Solutions of Chapter 8 Problems

Problem 8.1

We consider the 4-ary decision problem

$$H_k \,:\, Y(t) = s_k(t) + V(t)$$

for $0 \leq t \leq T$ and $k = 0,\,1,\,2,\,3$, with

$$\begin{array}{rcl} s_0(t) = A\sin(\omega_0 t) & , & s_1(t) = A\cos(\omega_0 t) \\ s_2(t) = -A\sin(\omega_0 t) & , & s_3(t) = -A\cos(\omega_0 t)\,. \end{array}$$

The signals $s_k(t)$ have all the same energy

$$E = \int_0^T s_k^2(t)dt = \frac{A^2}{2}\int_0^T [1 \mp \cos(\omega_0 t)]dt = \frac{A^2 T}{2}\,.$$

To formulate thid detection problem, we perform a Karhunen-Loève expansion of the observation and noise signals $Y(t)$ and $V(t)$ and of the signals $s_k(t)$ with respect to a basis such that

$$\phi_1(t) = (2/T)^{1/2}\sin(\omega_0 t) \;\;,\;\; \phi_2(t) = (2/T)^{1/2}\cos(\omega 0 t)\,,$$

and where functions $\phi_j(t)$ for $j \geq 3$ are arbitrary and orthogonal to $\phi_1(t)$ and $\phi_2(t)$. Since the signals

$$\begin{array}{rcl} s_0(t) = E^{1/2}\phi_1(t) & , & s_1(t) = E^{1/2}\phi_2(t) \\ s_2(t) = -E^{1/2}\phi_1(t) & , & s_3(t) = -E^{1/2}\phi_2(t) \end{array}$$

depend only on the first two basis functions, only the first two coefficients of the Karhunen-Loève (KL) expansions

$$\begin{aligned} Y(t) &= \sum_{j=1}^{\infty} Y_j \phi_j(t) \\ V(t) &= \sum_{j=1}^{\infty} V_j \phi_j(t) \end{aligned}$$

play a role in solving the 4-ary detection problem.
a) In KL coordinates, if

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad , \quad \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\,,$$

the detection problem becomes

$$H_k \,:\, \mathbf{Y} = \mathbf{S}_k + \mathbf{V}$$

for $0 \leq k \leq 3$, with

$$\begin{aligned} \mathbf{S}_0 &= -\mathbf{S}_2 = E^{1/2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{S}_1 &= -\mathbf{S}_3 = E^{1/2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\,. \end{aligned}$$

The signal space representation is shown in Fig. 27 below, where the decision region $\mathcal{Y}_0$ for hypothesis H_0 is represented in gray. The decision regions for the other three hypotheses are obtained by symmetry.

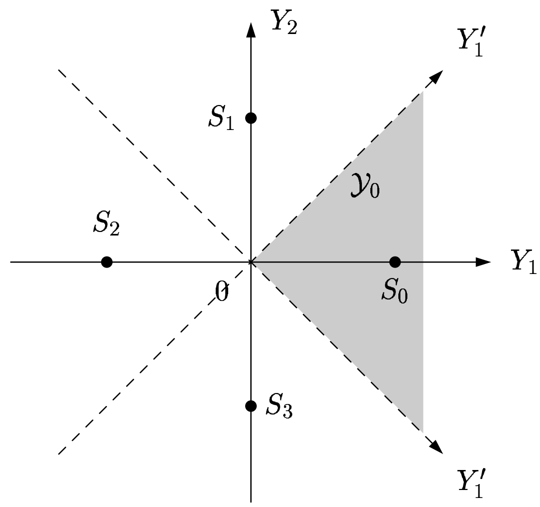


Figure 27: Signal space representation of the QPSK detection problem.

Since the signals $s_k(t)$ are expressible in terms of only two basis functions, two correlators/integrators or two matched filters are required to implement the optimum receiver.

b) To evaluate the probability of a correct decision, we note that because of the symmetry between the 4 signals

$$P[C] = P[C|H_0] \,.$$

To compute $P[C|H_0]$, it is convenient to rotate the axes by $-\pi/4$ and consider the coordinate system

$$\mathbf{Y}' \triangleq \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{Y} \,.$$

In this coordinate system, hypothesis H_0 becomes

$$H_0 \,:\, \mathbf{Y}' = \mathbf{S}_0' + \mathbf{V}' \,,$$

where

$$\mathbf{S}_0' = (E/2)^{1/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \,,$$

and $\mathbf{V}'$ is $N(0, \sigma^2 \mathbf{I}_2)$ distributed, since the distribution of a WGN is not affected by the choice of orthonormal coordinates. Under H_0, the probability density of vector $\mathbf{Y}'$ is given by

$$f_{\mathbf{Y}'}(\mathbf{y}'|H_0) = \frac{1}{2\pi\sigma^2} \exp\Big(-\frac{1}{2\sigma^2}[\big(y_1' - (\frac{E}{2})^{1/2}\big)^2 + \big(y_2' - (\frac{E}{2})^{1/2}\big)^2]\Big) \,,$$

and we select H_0 whenever $Y_1' \geq 0$ and $Y_2' \geq 0$. The probability of a correct decision can therefore be expressed as

$$P[C] = \int_0^\infty \int_0^\infty f_{\mathbf{Y}'}(\mathbf{y}'|H_0) dy_1' dy_2' = (P_1[C])^2 \,,$$

with

$$\begin{aligned} P_1[C] &= \int_0^\infty \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\Big(-\frac{1}{2\sigma^2}(y_1' - (\frac{E}{2})^{1/2})\Big) dy_1' \\ &= 1 - Q(d_1/2) \,. \end{aligned}$$

Here

$$d_1 = \frac{(E/2)^{1/2} - (-(E/2)^{1/2})}{\sigma} = \frac{(2E)^{1/2}}{\sigma}$$

represents the scaled distance between points $(E/2)^{1/2}$ and $-(E/2)^{1/2}$ corresponding to the y_1' coordinates of signal vectors $\mathbf{S}_0$ and $\mathbf{S}_1$. The scaling unit employed for evaluating the distance d_1 is the standard deviation σ of the WGN. Thus

$$P[C] = \big(1 - Q(\frac{(2E)^{1/2}}{\sigma}\big)^2 \,.$$

Note that since each quadriphase signal carries two bits, the signal energy $E = 2E_b$, where E_b denotes the bit enery, so that in terms of E_b

$$P_{\text{QPSK}}[C] = \big(1 - Q(E_b^{1/2}/\sigma)\big)^2 \,.$$

c) For a BPSK signal we have

$$P_{\text{BPSK}}[C] = 1 - Q(d/2) \,,$$

where

$$d = (E_b^{1/2} - (-E_b^{1/2}))/\sigma = 2E_b^{1/2}/\sigma \,,$$

so that

$$P_{\text{BPSK}}[C] = 1 - Q(E_b^{1/2}/\sigma) \,.$$

Comparing with the result of part b), we find

$$P_{\text{QPSK}}[C] = \big(P_{\text{BPSK}}[C]\big)^2 < P_{\text{BPSK}}[C] \,,$$

so the probability of a correct decision is smaller for QPSK than BPSK. However, for QPSK two bits are transamitted per baud, and the probability of a correct decision for a single bit is $P_1[C]$, which equals $P_{\text{BPSK}}[C]$, so the bit probability of error is the same for BPSK and QPSK, and equals

$$P_b = Q(E_b^{1/2}/\sigma) \,.$$

Problem 8.2

a) Let $s_k(t) = s(t - kT_c)$. Obviously, the signals $s_k(t)$ with $0 \leq k \leq M-1$ are orthogonal with equal energy

$$E = ||s_k||^2 = \int_0^{T_c} s^2(t)dt = \frac{A^2 T_c}{2} .$$

Then if we select $\phi_k(t) = s_k(t)/E^{1/2}$ with $1 \leq k \leq M$ as the first M basis functions of a COB, and if

$$\begin{aligned} Y_k &= < Y, \phi_k >= \int_0^T Y(t)\phi_k(t)dt \\ V_k &= < V, \phi_k >= \int_0^T V(t)\phi_k(t)dt \end{aligned}$$

denote the Karhunen-Loeve coefficients of the observation and noise processes with respect to this basis, the detection problem can be expressed as an M-ary detection problem in M-dimensional space. Let

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \\ \vdots \\ Y_M \end{bmatrix} \quad , \quad \mathbf{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_k \\ \vdots \\ V_M \end{bmatrix} .$$

The vector $\mathbf{V}$ is $N(\mathbf{0}, \sigma^2 \mathbf{I}_M)$ distributed, and since $s_k(t) = E^{1/2}\phi_k(t)$, the detection problem can be expressed as

$$H_k \; : \; \mathbf{Y} = \mathbf{S}_k + \mathbf{V}$$

where $\mathbf{S}_k = E^{1/2}\mathbf{e}_k$ is colinear with the k-th basis vector $\mathbf{e}_k$ of $\mathbb{R}^M$. The entries of $\mathbf{e}_k$ are all zero, except the k-th entry, which equals one. The vectors $\mathbf{S}_k$ are all orthogonal, and according to Example 8.2, the optimum decision rule is the largest component rule

$$\delta(\mathbf{Y}) = \arg \max_{0 \leq k \leq M-1} Y_{k+1} = \arg \max_{0 \leq k \leq M-1} E^{1/2} Y_{k+1} .$$

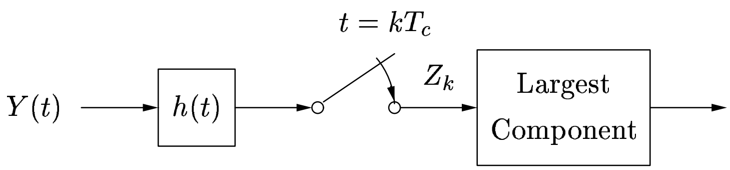
noindent b) Let $Z_k = E^{1/2} Y_k$. Since

$$Z_k = \int_0^T Y(t)s_k(t)dt = \int_0^T Y(t)s(t - kT_c)dt$$

by introducing the matched filter $h(t) = s(T_c - t)$ we obtain

$$Z_k = \int_0^T Y(t)h((k+1)T_c - t)dt = h(t) * Y(t) \,|_{t=(k+1)T_c} \; .$$

Thus the optimum receiver can be implemented by passing the observed signal $Y(t)$ through the matched filter $h(t)$ sampled at the cycle period T_c instead of the baud period T, and then by selecting the largest component among all samples Z_k, $0 \leq k \leq M-1$ obtained for each baud period. The corresponding block diagram is shown below.



c) According to equation (8.62) of Example 8.2, the probability of error is given by

$$P[E] = 1 - P[C] = 1 - \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \exp\Big(-\frac{(u-(E^{1/2}/\sigma))^2}{2}\Big)[1-Q(u)]^{M-1} du\,.$$

Problem 8.3

a) Let

$$Y = \int_0^T Y(t)p(t)dt \quad , \quad V = \int_0^T V(t)p(t)dt\,.$$

The PAM detection problem reduces to a one-dimensional problem of the form

$$H_i \,:\, Y = A_i + V \tag{8.1}$$

for $1 \leq i \leq M = 2(n+1)$, with

$$A_i = \frac{A}{n}(i-(n+1))$$

and $V \sim N(0, \sigma^2)$. The signal space representation of (8.1) is depicted in Fig. 28 below.

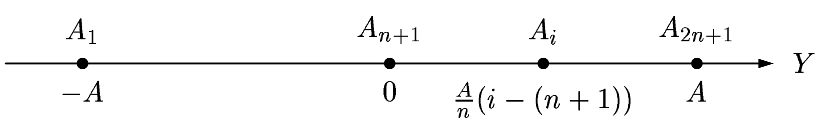


Figure 28: One dimensional representation of the PAM detection problem.

By applying the minimum distance rule, the optimum receiver selects H_i with $2 \leq i \leq 2n$ if

$$A_i - \frac{A}{2n} < Y \leq A_i + \frac{A}{2n}\,;$$

it selects H_1 for

$$Y \leq A_1 + \frac{A}{2n}\,,$$

and H_{2n+1} for

$$Y \geq A_{2n+1} - \frac{A}{2n}\,.$$

b) Under hypothesis H_i with $2 \leq i \leq 2n$, an error occurs if observation Y falls either below $A_i - A/2n$, or above $A_i + 1/2n$, so the probability of error can be expressed as

$$P[E|H_i] = P[Y \leq A_i - \frac{A}{2n}|H_i] + P[Y > A_i + \frac{A}{2n}|H_i]\,. \tag{8.2}$$

By symmetry, the two terms on the right-hand side of (8.2) are equal, and

$$P[Y > A_i + \frac{A}{2n}|H_i] = P[V > \frac{A}{2n}] = Q\big(\frac{h}{2\sigma}\big)$$

where

$$h = \frac{A}{n} = \frac{2A}{(M-1)}$$

represents the distance between two neighboring signal points A_i. scaled by the standard deviation of the noise. Thus we have

$$P[E|H_i] = 2Q\big(\frac{h}{2\sigma}\big)$$

for $2 \leq i \leq 2n$. For $i = 1$ and $i = 2n+1$, the probability of error can be expressed as

$$\begin{aligned} P[E|H_1] &= P[Y \leq A_1 - A/2n] = Q\big(\frac{h}{2\sigma}\big) \\ P[E|H_{2n+1}] &= P[Y > A_{2n+1} + A/2n] = Q\big(\frac{h}{2\sigma}\big) . \end{aligned}$$

Since the probability of each hypothesis is $P[H_i] = 1/M$, the average probability of error is given by

$$\begin{aligned} P[E] &= \sum_{i=1}^{2n+1} P[E|H_i]P[H_i] \\ &= \frac{M-2}{M} 2Q\big(\frac{h}{2\sigma)}\big) + \frac{2}{M}Q\big(\frac{h}{2\sigma}\big) = 2\big(1 - \frac{1}{M}\big)Q\big(\frac{h}{2\sigma}\big) . \end{aligned}$$

c) The average energy of the PAM signal is given by

$$E_{\text{ave}} = \frac{1}{M}\sum_{i=1}^{2n+1} E_i$$

where the energy E_i of signal $s_i(t)$ can be expressed as

$$E_i = A_i^2 = \Big(\frac{A}{n}\Big)^2 (i - (n+1))^2 .$$

This gives

$$E_{ave} = \frac{2}{M}\Big(\frac{A}{n}\Big)^2 \sum_{j=0}^{n} j^2$$

and taking into account the identity

$$\sum_{j=0}^{n} j^2 = \frac{n(n+1/2)(n+1)}{3} ,$$

we obtain

$$E_{\text{ave}} = \frac{A^2}{3}\frac{M+1}{M-1}\,.$$

Since

$$(h/2)^2 = \frac{A^2}{(M-1)^2} = \frac{3E_{\text{ave}}}{(M-1)(M+1)}$$

we have therefore

$$P[E] = 2\big(1-\frac{1}{M}\big)Q\left(\sqrt{\frac{3E_{\text{ave}}}{(M^2-1)\sigma^2}}\right)\,.$$

Problem 8.4

a) We have

$$s(t) = (2E/T)^{1/2}[\cos(\omega_c t)\cos\Theta(t) - \sin(\omega_c t)\sin\Theta(t)]\,.$$

In this expression, by observing that

$$\begin{aligned}\Theta(t) &= \Theta(2kT) + I_{2k}\frac{\pi}{2T}(t-2kT) \quad 2kT \le t \le (2k+1)T\\ \Theta(t) &= \Theta(2kT) + I_{2k-1}\frac{\pi}{2T}(t-2kT) \quad (2k-1)T \le t \le 2kT\end{aligned}$$

where I_{2k}, $I_{2k-1} \in \{1, -1\}$, and $\Theta(2kT) \in \{0, \pi\}$. we conclude that

$$\cos\Theta(t) = c(k)\cos(\frac{\pi}{2T}(t-2kT)) \tag{8.3}$$

for $(2k-1)T \le t \le (2k+1)T$, with

$$c(k) \triangleq \cos\Theta(2kT) = \cos\left(\frac{\pi}{2}\sum_{\ell=0}^{2k-1} I_\ell\right)\,.$$

Similarly, we observe that

$$\begin{aligned}\Theta(t) &= \Theta((2k+1)T) + I_{2k}\frac{\pi}{2T}(t-(2k+1)T) \quad 2kT \le t \le (2k+1)T\\ \Theta(t) &= \Theta((2k+1)T) + I_{2k+1}\frac{\pi}{2T}(t-(2k+1)T) \quad (2k+1)T \le t \le 2(k+1)T\end{aligned} \tag{8.4}$$

where I_{2k}, $I_{2k+1} \in \{1, -1\}$, and $\Theta((2k+1)T) \in \{0, \pi\}$, so that

$$\sin\Theta(t) = s(k+1)\cos(\frac{pi}{2T}(t-(2k+1)T)) \tag{8.5}$$

for $2kT \le t \le 2(k+1)T$, with

$$s(k+1) \triangleq \sin\Theta((2k+1)T) = \sin\left(\frac{\pi}{2}\sum_{\ell=0}^{2k} I_\ell\right) = I_{2k}c(k)\,.$$

Combining identities (), (8.3) and (8.5), we conclude that

$$s(t) = s_c(t)\cos(\omega_c t) - s_s(t)\sin(\omega_c t) \tag{8.6}$$

where

$$\begin{aligned} s_c(t) &= \sum_{k=0}^{\infty} c(k)p(t-2kT) \\ s_s(t) &= \sum_{k=1}^{\infty} s(k)p(t-(2k-1)T) \end{aligned}$$

where

$$p(t) = \begin{cases} \left(\frac{2E}{T}\right)^{1/2}\cos\left(\frac{\pi t}{2T}\right) & T \leq t \leq T \\ 0 & \text{otherwise}\ . \end{cases}$$

The expression (8.6) indicates that $s(t)$ can be viewed as an offset QPSK signal with baud rate $1/(2T)$ transmitting in-phase and quadrature pseudo-symbols $c(k)$ and $s(k)$.

b) The observations $Y(t)$ and noise $V(t)$ admit bandpass representations of the form

$$\begin{aligned} Y(t) &= Y_c(t)\cos(\omega_c t) - Y_s(t)\sin(\omega_c t) \\ V(t) &= V_c(t)\cos(\omega_c t) - V_s(t)\sin(\omega_c t)\ , \end{aligned}$$

where $V_c(t)$ and $V_s(t)$ are independent WGNs with intensity $2\sigma^2$. Accordingly, by extracting the in-phase and quadrature components of $Y(t)$, we obtain observations

$$\begin{aligned} Y_c(t) &= s_c(t) + V_c(t) \\ Y_s(t) &= s_s(t) + V_s(t)\ . \end{aligned}$$

Over interval $(2k-1)T \leq t \leq (2k+1)T$, we have

$$Y_c(t) = c(k)p(t-2kT) + V_c(t)$$

with $c(k) = \pm 1$, so that if $h(t) = p(-t) = p(t)$, the optimum receiver can be implemented as the matched filter shown in Fig. 29 shown below.

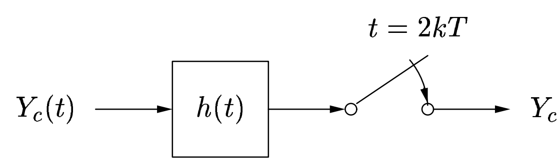


Figure 29: Block diagram of the optimum receiver for the in-phase component $Y_c(t)$ of the received signal.

Note that $h(t)$ is noncausal, so that a causal implementation would actually require implementing $h(t-T)$ and sampling at $(2k+1)T$.

The energy of pulse $p(t)$ is given by

$$\begin{aligned} E_p &= \int_{-T}^{T} p^2(t)dt = \frac{2E}{T}\int_{-T}^{T}\cos^2(\pi t/T)dt \\ &= \frac{E}{T}\int_{-T}^{T}[1+\cos(2\pi t/T)]dt = 2E\,, \end{aligned} \tag{8.7}$$

and the output Y_c of the matched filter in Fig. 29 can be expressed as

$$Y_c = E_p c(k) + V_c\,,$$

where $V_c \sim N(0, 2\sigma^2 E_p)$. The optimum decision rule is therefore given by

$$\hat{c}(k) = \mathrm{sgn}\,(Y_c)\,,$$

and the probability of error is $Q(d/2)$ where

$$d = \frac{2E_p}{(2\sigma^2 E_p)^{1/2}} = \frac{(2E_p)^{1/2}}{\sigma} = \frac{2E^{1/2}}{\sigma} \tag{8.8}$$

represents the distance between the two signal points $\pm E_p$ scaled by the standard deviation of noise V_c.

Similarly, over $2kT \leq t \leq 2(k+1)T$ we have

$$Y_s(t) = s(k+1)p(t-(2k+1)T) + V_s(t)$$

with $s(k+1) = \pm 1$, so that if $h(t) = p(-t) = p(t)$, the optimum receiver can be implemented as the matched filter shown in Fig. 30 shown below.

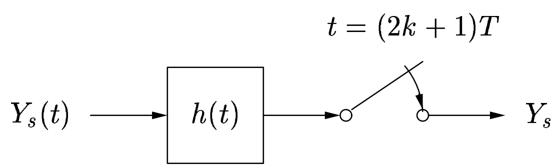


Figure 30: Block diagram of the optimum receiver for the auadrature component $Y_s(t)$ of the received signal.

We have therefore

$$Y_s = E_p s(k+1) + V_s\,,$$

with $V_s \sim N(0, 2\sigma^2 E_p)$, so that the optimum decision rule is given by

$$\hat{s}(k+1) = \mathrm{sgn}\,(Y_s)\,,$$

and the corresponding probability of error is $Q(d/2)$, where d is given by (8.8).

c) Since $s(k+1) = I_{2k}c(k)$, I_{2k} can be decoded according to the rule $\hat{I}_{2k} = \hat{s}(k+1)\hat{c}(k)$. Similarly, we have

$$c(k+1) = \cos\big(\frac{\pi}{2}\sum_{\ell=0}^{2k+1} I_\ell\big) = -I_{2k+1}s(k+1)\ ,$$

so that $\hat{I}_{2k+1} = -\hat{c}(k+1)\hat{s}(k+1)$.

From the above expressions, we see that a correct decision is reached for I_{2k} or I_{2k+1} if either the decisions for $s(k+1)$ and $c(k)$ (or $s(k+1)$ and $c(k+1)$) are both correct or both wrong, i.e.,

$$\begin{aligned} P[C] &= (1-P[E])^2 + P^2[E] \\ &= (1-Q(d/2))^2 + Q^2(d/2) \\ &= (1-Q(E^{1/2}/\sigma))^2 + Q(E^{1/2}/\sigma) \end{aligned}$$

so that the probability of error of the suboptimal MSK detection scheme considered here is given by

$$P[E] = 1 - P[C] = 2Q(E^{1/2}/\sigma)(1-Q(E^{1/2}/\sigma))\ .$$

For high SNR the second term is negligible, and

$$P[E] \approx 2Q(E^{1/2}/\sigma)\ .$$

Problem 8.5

Consider the two orthonormal functions

$$\phi_1(t) = (2/T)^{1/2}\cos(2\pi t/T) \quad , \quad \phi_2(t) = (2/T)^{1/2}\cos(4\pi t/T)\ .$$

Then Mercer's expansion of $K_V(t,s)$ takes the form

$$K_V(t,s) = \lambda_1\phi_1(t) + \lambda_2\phi_2(t)\ ,$$

where

$$\lambda_1 = \frac{CT}{2} \quad , \quad \lambda_2 = \frac{DT}{2}\ .$$

Thus $V(t)$ admits the Karhunen-Loève (KL) expansion

$$V(t) = V_1\phi_1(t) + V_2\phi_2(t)\ ,$$

where the coefficients V_1 and V_2 are independent with $V_j \sim N(0,\lambda_j)$ for $j = 1, 2$. Since

$$s(t) = S_1\phi_1(t) + S_2\phi_2(t)$$

with

$$S_1 = A(T/2)^{1/2} \quad , \quad S_2 = B(T/2)^{1/2}\ ,$$

the observation signal $Y(t)$ admits a KL expansion

$$Y(t) = Y_1\phi_1(t) + Y_2\phi_2(t)\ .$$

Accordingly the detection problem can be expressed in KL coefficient space as

$$\begin{aligned} H_0 &: \mathbf{Y} = \mathbf{V} \\ H_0 &: \mathbf{Y} = \mathbf{S} + \mathbf{V} \,, \end{aligned}$$

with

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad , \quad \mathbf{S} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} ,$$

and where

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_1 \end{bmatrix} \sim N\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix} , \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Big) .$$

The probability densities of $\mathbf{Y}$ under H_0 and H_1 can be expressed as

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}|H_0) &= \frac{1}{2\pi(\lambda_1\lambda_2)^{1/2}} \exp\Big[- \Big(\frac{y_1^2}{2\lambda_1} + \frac{y_2^2}{2\lambda_2} \Big) \Big] \\ f_{\mathbf{Y}}(\mathbf{y}|H_1) &= \frac{1}{2\pi(\lambda_1\lambda_2)^{1/2}} \exp\Big[- \Big(\frac{(y_1 - S_1)^2}{2\lambda_1} + \frac{(y_2 - S_2)^2}{2\lambda_2} \Big) \Big] , \end{aligned}$$

so that the LRT takes the form

$$\begin{aligned} L(\mathbf{y}) &= \frac{f_{\mathbf{Y}}(\mathbf{y}|H_1)}{f_{\mathbf{Y}}(\mathbf{y}|H_0)} \\ &= \exp(-E/2) \exp\Big(\frac{y_1 S_1}{\lambda_1} + \frac{y_2 S_2}{\lambda_2} \Big) \underset{H_0}{\overset{H_1}{\gtrless}} \tau \end{aligned} \tag{8.9}$$

with

$$E \triangleq \frac{S_1^2}{\lambda_1} + \frac{S_2^2}{\lambda_2} = \frac{A^2}{C} + \frac{B^2}{D} .$$

Taking logarithms on both sides of (8.9, the LRT can be expressed as

$$R \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \ln(\tau) + E/2 \,, \tag{8.10}$$

where the sufficient statistic

$$R \triangleq \frac{Y_1 S_1}{\lambda_1} + \frac{Y_2 S_2}{\lambda_2} = \frac{2}{T} \int_0^T Y(t) \Big[\frac{A}{C} \cos(2\pi t) + \frac{B}{D} \cos(4\pi t) \Big] dt \,.$$

Under H_0, we have $R \sim N(0, E)$ and under H_1, $R \sim N(E, E)$. The probability of false alarm for test can be expressed as

$$\begin{aligned} P_F &= \int_\eta^\infty f_R(r|H_0) dr \\ &= \frac{1}{(2\pi E)^{1/2}} \int_\eta^\infty \exp\big(- \frac{\ell^2}{2E} \big) d\ell = Q(\eta/E^{1/2}) \,, \end{aligned} \tag{8.11}$$

so that given a desired probability of false alarm P_F, the NP test threshold

$$\eta = E^{1/2} Q^{-1}(P_F) \, .$$

Similary the probability of detection is given by

$$\begin{aligned} P_D &= \int_\eta^\infty f_R(r|H_1) dr \\ &= \frac{1}{(2\pi E)^{1/2}} \int_\eta^\infty \exp\big(-\frac{(\ell - E)^2}{2E}\big) d\ell = Q((\eta - E)/E^{1/2}) \\ &= 1 - Q(E^{1/2} - \eta/E^{1/2}) = 1 - Q(E^{1/2} - Q^{-1}(P_F)) \, , \end{aligned} \tag{8.12}$$

where to go from the second to the third line we have used the identity $Q(-x) = 1 - Q(x)$.

Problem 8.6

Consider the zero-mean colored Gaussian noise $Z(t)$ with covariance function

$$K_Z(t,s) = \cos(\omega_0(t-s)) \, .$$

According to Example 7.3, K_Z has only two nonzero eigenvalues $\lambda_1 = \lambda_2 = T/2$ corresponding to eigenfunctions

$$\phi_1(t) = (2/T)^{1/2} \cos(\omega_0 t) \quad , \quad \phi_2(t) = (2/T)^{1/2} \sin(\omega_0 t) \, .$$

By completing this basis with othonormal functions $\phi_k(t)$ with $k \geq 3$, and expressing the detection problem in the KL domain, it can be reduced to a two-dimensional problem of the form

$$\begin{aligned} H_0 &: \mathbf{Y} = \mathbf{S}_0 + \mathbf{Z} + \mathbf{V} \\ H_1 &: \mathbf{Y} = \mathbf{S}_1 + \mathbf{Z} + \mathbf{V} \, , \end{aligned} \tag{8.13}$$

where we have

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad \mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

with

$$\begin{aligned} Y_k &= \int_0^T Y(t)\phi_k(t) dt \\ Z_k &= \int_0^T Z(t)\phi_k(t) dt \\ V_k &= \int_0^T V(t)\phi_k(t) dt \, . \end{aligned}$$

The formulation (8.13) of the detection problem relies on the observation that under H_0 and H_1, the KL coefficients $Y_k = V_k$ for $k \geq 3$ provide no information about the detection problem and can therefore be discarded. In (8.13) the vectors $\mathbf{Z}$ and $\mathbf{V}$ are independent with

$$\mathbf{Z} \sim N(\mathbf{0}, (T/2)\mathbf{I}_2) \quad \text{and} \quad \mathbf{V} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_2) \,.$$

The signal vectors are given by

$$\mathbf{S}_0 = E^{1/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{S}_1 = E^{1/2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with $E = A^2 T/2$, and are therefore orthogonal. Since the hypotheses are equally likely and we seek to minimize the probability of error, the Bayesian theshold $\tau = 1$. Then by using the results of Section 8.2.2, we find that the optimum decision rule is

$$\mathbf{Y}^T (\mathbf{S}_1 - \mathbf{S}_0) \underset{H_0}{\overset{H_1}{\gtrless}} 0 \,,$$

or equivalently

$$Y_2 \underset{H_0}{\overset{H_1}{\gtrless}} Y_1 \,.$$

The probability of error is $P[E] = Q(d/2)$ with

$$d = \frac{||\mathbf{S_1} - \mathbf{S}_0||_2}{(T/2 + \sigma^2)^{1/2}} = \Big(\frac{2E}{T/2 + \sigma^2} \Big)^{1/2} \,.$$

Problem 8.7

a) Since $s(t) = A$ for $0 \leq t \leq T$, it follows from Example 8.3 that the distorted function $g(t)$ can be expressed as

$$g(t) = q^{-1}((1 + a^2) - 2a)A = q^{-1} A (1 - a)^2$$

for $1 \leq t \leq T - 1$, where $q = (1 - a^2)\sigma^2$, and

$$g(0) = g(T) = q^{-1} A (1 - a) \,.$$

The optimum test can be expressed as

$$S \underset{H_0}{\overset{H_1}{\gtrless}} \frac{E}{2} + \ln(\tau) \,,$$

where for equally likely hypotheses and minimum probability of error detector we can set $\tau = 1$. In this test

$$S = \sum_{t=0}^{T} Y(t) g(t) = \frac{(1 - a)A}{(1 + a)\sigma^2} \sum_{t=1}^{T-1} Y(t) + \frac{A}{(1 + a)\sigma^2} (Y(0) + Y(T)) \,.$$

and

$$E = \sum_{t=0}^{T} s(t) g(t) = \frac{(1 - a)A^2}{(1 + a)\sigma^2} [(T + 1) - a(T - 1)] \,.$$

b) Under H_1 we have

$$S = \sum_{t=0}^{T} g(t)(A + V(t)) = E + N$$

where

$$N = \sum_{t=0}^{T} g(t)V(t)$$

is a zero-mean Gaussian random variable with variance

$$\begin{aligned} E[N^2] &= \sum_{t=0}^{T}\sum_{s=0}^{T} g(t)K_V(t,s)g(s) \\ &= \mathbf{g}^T\mathbf{K}\mathbf{g} = \mathbf{g}^T\mathbf{s} = E\,, \end{aligned}$$

where on the second line we have used the fact that $\mathbf{g} = \mathbf{K}^{-1}\mathbf{s}$. So under H_1, S is $N(E,E)$ distributed. Under H_0 we have

$$S = N \sim N(0,E)\,.$$

Consequently, the probability of error is $P[E] = Q(d/2)$ with $d = E/E^{1/2} = E^{1/2}$.

Problem 8.8

a) Let $q = \sigma^2(1-a^2)$. Then according to Example 8.6, under the noise whitening transformation

$$\check{V}(t) = \begin{cases} \sigma^{-1}V(0) & t = 0 \\ q^{-1/2}(V(t) - aV(t-1) & t \geq 1\,, \end{cases}$$

the transformed noise $\check{V}(t)$ with $0 \leq t \leq T$ is a WGN with unit variance. For a constant signal $s(t) = A$ with $0 \leq t \leq T$, the corresponding transformed signal and observations are given by

$$\check{s}(t) = \begin{cases} \sigma^{-1}A & t = 0 \\ q^{-1/2}(1-a)A & t \geq 1 \end{cases} \tag{8.14}$$

and

$$\check{Y}(t) = \begin{cases} \sigma^{-1}Y(0) & t = 0 \\ q^{-1/2}(Y(t) - aY(t-1)) & t \geq 1\,. \end{cases} \tag{8.15}$$

b) Let

$$\begin{aligned} E &= \sum_{t=0}^{T} \check{s}^2(t) = \frac{A^2}{\sigma^2} + \frac{(1-a)^2}{q}A^2T \\ &= \Big[1 + \frac{(1-a)T}{(1+a)}\Big]\frac{A^2}{\sigma^2} = \frac{A^2}{(1+a)\sigma^2}[(T+1) - a(T-1)] \end{aligned}$$

denote the energy of transformed signal $\check{s}(t)$, and consider the sufficient statistic

$$S \triangleq \sum_{t=0}^{T} \check{Y}(t)\check{s}(t)\,.$$

Substituting (8.14) and (8.15), we find

$$\begin{aligned} S &= \frac{A}{\sigma^2}Y(0) + \frac{(1-a)A}{q}\sum_{t=1}^{T}(Y(t) - aY(t-1)) \\ &= \frac{A}{\sigma^2}Y(0) + \frac{A}{(1+a)\sigma^2}\sum_{t=1}^{T}(Y(t) - aY(t-1)) \,. \end{aligned} \tag{8.16}$$

By reorganizing sums, we see it can be rewritten as

$$S = \frac{A}{(1+a)\sigma^2}(Y(0) + Y(T)) + \frac{(1-a)A}{(1+a)\sigma^2}\sum_{t=1}^{T-1} Y(t) \,,$$

which is identical to the expression obtained for S in Problem 8.7. Then, as indicated by equation (8.126) of the text, the optimum detector takes the form

$$S \underset{H_0}{\overset{H_1}{\gtrless}} \gamma = E/2 + \ln(\tau) \,.$$

In this expression $\tau = 1$ whenever the two hypotheses are equally likely and we seek to minimize the probability of error. A block diagram for the noise whitening receiver based on expression (8.16) is shown in Fig. 31 below.

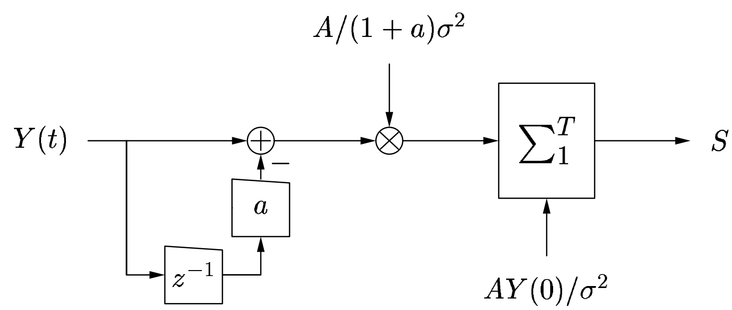


Figure 31: Block diagram for the noise whitening implementation of the detector for a constant signal of amplitude A in Ornnstein-Uhlenbeck noise.

c) Under H_0, we have

$$S = \sum_{t=0}^{T} \check{V}(t)\check{s}(t) \sim N(0, E) \,,$$

and under H_1,

$$S = \sum_{t=0}^{T} (\check{s}(t) + \check{V}(t))\check{s}(t) \sim N(E, E) \,.$$

The distance between H_0 and H_1 scaled by the standard deviation of S is therefore

$$d = \frac{E[S|H_1] - E[S|H_0]}{E^{1/2}} = E^{1/2} \,.$$

Then according to expressions (8.15) of the text, for the case when the two hypotheses are equally likely and we seek to minimize the probability of error, the probability of error is given by

$$P[E] = Q(d/2) = Q(E^{1/2}/2) \,.$$

Problem 8.9

a) Let $q = (1-a^2)\sigma^2$. As indicated in Example 8.3, the distorted signal $g(t)$ is given by

$$g(t) = q^{-1}[(1+a^2)s(t) - a(s(t-1) + s(t+1))]$$

for $1 \leq t \leq T-1$ and

$$\begin{aligned} g(0) &= q^{-1}[s(0) - as(1)] \\ g(T) &= q^{-1}[s(T) - as(T-1)] \,. \end{aligned}$$

By substituting $s(t) = A\cos(\omega_0 t)$ and using trigonometric identities

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \,,$$

we find

$$g(t) = \frac{A}{q}[(1+a^2) - 2a\cos(\omega_0)]\cos(\omega_0 t) \tag{8.17}$$

for $1 \leq t \leq T-1$, and

$$\begin{aligned} g(0) &= \frac{A}{q}[1 - a\cos(\omega_0)] \\ g(T) &= \frac{A}{q}[1 - a\cos(\omega_0)](-1)^k \,. \end{aligned}$$

In the last expression we have used the fact that since $T = k\pi/omega_0$,

$$\cos(\omega_0 T) = \cos(k\pi) = (-1)^k \,.$$

If we introduce the power spectral density

$$S_V(\omega) = \frac{q}{(1+a^2) - 2a\cos(\omega)}$$

of the DT Ornstein-Uhlenbeck process $V(t)$, it is interesting to note that expression (8.17) for $g(t)$ with $1 \leq t \leq T-1$ can be rewritten as

$$g(t) = S_V^{-1}(\omega_0)s(t) = \frac{A}{S_V(\omega_0)}\cos(\omega_0 t) \,.$$

The optimal detector is then given by

$$S \underset{H_0}{\overset{H_1}{\gtrless}} \gamma = \frac{E}{2} + \ln(\tau) \,,$$

where

$$\begin{aligned} S &= \sum_{t=0}^{T} Y(t)g(t) \\ &= S_V^{-1}(\omega_0)\sum_{t=1}^{T-1} Y(t)s(t) + \frac{A}{q}(1 - a\cos(\omega_0))(Y(0) + (-1)^k Y(T)) , \end{aligned}$$

and

$$\begin{aligned} E &= \sum_{t=0}^{T} s(t)g(t) \\ &= \frac{A^2}{S_V(\omega_0)}\sum_{t=1}^{T-1}\cos^2(\omega_0 t) + \frac{2A^2}{q}(1 - a\cos(\omega_0)) . \end{aligned} \tag{8.18}$$

But

$$\cos^2(\omega_0 t) = \frac{1}{2}[1 + \cos(2\omega_0 t)] ,$$

so

$$\sum_{t=1}^{T-1}\cos^2(\omega_0 t) = \frac{T-1}{2} + \frac{1}{2}[-1 + \sum_{t=0}^{T-1}\cos(2\omega_0 t)] , \tag{8.19}$$

where

$$\begin{aligned} \sum_{t=0}^{T-1}\cos(2\omega_0 t) &= \Re\Big\{\sum_{t=0}^{T-1}\exp(j2\omega_0 t)\Big\} \\ &= \Re\Big\{\frac{(1-\exp(j2\omega_0 T))}{(1-\exp(j2\omega_0))}\Big\} = 0 . \end{aligned} \tag{8.20}$$

Combining (8.18)–(8.20) gives

$$E = \frac{A^2}{2S_V(\omega_0)}(T-2) + \frac{2A^2}{q}(1 - a\cos(\omega_0)) . \tag{8.21}$$

b) The probability of error is

$$P[E] = Q(E^{1/2}/2) ,$$

where expression (8.21) for the energy E depends on A, ω_0, a, σ^2 and T.

Problem 8.10

a) Let $q = (1-a^2)\sigma^2$. As indicated in Example 8.3, the distorted signal $g(t)$ is given by

$$g(t) = q^{-1}[(1+a^2)s(t) - a(s(t-1) + s(t+1))]$$

for $1 \leq t \leq T-1$ and

$$\begin{aligned} g(0) &= q^{-1}[s(0) - as(1)] \\ g(T) &= q^{-1}[s(T) - as(T-1)] . \end{aligned}$$

By substituting $s(t) = A\sin(\omega_0 t)$ and using trigonometric identities

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) ,$$

we find

$$g(t) = \frac{A}{q}[(1+a^2) - 2a\cos(\omega_0)]\sin(\omega_0 t) \tag{8.22}$$

for $1 \leq t \leq T-1$, and

$$\begin{aligned} g(0) &= -\frac{Aa}{q}\sin(\omega_0)] \\ g(T) &= \frac{Aa}{q}(-1)^k \sin(\omega_0) . \end{aligned}$$

In the last expression we have used the fact that since $T = k\pi/\omega_0$,

$$\begin{aligned} \sin(\omega_0 T) &= \sin(k\pi) = 0 \\ \cos(\omega_0 T) &= \cos(k\pi) = (-1)^k . \end{aligned}$$

If we introduce the power spectral density

$$S_V(\omega) = \frac{q}{(1+a^2) - 2a\cos(\omega)}$$

of the DT Ornstein-Uhlenbeck process $V(t)$, it is interesting to note that expression (8.22) for $g(t)$ with $1 \leq t \leq T-1$ can be rewritten as

$$g(t) = S_V^{-1}(\omega_0)s(t) = \frac{A}{S_V(\omega_0)}\sin(\omega_0 t) .$$

The optimal detector is then given by

$$S \underset{H_0}{\overset{H_1}{\gtrless}} \gamma = \frac{E}{2} + \ln(\tau) ,$$

where

$$\begin{aligned} S &= \sum_{t=0}^{T} Y(t)g(t) \\ &= S_V^{-1}(\omega_0)\sum_{t=1}^{T-1} Y(t)s(t) - \frac{Aa}{q}\sin(\omega_0))(Y(0) + (-1)^{k+1}Y(T)) , \end{aligned}$$

and

$$\begin{aligned} E &= \sum_{t=0}^{T} s(t)g(t) \\ &= \frac{A^2}{S_V(\omega_0)}\sum_{t=1}^{T-1}\sin^2(\omega_0 t) . \end{aligned} \tag{8.23}$$

But

$$\sin^2(\omega_0 t) = \frac{1}{2}[1 - \cos(2\omega_0 t)] \, ,$$

so

$$\sum_{t=1}^{T-1} \sin^2(\omega_0 t) = \frac{T-1}{2} - \frac{1}{2}[-1 + \sum_{t=0}^{T-1} \cos(2\omega_0 t)] \, , \tag{8.24}$$

where

$$\begin{aligned} \sum_{t=0}^{T-1} \cos(2\omega_0 t) &= \Re\Big\{ \sum_{t=0}^{T-1} \exp(j2\omega_0 t) \Big\} \\ &= \Re\Big\{ \frac{(1 - \exp(j2\omega_0 T))}{(1 - \exp(j2\omega_0))} \Big\} = 0 \, . \end{aligned} \tag{8.25}$$

Combining (8.23) and (8.25) gives

$$E = \frac{A^2 T}{2 S_V(\omega_0)} \, . \tag{8.26}$$

b) The probability of error is

$$P[E] = Q(E^{1/2}/2) \, ,$$

where expression (8.26) for the energy E depends on A, ω_0, a, σ^2 and T.

Springer

http://www.springer.com/978-0-387-76542-6

Principles of Signal Detection and Parameter Estimation
Levy, B.C.
2008, 664 p. 101 illus., Hardcover
ISBN: 978-0-387-76542-6