2

Linear Programming

Linear programming (LP) problems involve linear objective function and linear constraints, as shown below in Example 2.1.

Example 2.1: Solvents are extensively used as process materials (e.g., extractive agents) or process fluids (e.g., CFC) in chemical process industries. Cost is a main consideration in selecting solvents. A chemical manufacturer is accustomed to a raw material X_1 as the solvent in his plant. Suddenly, he found out that he can effectively use a blend of X_1 and X_2 for the same purpose. X_1 can be purchased at \$4 per ton, however, X_2 is an environmentally toxic material which can be obtained from other manufacturers. With the current environmental policy, this results in a credit of \$1 per ton of X_2 consumed. He buys the material a day in advance and stores it. The daily availability of these two materials is restricted by two constraints: (1) the combined storage (intermediate) capacity for X_1 and X_2 is 8 tons per day. The daily availability for X_1 is twice the required amount. X_2 is generally purchased as needed. (2) The maximum availability of X_2 is 5 tons per day. Safety conditions demand that the amount of X_1 cannot exceed the amount of X_2 by more than 4 tons. The manufacturer wants to determine the amount of each raw material required to reduce the cost of solvents to a minimum. Formulate the problem as an optimization problem.
Solution: Let x_1 be the amount of X_1 and x_2 be the amount of X_2 required per day in the plant. Then, the problem can be formulated as a linear programming problem as given below.

$$\underset{x_1,\ x_2}{\text{Minimize}} \quad Z = 4x_1 - x_2 \tag{2.1}$$

subject to

$$2x_1 + x_2 \leq 8 \qquad \text{Storage Constraint} \tag{2.2}$$

$$x_2 \leq 5 \qquad \text{Availability Constraint} \tag{2.3}$$

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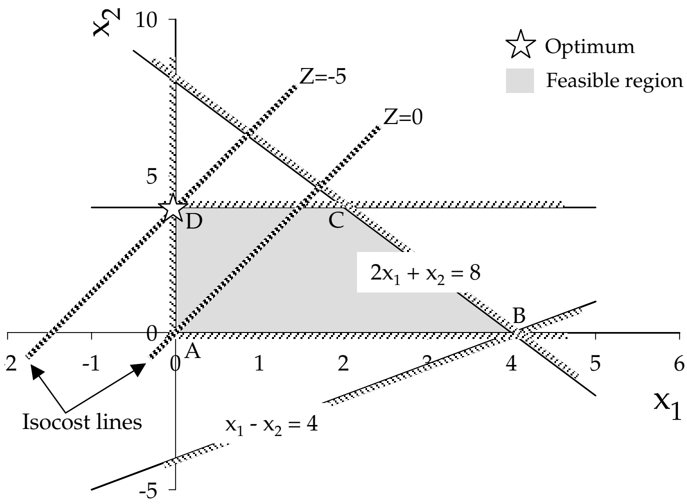

Fig. 2.1. Linear programming graphical representation, Exercise 2.1.

$$x_1 - x_2 \leq 4 \qquad \text{Safety Constraint} \tag{2.4}$$

$$x_1 \geq 0; x_2 \geq 0$$

As shown above, the problem is a two-variable LP problem, which can be easily represented in a graphical form. Figure 2.1 shows constraints (2.2) through (2.4), plotted as three lines by considering the three constraints as equality constraints. Therefore, these lines represent the boundaries of the inequality constraints. In the figure, the inequality is represented by the points on the other side of the hatched lines. The objective function lines are represented as dashed lines (isocost lines). It can be seen that the optimal solution is at the point $x_1 = 0;\ x_2 = 5$, a point at the intersection of constraint (2.3) and one of the isocost lines. All isocost lines intersect constraints either once or twice. The LP optimum lies at a vertex of the feasible region, which forms the basis of the simplex method. The simplex method is a numerical optimization method for solving linear programming problems developed by George Dantzig in 1947.

2.1 The Simplex Method

The graphical method shown above can be used for two-dimensional problems; however, real-life LPs consist of many variables, and to solve these linear programming problems, one has to resort to a numerical optimization method such as the simplex method.

The generalized form of an LP can be written as follows.

$$\underset{x_i}{\text{Optimize}} \quad Z = \sum_{i=1}^{n} C_i x_i \tag{2.5}$$

subject to

$$\sum_{i=1}^{n} a_{ji} x_i \leq b_j \tag{2.6}$$

$$j = 1, 2, \ldots, m$$

$$x_j \in R$$

As stated in Chapter 1, a numerical optimization method involves an iterative procedure. The simplex method involves moving from one extreme point on the boundary (vertex) of the feasible region to another along the edges of the boundary iteratively. This involves identifying the constraints (lines) on which the solution will lie. In simplex, a slack variable is incorporated in every constraint to make the constraint an equality. Now, the aim is to solve the linear equations (equalities) for the decision variables x, and the slack variables s. The active constraints are then identified based on the fact that, for these constraints, the corresponding slack variables are zero.

The simplex method is based on the Gauss elimination procedure of solving linear equations. However, some complicating factors enter in this procedure: (1) all variables are required to be nonnegative because this ensures that the feasible solution can be obtained easily by a simple ratio test (Step 4 of the iterative procedure described below); and (2) we are optimizing the linear objective function, so at each step we want ensure that there is an improvement in the value of the objective function (Step 3 of the iterative procedure given below).

The simplex method uses the following steps iteratively.

1. *Convert the LP into the standard LP form.*
   Standard LP
   - All the constraints are equations with a nonnegative right-hand side.
   - All variables are nonnegative.
     - Convert all negative variables x to nonnegative variables using two variables (e.g., $x = x^+ - x^-$); this is equivalent to saying if $x = -5$ then $-5 = 5 - 10$, $x^+ = 5$, and $x^- = 10$.
     - Convert all inequalities into equalities by adding slack variables (nonnegative) for less than or equal to constraints ($\leq$) and by subtracting surplus variables for greater than or equal to constraints ($\geq$).
   - The objective function must be minimization or maximization.
   - The standard LP involving m equations and n unknowns has m basic variables and $n-m$ nonbasic or zero variables. This is explained below using Example 2.1.

Consider Example 2.1 in the standard LP form with slack variables, as given below.
Standard LP:

$$\text{Maximize} \quad -Z \tag{2.7}$$

$$-Z + 4x_1 - x_2 = 0 \tag{2.8}$$
$$2x_1 + x_2 + s_1 = 8 \tag{2.9}$$
$$x_2 + s_2 = 5 \tag{2.10}$$
$$x_1 - x_2 + s_3 = 4 \tag{2.11}$$

$$x_1 \geq 0; \quad x_2 \geq 0$$
$$s_1 \geq 0; \quad s_2 \geq 0; \quad s_3 \geq 0$$

The feasible region for this problem is represented by the region ABCD in Figure 2.1. Table 2.1 shows all the vertices of this region and the corresponding slack variables calculated using the constraints given by Equations (2.9)–(2.11) (note that the nonnegativity constraint on the variables is not included).

It can be seen from Table 2.1 that at each extreme point of the feasible region, there are $n - m = 2$ variables that are zero and $m = 3$ variables that are nonnegative. An extreme point of the linear program is characterized by these m basic variables.

In simplex the feasible region shown in Table 2.1 gets transformed into a tableau (Table 2.2).

Table 2.1. Feasible region in Figure 2.1 and slack variables.

Point	x_1	x_2	s_1	s_2	s_3
A	0.0	0.0	8.0	5.0	4.0
B	4.0	0.0	0.0	5.0	0.0
C	1.5	5.0	0.0	0.0	7.5
D	0.0	5.0	3.0	0.0	9.0

Table 2.2. Simplex tableau from Table 2.1.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic
0	1	4	−1	0	0	0	0	$-Z = 0$
1	0	2	1	1	0	0	8	$s_1 = 8$
2	0	0	$\underline{1}$	0	1	0	5	$s_2 = 5$
3	0	1	−1	0	0	1	4	$s_3 = 4$

2. *Determine the starting feasible solution.* A basic solution is obtained by setting $n - m$ variables equal to zero and solving for the values of the remaining m variables.
3. *Select an entering variable* (in the list of nonbasic variables) using the optimality (defined as better than the current solution) condition; that is, choose the next operation so that it will improve the objective function. Stop if there is no entering variable.
   Optimality Condition:
   - Entering variable: The nonbasic variable that would increase the objective function (for maximization). This corresponds to the nonbasic variable having the most negative coefficient in the objective function equation or the row zero of the simplex tableau.
     In many implementations of simplex, instead of wasting the computation time in finding the most negative coefficient, any negative coefficient in the objective function equation is used.
4. *Select a leaving variable using the feasibility condition.*
   Feasibility Condition:
   - Leaving variable: The basic variable that is leaving the list of basic variables and becoming nonbasic. The variable corresponding to the smallest nonnegative ratio (the right-hand side of the constraint divided by the constraint coefficient of the entering variable).
5. *Determine the new basic solution by using the appropriate Gauss–Jordan Row Operation.*
   Gauss–Jordon Row Operation:
   - Pivot Column: Associated with the row operation.
   - Pivot Row: Associated with the leaving variable.
   - Pivot Element: Intersection of Pivot row and Pivot Column.
     ROW OPERATION
   - Pivot Row = Current Pivot Row ÷ Pivot Element.
   - All other rows: New Row = Current Row - (its Pivot Column Coefficients × New Pivot Row).
6. *Go to Step 2.*

The following example illustrates the simplex method.

Example 2.2: Solve Example 2.1 using the simplex method.
Solution:

- Convert the LP into the standard LP form. For simplicity, we are converting this minimization problem to a maximization problem with $-Z$ as the objective function. Furthermore, nonnegative slack variables s_1, s_2, and s_3 are added to each constraint.
  Standard LP:

$$\text{Maximize} \quad -Z \tag{2.12}$$

$$-Z + 4x_1 - x_2 = 0 \tag{2.13}$$
$$2x_1 + x_2 + s_1 = 8 \quad \text{Storage Constraint} \tag{2.14}$$
$$x_2 + s_2 = 5 \quad \text{Availability Constraint} \tag{2.15}$$
$$x_1 - x_2 + s_3 = 4 \quad \text{Safety Constraint} \tag{2.16}$$

$$x_1 \geq 0; \quad x_2 \geq 0$$

The standard LP is shown in Table 2.3 below where x_1 and x_2 are nonbasic or zero variables and s_1, s_2, and s_3 are the basic variables. The starting solution is $x_1 = 0;\ x_2 = 0;\ s_1 = 8;\ s_2 = 5;\ s_3 = 4$ obtained from the RHS column.

Table 2.3. Initial tableau for Example 2.2.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	4	-1	0	0	0	0	$-Z = 0$	—
1	0	2	1	1	0	0	8	$s_1 = 8$	8
2	0	0	$\underline{1}$	0	1	0	5	$s_2 = 5$	$\underline{5}$
3	0	1	-1	0	0	1	4	$s_3 = 4$	—

- Determine the entering and leaving variables.
  Is the starting solution optimum? No, because Row 0 representing the objective function equation contains nonbasic variables with negative coefficients. This can also be seen from Figure 2.2. In this figure, the current basic solution is shown to be increasing in the direction of the arrow.

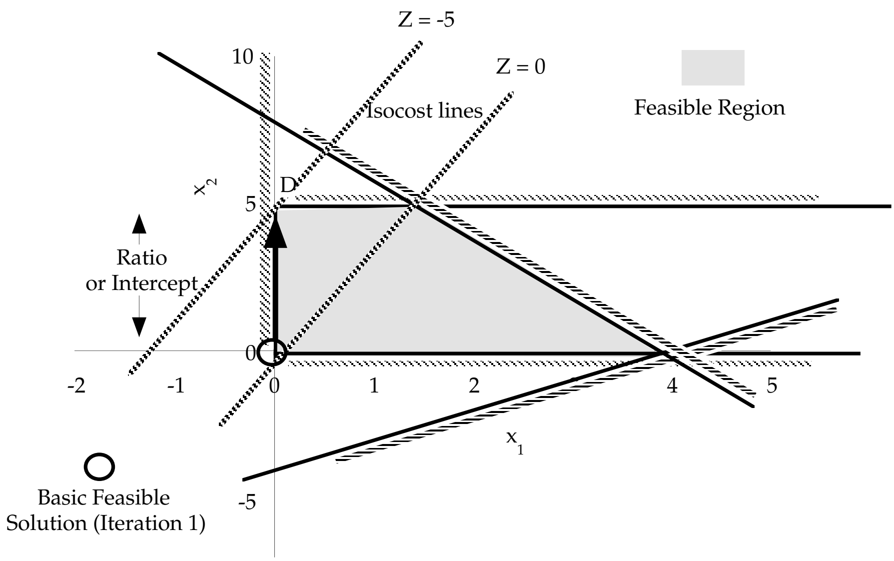

Fig. 2.2. Basic solution for Exercise 2.2.

Table 2.4. The simplex tableau, Example 2.2, iteration 2.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	4	0	0	1	0	5	$-Z = 5$	—
1	0	2	0	1	-1	0	3	$s_1 = 3$	—
2	0	0	1	0	1	0	5	$x_2 = 5$	—
3	0	1	0	0	1	1	9	$s_3 = 9$	—

Entering Variable: The most negative coefficient in Row 0 is x_2. Therefore, the entering variable is x_2. This variable must now increase in the direction of the arrow. How far can this increase the objective function? Remember that the solution has to be in the feasible region. Figure 2.2 shows that the maximum increase in x_2 in the feasible region is given by point D, which is on constraint (2.3). This is also the intercept of this constraint with the y-axis, representing x_2. Algebraically, these intercepts are the ratios of the right-hand side of the equations to the corresponding constraint coefficient of x_2. We are interested only in the nonnegative ratios, as they represent the direction of increase in x_2. This concept is used to decide the leaving variable.

Leaving Variable: The variable corresponding to the smallest nonnegative ratio (5 here) is s_2. Hence, the leaving variable is s_2.

So, the Pivot Row is Row 2 and Pivot Column is x_2.

- The two steps of the Gauss–Jordon Row Operation are given below.
  The pivot element is underlined in the Table 2.3 and is 1.
  Row Operation:
  Pivot: $(0, 0, 1, 0, 1, 0, 5)$
  Row 0: $(1, 4, -1, 0, 0, 0, 0) - (-1)(0, 0, 1, 0, 1, 0, 5) = (1, 4, 0, 0, 1, 0, 5)$
  Row 1: $(0, 2, 1, 1, 0, 0, 8) - (1)(0, 0, 1, 0, 1, 0, 5) = (0, 2, 0, 1, -1, 0, 3)$
  Row 3: $(0, 1, -1, 0, 0, 1, 4) - (-1)(0, 0, 1, 0, 1, 0, 5) = (0, 1, 0, 0, 1, 1, 9)$
  These steps result in the following table (Table 2.4).
  There is no new entering variable because there are no nonbasic variables with a negative coefficient in row 0. Therefore, we can assume that the solution is reached, which is given by (from the RHS of each row) $x_1 = 0$; $x_2 = 5$; $s_1 = 3$; $s_2 = 0$; $s_3 = 9$; $Z = -5$.
  Note that at an optimum, all basic variables (x_2, s_1, s_3) have a zero coefficient in Row 0.

2.2 Infeasible Solution

Now consider the same example, and change the right-hand side of Equation (2.2) to -8 instead of 8. We know that constraint (2.2) represents the storage capacity and physics tells us that the storage capacity cannot be negative. However, let us see what we get mathematically.

Example 2.3: Constraint (2.2) is changed to reflect a negative storage capacity.
Solution: This results in the following LP.

$$\underset{x_1,\ x_2}{\text{Maximize}} \quad -Z \tag{2.17}$$

subject to

$$-Z + 4x_1 - x_2 = 0 \tag{2.18}$$

$$2x_1 + x_2 \leq -8 \qquad \text{Storage Constraint} \tag{2.19}$$

$$x_2 \leq 5 \qquad \text{Availability Constraint} \tag{2.20}$$

$$x_1 - x_2 \leq 4 \qquad \text{Safety Constraint} \tag{2.21}$$

$$x_1 \geq 0; \quad x_2 \geq 0$$

From Figure 2.3, it is seen that the solution is infeasible for this problem. Applying the simplex Method results in Table 2.5 for the first step.

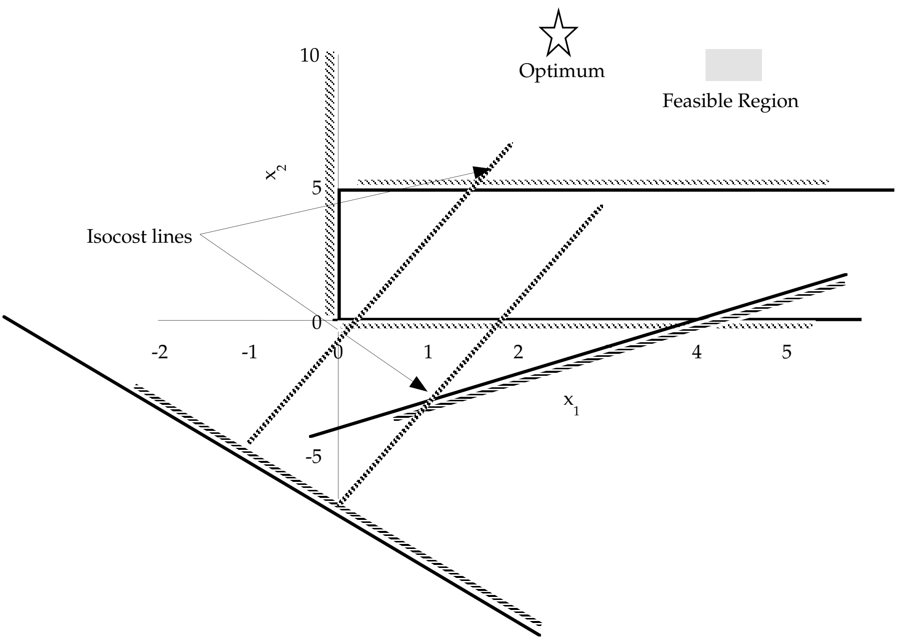

Fig. 2.3. Infeasible LP.

Table 2.5. Initial simplex tableau, Example 2.3.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	4	−1	0	0	0	0	$-Z = 0$	—
1	0	−2	−1	−1	0	0	8	$s_1 = -8$	—
2	0	0	$\underline{1}$	0	1	0	5	$s_2 = 5$	$\underline{5}$
3	0	1	−1	0	0	1	4	$s_3 = 4$	None

Table 2.6. The simplex tableau, Example 2.3, iteration 2.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	4	0	0	1	0	5	$-Z = 0$	—
1	0	−2	0	−1	1	0	13	$s_1 = -13$	—
2	0	0	1	0	1	0	5	$x_2 = 5$	—
3	0	1	0	0	1	1	9	$\underline{s_3 = 9}$	—

Standard LP:

$$\text{Maximize} \quad -Z + 4x_1 - x_2 \tag{2.22}$$

$$-2x_1 - x_2 - s_1 = 8 \tag{2.23}$$

$$x_2 + s_2 = 5 \tag{2.24}$$

$$x_1 - x_2 + s_3 = 4 \tag{2.25}$$

As can be seen, the entering variable with the most negative coefficient is x_2 and the leaving variable corresponding to the smallest nonnegative ratio is s_2.

Applying the Gauss–Jordan row operation results in Table 2.6.

The solution to this problem is the same as before: $x_1 = 0;\ x_2 = 5$. However, this solution is not a feasible solution because the slack variable (artificial variable defined to be always positive) s_1 is negative.

2.3 Unbounded Solution

If constraints (2.19) and (2.20) are removed in the above example, the solution is unbounded, as can be seen in Figure 2.4. This means there are points in the feasible region with arbitrarily large objective function values (for maximization).

Example 2.4: Constraints (2.19) and (2.20) removed.
Solution:

$$\underset{x_1,\ x_2}{\text{Minimize}} \quad Z = 4x_1 - x_2 \tag{2.26}$$

Table 2.7. The simplex tableau, Example 2.4.

Row	$-Z$	x_1	x_2	s_3	RHS	Basic	Ratio
0	1	4	-1	0	0	$-Z = 0$	—
1	0	1	-1	1	4	$s_3 = 4$	None

subject to

$$x_1 - x_2 \leq 4 \qquad \text{Safety Constraint} \tag{2.27}$$

$$x_1 \geq 0; \quad x_2 \geq 0$$

The simplex tableau for this problem is shown in Table 2.7.

The entering variable is x_2 as it has the most negative coefficient in row 0. However, there is no leaving variable corresponding to the binding constraint (the smallest nonnegative ratio or intercept). That means x_2 can take as high a value as possible. This is also apparent in the graphical solution shown in Figure 2.4.

The LP is unbounded when (for a maximization problem) a nonbasic variable with a negative coefficient in row 0 has a nonpositive coefficient in each constraint, as shown in the above table.

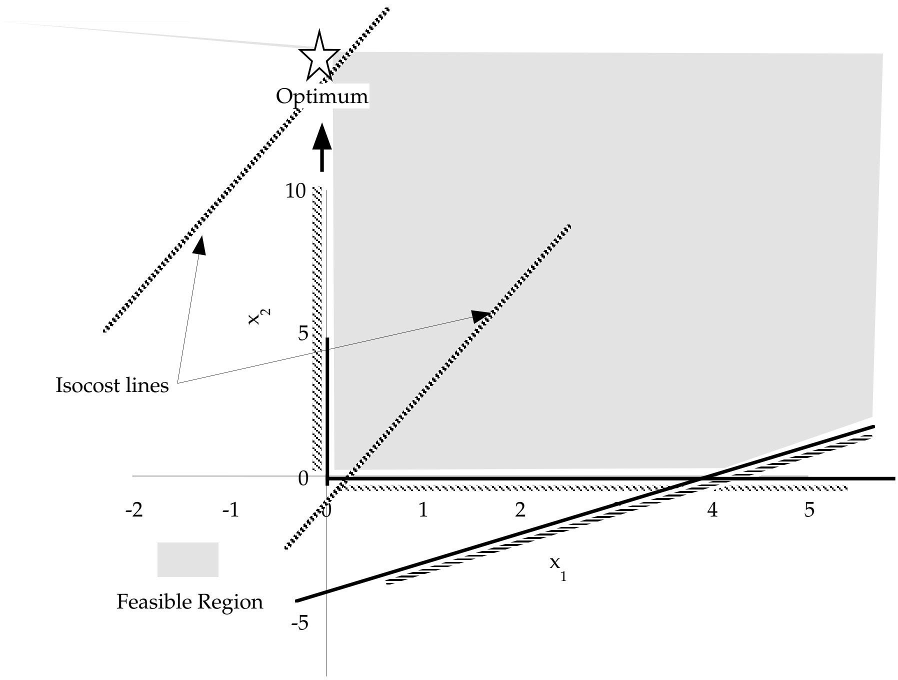

Fig. 2.4. Unbounded LP.

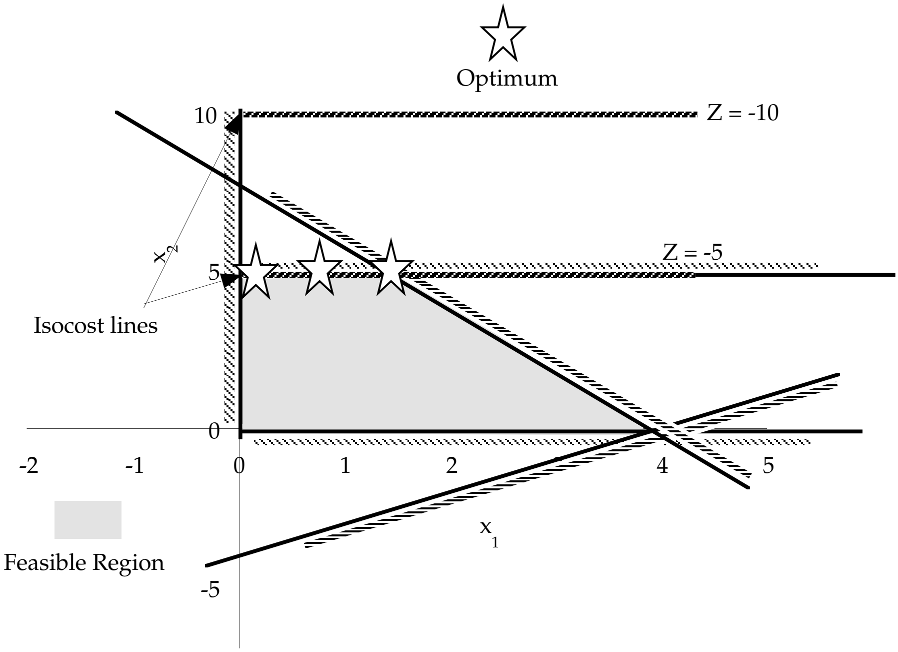

Fig. 2.5. LP with multiple solutions.

2.4 Multiple Solutions

In the following example, the cost of X_1 is assumed to be negligible as compared to the credit of X_2. This LP has infinite solutions given by the isocost line ($x_2 = 5$) in Figure 2.5. The simplex method generally finds one solution at a time. Special methods such as goal programming or multiobjective optimization can be used to find these solutions. These methods are described in Chapter 6.

Example 2.5: Assume that in Example 2.1, the cost of X_1 is negligible. Find the optimal solution.

$$\underset{x_1,\ x_2}{\text{Minimize}} \quad Z = -x_2 \tag{2.28}$$

subject to

$$2x_1 + x_2 \leq 8 \qquad \text{Storage Constraint} \tag{2.29}$$

$$x_2 \leq 5 \qquad \text{Availability Constraint} \tag{2.30}$$

$$x_1 - x_2 \leq 4 \qquad \text{Safety Constraint} \tag{2.31}$$

$$x_1 \geq 0; \quad x_2 \geq 0$$

Table 2.8. Initial tableau for Example 2.5.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	0	−1	0	0	0	0	$-Z = 0$	—
1	0	2	1	1	0	0	8	$s_1 = 8$	8
2	0	0	$\underline{1}$	0	1	0	5	$s_2 = 5$	$\underline{5}$
3	0	1	−1	0	0	1	4	$s_3 = 4$	—

Table 2.9. The simplex tableau, Example 2.5, iteration 2.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	0	0	0	1	0	5	$-Z = 5$	—
1	0	2	0	1	−1	0	3	$s_1 = 3$	—
2	0	0	1	0	1	0	5	$x_2 = 5$	—
3	0	1	0	0	1	1	9	$s_3 = 9$	—

Table 2.10. The simplex tableau, Example 2.5, iteration 3.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	0	0	0	1	0	5	$-Z = 5$	—
1	0	$\underline{2}$	0	1	−1	0	3	$s_1 = 3$	1.5
2	0	0	1	0	1	0	5	$x_2 = 5$	—
3	0	1	0	0	1	1	9	$s_3 = 9$	9

Table 2.11. The simplex tableau, Example 2.5, iteration 4.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	0	0	0	1	0	5	$-Z = 5$	—
1	0	1	0	0.5	−0.5	0	1.5	$x_1 = 1.5$	—
2	0	0	1	0	1	0	5	$x_2 = 5$	—
3	0	0	0	−0.5	1.5	1	7.5	$s_3 = 7.5$	—

Solution: The graphical solution to this problem is shown in Figure 2.5.

The simplex solution iteration summary is presented in Tables 2.8 and 2.9.

The simplex method found the first solution to the problem; that is, $x_1 = 0$, $x_2 = 5$. Can simplex recognize that there are multiple solutions? Note that in Example 2.2, we stated that in the final simplex tableau solution, all basic variables have a zero coefficient in row 0. However, in the optimal tableau, there is a nonbasic variable x_1, which also has a zero coefficient.

Let us see if we make x_1 as an entering variable from the list of basic variables (Table 2.10). From the ratio test, one can see that s_1 would be the leaving variable. This results in the simplex tableau presented in Table 2.11.

An alternate solution to the simplex is $x = (1.5, 5.0)$. Remember that this is also an optimum solution because there are only nonnegative coefficients left in row 0.

2.5 Sensitivity Analysis

The sensitivity of the linear programming solution is expressed in terms of shadow prices and opportunity (reduced) cost.

- Shadow Prices/Dual Prices/Simplex Multipliers: A shadow price is the rate of change (increase in the case of maximization and decrease in the case of minimization) of the optimal value of the objective function with respect to a particular constraint. Shadow prices are also called dual prices from the dual representations of LP problems used in the dual simplex method described in the next section.
  Figure 2.6 shows the shadow prices for various constraints in Example 2.1. As shown in the figure, if one changes the right-hand side of constraints (2.2) and (2.4) and uses the same basis, the optimal value is unchanged, so the shadow prices for these constraints are zero. This shows that if the management of the manufacturing company wants to increase their storage capacity, this decision will not have any implications as far as the solvent optimal cost is concerned. Similarly, if the company decides to relax the constraint on excess component volume (constraint (2.3)), that will also not affect their solvent costs. However, if they can have access to more chemical X_2 per day (please see the LP formulation and corresponding simplex iteration summary, Tables 2.12 and 2.13 given below then that reduces the cost (objective function), as the shadow price for this constraint is 1.
  Standard LP:

$$\text{Maximize} \quad -Z \tag{2.32}$$

$$-Z + 4x_1 - x_2 = 0 \tag{2.33}$$

$$2x_1 + x_2 + s_1 = 8 \qquad \text{Storage Constraint} \tag{2.34}$$

$$x_2 + s_2 = 6 \qquad \text{Availability Constraint} \tag{2.35}$$

$$x_1 - x_2 + s_3 = 4 \qquad \text{Safety Constraint} \tag{2.36}$$

$$x_1 \geq 0; \quad x_2 \geq 0$$

  Table 2.13 demonstrates that the slack variables for the two constraints with shadow prices of zero are positive (row 1 and 3). A less than or equal to ($\leq$) constraint will always have a nonnegative shadow price; a less than or equal to ($\leq$) constraint with positive slack variable (constraints 1 and 3) will have a zero shadow price; a greater than or equal to ($\geq$) constraint will always have a nonpositive shadow price; and an equality constraint may have a positive, a negative, or a zero shadow price.
  The shadow prices are important for the following reasons.
  - To identify which constraints might be the most beneficially changed, and to initiate these changes as a fundamental means to improve the solution

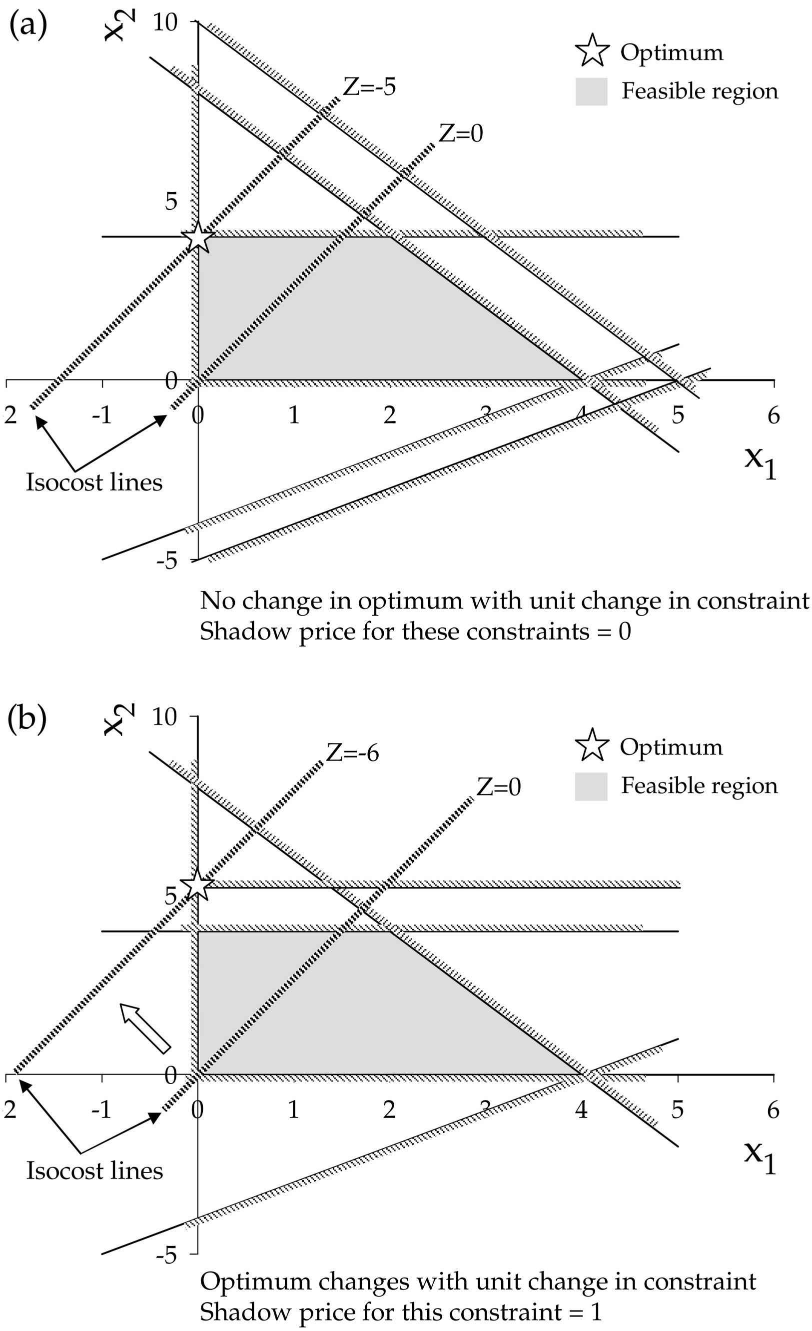

Fig. 2.6. Shadow prices.

Table 2.12. Initial tableau for the new LP.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	4	-1	0	0	0	0	$-Z = 0$	—
1	0	2	1	1	0	0	8	$s_1 = 8$	8
2	0	0	$\underline{1}$	0	1	0	6	$s_2 = 6$	$\underline{6}$
3	0	1	-1	0	0	1	4	$s_3 = 4$	—

Table 2.13. The simplex tableau, iteration 2.

Row	$-Z$	x_1	x_2	s_1	s_2	s_3	RHS	Basic	Ratio
0	1	4	0	0	1	0	6	$-Z = 6$	—
1	0	2	0	1	-1	0	2	$s_1 = 2$	—
2	0	0	1	0	1	0	6	$x_2 = 6$	—
3	0	1	0	0	1	1	10	$s_3 = 10$	—

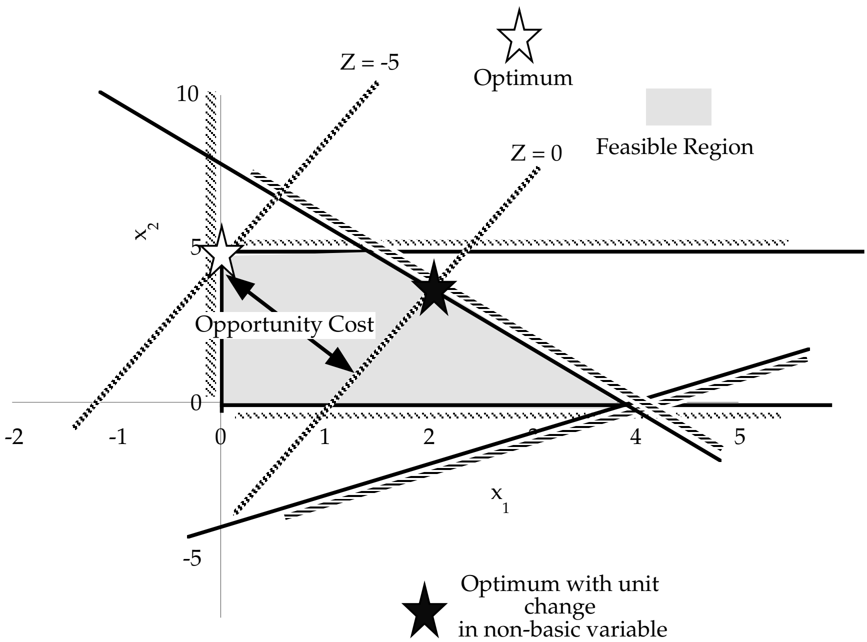

Fig. 2.7. Opportunity cost.

– To react appropriately when external circumstances create opportunities or threats to change the constraints

- Opportunity Cost/Reduced Cost: This is the rate of degradation of the optimum per unit use of a nonbasic (zero) variable in the solution.
  Figure 2.7 shows that the opportunity costs for the nonbasic variable x_1 is 5. It can be seen that with the unit change in x_1, the solution lies

Table 2.14. The primal and dual representation for an LP.

Primal	Dual
Maximize $Z = \sum_{i=1}^{n} C_i x_i$	Minimize $Z_d = \sum_{j=1}^{m} b_j \mu_j$
$x_i,\ i = 1, 2, \ldots, n$	$\mu_j\ j = 1, 2, \ldots, m$
$\sum_{i=1}^{n} a_{ij} x_i \leq b_j$	$\sum_{j=1}^{m} a_{ij} \mu_j \geq C_i$
$j = 1, 2, \ldots, m$	$i = 1, 2, \ldots, n$
$x_i \geq 0$	$\mu_j \geq 0$

on a different constraint (Equation (2.2)) changing the optimal objective function value from -5 to 0.

2.6 Other Methods

As a general rule, LP computational effort depends more on the number of constraints than the number of variables. The dual simplex method uses the dual representation of the original (primal) standard LP problem where the number of constraints is changed to the number of variables and vice versa. For large numbers of constraints, the dual simplex method is more efficient than the conventional simplex method. Table 2.14 shows the primal and dual representation of a standard LP. In the table, μ_j are the dual prices, or simplex multipliers. In nonlinear programming (Chapter 3) terminology, they are also known as the Lagrange multipliers. Using the NLP notations in Chapter 3, Example 3.8 shows the equivalence between the primal and dual representation shown in Table 2.14.

The simplex method requires the initial basic solution to be feasible. The Big M method and the two-phase simplex method circumvent the basic initial feasibility requirement of the simplex method. For details of these methods, please refer to Winston (1991).

Simplex methods move from boundary to boundary within the feasible region. On the other hand, interior point methods visit points within the interior of the feasible region, which is more in line with the nonlinear programming techniques described in the next chapter. These methods are derived from the nonlinear programming techniques developed and popularized in the 1960s by Fiacco and McCormick, but their application to linear programming dates back only to Karmarkar's innovative analysis in 1984. The following example provides the basic concepts behind the interior point method.

Example 2.6: Take Example 2.5 and eliminate constraints (2.29) and (2.31). This converts the problem into a one-dimensional LP. Provide the conceptual steps for the interior point method using this LP.

$$\underset{x_2}{\text{Minimize}} \quad Z = -x_2 \tag{2.37}$$

subject to

$$x_2 \leq 5 \quad \text{Availability Constraint} \tag{2.38}$$

Solution: Just as we did in the simplex method earlier, let us add a variable s_2 to constraint (2.38). This results in the following two-dimensional problem.

$$\underset{x_2, s_2}{\text{Maximize}} \quad -Z = x_2 + 0s_2 \tag{2.39}$$

subject to

$$x_2 + s_2 = 5 \tag{2.40}$$

This LP problem is shown in Figure 2.8. The constraint line represents the feasible region. Now consider a feasible point A on this constraint as a starting point. We need to take a step towards increasing the objective function (maximization) in the x_2 space, that is, the direction parallel to the x-axis. However, because this will be going out of the feasible region, this gradient is projected back to the feasible region at point B. As can be seen, this point is closer to the optimum than A. This gradient projection step is repeated until one reaches the optimum. Note that the step towards the

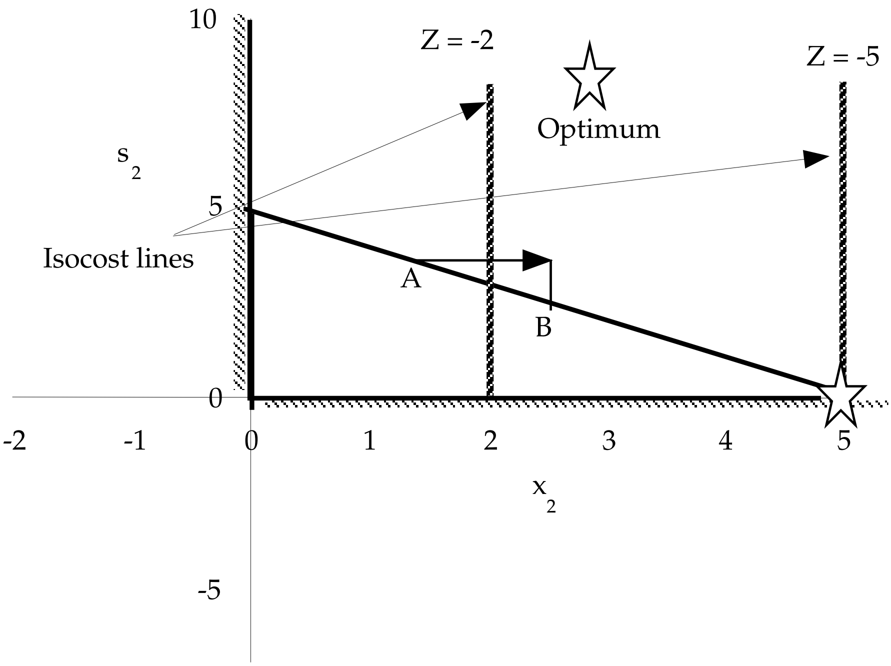

Fig. 2.8. The interior point method conceptual diagram.

gradient should not be too large to overshoot the optimum or too small to increase the number of iterations. It should also not get entrapped in the non-optimum solution. Karmarkar's interior point algorithm addresses these two concerns.

Prior to 1987, all of the commercial codes for solving general linear programs made use of the simplex algorithm. This algorithm, invented in the late 1940s, has fascinated optimization researchers for many years because its performance on practical problems is usually far better than the theoretical worst case. During the period of 1984–1995, the interior-point methods were the subject of intense theoretical and practical investigation, with practical code first appearing around 1989. These methods appear to be faster than the simplex method on large problems, but the advent of a serious rival spurred significant improvements in simplex codes. Today, the relative merits of the two approaches on any given problem depend strongly on the particular geometric or algebraic properties of the problem. In general, however, good interior point codes continue to perform as well as or better than good simplex codes on larger problems when no prior information about the solution is available. When such "warm start" information is available, simplex methods are able to make much better use of it than the interior point methods (Wright, 1999).

2.7 Hazardous Waste Blending Problem as an LP

The Hanford site in southeastern Washington has produced nuclear materials using various processes for nearly 50 years. Radioactive hazardous waste was produced as byproducts of the processes. This waste will be retrieved and separated into high-level and low-level portions. The high-level and low-level wastes will be immobilized for future disposal.

The high-level waste will be converted into a glass form for disposal. The glass must meet both processibility and durability restictions. The processibility conditions ensure that during processing, the glass melt has properties such as viscosity, electrical conductivity, and liquidus temperature, which lie within ranges known to be acceptable for the vitrification process. Durability restrictions ensure that the resultant glass meets the quantitative criteria for disposal in a repository. There are also bounds on the composition of the various components in the glass. In the simplest case, waste and appropriate glass forms (frit) are mixed and heated in a melter to form a glass that satisfies the constraints. It is desirable to keep the amount of frit added to a minimum for two reasons. First, this keeps the frit costs to a minimum. Second, the amount of waste per glass log formed is to be maximized, which keeps the waste disposal costs to a minimum. When there is only a single type of waste, the problem of finding the minimum amount of frit is relatively easy (Narayan et al., 1996).

Hanford has 177 tanks (50,000 to 1 million gallons) containing radioactive waste. Because these wastes result from a variety of processes, these wastes vary widely in composition, and the glasses produced from these wastes will be limited by a variety of components. The minimum amount of frit would be used if all the high-level wastes were combined to form a single feed to the vitrification process. Because of the volume of waste involved and the time span over which it will be processed, this is logistically impossible. However, much of the same benefit can be obtained by forming blends from sets of tanks. The problem is how to divide all the tanks into sets to be blended together so that a minimal amount of frit is required.

In this discrete blending problem, there are N different sources of waste that have to form a discrete number of blends B, with the number of blends being less than the number of sources or tanks. All the waste from any given tank is required to go to a single blend, and each blend contains waste from N/B sources. Blends of equal size (same number of wastes per blend) were specified; alternatively, blends could be formulated to have approximately the same waste masses. Figure 2.9 shows a set of four wastes that needs to be partitioned into two parts to form two blends. If neither of these were specified as constraints, all the waste would go to a single-blend. In this chapter, we look at the single-blend problem. Table 2.15 shows the chemical composition of the high-level waste in three different tanks to be combined to form a single-blend. The table shows the waste mass expressed as a total of the first ten chemicals, including the chemical termed as "other." Frit added to the blend consists of these ten chemicals. The waste mass is scaled down by dividing it by 1000 so as to numerically simplify the solution process. The rest of the chemicals are expressed as the fraction of the total.

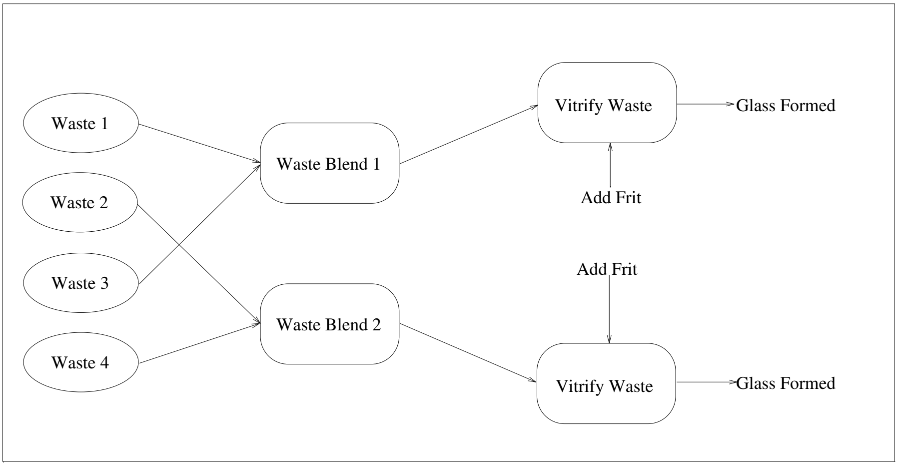

Fig. 2.9. Conversion of waste to glass.

Table 2.15. Waste composition.

Fractional Composition of Wastes				
Component	Comp. ID	AY-102 Tank 1	AZ-101 Tank 2	AZ-102 Tank 3
SiO_2	1	0.072	0.092	0.022
B_2O_3	2	0.026	0.000	0.006
Na_2O	3	0.105	0.264	0.120
Li_2O	4	0.000	0.000	0.000
CaO	5	0.061	0.012	0.010
MgO	6	0.040	0.000	0.003
Fe_2O_3	7	0.328	0.323	0.392
Al_2O_3	8	0.148	0.157	0.212
ZrO_2	9	0.002	0.057	0.063
Other	10	0.217	0.096	0.173
Total	—	1.000	1.000	1.000
Cr_2O_3	11	0.016	0.007	0.005
F	12	0.006	0.001	0.001
P_2O_5	13	0.042	0.001	0.021
SO_3	14	0.001	0.018	0.009
Noble Metals	15	0.000	0.000	0.000
Waste Mass (kgs)		**59772**	**40409**	**143747**

In order to form glass, the blend must satisfy certain constraints. These constraints are briefly described below.

1. *Individual Component Bounds:* There are upper ($p_{UL}^{(i)}$) and lower ($p_{LL}^{(i)}$) limits on the fraction of each component $p^{(i)}$ in glass. Therefore,

$$p_{LL}^{(i)} \leq p^{(i)} \leq p_{UL}^{(i)} \tag{2.41}$$

   These bounds are shown in Table 2.16.

2. *Crystallinity Constraints:* The crystallinity constraints, or multiple component constraints, specify the limits on the combined fractions of different components. There are five such constraints.
   a) The ratio of the mass fraction of SiO_2 to the mass fraction of Al_2O_3 should be greater than C_1 ($C_1 = 3.0$).
   b) The sum of the mass fraction of MgO and the mass fraction of CaO should be less than C_2 ($C_2 = 0.08$).
   c) The combined sum of the mass fractions of Fe_2O_3, Al_2O_3, ZrO_2 and *Other* should be less than C_3 ($C_3 = 0.225$).
   d) The sum of the mass fraction of Al_2O_3 and the mass fraction of ZrO_2 should be less than C_4 ($C_4 = 0.18$).

Table 2.16. Component bounds.

Component	Lower Bound, $p_{LL}^{(i)}$	Upper Bound, $p_{UL}^{(i)}$
SiO_2	0.42	0.57
B_2O_3	0.05	0.20
Na_2O	0.05	0.20
Li_2O	0.01	0.07
CaO	0.00	0.10
MgO	0.00	0.08
Fe_2O_3	0.02	0.15
Al_2O_3	0.00	0.15
ZrO_2	0.00	0.13
Other	0.01	0.10

e) The combined sum of the mass fractions of MgO, CaO, and ZrO_2 should be less than C_5 ($C_5 = 0.18$).

3. *Solubility Constraints:* These constraints limit the maximum value for the mass fraction of one or a combination of components.
   a) The mass fraction of Cr_2O_3 should be less than 0.005.
   b) The mass fraction of F should be less than 0.017.
   c) The mass fraction of P_2O_5 should be less than 0.01.
   d) The mass fraction of SO_2 should be less than 0.005.
   e) The combined mass fraction of Rh_2O_3, PdO, and Ru_2O_2 should be less than 0.025.

4. *Glass Property Constraints:* Additional constraints govern the properties of viscosity, electrical conductivity, and durability but are not considered here.

Blending is most effective when the limiting constraint is one of the first three types, and for the LP formulation these three types of constraints are considered here.

Solution:

Hanford scientists have to decide the amount of each component to be added in the blend to obtain the minimum amount of glass satistifying the first three constraints. We define the decision variables first.

- w_{ij} = amount of component i (where i corresponds to the component ID) in the tank j.
- $W^{(i)}$ = amount of component i in the waste blend.
- $f^{(i)}$ = mass of i^{th} component in the frit.
- $g^{(i)}$ = mass of i^{th} component in the glass.
- G = total mass of glass.
- $p^{(i)}$ = fraction of i^{th} component in the glass.

Definition of the above decision variables implies that

$$W^{(i)} = \sum_{j=1}^{3} w_{ij} \tag{2.42}$$

$$g^{(i)} = W^{(i)} + f^{(i)} \tag{2.43}$$

$$G = \sum_{i=1}^{n} g^{(i)} \tag{2.44}$$

$$p^{(i)} = g^{(i)}/G \tag{2.45}$$

Note that G is composed of a known component $W^{(i)}$ and an unknown component $f^{(i)}$ representing degrees of freedom. Also, all these variables are nonnegative because frit can only be added to comply with the constraint. The objective is to minimize the total amount of waste to be vitrified. This can be formulated as:

$$\text{Min } G \equiv \text{Min} \sum_{i=1}^{n} f^{(i)} \tag{2.46}$$

Subject to the following constraints.

1. Component bounds:
   a) $0.42 \le p^{(\mathrm{SiO_2})} \le 0.57$
   b) $0.05 \le p^{(\mathrm{B_2O_3})} \le 0.20$
   c) $0.05 \le p^{(\mathrm{Na_2O})} \le 0.20$
   d) $0.01 \le p^{(\mathrm{Li_2O})} \le 0.07$
   e) $0.0 \le p^{(\mathrm{CaO})} \le 0.10$
   f) $0.0 \le p^{(\mathrm{MgO})} \le 0.08$
   g) $0.02 \le p^{(\mathrm{Fe_2O_3})} \le 0.15$
   h) $0.0 \le p^{(\mathrm{Al_2O_3})} \le 0.15$
   i) $0.0 \le p^{(\mathrm{ZrO_2})} \le 0.13$
   j) $0.01 \le p^{(\mathrm{other})} \le 0.10$
2. Five glass crystallinity constraints:
   a) $p^{(\mathrm{SiO_2})} > p^{(\mathrm{Al_2O_3})} * C_1$
   b) $p^{(\mathrm{MgO})} + p^{(\mathrm{CaO})} < C_2$
   c) $p^{(\mathrm{Fe_2O_3})} + p^{(\mathrm{Al_2O_3})} + p^{(\mathrm{ZrO_2})} + p^{(\mathrm{Other})} < C_3$
   d) $p^{(\mathrm{Al_2O_3})} + p^{(\mathrm{ZrO_2})} < C_4$
   e) $p^{(\mathrm{MgO})} + p^{(\mathrm{CaO})} + p^{(\mathrm{ZrO_2})} < C_5$
3. Solubility Constraints:
   a) $p^{(\mathrm{Cr_2O_3})} < 0.005$
   b) $p^{(\mathrm{F})} < 0.017$
   c) $p^{(\mathrm{P_2O_5})} < 0.01$
   d) $p^{(\mathrm{SO_3})} < 0.005$
   e) $p^{(\mathrm{Rh_2O_3})} + p^{(\mathrm{PdO})} + p^{(\mathrm{Ru_2O_3})} < 0.025$
4. Nonnegativity Constraint:
   a) $f^{(i)} \ge 0$

Note that Equation (2.45) is a nonlinear equation, making the problem an NLP. We can eliminate this constraint if we can write all four types of constraint equations in terms of the mass of the component $g^{(i)}$ instead of the fraction $p^{(i)}$.

The LP Formulation

$$\text{Min} \sum_{i=1}^{n} f^{(i)} \tag{2.47}$$

$$W^{(i)} = \sum_{j=1}^{3} w_{ij} \tag{2.48}$$

$$g^{(i)} = W^{(i)} + f^{(i)} \tag{2.49}$$

$$G = \sum_{i=1}^{n} g^{(i)} \tag{2.50}$$

1. Component bounds:
   a) $0.42G \leq g^{(\mathrm{SiO_2})} \leq 0.57G$
   b) $0.05G \leq g^{(\mathrm{B_2O_3})} \leq 0.20G$
   c) $0.05G \leq g^{(\mathrm{Na_2O})} \leq 0.20G$
   d) $0.01G \leq g^{(\mathrm{Li_2O})} \leq 0.07G$
   e) $0.0 \leq g^{(\mathrm{CaO})} \leq 0.10G$
   f) $0.0 \leq g^{(\mathrm{MgO})} \leq 0.08G$
   g) $0.02G \leq g^{(\mathrm{Fe_2O_3})} \leq 0.15G$
   h) $0.0 \leq g^{(\mathrm{Al_2O_3})} \leq 0.15G$
   i) $0.0 \leq g^{(\mathrm{ZrO_2})} \leq 0.13G$
   j) $0.01G \leq g^{(\mathrm{other})} \leq 0.10G$
2. Five Glass crystallinity constraints:
   a) $g^{(\mathrm{SiO_2})} > g^{(\mathrm{Al_2O_3})} * C_1$
   b) $g^{(\mathrm{MgO})} + g^{(\mathrm{CaO})} < C_2 * G$
   c) $g^{(\mathrm{Fe_2O_3})} + g^{(\mathrm{Al_2O_3})} + g^{(\mathrm{ZrO_2})} + g^{(\mathrm{Other})} < C_3 * G$
   d) $g^{(\mathrm{Al_2O_3})} + g^{(\mathrm{ZrO_2})} < C_4 * G$
   e) $g^{(\mathrm{MgO})} + g^{(\mathrm{CaO})} + g^{(\mathrm{ZrO_2})} < C_5 * G$
3. Solubility Constraints:
   a) $g^{(\mathrm{Cr_2O_3})} < 0.005G$
   b) $g^{(\mathrm{F})} < 0.017G$
   c) $g^{(\mathrm{P_2O_5})} < 0.01G$
   d) $g^{(\mathrm{SO_3})} < 0.005G$
   e) $g^{(\mathrm{Rh_2O_3})} + g^{(\mathrm{PdO})} + g^{(\mathrm{Ru_2O_3})} < 0.025G$
4. Nonnegativity Constraint:
   a) $f^{(i)} \geq 0$

This problem is then solved using iterative solution procedures using GAMS, and the solution to the LP is given in the Table 2.17. The GAMS

Table 2.17. Composition for the optimal solution.

Component	**Mass in the Waste, $W^{(i)}$**	**Mass in Frit $f^{(i)}$**
SiO_2	11.2030	464.2909
B_2O_3	2.4111	110.1268
Na_2O	34.1980	7.5120
Li_2O	0.0000	8.3420
CaO	5.5436	
MgO	2.8776	
Fe_2O_3	89.0097	
Al_2O_3	45.5518	
ZrO_2	11.4111	
Other	41.7223	
Total	243.9281	590.2718

input files for this problem and the solution can be found online on Springer website with the book link. Thus, Hanford should add approximately 590 kgs of frit to the blend of these three tanks. Although this appears to be a small amount as compared to the total mass of the glass, when all the tanks are considered, blending and optimization can reduce the amount of total glass formed by more than half.

2.8 Summary

Linear programming problems involve linear objective functions and linear constraints. The LP optimum lies at a vertex of the feasible region, which is the basis of the simplex method. LP can have 0 (infeasible), 1, or infinite (multiple) solutions. LPs do not have multiple local minima. As a general rule, LP computational effort depends on the number of constraints rather than the number of variables. Many of the LP methods are derived from the simplex method, and special classes of problems can be solved efficiently with special LP methods. The interior point method is based on the transformation of variables and using a search direction similar to nonlinear programming methods discussed in the next chapter. This method is polynomially bounded, but only large-scale problems where no prior information is available show computational savings.

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Exercises

2.1 Write the following problems in standard form and solve using the simplex method. Verify your solutions graphically (where possible).

1.

$$\max 6x_1 + 4x_2$$
$$3x_1 + 2x_2 \leq 8$$
$$-4x_1 + 9x_2 \leq 20$$
$$x_1, x_2 \geq 0$$

2.

$$\max 3x_1 + 2x_2$$
$$-2x_1 + x_2 \leq 1$$
$$x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

3.

$$\min 2x_1 - 4x_2$$
$$3x_1 + x_2 \leq 1$$

$$-2x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

4.

$$\max x_1 + 5x_2$$
$$x_1 + 3x_2 \leq 5$$
$$2x_1 + x_2 = 4$$
$$x_1 - 2x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

5.

$$\min 3x_1 + 4x_2 - x_3$$
$$x_1 + 3x_2 - x_3 \geq 1$$
$$2x_1 + x_2 + 0.5x_3 \geq 4$$
$$x_1, x_2 \geq 0; x_3 \text{ is unconstrained}$$

6.

$$\min 8x_1 - 3x_2 + 10x_3$$
$$5x_1 - 2x_2 - 4x_3 \geq 3$$
$$3x_1 + 6x_2 + 8x_3 \geq 4$$
$$2x_1 - 4x_2 + 8x_3 \geq -4$$
$$-x_2 + 5x_3 \geq 1$$
$$x_1, x_2, x_3 \geq 0$$

Also solve this problem using a dual formulation.

2.2 A refinery has two crude oil materials with which to create gasoline and lube oil:

1. Crude A costs \$28/bbl and 18,000 bbl are available.
2. Crude B costs \$38/bbl and 32,000 bbl are available.

The yield and sale price per barrel of the products and the associated markets are shown in Table 2.18.

How much crude A and B should be used to maximize the profit of the company? Formulate and solve the problem using the simplex algorithm.

Table 2.18. Yield and sale prices of products.

Product	Yield/bbl Crude A	Yield/bbl Crude B	Sale Price per bbl	Market (bbl)
gasoline	0.6	0.85	\$60	20,000
lube oil	0.4	0.15	\$130	12,000

Verify your solution graphically. How would the optimal solution be affected if

1. The market for lube oil increased to 14,000. bbl
2. The cost of crude A decreased to \$20/bbl.

2.3 A manufacturer sells products A and B. The profit from A is \$12/kg and from B \$7/kg. The available raw materials for the products are 100 kg of C and 80 kg of D. To produce one kilogram of A the manufacturer needs 0.5 kg of C and 0.5 kg of D. To produce one kilogram of B the manufacturer needs 0.4 kg of C and 0.6 kg of D. The market for product A is 60 kg and for B 120 kg. How much raw material should be used to maximize the manufacturer's profit? Formulate and solve the problem using the simplex algorithm. Verify your solution graphically. How would the optimal solution be affected if:

1. The availability of C were increased to 120 kg.
2. The availability of D were increased to 100 kg.
3. The market for A were decreased to 40 kg.
4. The profit of A were \$10/kg.

2.4 On the bank of a river there are three neighboring cities that are discharging two kinds of pollutants A and B into the river. Now the state government has set up a treatment plant that treats pollutants from City 1 for \$15/ton which reduces pollutants A and B by the amount of 0.10 and 0.45 tons per ton of waste, respectively. It costs \$10/ton to process a ton of City 2 waste and consequentially reducing pollutants A and B by 0.20 and 0.25 tons per ton of waste, respectively. Similarly City 3 waste is treated for \$20 reducing A by 0.40 and B by 0.30 tons per ton of waste. The state wishes to reduce the amount of pollutant A by at least 30 and B by 40 tons. Formulate the LP that will minimize the cost of reducing pollutants by desired amount.

2.5 Products I and II that are manufactured by a firm are sold at the rate of \$2 and \$3, respectively. Both products have to be processed on machines A and B. Product I requires 1 minute on A and 2 minutes on B where as Product II requires 1 minute on each machine. Machine A is not available for more than 6 hours 40 minutes/day, where as machine B is not available for more than 10 hours. Formulate the problem for profit maximization. Solve this problem using the simplex method.

2.6 There are many drug manufacturers producing various combinations for a similar ailment. Now a doctor wishes to prescribe a combination dosage such that the cost is minimum so that it could be given to poor patients. Drug A costs 50 cents, Drug B costs 20 cents, Drug C 30 cents, and Drug D 80 cents per tablet, respectively. Daily requirements are 5 mg of Medicine 1, 6 mg Medicine 2, 10 mg Medicine 3, and 8 mg Medicine 4.

The prescribed composition of each drug is given in Table 2.19. Write the prescription that satisfies the medicinal requirements at minimum cost.

Table 2.19. Prescribed composition of each drug.

Drug	Medicine 1	Medicine 2	Medicine 3	Medicine 4
A	4	3	2	2
B	2	2	2	4
C	1.5	0	4	1
D	5	0	4	5

2.7 A manufacturing firm has discontinued production of a certain profitable product line. This created considerable excess production capacity. Management is considering devoting their excess capacity to one or more of three products 1, 2, and 3. The available capacity on the machines and the number of machine-hours required for each unit of the respective product, is given in Table 2.20.

Table 2.20. Available machine capacities.

Machine	Available Time(hrs/week)	Productivity(hrs/unit)		
		Product 1	Product 2	Product 3
Milling	250	8	2	3
Lathe	150	4	3	0
Grinder	50	2	0	1

The unit profit would be \$20, \$6, and \$8 respectively for products 1, 2, and 3. Find how much of each product the firm should produce in order to maximize profit.

2.8 Four professors are each capable of teaching any of four different courses. Class preparation time in hours for different topics varies from professor to professor and is given in Table 2.21. Each professor is assigned only one course. Find the assignment policy schedule so as to minimize the total course preparation time for all the courses.

Table 2.21. Course preparation times in hours.

Professor	LP	Queueing Theory	Dynamic Programming	Regression Analysis
1	2	10	9	7
2	15	4	14	8
3	13	14	16	11
4	3	15	13	8

2.9 Three investment opportunities are available (Table 2.22) with their cash flow and net present value (million dollars) for a firm. It at the start has 30 million dollars and estimates that at the end of one year it will have 15 million dollars. The firm can purchase any fraction of any investment, the cash flow, and net present value accordingly. The firm's objective is to maximize the NPV. Assumption is that any funds left over time at time zero cannot be used at time one.

Table 2.22. Investment opportunities.

	Investment 1	Investment 2	Investment 3
Time 0 cash flow	\$11	\$297	\$5
Time 1 cash flow	\$3	\$34	\$5
NPV	\$13	\$39	\$16

2.10 The engineering department for Alash Inc. has their computers distributed to their employees according to Table 2.23. The designer and analysts (grade 1) are responsible for generating engineering designs, whereas the analysts (grade 2) and engineers are responsible for generating repair item reports. Currently, all of the designers and analysts (grade 1) utilize Autocad software on their computers for generating the designs. The analysts (grade 2) and engineers utilize software M (name changed for confidentiality reasons) on their computers. Autocad software requires more Pentium and more RAM than software M. With a computer with 266 MHz Pentium and 64 MB of RAM, it takes a designer or analyst (grade 1) an average of 40 man-hours to produce one drawing. A difference in one MHz of Pentium changes the speed of producing a drawing on Autocad 0.02% and an increase of 32 MB of RAM allows the computer 0.15% faster. With a computer with 166 MHz and 32 MB of RAM it takes an engineer an average of 20 man-hours to produce one repair item report. Find the distribution that will minimize the cost to finish the required amount of work.

Table 2.23. Computer distribution.

Computer(RAM)	Designer	Analysts 1	Analysts 2	Engineers
266 MHz 64 MB	10	7	7	14
200 MHz 64 MB	8	2	2	6
166 MHz 32 MB	18	2	2	34
133 MHz 32 MB	7	0	0	17
350 MHz 128 MB	0	0	0	0

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