must have $T_*(t) \to \infty$ as $t \to t_1$, and assuming T is sufficiently regular,

$$\int_0^{t_1} T_*^\gamma(s)ds = \infty. \tag{2.257}$$

The next step is to raise both sides of inequality (2.256) to the power γ and then, provided $t \le t_1$, this inequality yields

$$T_*^\gamma(t) \exp\left[-\gamma \int_0^t T_*^\gamma(s)ds\right] \le T_*^\gamma(0).$$

A further integration of this inequality over $0, t < t_1$, leads to

$$1 - \exp\left[-\gamma \int_0^t T_*^\gamma(s)ds\right] \le \gamma t T_*^\gamma(0).$$

Let now $t \to t_1$ and employ condition (2.257). In this way we find

$$\frac{1}{\gamma T_*^\gamma(0)} \le t_1\,. \tag{2.258}$$

Inequality (2.258) represents an alternative lower bound for the blow-up time t_1 to the estimate (2.250).

The above proof is a straightforward adaptation of the demonstration of (Payne, 1975), pp. 49, 50.

2.12 Uniqueness in compressible porous flows

So far in this book we have concentrated on fluid flow in a porous medium where the fluid may be treated as incompressible. However, sound propagation through a porous medium is one important example of a situation where flow of a compressible gas in a porous material is necessary. We study in detail wave motion of a compressible fluid in a porous medium in chapter 8 with related material given in chapter 7. Therefore, in this chapter we commence a study of the well posedness of a theory for compressible flow in a porous medium by establishing a uniqueness theorem. Since the wave motion in chapter 8 is typically for sound waves propagating in an infinite medium we here establish a uniqueness theorem for flow in an infinite spatial region. To establish our theorem we appeal to a beautiful result of Dario Graffi, (Graffi, 1960) although Graffi's paper is conveniently found in the selected works, (Graffi, 1999), pages 273 – 280.

The model for compressible flow in a porous material is taken from (De Ville, 1996). It consists of the equations for flow of a barotropic perfect fluid, cf. (Fabrizio, 1994), to which have been added a Darcy term and a Forchheimer term to represent the interaction with the porous matrix. This model is one of equivalent fluid type, and these are discussed in greater detail in section 8.1. The equations we employ are those of (De Ville, 1996), equations (4) and (5), although we assume the fluid is polytropic so that the pressure - density relation is of form $p = a\rho^\gamma$, where p and ρ are pressure and density, a is a positive constant, and γ is a constant with $1 < \gamma < 2$. With v_i being the fluid velocity, k, λ, b_1 positive constants the model of (De Ville, 1996) may be written

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0, \\
&\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + b_1 v v_i + \frac{k}{\rho}\, v_i = -\frac{a\gamma}{\lambda}\, \rho^{\gamma-2} \rho_{,i}\,,
\end{aligned} \tag{2.259}$$

where we adopt the (Graffi, 1960; Graffi, 1999) notation $v = |\mathbf{v}|$.

(Graffi, 1960; Graffi, 1999) establishes uniqueness for (2.259) when $b_1 = 0, k = 0$. The extension to include these terms is non-trivial and given below. Nevertheless, we extend the (Graffi, 1999) method and employ his notation. Henceforth, we employ the notation (Graffi, 1999) to denote paper 22 of the selected works, pages 273 – 280.

Equations (2.259) are defined on a space - time domain. The time domain is $(0, \mathcal{T})$ and the spatial domain D is either $\mathbb{R}^3$ or the exterior of a bounded domain σ_0 in $\mathbb{R}^3$. In either case D is an unbounded domain. We impose the same hypotheses as (Graffi, 1999), and in particular his hypotheses (a) – (g). However, we have already mentioned hypothesis (b) which states that the pressure is polytropic and we have no need for hypothesis (c) which concerns the body force, since one may regard equation $(2.259)_2$ as defining a particular form for the body force. The remaining hypotheses (a) and (d) – (g) are stated below.

(a) In the domain $D \times (0, \mathcal{T})$ the velocity $\mathbf{v}$ and density ρ are uniformly bounded together with their first derivatives in space and time.

(d) If D has an interior boundary $\partial\sigma_0$, then on $\partial\sigma_0$ we assign $\mathbf{v}\cdot\mathbf{n}$, $\mathbf{n}$ being the unit outward normal to $\partial\sigma_0$, and where the fluid enters so that $\mathbf{v}\cdot\mathbf{n} < 0$ we assign ρ and $\mathbf{v}$.

(e) The values of $\rho(\mathbf{x}, 0)$ and $v_i(\mathbf{x}, 0)$ are assigned.

(f) The density ρ is positive and $|\nabla\rho|/\rho$ is bounded in $D \times (0, \mathcal{T})$.

(g) Let R denote the distance from the origin in D, then $\rho \geq c/R^\beta$, where c is a positive constant and $\beta \geq 0$ is a constant.

Let us observe that the last relation is physically necessary. It allows the density to vanish as $R \to \infty$ although not in an arbitrary way. In fact, it is condition (g) which makes the extension of the (Graffi, 1999) result to system (2.259) non-trivial. One now has to also handle the terms $b_1 v v_i$ and $k v_i/\rho$.

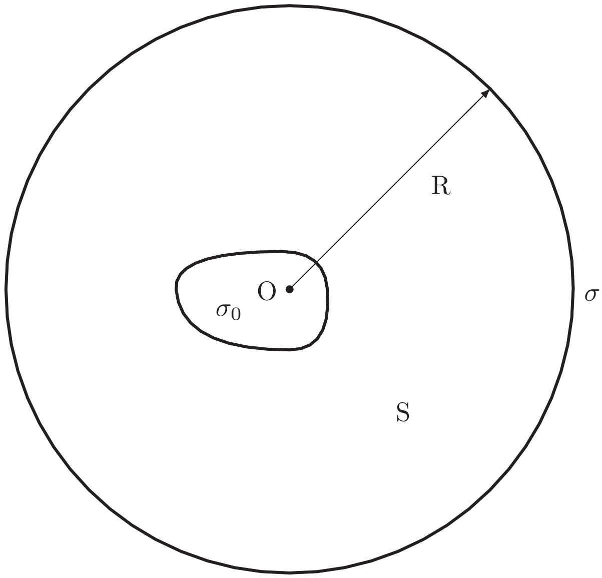

Figure 2.1. Geometry for uniqueness proof

We now denote by S the intersection of the ball of radius R with D. The geometrical configuration is shown in figure 2.1.

The outer boundary of S, i.e. the spherical surface of radius R, is denoted by σ.

To study uniqueness we follow (Graffi, 1999) and let ρ, v_i and $\rho+\rho_1, v_i+v_i^1$ be two solutions to equations (2.259) which both satisfy hypotheses (a) and (d) – (g). By subtraction we find ρ_1 and v_i^1 satisfy the equations

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x_i}\big[(\rho+\rho_1)(v_i+v_i^1) - \rho v_i\big] = 0,\\
&\frac{\partial v_i^1}{\partial t} + v_j^1 \frac{\partial (v_i+v_i^1)}{\partial x_j} + v_j \frac{\partial v_i^1}{\partial x_j}\\
&= -b\bigg\{(\rho+\rho_1)^{\gamma-2}\frac{\partial}{\partial x_i}(\rho+\rho_1) - \rho^{\gamma-2}\frac{\partial \rho}{\partial x_i}\bigg\}\\
&\quad - k\bigg\{\Big(\frac{v_i+v_i^1}{\rho+\rho_1}\Big) - \frac{v_i}{\rho}\bigg\}\\
&\quad - b_1\big\{(v+v_1)(v_i+v_i^1) - v v_i\big\},
\end{aligned} \tag{2.260}$$

where we have put $b = a\gamma/\lambda$. The proof of (Graffi, 1999) is very clever and balances the $\mathbf{v}_1 \cdot \nabla\rho_1$ term which arises from $(2.260)_1$ with an equivalent term from $(2.260)_2$. This necessitates the use of a weighted L^2 energy for ρ_1, weighted by both $\rho^{\gamma-3}$ and $(\rho+\rho_1)^{\gamma-3}$.

We begin by multiplying $(2.260)_2$ by v_i^1 and find

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}v_1^2 = &- v_i^1 v_j^1(v_i+v_i^1)_{,j} - v_i^1 v_j v_{i,j}^1\\ &- bv_i^1\big\{(\rho+\rho_1)^{\gamma-2}(\rho+\rho_1)_{,i} - \rho^{\gamma-2}\rho_{,i}\big\}\\ &- kv_i^1\bigg\{\Big(\frac{v_i+v_i^1}{\rho+\rho_1}\Big) - \frac{v_i}{\rho}\bigg\}\\ &- b_1 v_i^1\big\{(v+v_1)(v_i+v_i^1) - vv_i\big\}\,.\end{aligned}\tag{2.261}$$

Employ the rearrangement (5) of (Graffi, 1999),

$$\begin{aligned}&(\rho+\rho_1)^{\gamma-2}(\rho+\rho_1)_{,i} - \rho^{\gamma-2}\rho_{,i}\\ &= \frac{1}{2}\big\{(\rho+\rho_1)^{\gamma-2} - \rho^{\gamma-2}\big\}\big\{(\rho+\rho_1)_{,i} + \rho_{,i}\big\}\\ &\qquad + \frac{1}{2}\rho^{\gamma-2}\rho_{1,i} + (\rho+\rho_1)^{\gamma-2}\rho_{1,i}\,.\end{aligned}\tag{2.262}$$

The first term on the right of (2.262) is handled by firstly noting that from hypothesis (f) there is a positive constant n such that

$$|(\rho+\rho_1)_{,i}| \le n(\rho+\rho_1), \qquad |\rho_{,i}| \le n\rho,$$

then

$$\begin{aligned}&\big|\big\{(\rho+\rho_1)^{\gamma-2} - \rho^{\gamma-2}\big\}\big\{(\rho+\rho_1)_{,i} + \rho_{,i}\big\}\big|\\ &\le n(\rho+\rho_1+\rho)|(\rho+\rho_1)^{\gamma-2} - \rho^{\gamma-2}|\\ &= n|(\rho+\rho_1)^{\gamma-1} - \rho^{\gamma-1} + \rho(\rho+\rho_1)^{\gamma-2} - (\rho+\rho_1)\rho^{\gamma-2}|\,.\end{aligned}\tag{2.263}$$

To bound the terms on the right of (2.263) one uses the intermediate value theorem, and for $0<\theta<1$, and $0<\theta'<1$, one finds

$$|(\rho+\rho_1)^{\gamma-1} - \rho^{\gamma-1}| \le |(\gamma-1)(\rho+\theta\rho_1)^{\gamma-2}\rho_1|,\tag{2.264}$$

and

$$\begin{aligned}|\rho(\rho+\rho_1)^{\gamma-2} - (\rho+\rho_1)\rho^{\gamma-2}| &= \rho(\rho+\rho_1)\bigg|\frac{1}{(\rho+\rho_1)^{3-\gamma}} - \frac{1}{\rho^{3-\gamma}}\bigg|\\ &= \rho(\rho+\rho_1)\bigg|\frac{(\rho+\rho_1)^{3-\gamma} - \rho^{3-\gamma}}{(\rho+\rho_1)^{3-\gamma}\rho^{3-\gamma}}\bigg|\\ &= \bigg|\frac{(3-\gamma)(\rho+\theta'\rho_1)^{2-\gamma}\rho_1}{(\rho+\rho_1)^{2-\gamma}\rho^{2-\gamma}}\bigg|\,.\end{aligned}\tag{2.265}$$

Combining (2.264) and (2.265) in (2.263) one then obtains

$$\begin{aligned} &\big|\{(\rho+\rho_1)^{\gamma-2}-\rho^{\gamma-2}\}\{(\rho+\rho_1)_{,i}+\rho_{,i}\}\big| \\ &\leq n\left|(\gamma-1)(\rho+\theta\rho_1)^{\gamma-2}+\frac{(3-\gamma)(\rho+\theta'\rho_1)^{2-\gamma}}{(\rho+\rho_1)^{2-\gamma}\rho^{2-\gamma}}\right||\rho_1|\,. \end{aligned} \tag{2.266}$$

If ρ_1 is positive then the greater value of the right of (2.266) is achieved with $\theta=0,\theta'=1$ (since $\gamma-2<0$) whereas if ρ_1 is negative we select $\theta=1,\theta'=0$, which in turn yield,

$$\begin{aligned} &\big|\{(\rho+\rho_1)^{\gamma-2}-\rho^{\gamma-2}\}\{(\rho+\rho_1)_{,i}+\rho_{,i}\}\big| \leq 2n\rho^{\gamma-2}|\rho_1|, \\ &\big|\{(\rho+\rho_1)^{\gamma-2}-\rho^{\gamma-2}\}\{(\rho+\rho_1)_{,i}+\rho_{,i}\}\big| \leq 2n(\rho+\rho_1)^{\gamma-2}|\rho_1|. \end{aligned}$$

These results together in (2.266) lead to

$$\begin{aligned} &\big|\{(\rho+\rho_1)^{\gamma-2}-\rho^{\gamma-2}\}\{(\rho+\rho_1)_{,i}+\rho_{,i}\}\big| \\ &\leq 2n\{(\rho+\rho_1)^{\gamma-2}+\rho^{\gamma-2}\}|\rho_1|. \end{aligned} \tag{2.267}$$

We now see that from (2.267)

$$\begin{aligned} &\left|-\frac{b}{2}\,v_i^1\{(\rho+\rho_1)^{\gamma-2}(\rho+\rho_1)_{,i}-\rho^{\gamma-2}\rho_{,i}\}\right| \\ &\leq bn\{(\rho+\rho_1)^{\gamma-2}+\rho^{\gamma-2}\}\,|\rho_1\mathbf{v}_1| \\ &\leq \frac{bn}{2}\{(\rho+\rho_1)^{2\gamma-4}\rho_1^2+\rho^{2\gamma-4}\rho_1^2+2v_1^2\}\,, \end{aligned} \tag{2.268}$$

where in the last line the arithmetic-geometric mean inequality has been employed. Thus, from (2.268), (2.262) and (2.261) we obtain

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\,v_1^2 \leq &-v_i^1v_j^1(v_i+v_i^1)_{,j}-v_i^1v_jv_{i,j}^1 \\ &+\frac{bn}{2}\{(\rho+\rho_1)^{2\gamma-4}\rho_1^2+\rho^{2\gamma-4}\rho_1^2+2v_1^2\} \\ &-\frac{b}{2}v_i^1\{\rho^{\gamma-2}\rho_{1,i}+(\rho+\rho_1)^{\gamma-2}\rho_{1,i}\} \\ &-kv_i^1\left\{\left(\frac{v_i+v_i^1}{\rho+\rho_1}\right)-\frac{v_i}{\rho}\right\} \\ &-b_1v_i^1\{(v+v_1)(v_i+v_i^1)-vv_i\}\,. \end{aligned} \tag{2.269}$$

The first two terms on the right of (2.269) are written as

$$-v_i^1v_j^1(v_i+v_i^1)_{,j}-\frac{1}{2}\frac{\partial}{\partial x_j}(v_jv_1^2)+\frac{1}{2}v_{j,j}v_1^2\,. \tag{2.270}$$

The final term of (2.269) is handled with the identity, (Payne and Straughan, 1999a),

$$\big[(v+v_1)(v_i+v_i^1)-vv_i\big]v_i^1=\frac{1}{2}(v+v_1+v)v_1^2$$
$$+\frac{1}{2}(|\mathbf{v}+\mathbf{v}_1|-|\mathbf{v}|)^2(v+v_1+v)\,. \qquad (2.271)$$

Thus, recalling hypothesis (a) the gradients of $\mathbf{v}$ and $\mathbf{v}+\mathbf{v}^1$ are bounded in (2.270) and then from employment of (2.271) and (2.270) in inequality (2.269) we deduce that, after integration over S,

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_S v_1^2dx &\le \frac{3}{2}N_1\int_S v_1^2dx-\frac{1}{2}\int_S\frac{\partial}{\partial x_j}(v_jv_1^2)dx\\
&+\frac{bn}{2}\int_S\big[(\rho+\rho_1)^{2\gamma-4}\rho_1^2+\rho^{2\gamma-4}\rho_1^2+2v_1^2\big]dx\\
&-\frac{b}{2}\int_S v_i^1\big[\rho^{\gamma-2}\rho_{1,i}+(\rho+\rho_1)^{\gamma-2}\rho_{1,i}\big]dx\\
&-k\int_S v_i^1\left[\left(\frac{v_i+v_i^1}{\rho+\rho_1}\right)-\frac{v_i}{\rho}\right]dx\,,\end{aligned} \qquad (2.272)$$

where N_1 is a bound for $|\nabla\mathbf{v}|$ and $|\nabla(\mathbf{v}+\mathbf{v}^1)|$ and we have discarded the last term of (2.269) thanks to (2.271). (In studying continuous dependence one may desire to retain the right hand side of (2.271) and then use the effect of v_1 in L^3, cf. section 4.6.2.)

To handle the last term in (2.272) we note

$$\begin{aligned}&-k\int_S v_i^1\left[\left(\frac{v_i+v_i^1}{\rho+\rho_1}\right)-\frac{v_i}{\rho}\right]dx\\
&=-k\int_S\frac{v_1^2}{(\rho+\rho_1)}\,dx+k\int_S\frac{\rho_1v_iv_i^1}{\rho(\rho+\rho_1)}\,dx\,.\end{aligned} \qquad (2.273)$$

The arithmetic-geometric mean inequality is used on the last term in the form

$$k\int_S\frac{\rho_1v_iv_i^1}{\rho(\rho+\rho_1)}\,dx\le k\int_S\frac{v_1^2}{(\rho+\rho_1)}\,dx+\frac{k}{4}\int_S\frac{v^2\rho_1^2}{\rho^2(\rho+\rho_1)}\,dx\,.$$

This inequality is inserted in (2.273) and the result employed in (2.272) to find

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_S v_1^2 dx \le{}& \frac{3}{2}N_1\int_S v_1^2 dx - \frac{1}{2}\int_{\sigma\cup\partial\sigma_0} n_i v_1^2 v_i dS\\&+\frac{bn}{2}\int_S\big[(\rho+\rho_1)^{2\gamma-4}\rho_1^2+\rho^{2\gamma-4}\rho_1^2+2v_1^2\big]dx\\&-\frac{b}{2}\int_S v_i^1\big[\rho^{\gamma-2}\rho_{1,i}+(\rho+\rho_1)^{\gamma-2}\rho_{1,i}\big]dx\\&+\frac{k}{4}\int_S\frac{v^2\rho_1^2}{\rho^2(\rho+\rho_1)}\,dx\,.\end{aligned}\tag{2.274}$$

To continue we note that from $(2.259)_2$,

$$kv_i=-\rho v_{i,t}-\rho v_j v_{i,j}-b_1\rho v v_i - b\rho^{\gamma-1}\rho_{,i}\,.$$

Thus, recollecting hypotheses (a) and (f) we see there are constants n_1, n_2 such that

$$v\le n_1\rho+n_2\rho^\gamma.$$

Thus, there are further constants $m_1, m_2, m_3, \ell_1, \ell_2$ and ℓ_3 such that

$$v^2\le m_1\rho^2+m_2\rho^{\gamma+1}+m_3\rho^{2\gamma},$$

and

$$\frac{kv^2}{4\rho^2(\rho+\rho_1)}\le\frac{\ell_1+\ell_2\rho^{\gamma-1}+\ell_3\rho^{2\gamma-2}}{(\rho+\rho_1)}\,.\tag{2.275}$$

To incorporate the (Graffi, 1999) weights $\rho^{\gamma-3}$, $(\rho+\rho_1)^{\gamma-3}$ we write

$$\frac{\ell_1}{\rho+\rho_1}\;=\;(\rho+\rho_1)^{\gamma-3}\cdot\ell_1(\rho+\rho_1)^{2-\gamma}\,.\tag{2.276}$$

Then we use Young's inequality for arbitrary $\alpha>0$,

$$\frac{\ell_3\rho^{2\gamma-2}}{\rho+\rho_1}\;\le\;\ell_3\left[\frac{(\alpha\rho^{2\gamma-2})^p}{p}+\frac{\big[\alpha^{-1}(\rho+\rho_1)^{-1}\big]^q}{q}\right]$$

$p^{-1}+q^{-1}=1$. Pick $q=3-\gamma>1$, then $p=(3-\gamma)/(2-\gamma)>1$. Thus,

$$\begin{aligned}\frac{\ell_3\rho^{2\gamma-2}}{\rho+\rho_1}\;\le{}&\;\frac{\ell_3}{(3-\gamma)\alpha^{(3-\gamma)}}\,(\rho+\rho_1)^{\gamma-3}\\&+\frac{\ell_3(2-\gamma)\alpha^{(3-\gamma)/(2-\gamma)}}{(3-\gamma)}\,\rho^{\gamma(3-\gamma)/(2-\gamma)}\cdot\rho^{\gamma-3}\,.\end{aligned}\tag{2.277}$$

A similar calculation utilizing Young's inequality shows

$$\begin{aligned}\frac{\ell_2\rho^{\gamma-1}}{\rho+\rho_1}\;\le{}&\;\frac{\ell_2}{(3-\gamma)\beta^{(3-\gamma)}}\,(\rho+\rho_1)^{\gamma-3}\\&+\frac{\ell_2(2-\gamma)\beta^{(3-\gamma)/(2-\gamma)}}{(3-\gamma)}\,\rho^{(3-\gamma)/(2-\gamma)}\cdot\rho^{\gamma-3}\,.\end{aligned}\tag{2.278}$$

Thus, (2.275) – (2.278) in inequality (2.274) give

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_S v_1^2 dx \le \frac{3}{2}N_1\int_S v_1^2 dx - \frac{1}{2}\int_{\sigma\cup\partial\sigma_0} n_i v_1^2 v_i dS\\
&+\frac{bn}{2}\int_S\big[(\rho+\rho_1)^{2\gamma-4}\rho_1^2+\rho^{2\gamma-4}\rho_1^2+2v_1^2\big]dx\\
&-\frac{b}{2}\int_S v_i^1\big[\rho^{\gamma-2}\rho_{1,i}+(\rho+\rho_1)^{\gamma-2}\rho_{1,i}\big]dx\\
&+\int_S(\rho+\rho_1)^{\gamma-3}\Bigg[\ell_1(\rho+\rho_1)^{2-\gamma}\\
&+\frac{\ell_2}{(3-\gamma)\beta^{3-\gamma}}+\frac{\ell_3}{(3-\gamma)\alpha^{3-\gamma}}\Bigg]\rho_1^2 dx\\
&\int_S\rho^{\gamma-3}\frac{(2-\gamma)}{(3-\gamma)}\Bigg[\ell_2\beta^{(3-\gamma)/(2-\gamma)}\rho^{(3-\gamma)/(2-\gamma)}\\
&+\ell_3\alpha^{(3-\gamma)/(2-\gamma)}\rho^{\gamma(3-\gamma)/(2-\gamma)}\Bigg]\rho_1^2 dx\,.
\end{aligned}
\tag{2.279}
$$

The next step is to multiply equation $(2.260)_1$ by $\rho^{\gamma-3}\rho_1$ and then by $(\rho+\rho_1)^{\gamma-3}\rho_1$ and integrate over S. This part of the calculation follows that of (Graffi, 1999).

Upon multiplying $(2.260)_1$ by $\rho^{\gamma-3}\rho_1$ one may show that

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}(\rho^{\gamma-3}\rho_1^2) &= \rho^{\gamma-2}\rho_{1,i}v_i^1\\
&-\frac{\partial}{\partial x_i}\big[\rho^{\gamma-2}\rho_1 v_i^1+\frac{1}{2}\rho^{\gamma-3}\rho_1^2(v_i+v_i^1)\big]\\
&+\frac{\rho_1^2}{2}\frac{\partial}{\partial t}\rho^{\gamma-3}+\big[(\rho^{\gamma-2})_{,i}-\rho^{\gamma-3}\rho_{,i}\big]\rho_1 v_i^1\\
&-\frac{1}{2}\rho^{\gamma-3}\rho_1^2(v_i+v_i^1)_{,i}+\frac{1}{2}(\rho^{\gamma-3})_{,i}(v_i+v_i^1)\rho_1^2.
\end{aligned}
\tag{2.280}
$$

For the fourth term on the right,

$$
\begin{aligned}
\big|\big[(\rho^{\gamma-2})_{,i}-\rho^{\gamma-3}\rho_{,i}\big]\rho_1 v_i^1\big| &= |(3-\gamma)\rho^{\gamma-3}\rho_{,i}v_i\rho_1|\\
&\le n(3-\gamma)\rho^{\gamma-2}|\rho_1 v_1|\\
&\le \frac{n}{2}(3-\gamma)\big[\rho^{2\gamma-4}\rho_1^2+v_1^2\big]\,,
\end{aligned}
\tag{2.281}
$$

where hypothesis (f) has been employed. Further, using hypotheses (a) and (f), we have for constants q_1, q_2,

$$
\big|-\frac{1}{2}\rho^{\gamma-3}\rho_1^2(v_i+v_i^1)_{,i}\big| \le q_1\rho^{\gamma-3}\rho_1^2, \tag{2.282}
$$

$$
\big|\frac{1}{2}(\rho^{\gamma-3})_{,i}(v_i+v_i^1)\rho_1^2\big| \le q_2\rho^{\gamma-3}\rho_1^2. \tag{2.283}
$$

The third term on the right of (2.280) is handled by noting

$$\frac{\rho_1^2}{2}\frac{\partial}{\partial t}\rho^{\gamma-3} \le \frac{(3-\gamma)}{2}\rho_1^2\left|\frac{\rho_t}{\rho}\right|\rho^{\gamma-3}.$$

Then, from $(2.259)_1$,

$$\frac{\rho_t}{\rho} = -\frac{v_i}{\rho}\rho_{,i} - v_{i,i}\,.$$

Using hypotheses (a) and (f),

$$\left|\frac{\rho_t}{\rho}\right| \le q_3,$$

for a constant q_3. Then, for a further constant $q_4 > 0$,

$$\frac{\rho_1^2}{2}\frac{\partial}{\partial t}\rho^{\gamma-3} \le q_4\rho^{\gamma-3}\rho_1^2. \tag{2.284}$$

Thus, combining (2.281) – (2.284) in equation (2.280) we find

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}(\rho^{\gamma-3}\rho_1^2) &= \rho^{\gamma-2}\rho_{1,i}v_i^1\\ &\quad - \frac{\partial}{\partial x_i}\big[\rho^{\gamma-2}\rho_1 v_i^1 + \frac{1}{2}\rho^{\gamma-3}\rho_1^2(v_i+v_i^1)\big]\\ &\quad + m_2\rho^{2\gamma-4}\rho_1^2 + m_3 v_1^2 + m_4\rho^{\gamma-3}\rho_1^2\,.\end{aligned} \tag{2.285}$$

Similarly, we multiply equation $(2.260)_1$ by $(\rho+\rho_1)^{\gamma-3}\rho_1$ and obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\big[(\rho+\rho_1)^{\gamma-3}\rho_1^2\big] &= -\frac{\rho_1^2}{2}\frac{\partial}{\partial t}(\rho+\rho_1)^{\gamma-3}\\ &\quad + (\rho+\rho_1)^{\gamma-3}\rho_1\big[\rho v_i - (\rho+\rho_1)(v_i+v_i^1)\big]_{,i}\,.\end{aligned}$$

The last term of this expression may be rewritten

$$\begin{aligned}&(\rho+\rho_1)^{\gamma-3}\rho_1\Big\{-\big[(\rho+\rho_1)v_i^1\big]_{,i} - (\rho_1 v_i)_{,i}\Big\}\\ &= -\big[(\rho+\rho_1)^{\gamma-2}\rho_1 v_i^1\big]_{,i} + (\rho+\rho_1)^{\gamma-2}v_i^1\rho_{1,i}\\ &\quad - (\rho+\rho_1)^{\gamma-3}\rho_1^2 v_{i,i} - (\rho+\rho_1)^{\gamma-3}\frac{v_i}{2}(\rho_1^2)_{,i}\\ &= -\frac{\partial}{\partial x_i}\big[(\rho+\rho_1)^{\gamma-2}\rho_1 v_i^1 + \frac{1}{2}(\rho+\rho_1)^{\gamma-3}v_i\rho_1^2\big]\\ &\quad - \frac{1}{2}(\rho+\rho_1)^{\gamma-3}v_{i,i}\rho_1^2\\ &\quad + \frac{(\gamma-3)}{2}v_i\frac{(\rho+\rho_1)_{,i}}{(\rho+\rho_1)}(\rho+\rho_1)^{\gamma-3}\rho_1^2 + (\rho+\rho_1)^{\gamma-2}v_i^1\rho_{1,i}\,.\end{aligned}$$

Hence, we find

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\big[(\rho+\rho_1)^{\gamma-3}\rho_1^2\big] = & -\frac{\rho_1^2}{2}\frac{\partial}{\partial t}(\rho+\rho_1)^{\gamma-3} \\
& - \frac{\partial}{\partial x_i}\big[(\rho+\rho_1)^{\gamma-2}\rho_1 v_i^1 + \frac{1}{2}(\rho+\rho_1)^{\gamma-3}v_i\rho_1^2\big] \\
& - \frac{1}{2}(\rho+\rho_1)^{\gamma-3}v_{i,i}\rho_1^2 \\
& + \frac{(\gamma-3)}{2}v_i\frac{(\rho+\rho_1)_{,i}}{(\rho+\rho_1)}(\rho+\rho_1)^{\gamma-3}\rho_1^2 + (\rho+\rho_1)^{\gamma-2}v_i^1\rho_{1,i}\,.
\end{aligned}
\tag{2.286}
$$

From equation $(2.259)_1$

$$
\frac{(\rho+\rho_1)_t}{(\rho+\rho_1)} = -\frac{(v_i+v_i^1)}{(\rho+\rho_1)}(\rho+\rho_1)_{,i} - (v_i+v_i^1)_{,i}
$$

and hence recollecting hypotheses (a) and (f) we find from (2.286) that there is a constant m_5 such that

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\big[(\rho+\rho_1)^{\gamma-3}\rho_1^2\big] \le & \, m_5(\rho+\rho_1)^{\gamma-3}\rho_1^2 \\
& - \frac{\partial}{\partial x_i}\big[(\rho+\rho_1)^{\gamma-2}\rho_1 v_i^1 + \frac{1}{2}(\rho+\rho_1)^{\gamma-3}v_i\rho_1^2\big] \\
& + (\rho+\rho_1)^{\gamma-2}v_i^1\rho_{1,i}\,.
\end{aligned}
\tag{2.287}
$$

Upon adding (2.285) and (2.287) and integrating over S we may derive,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_S \Big\{\big[\rho^{\gamma-3} + (\rho+\rho_1)^{\gamma-3}\big]\rho_1^2\Big\}dx \le & \int_S \big[\rho^{\gamma-2} + (\rho+\rho_1)^{\gamma-2}\big]v_i^1\rho_{1,i}dx \\
- \int_{\sigma\cup\partial\sigma_0} n_i\Big[\{\rho^{\gamma-2} + (\rho+\rho_1)^{\gamma-2}\}\rho_1 v_i^1 & \\
& + \frac{1}{2}\rho^{\gamma-3}\rho_1^2(v_i+v_i^1) + \frac{1}{2}(\rho+\rho_1)^{\gamma-3}v_i\rho_1^2\Big]dS \\
+ m_2\int_S \rho^{2\gamma-4}\rho_1^2 dx + m_3\int_S v_1^2 dx & \\
+ \int_S \big[m_4\rho^{\gamma-3} + m_5(\rho+\rho_1)^{\gamma-3}\big]\rho_1^2 dx. &
\end{aligned}
\tag{2.288}
$$

The idea is now to add (2.279) and (2.288) together in such a way that the terms involving $v_i^1\rho_{1,i}$ add to zero. So, we add (2.279)+$(b/2)$(2.288) to derive

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\int_S v_1^2 dx + \frac{b}{4}\int_S \rho^{\gamma-3}\rho_1^2 dx + \frac{b}{4}\int_S (\rho+\rho_1)^{\gamma-3}\rho_1^2 dx\right]\\
&\leq \int_\sigma \frac{1}{2}\big[\rho^{2\gamma-4} + (\rho+\rho_1)^{2\gamma-4}\big]\rho_1^2 dS + \frac{1}{2}\int_\sigma (1+n_1)v_1^2 dS\\
&\quad + \int_\sigma \frac{n_1}{2}\big[\rho^{\gamma-3} + (\rho+\rho_1)^{\gamma-3}\big]\rho_1^2 dS\\
&\quad + \int_S \rho_1^2\rho^{\gamma-3}[m_4 + r_1\rho^{\gamma-1} + r_2\rho^{(3-\gamma)/(2-\gamma)} + r_3\rho^{\gamma(3-\gamma)/(2-\gamma)}]dx\\
&\quad + \int_S \rho_1^2(\rho+\rho_1)^{\gamma-3}\left[r_4 + \frac{bn}{2}(\rho+\rho_1)^{\gamma-1}\right] dx\\
&\quad + r_4\int_S v_1^2 dx\,,
\end{aligned} \tag{2.289}$$

where

$$\begin{aligned}
&r_1 = m_2 + \frac{bn}{2}\,, \qquad r_2 = \left(\frac{2-\gamma}{3-\gamma}\right)\ell_2\beta^{(3-\gamma)/(2-\gamma)},\\
&r_3 = \left(\frac{2-\gamma}{3-\gamma}\right)\ell_3\alpha^{(3-\gamma)/(2-\gamma)},\\
&r_4 = m_5 + \frac{\ell_2}{(3-\gamma)\beta^{3-\gamma}} + \frac{\ell_3}{(3-\gamma)\alpha^{3-\gamma}}\,, \qquad r_5 = m_3 + \frac{3N_1}{2} + bn\,.
\end{aligned}$$

Now invoke hypothesis (a), let n_1 be a bound for $\rho, \rho+\rho_1$, and integrate (2.289) twice over the time interval $(0,h)$ to see that

$$\begin{aligned}
&\int_0^h dt\int_S \Big[\frac{1}{2}v_1^2 + \frac{b}{4}\rho^{\gamma-3}\rho_1^2 + \frac{b}{4}(\rho+\rho_1)^{\gamma-3}\rho_1^2\Big]dx\\
&\leq \int_0^h dt\int_\sigma h(n_1^{\gamma-1} + n_1)\big[\rho^{\gamma-3} + (\rho+\rho_1)^{\gamma-3}\big]\rho_1^2 dS\\
&\quad + \int_0^h dt\int_\sigma h\frac{(1+n_1)}{2}v_1^2 dS + \int_0^h dt\int_S hk_1\big[\rho^{\gamma-3} + (\rho+\rho_1)^{\gamma-3}\big]\rho_1^2 dx\\
&\quad + r_4 h\int_0^h dt\int_S v_1^2 dx\,,
\end{aligned} \tag{2.290}$$

for a constant k_1 independent of h. Let $n_1^{\gamma-1}+n_1, (1+n_1)/2$ be denoted by constants r_5, r_6. Then we rewrite (2.290) as

$$\begin{aligned}
&\int_0^h dt \int_S v_1^2(1-2r_4h)dx + \int_0^h dt \int_S \rho_1^2\rho^{\gamma-3}\Big(\frac{b}{2}-2k_1h\Big)dx \\
&+\int_0^h dt \int_S \rho_1^2(\rho+\rho_1)^{\gamma-3}\Big(\frac{b}{2}-2k_1h\Big)dx \\
&\le 2r_6h\int_0^h dt \int_\sigma v_1^2 dS + 2r_5h\int_0^h dt\int_\sigma \rho^{\gamma-3}\rho_1^2 dS \\
&\quad + 2r_5h\int_0^h dt\int_\sigma (\rho+\rho_1)^{\gamma-3}\rho_1^2 dS\,. \qquad (2.291)
\end{aligned}$$

Now suppose h is such that

$$1-2r_4h\ge\frac{1}{2}\,,\qquad \frac{b}{2}-2k_1h\ge\frac{b}{4}\,,$$

then define the Graffi function $G(R)$ by

$$G(R)=\int_0^h dt\int_S\Big(v_1^2+\frac{b}{2}\big[\rho^{\gamma-3}+(\rho+\rho_1)^{\gamma-3}\big]\rho_1^2\Big)dx\,. \qquad (2.292)$$

Then from (2.291) we see that for a constant $A=\max\{8r_6h, 8r_5h/b\}$,

$$G(R)\le AG'(R).$$

This inequality integrates to see that for $R\ge R_0>0$,

$$G(R)\ge G(R_0)\,\exp\left(\frac{R-R_0}{A}\right). \qquad (2.293)$$

Now, $|\rho_1|=|\rho+\rho_1-\rho|\le|\rho+\rho_1|+|\rho|$ and so by hypothesis (a), $|\rho_1|$ and v_1 are bounded then $G(R)$ has maximum growth in R like $R^{\beta(3-\gamma)+3}$ using also hypothesis (g). Thus,

$$\lim_{R\to 0}\frac{G(R)}{R^{\beta(3-\gamma)+3+\epsilon}}=0.$$

This contradicts (2.293) and so $v_i^1\equiv 0$, $\rho_1\equiv 0$ on $S\times(0,h)$. Since the bounds in hypotheses (a), (d)-(g) are independent of h we may reapply the argument on $(h,2h)$ etc., to conclude uniqueness on $S\times(0,\mathcal{T})$.

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Stability and Wave Motion in Porous Media
straughan, b.
2008, XIV, 439 p., Hardcover
ISBN: 978-0-387-76541-9