

Chapter 2

Schechter–Tintarev Linking

The relationship between the classical linking theorem and the sign-changing critical point is established. The abstract theory is applied to elliptic equations with miscellaneous resonance.

2.1 Schechter–Tintarev Linking

Let E be a Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Define a class of contractions of E as follows.

$$(2.1) \quad \Phi := \left\{ \Gamma : \begin{array}{l} \Gamma(\cdot, \cdot) \in \mathbf{C}([0, 1] \times E, E), \quad \Gamma(0, \cdot) = \mathbf{id}, \\ \text{for each } t \in [0, 1], \Gamma(t, \cdot) \text{ is a homeomorphism of } E \\ \text{onto itself and } \Gamma^{-1}(t, \cdot) \text{ is continuous on } [0, 1] \times E; \\ \text{there exists a } x_0 \in E \text{ such that } \Gamma(1, x) = x_0 \\ \text{for each } x \in E \text{ and that } \Gamma(t, x) \rightarrow x_0 \text{ as } t \rightarrow 1 \\ \text{uniformly on bounded subsets of } E \end{array} \right\}.$$

Obviously, $\Gamma(t, u) = (1 - t)u \in \Phi$.

Definition 2.1. A subset A of E is linked to a subset B of E if $A \cap B = \emptyset$ and, for every $\Gamma \in \Phi$, there is a $t \in [0, 1]$ such that $\Gamma(t, A) \cap B \neq \emptyset$.

Proposition 2.2. Let $H \in \mathbf{C}(E, \mathbf{R}^N)$ and $Q \subset E$ be such that $H_0 = H|_Q$ is a homeomorphism of Q onto the closure of a bounded open subset Ω of $\mathbf{R}^N$. Let $p \in \Omega$; then $H_0^{-1}(\partial\Omega)$ links $H^{-1}(p)$.

Proof. Assume by negation that $H_0^{-1}(\partial\Omega)$ does not link $H^{-1}(p)$. Then there is a $\Gamma \in \Phi$ such that

$$(2.2) \quad \Gamma(t, H_0^{-1}(\partial\Omega)) \cap H^{-1}(p) = \emptyset, \quad t \in [0, 1].$$

That is,

$$(2.3) \quad H(\Gamma(t, H_0^{-1}(\partial\Omega))) \cap \{p\} = \emptyset, \quad t \in [0, 1].$$

Set

$$(2.4) \quad \theta(t) := H \circ \Gamma(t, \cdot) \circ H_0^{-1}.$$

Then we see that $\theta(t) \in \mathbf{C}(\bar{\Omega}, \mathbf{R}^N)$ for each $t \in [0, 1]$ and $\theta(0) = \mathbf{id}$ on Ω . If $\Gamma(1, E) = x_0$, then $\theta(1)x = Hx_0 \neq p$ for all $x \in \bar{\Omega}$ because by (2.2)–(2.4),

$$H(\Gamma(1, H_0^{-1}(\partial\Omega))) \cap \{p\} = \emptyset.$$

By Brouwer's degree,

$$\deg(\theta(t), \Omega, p) = \deg(\theta(1), \Omega, p) = 1, \quad \forall t \in [0, 1].$$

This is a contradiction. $\square$

Proposition 2.3. *Let A, B be two closed bounded subsets of E such that $E \setminus A$ is path connected. If A links B , then B links A as well.*

Proof. Assume by negation that B does not link A . Then we may find a $\Gamma \in \bar{\Phi}$ such that

$$(2.5) \quad \Gamma(t, B) \cap A = \emptyset, \quad \forall t \in [0, 1].$$

By the definition of $\bar{\Phi}$, we assume that $\Gamma(1, E) = x_0$, hence $x_0 \notin A$. Let Ω be a closed ball such that $A \subset \Omega$. Note that $E \setminus A$ is path connected; then there is a path γ connecting $x_0 \notin A$ to a point $x_1 \notin \Omega$. Let $t_0 \in [0, 1]$ be such that the diameter of $\Gamma(t_0, B) - x_0$ is less than $\min\{\text{dist}(\gamma, A), \text{dist}(x_1, \Omega)\}$. Parameterize γ in such a way that it is given by $\gamma(t), t_0 \leq t \leq 1, \gamma(t_0) = x_0, \gamma(1) = x_1$. Then

$$(2.6) \quad (\Gamma(t_0, B) + \gamma(t) - x_0) \cap A = \emptyset, \quad t_0 \leq t \leq 1$$

and

$$(2.7) \quad (\Gamma(t_0, B) + x_1 - x_0) \cap \Omega = \emptyset.$$

Define

$$\Gamma_1(t, x) = \begin{cases} \Gamma(t, x) & \text{for } t \in [0, t_0], \\ \Gamma(t_0, x) - x_0 + \gamma(t) & \text{for } t \in [t_0, 1]. \end{cases}$$

By (2.5), we see that $\Gamma_1(t, B) \cap A = \emptyset$ for all $t \in [0, 1]$. Hence,

$$(2.8) \quad B \cap \Gamma_1^{-1}(t, A) = \emptyset, \quad \forall t \in [0, 1].$$

By (2.7),

$$(2.9) \quad \Gamma_1(1, B) \cap \Omega = \emptyset.$$

Let Γ_2 be any map in Φ such that $\Gamma_2(t, \Omega) \subset \Omega$ for all t . Take

$$\Gamma_3(t, \cdot) = \begin{cases} \Gamma_1(2t, \cdot)^{-1} & \text{for all } t \in [0, 1/2], \\ \Gamma_1(1, \cdot)^{-1} \circ \Gamma_2(2t - 1, \cdot) & \text{for all } (1/2, 1]. \end{cases}$$

It is easy to check that $\Gamma_3 \in \Phi$. But Equations (2.8)–(2.9) imply that

$$B \cap \Gamma_3(t, A) = \emptyset, \quad t \in [0, 1].$$

It contradicts the fact that A links B . $\square$

Proposition 2.4. *Let $E = M \oplus Y$, where M, Y are closed subspaces with one of them finite-dimensional. If $y_0 \in M \setminus \{0\}$ and $0 < \rho < R$, then the sets*

$$\begin{aligned} A &:= \{u = v + sy_0 : v \in Y, s \geq 0, \|u\| = R\} \cup [Y \cap \bar{B}_R], \\ B &:= M \cap \partial B_\rho \end{aligned}$$

link each other in the sense of Definition 2.1, where $B_r := \{u \in E : \|u\| < r\}$.

Proof. We first consider the case of $\dim Y < \infty$ and identify Y with some $\mathbf{R}^N$. We may assume that $\|y_0\| = 1$. Let

$$Q = \{sy_0 + v : v \in Y, s \geq 0, \|sy_0 + v\| \leq R\}.$$

Then $A = \partial Q$ in $\mathbf{R}^{N+1}$. Let $u = v + w$ with $v \in Y, w \in M$; we define $Fu = v + \|w\|y_0$. Then $F|_Q = \text{id}$ and $B = F^{-1}(\rho y_0)$. We can apply Proposition 2.2 to conclude that A links B . Because A and B are bounded and $E \setminus A$ is path connected, B links A as well. $\square$

Proposition 2.5. *Let $E = M \oplus Y$, where M, Y are closed subspaces with $\dim Y < \infty$. Let $B_R = \{u \in E : \|u\| < R\}$ and let $A = \partial B_R \cap Y, B = M$. Then A links B .*

Proof. We identify Y with some $\mathbf{R}^N$ and take $\Omega = B_R \cap Y, Q = \bar{\Omega}$. For $u = v + w, v \in Y, w \in M$, define the projection $Fu = v$. Because $F|_Q = \text{id}$ and $M = F^{-1}(0)$, we observe by Proposition 2.2 that A links B . $\square$

Proposition 2.6. *Let B be an open set in E and $A = \{a, b\}$ such that $a \in B, b \notin \bar{B}$. Then A links ∂B .*

Proof. Let $\Gamma \in \Phi$. If $\Gamma(1, E) = u_0$, then $\Gamma(t, a) (\Gamma(t, b))$ is a curve in E connecting a (b , respectively) with u_0 . If $u_0 \notin \bar{B}$, then $\Gamma(t, a)$ intersects ∂B . If $u_0 \in B$, then $\Gamma(t, b)$ intersects ∂B . Hence A links ∂B . $\square$

Proposition 2.7. *Let $E = M \oplus Y$, where M, Y are closed subspaces with $\dim Y < \infty$. Let $B_R := \{u \in E : \|u\| < R\}$ and take $A = \partial B_R \cap Y$. Choose $z_0 \neq 0, z_0 \in Y$ and let*

$$B = \{u \in M : \|u\| \geq \delta\} \cup \{u = sz_0 + v : v \in M, s \geq 0, \|sz_0 + v\| = \delta\}.$$

Then A links B with respect to Φ for any $R > \delta > 0$.

Proof. Let $Q = \bar{B}_R \cap Y$. For simplicity, we may assume that $\|z_0\| = 1$ and that

$$E = \tilde{Y} \oplus \mathbf{R}z_0 \oplus M,$$

for each $u = \tilde{u} + sz_0 + v$ with $\tilde{u} \in \tilde{Y}, v \in M$. Define

$$H(u) = \begin{cases} \tilde{u} + (s + \delta - (\delta^2 - \|v\|^2)^{1/2})z_0, & \text{for } \|v\| \leq \delta, \\ \tilde{u} + (s + \delta)z_0, & \text{for } \|v\| > \delta. \end{cases}$$

We observe that $H|_Q = \mathbf{id}$ and that $H^{-1}(\delta z_0)$ is precisely the set B . We hence conclude by Proposition 2.2 that A links B . $\square$

Proposition 2.8. *Let $E = M \oplus Y$, where M, Y are closed subspaces with $\dim N < \infty$. Let $B_R := \{u \in E : \|u\| < R\}$ and take $A = \partial B_R \cap Y$. Choose $z_0 \neq 0, z_0 \in Y, R_1 > \delta$ and let*

$$B = \{sz_0 + v : s \leq 0; v \in M, R_1 \geq \|v\| \geq \delta\} \\ \cup \{u = sz_0 + v : v \in M, s \geq 0, \|sz_0 + v\| = \delta\}.$$

Then A links B with respect to Φ for any $R > \delta > 0$.

Proof. Let $Q = \bar{B}_R \cap Y$. For simplicity, we may assume that $\|z_0\| = 1$ and that

$$E = \tilde{Y} \oplus \mathbf{R}z_0 \oplus M.$$

For each $u = \tilde{u} + sz_0 + v$ with $\tilde{u} \in \tilde{Y}, v \in M$. Define

$$H(u) = \begin{cases} \tilde{u} + (s + \delta - (\delta^2 - \|v\|^2)^{1/2})z_0, & \text{for } \|v\| \leq \delta, \\ \tilde{u} + (s + \delta)z_0 + \frac{\|v\| - \delta}{R_1 - \delta}(s^2 + 1)z_0, & \text{for } R_1 \geq \|v\| > \delta, \\ \tilde{u} + (s^2 + s + 1 + \delta)z_0, & \text{for } \|v\| \geq R_1. \end{cases}$$

We observe that $H|_Q = \mathbf{id}$ and that $H^{-1}(\delta z_0)$ is precisely the set B . We hence conclude by Proposition 2.2 that A links B . $\square$

Proposition 2.9. *If B is any subset of a bounded open set $\Omega \subset E$, then $\partial\Omega$ links B .*

Proof. Assume that $\partial\Omega$ does not link B , and we seek a contradiction. In this case, there is a $\Gamma \in \Phi$ such that

$$(2.10) \quad \Gamma(s, \partial\Omega) \cap B = \emptyset, \quad \forall s \in [0, 1].$$

Hence,

$$(2.11) \quad \Gamma(1, \partial\Omega) := u \notin B.$$

Let v be any point in B . We now show that

$$(2.12) \quad \|\Gamma^{-1}(s, v)\| \rightarrow \infty, \quad \text{as } s \rightarrow 1.$$

If this were not true, we would have by the definition of Φ that

$$v = \Gamma(s, v)[\Gamma^{-1}(s, v)] \rightarrow u,$$

which contradicts (2.11). Therefore, (2.12) is correct and thus

$$(2.13) \quad \Gamma^{-1}(s, v) \notin \bar{\Omega}$$

for s close to 1 because Ω is bounded. Note that $v = \Gamma^{-1}(0, v) \in B \subset \Omega$; we may have an $s_0 \in (0, 1)$ such that

$$(2.14) \quad \Gamma^{-1}(s_0, v) \in \partial\Omega.$$

To see this, let

$$s_0 = \sup\{s \in [0, 1] : \Gamma^{-1}(t, v) \in \Omega, \forall t \in [0, s]\}.$$

Then $\Gamma^{-1}(s_0, v) \notin \Omega$. Otherwise, there would be an interval containing s_0 in which $\Gamma^{-1}(s, v) \in \Omega$; this contradicts the definition of s_0 . But

$$\Omega \ni \Gamma^{-1}(s, v) \rightarrow \Gamma^{-1}(s_0, v), \quad \text{as } s \rightarrow s_0.$$

Hence, (2.14) holds and this means that $v \in \Gamma(s_0, \partial\Omega)$ which contradicts (2.10). Thus, $\partial\Omega$ links B . $\square$

Proposition 2.10. *Let E be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$. We assume that there is another norm $\|\cdot\|_*$ of E such that $\|u\|_* \leq C_* \|u\|$ for all $u \in E$; here $C_* > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_* \rightarrow 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. Write $E = M \oplus Y$, where M, Y are closed subspaces with $\dim Y < \infty$. If $y_0 \in M \setminus \{0\}$ with $\|y_0\| = 1$ and $0 < \rho < R$ with*

$$R^{p-2} \|y_0\|_*^p + \frac{R \|y_0\|_*}{1 + D_* \|y_0\|_*} > \rho, \quad D_* > 0, p > 2 \text{ are constants.}$$

Let

$$A := \{u = v + sy_0 : v \in Y, s \geq 0, \|u\| = R\} \cup [Y \cap \bar{B}_R],$$

$$B := \left\{ u \in M : \frac{\|u\|_*^p}{\|u\|^2} + \frac{\|u\|\|u\|_*}{\|u\| + D_*\|u\|_*} = \rho \right\},$$

where $\bar{B}_R$ denotes the closed ball of E centered at zero with radius R . Then A links B in the sense of Definition 2.1.

Proof. It is easy to check that $A \cap B = \emptyset$. We identify Y with some $\mathbf{R}^N$. Let

$$Q = \{sy_0 + v : v \in Y, s \geq 0, \|sy_0 + v\| \leq R\}.$$

Then $A = \partial Q$ in $\mathbf{R}^{N+1}$. Let $u = v + w$ with $v \in Y, w \in M$; we define

$$(2.15) \quad \xi_0(u) := \begin{cases} \frac{\|u\|_*^p}{\|u\|^2} + \frac{\|u\|\|u\|_*}{\|u\| + D_*\|u\|_*}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

Then $\xi_0 : E \rightarrow E$ is continuous. We define

$$Fu = v + \xi_0(w)y_0.$$

Then for any $u = v + sy_0 \in Q$, we see that $s \in [0, R]$ and $F(u) = v + \xi_0(sy_0)y_0$. Here $\xi_0(sy_0) = 0$ iff $s = 0$. Otherwise

$$\xi_0(sy_0) = s^{p-2} \frac{\|y_0\|_*^p}{\|y_0\|^2} + s \frac{\|y_0\|_*\|y_0\|}{D_*\|y_0\|_* + \|y_0\|} := as^{p-2} + sb,$$

where $a, b > 0$ are two constants depending on y_0 only. Therefore, $F_0 = F|_Q$ is a homeomorphism of Q onto the closure of a bounded open subset Ω of $\mathbf{R}^{N+1}$. Let $\rho y_0 \in \Omega$; then by Proposition 2.2, $F_0^{-1}(\partial\Omega) = \partial Q$ links $F^{-1}(\rho y_0) = B$. That is,

$$A \text{ links } B.$$

Proposition 2.11. *In Proposition 2.10, if we choose $y_0 \in M$ with $\|y_0\|_* = 1$ and $R > \rho\|y_0\|$, then A links B where B is replaced by $B := \{u \in M : \|u\|_* = \rho\}$.* □

Proof. It is obvious. □

Notes and Comments. Definition 2.1 was introduced by Schechter and Tintarev [278] and Propositions 2.2–2.4 were proved there (see also Schechter [275]). Propositions 2.5–2.7 and 2.9 can be found in Schechter [275]. Proposition 2.8 is a new one. More examples can be seen in Schechter [273, 275] and Schechter and Zou [280, 283]. The original approach to linking required A to be of a special nature (e.g., the boundary of a manifold) in order to

link a set B . This severely restricted the kind of sets that could be used. The linking in the current book seems more general and realistic. It only means that A cannot be continuously shrunk to a point without intersecting B . We refer readers to Benci and Rabinowitz's linking (infinite-dimensional and compact maps) in [55], Silva's linking of deformation type in [295], Corvellec's linking on metric spaces in [109], Tintarev's isotopic linking in [328], Li and Liu's local linking at zero in [197] (see also Chang [93], Brézis and Nirenberg [70], and Li and Willem [200]), Benci's homological linking in [51] (see also Perera [241] and Liu [207]), and Ramos and Sanchez's homotopical linking in [261].

2.2 Sign-Changing Critical Points via Linking

Let E be a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. Consider the following type of functional $G \in \mathbf{C}^1(E, \mathbf{R})$. Its gradient G' is of the form

$$(2.16) \quad G'(u) = \kappa(u)u - \Theta_G u,$$

where $\kappa(u) : E \rightarrow [1/2, 1]$ is a locally Lipschitz continuous function; $\Theta_G : E \rightarrow E$ is a continuous operator. Let $\mathcal{K} := \{u \in E : G'(u) = 0\}$ and $\tilde{E} := E \setminus \mathcal{K}$.

Let $V : \tilde{E} \rightarrow E$ be a pseudo-gradient vector field for G (cf. Definition 1.53); that is,

- (1) $\langle G'(u), V(u) \rangle \geq \frac{1}{2} \|G'(u)\|^2$ for all $u \in \tilde{E}$.
- (2) $\|V(u)\| \leq 2 \|G'(u)\|$ for all $u \in \tilde{E}$.

By Theorem 1.43, the following initial value problem

$$\begin{cases} \frac{d\sigma(t, u)}{dt} = -V(\sigma(t, u)), \\ \sigma(0, u) = u \in \tilde{E}, \end{cases}$$

has a unique solution (called flow or trajectory) $\sigma : [0, T(u)) \rightarrow E$, where $T(u) \in (0, \infty]$ is the maximal time of the existence of the flow with initial value u .

Let $\mathcal{P}$ be a closed convex and weakly closed subset of E such that $(\mathcal{P}) \setminus \{0\} \neq \emptyset$.

For any $\delta > 0$, define

$$(2.17) \quad \pm \mathcal{D}(\delta) := \{u \in E : \text{dist}(u, \pm \mathcal{P}) < \delta\},$$

$$(2.18) \quad \mathcal{D}^* := \mathcal{D}(\delta) \cup (-\mathcal{D}(\delta)), \quad \mathcal{S} = E \setminus \mathcal{D}^*.$$

Then $\pm\mathcal{D}(\delta)$ are open convex, $\mathcal{D}^*$ is open, $\pm\mathcal{P} \subset \pm\mathcal{D}(\delta/2) \subset \pm\mathcal{D}(\delta)$, and $\mathcal{S}$ is closed. We make the following assumption.

(A₁) There exists a $\delta > 0$ (small enough) such that $\Theta_G(\pm\mathcal{D}(\delta)) \subset \pm\mathcal{D}(\delta/2)$.

Lemma 2.11. *Assume (A₁). Then there exists a locally Lipschitz continuous map $L_0 : \tilde{E} \rightarrow E$ such that $L_0(\pm\mathcal{D}(\delta) \cap \tilde{E}) \subset \pm\mathcal{D}(\delta/2)$ and that $V(u) := \kappa(u)u - L_0(u)$ is a pseudo-gradient vector field of G . Furthermore, V and L_0 can be chosen to be odd if G, κ are even.*

Proof. Note that $\|G'(v)\| \neq 0$ for any $v \in \tilde{E}$. Define

$$\Omega(v) := \left\{ u \in \tilde{E} : \|G'(u)\| > \frac{1}{2}\|G'(v)\|, \|\Theta_G u - \Theta_G v\| < \frac{1}{8}\|G'(v)\| \right\}.$$

Then $\{\Omega(v) : v \in \tilde{E}\}$ is an open covering of $\tilde{E}$ in the topology of E , and we can find a locally finite refinement open covering $\{\tilde{\Omega}(\lambda) : \lambda \in \Lambda\}$ of $\tilde{E}$, where Λ is the index set. For any $\lambda \in \Lambda$, only one of the following cases occurs.

- (1) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) = \emptyset, \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta)) = \emptyset;$
- (2) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \neq \emptyset, \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta)) = \emptyset;$
- (3) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) = \emptyset, \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta)) \neq \emptyset;$
- (4) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta)) \neq \emptyset;$
- (5) $\tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \neq \emptyset, \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta)) \neq \emptyset, \tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta)) = \emptyset.$

If the last case happens, we remove $\tilde{\Omega}(\lambda)$ from the covering and replace it with $\tilde{\Omega}(\lambda) \setminus \bar{\mathcal{D}}(\delta)$ and $\tilde{\Omega}(\lambda) \setminus (-\bar{\mathcal{D}}(\delta))$. In this way, we rearrange them so that the new covering has only the properties (1)–(4). In particular, the new covering is still a covering of $\tilde{E}$. To see this, we take any $w \in \tilde{E}$; then we have a $\tilde{\Omega}(\lambda)$ in the “old” covering $\{\tilde{\Omega}(\lambda) : \lambda \in \Lambda\}$ of $\tilde{E}$ such that $w \in \tilde{\Omega}(\lambda)$. If $\tilde{\Omega}(\lambda)$ is one of the cases (1)–(4), then w is covered by $\tilde{\Omega}(\lambda)$ which is also in the new covering. If $\tilde{\Omega}(\lambda)$ is of case (5), then we may distinguish the following cases.

- If $w \notin \bar{\mathcal{D}}(\delta)$, then $w \in \tilde{\Omega}(\lambda) \setminus \bar{\mathcal{D}}(\delta)$; hence, w is covered by the new covering.
- If $w \notin -\bar{\mathcal{D}}(\delta)$, then $w \in \tilde{\Omega}(\lambda) \setminus (-\bar{\mathcal{D}}(\delta))$; hence, w is covered by the new covering.
- The remaining case is $w \in \bar{\mathcal{D}}(\delta) \cap (-\bar{\mathcal{D}}(\delta))$; we now show that this will not happen. Firstly, we observe that $w \notin \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta))$. Otherwise, it contradicts the fact that $\tilde{\Omega}(\lambda)$ is of Case (5). So, we must have the following cases.

- (a) $w \notin \mathcal{D}(\delta), w \in -\mathcal{D}(\delta)$; hence, $w \in \partial\mathcal{D}(\delta) \cap (-\mathcal{D}(\delta))$.
- (b) $w \notin \mathcal{D}(\delta), w \notin -\mathcal{D}(\delta)$; hence, $w \in \partial\mathcal{D}(\delta) \cap (-\partial\mathcal{D}(\delta))$.
- (c) $w \in \mathcal{D}(\delta), w \notin -\mathcal{D}(\delta)$; hence, $w \in \mathcal{D}(\delta) \cap (-\partial\mathcal{D}(\delta))$.

Because $0 \in \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta))$, which is open convex, then for these cases (a)–(c),

$$tw = tw + (1-t)0 \in \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta)), \quad t \in (0, 1).$$

However, $tw \in \tilde{\Omega}(\lambda)$ for $t \rightarrow 1^-$ because $\tilde{\Omega}(\lambda)$ is open and $w \in \tilde{\Omega}(\lambda)$. This also contradicts the fact that $\tilde{\Omega}(\lambda)$ is of Case (5).

Therefore, we indeed get a new covering which has only the properties (1)–(4). For each $\lambda \in \Lambda$, define

$$\alpha_\lambda(u) := \text{dist}(u, \tilde{E} \setminus \tilde{\Omega}_\lambda), \quad \phi_\lambda(u) := \frac{\alpha_\lambda(u)}{\sum_{\lambda \in \Lambda} \alpha_\lambda(u)}, \quad u \in \tilde{E};$$

then $0 \leq \phi_\lambda(u) \leq 1$ and $\phi_\lambda : \tilde{E} \rightarrow E$ is locally Lipschitz continuous. For each $\lambda \in \Lambda$, choose $a_\lambda \in \tilde{\Omega}(\lambda)$ such that a_λ is arbitrary in Case (1); $a_\lambda \in \tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta)$ in Case (2); $a_\lambda \in \tilde{\Omega}(\lambda) \cap (-\mathcal{D}(\delta))$ in Case (3); and $a_\lambda \in \tilde{\Omega}(\lambda) \cap \mathcal{D}(\delta) \cap (-\mathcal{D}(\delta))$ in Case (4). Define

$$L_0(u) = \sum_{\lambda \in \Lambda} \phi_\lambda(u) \Theta_G a_\lambda, \quad u \in \tilde{E}.$$

Then $L_0 : \tilde{E} \rightarrow E$ is locally Lipschitz continuous. Let

$$V(u) := \kappa(u)u - L_0 u.$$

We prove that L_0 and V are what we want.

For any $u \in \tilde{E}$, there are only finitely many numbers $\lambda_1, \dots, \lambda_s \in \Lambda$ such that $u \in \tilde{\Omega}(\lambda_1) \cap \dots \cap \tilde{\Omega}(\lambda_s)$. Moreover, there are $w_1, \dots, w_s \in \tilde{E}$ such that $\tilde{\Omega}(\lambda_i) \subset \Omega(w_i)$ for $i = 1, \dots, s$. Then

$$L_0(u) = \sum_{i=1}^s \phi_{\lambda_i}(u) \Theta_G a_{\lambda_i},$$

where $a_{\lambda_i} \in \tilde{\Omega}(\lambda_i)$ for $i = 1, \dots, s$. Note that

$$\begin{aligned} & \|\Theta_G u - \Theta_G a_{\lambda_i}\| \\ & \leq \|\Theta_G u - \Theta_G w_i\| + \|\Theta_G w_i - \Theta_G a_{\lambda_i}\| \\ & \leq \frac{1}{4} \|G'(w_i)\| \\ & \leq \frac{1}{2} \|G'(u)\| \end{aligned}$$

for $i = 1, \dots, s$; it follows that

$$\begin{aligned}
& \|\Theta_G u - L_0 u\| \\
&= \left\| \Theta_G u - \sum_{i=1}^s \phi_{\lambda_i}(u) \Theta_G a_{\lambda_i} \right\| \\
&= \left\| \sum_{i=1}^s \phi_{\lambda_i}(u) (\Theta_G u - \Theta_G a_{\lambda_i}) \right\| \\
&\leq \frac{1}{2} \|G'(u)\|.
\end{aligned}$$

Hence

$$\begin{aligned}
(2.19) \quad & \|V(u)\| \\
&= \|\kappa(u)u - L_0(u)\| \\
&\leq \|\kappa(u)u - \Theta_G u\| + \|\Theta_G u - L_0(u)\| \\
&\leq \|G'(u)\| + \left\| \sum_{i=1}^s \phi_{\lambda_i}(u) \Theta_G u - \sum_{i=1}^s \phi_{\lambda_i}(u) \Theta_G a_{\lambda_i} \right\| \\
(2.20) \quad & \leq \frac{3}{2} \|G'(u)\|.
\end{aligned}$$

and $|\langle G'(u), \Theta_G u - L_0 u \rangle| \leq \frac{1}{2} \|G'(u)\|^2$. Furthermore,

$$(2.21) \quad \langle G'(u), V(u) \rangle = \|G'(u)\|^2 + \langle G'(u), \Theta_G u - L_0 u \rangle \geq \frac{1}{2} \|G'(u)\|^2.$$

Inequalities (2.20) and (2.21) imply that $V(u) := \kappa(u)u - L_0 u$ is a pseudo-gradient vector field for G . Next, we show that $L_0(\pm \mathcal{D}(\delta) \cap \tilde{E}) \subset \pm \mathcal{D}(\delta/2)$. In fact, for any $u \in \mathcal{D}(\delta) \cap \tilde{E}$, there are finitely many $\phi_\lambda(u)$, say $\phi_{\lambda_i}(u)$ ($i = 1, \dots, s$), which are nonzero. Then

$$L_0 u = \sum_{i=1}^s \phi_{\lambda_i}(u) \Theta_G a_{\lambda_i}$$

and $u \in \tilde{\Omega}(\lambda_i) \cap \mathcal{D}(\delta)$ for $i = 1, \dots, s$. Hence, $a_{\lambda_i} \in \tilde{\Omega}(\lambda_i) \cap \mathcal{D}(\delta)$ by the definition of a_λ . It follows that $\Theta_G a_{\lambda_i} \in \mathcal{D}(\delta/2)$ by recalling the condition (A_1) . It implies that $L_0(u) \in \mathcal{D}(\delta/2)$, because $\mathcal{D}(\delta/2)$ is also convex. This proves that $L_0(\mathcal{D}(\delta) \cap \tilde{E}) \subset \mathcal{D}(\delta/2)$. Similarly, we have that $L_0(-\mathcal{D}(\delta) \cap \tilde{E}) \subset -\mathcal{D}(\delta/2)$.

Finally, we show that V and L_0 can be chosen to be odd if G and κ are even. Let $\bar{L}_0(u) = \frac{1}{2}(L_0 u - L_0(-u))$; then $\bar{L}_0 : \tilde{E} \rightarrow E$ is odd and locally

Lipschitz continuous. Define $\bar{V} := \kappa(\cdot) - \bar{L}_0$; then $\bar{V} : \tilde{E} \rightarrow E$ is also odd and locally Lipschitz continuous. We may show that $\bar{L}_0(\pm\mathcal{D}(\delta) \cap \tilde{E}) \subset \pm\mathcal{D}(\delta/2)$. Indeed, for any $u \in \pm\mathcal{D}(\delta) \cap \tilde{E}$, then $-u \in \pm\mathcal{D}(\delta) \cap \tilde{E}$ and hence

$$L_0(\pm u) \in \pm\mathcal{D}(\delta/2), \quad -L_0(-u) \in \pm\mathcal{D}(\delta/2).$$

Therefore,

$$\bar{L}_0 u = \frac{1}{2}L_0 u + \frac{1}{2}(-L_0(-u)) \in \pm\mathcal{D}(\delta/2),$$

because $\pm\mathcal{D}(\delta/2)$ is convex. $\square$

By checking the proof of Lemma 2.11, we readily have the following variant of Lemma 2.11.

Lemma 2.12. *Consider the functional $G \in \mathbf{C}^1(E, \mathbf{R})$. Its gradient G' is of the form*

$$G'(u) = u - \Theta_G u.$$

Let M_1, M_2 be two closed convex (or open convex) subsets of E . Suppose that $(\mathbf{A}_2) \Theta_G(M_i) \subset M_i, i = 1, 2$.

Then there exists a locally Lipschitz continuous map $L_0 : \tilde{E} \rightarrow E$ such that $L_0(M_i \cap \tilde{E}) \subset M_i, i = 1, 2$ and that $V(u) := u - L_0(u)$ is a pseudo-gradient vector field of G . Furthermore, V and L_0 can be chosen to be odd if G is even and $M_2 = -M_1$.

From now on, let $\mathcal{P}$ denote a positive cone of E ; that is, $\mathcal{P}$ is a closed convex subset of E such that $t\mathcal{P} \subset \mathcal{P}$ for all $t \geq 0$ and $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$. We always assume implicitly that $\mathcal{P} \neq \{0\}$. We call $-\mathcal{P}$ a negative cone. Consider the following vector field,

$$W(u) := \frac{(1 + \|u\|)^2 V(u)}{(1 + \|u\|)^2 \|V(u)\|^2 + 1}.$$

Then W is a locally Lipschitz continuous vector field on $\tilde{E}$. Obviously,

$$\|W(u)\| \leq \|u\| + 1$$

for all $u \in \tilde{E}$. We denote

$$\mathcal{K}[a, b] := \{u \in E : G'(u) = 0, a \leq G(u) \leq b\},$$

$$G^c := \{u \in E : G(u) \leq c\}, \quad B_R(0) := \{u \in E : \|u\| \leq R\}.$$

Define

$$(2.22) \quad \Phi^* := \{\Gamma \in \Phi : \Gamma(t, \mathcal{D}^*) \subset \mathcal{D}^*\}.$$

Then $\Gamma(t, u) = (1 - t)u \in \Phi^*$.

Theorem 2.13. *Suppose that (2.16) and (A_1) hold. Assume that a compact subset A of E links a closed subset B of $\mathcal{S}$ and*

$$a_0 := \sup_A G \leq b_0 := \inf_B G.$$

Define

$$d^* := \inf_{\Gamma \in \Phi^*} \sup_{\Gamma \cap ([0,1], A) \cap \mathcal{S}} G(u);$$

then

$$d^* \in \left[b_0, \sup_{(t,u) \in [0,1] \times A} G((1-t)u) \right].$$

Furthermore, if G satisfies the $(w\text{-PS})_c$ condition for any c with

$$c \in \left[b_0, \sup_{(t,u) \in [0,1] \times A} G((1-t)u) \right],$$

then

$$\mathcal{K}[d^* - \varepsilon, d^* + \varepsilon] \cap (E \setminus (-\mathcal{P} \cup \mathcal{P})) \neq \emptyset$$

for all $\varepsilon > 0$ small. Moreover, if $d^* = b_0$, then $\mathcal{K}[d^*, d^*] \subset B$.

Proof. Obviously, d^* is well defined because A links B and $B \subset \mathcal{S}$. Moreover,

$$d^* \in \left[b_0, \sup_{(t,u) \in [0,1] \times A} G((1-t)u) \right].$$

We first consider the case of $d^* > b_0$. By contradiction, we assume that

$$(2.23) \quad \mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0] \cap (E \setminus (-\mathcal{P} \cup \mathcal{P})) = \emptyset$$

for some $\varepsilon_0 > 0$ small enough. Then

$$(2.24) \quad \mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0] \subset (-\mathcal{P} \cup \mathcal{P}).$$

Case 1. Assume $\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0] \neq \emptyset$.

$\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0]$ is compact, thus we may assume that $\text{dist}(\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0], \mathcal{S}) := \delta_0 > 0$.

By the $(w\text{-PS})$ condition, there is an $\bar{\varepsilon} > 0$ such that

$$(2.25) \quad \frac{(1 + \|u\|)^2 \|G'(u)\|^2}{(1 + \|u\|)^2 \|G'(u)\|^2 + 1} \geq \bar{\varepsilon}$$

for

$$u \in G^{-1}[d^* - \bar{\varepsilon}, d^* + \bar{\varepsilon}] \setminus (\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/2},$$

where $(T)_c := \{u \in E : \text{dist}(u, T) \leq c\}$. By decreasing $\bar{\varepsilon}$, we may assume that $\bar{\varepsilon} < d^* - b_0, \bar{\varepsilon} < \varepsilon_0/3$; then

$$\langle G'(u), W(u) \rangle \geq \bar{\varepsilon}/8$$

for any

$$u \in G^{-1}[d^* - \bar{\varepsilon}, d^* + \bar{\varepsilon}] \setminus (\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/2}.$$

Let

$$\Omega_1 = \{u \in E : |G(u) - d^*| \geq 3\bar{\varepsilon}\}, \quad \Omega_2 = \{u \in E : |G(u) - d^*| \leq 2\bar{\varepsilon}\}$$

and

$$\vartheta(u) = \frac{\text{dist}(u, \Omega_1)}{\text{dist}(u, \Omega_1) + \text{dist}(u, \Omega_2)}.$$

Let $\beta(u) : E \rightarrow [0, 1]$ be locally Lipschitz continuous such that

$$(2.26) \quad \beta(u) = \begin{cases} 1 & \text{for all } u \in E \setminus (\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/2}, \\ 0 & \text{for all } u \in (\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/3}. \end{cases}$$

Let $\bar{W}(u) = \vartheta(u)\beta(u)W(u)$ for $u \in \tilde{E}$; $\bar{W}(u) = 0$ otherwise. Then $\bar{W}$ is a locally Lipschitz vector field on E . We consider the following Cauchy initial value problem,

$$(2.27) \quad \begin{cases} \frac{d\varphi(t, u)}{dt} = -\bar{W}(\varphi(t, u)), \\ \varphi(0, u) = u, \end{cases}$$

which has a unique continuous solution $\varphi(t, u)$ in E . Evidently,

$$\frac{dG(\varphi(t, u))}{dt} \leq 0.$$

By the definition of d^* , there exists a $\Gamma \in \Phi^*$ such that

$$\Gamma([0, 1], A) \cap \mathcal{S} \subset G^{d^* + \bar{\varepsilon}}.$$

Therefore, $\Gamma([0, 1], A)$ is a subset of $G^{d^* + \bar{\varepsilon}} \cup \mathcal{D}^*$. Denote

$$A_1 := \Gamma([0, 1], A).$$

We claim that there exists a $T_1 > 0$ such that $\varphi(T_1, A_1) \subset G^{d^* - \bar{\varepsilon}/4} \cup \mathcal{D}^*$.

First, if $u \in \mathcal{D}^*$, we show that $\varphi(t, u) \in \mathcal{D}^*$ for all $t \geq 0$. Without loss of generality, we may assume that $u \in \mathcal{D}(\delta)$. Suppose there exists a $t_0 > 0$ such that $\varphi(t_0, u) \notin \mathcal{D}(\delta)$. We may choose a neighborhood $\mathcal{N}_u$ of u such that $\mathcal{N}_u \subset \mathcal{D}(\delta)$ because $\mathcal{D}(\delta)$ is open. By the theory of ordinary differential equations in Banach space, we can find a neighborhood $\mathcal{N}_{t_0}$ of $\varphi(t_0, u)$ such that $\varphi(t_0, \cdot) : \mathcal{N}_u \rightarrow \mathcal{N}_{t_0}$ is a homeomorphism. Because $\varphi(t_0, u) \notin \mathcal{D}(\delta)$, we can take a $w \in \mathcal{N}_{t_0} \setminus \bar{\mathcal{D}}(\delta)$. Correspondingly, we find a $v \in \mathcal{N}_u$ such that

$\varphi(t_0, v) = w$. Hence, we may find a $t_1 \in (0, t_0)$ such that $\varphi(t_1, v) \in \partial\mathcal{D}(\delta)$ and $\varphi(t, v) \notin \bar{\mathcal{D}}(\delta)$ for all $t \in (t_1, t_0]$.

On the other hand, for any $z \in \bar{\mathcal{D}}(\delta) \cap \mathcal{K}$, $\bar{W}(z) = 0$, hence

$$(2.28) \quad \text{dist}(z + \lambda(-\bar{W}(z)), \bar{\mathcal{D}}(\delta)) = 0, \quad \text{for all } \lambda > 0.$$

For any $z \in \bar{\mathcal{D}}(\delta) \cap \tilde{E}$, we have $L_0(z) \in \bar{\mathcal{D}}(\delta/2)$ because $L_0(\mathcal{D}(\delta) \cap \tilde{E}) \subset \mathcal{D}(\delta/2)$ in view of Lemma 2.11. Therefore, by a property of the cone $\mathcal{P}$: $x\mathcal{P} + y\mathcal{P} \subset \mathcal{P}$ for all $x, y \geq 0$, we have

$$\begin{aligned} & \text{dist}(z + \lambda(-\bar{W}(z)), \mathcal{P}) \\ &= \text{dist}(z - \lambda\vartheta(z)\beta(z)W(z), \mathcal{P}) \\ &= \text{dist}\left(\left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right)z + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1}L_0(z), \mathcal{P}\right) \\ &\leq \text{dist}\left(\left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right)z + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1}L_0(z), \right. \\ &\quad \left.\left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right)\mathcal{P} + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\mathcal{P}\right) \\ &\leq \left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right)\text{dist}(z, \mathcal{P}) \\ &\quad + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\text{dist}(L_0(z), \mathcal{P}) \\ &\leq \left(1 - \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2\kappa(z)}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\right)\mu_0 + \frac{\lambda\vartheta(z)\beta(z)(1 + \|z\|)^2}{(1 + \|z\|)^2\|V(z)\|^2 + 1}\frac{\mu_0}{2} \\ &\leq \mu_0 \end{aligned}$$

for $\lambda > 0$ small enough because $\kappa(z) \geq \frac{1}{2}$. That is, $z + \lambda(-\bar{W}(z)) \in \bar{\mathcal{D}}(\delta)$ for λ small. Once again, we get

$$(2.29) \quad \text{dist}(z + \lambda(-\bar{W}(z)), \bar{\mathcal{D}}(\delta)) = 0, \quad \text{for all } \lambda > 0 \text{ small enough.}$$

Combining (2.28) and (2.29), we thus obtain

$$\lim_{\lambda \rightarrow 0+} \frac{\text{dist}(z + \lambda(-\bar{W}(z)), \bar{\mathcal{D}}(\delta))}{\lambda} = 0, \quad \forall z \in \bar{\mathcal{D}}(\delta).$$

Consider the following initial value problem

$$\begin{cases} \frac{d\varphi(t, \varphi(t_1, v))}{dt} = -\bar{W}(\varphi(t, \varphi(t_1, v))), \\ \varphi(0, \varphi(t_1, v)) = \varphi(t_1, v) \in \bar{\mathcal{D}}(\delta). \end{cases}$$

It has a unique solution $\varphi(t, \varphi(t_1, v))$. By Theorem 1.49, there is a $\tilde{\delta} > 0$ such that

$$\varphi(t, \varphi(t_1, v)) \in \bar{\mathcal{D}}(\delta) \quad \text{for all } t \in [0, \tilde{\delta}).$$

Hence, by the semigroup property, $\varphi(t, v) \in \bar{\mathcal{D}}(\delta)$ for all $t \in [t_1, t_1 + \tilde{\delta})$, which contradicts the definition of t_1 . Therefore, $\varphi(t, u) \in \mathcal{D}^*$ for all $t \geq 0$.

If $u \in A_1, u \notin \mathcal{D}^*$, then we observe that $G(u) \leq d^* + \bar{\varepsilon}$. If $G(u) \leq d^* - \bar{\varepsilon}$, then

$$G(\varphi(t, u)) \leq G(u) \leq d^* - \bar{\varepsilon}$$

for all $t \geq 0$. Assume $G(u) > d^* - \bar{\varepsilon}$. Then $u \in G^{-1}[d^* - \bar{\varepsilon}, d^* + \bar{\varepsilon}]$. If

$$(2.30) \quad \text{dist}(\varphi([0, \infty), u), \mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0]) \leq \delta_0/2,$$

then there exists a t_m such that $\text{dist}(\varphi(t_m, u), \mathcal{S}) \geq \delta_0/4$; that is, $\varphi(t_m, u) \in \mathcal{D}$. Assume that

$$(2.31) \quad \text{dist}(\varphi([0, \infty), u), \mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0]) > \delta_0/2 > 0.$$

Similarly, we assume that $G(\varphi(t, u)) > d^* - \bar{\varepsilon}$ for all $t \geq 0$ (otherwise, we are done). Then, by (2.26)–(2.31), we have that

$$(2.32) \quad \frac{(1 + \|\varphi(t, u)\|)^2 \|G'(\varphi(t, u))\|^2}{(1 + \|\varphi(t, u)\|)^2 \|G'(\varphi(t, u))\|^2 + 1} \geq \bar{\varepsilon}, \quad \vartheta(\varphi(t, u)) = \beta(\varphi(t, u)) = 1$$

for all $t \geq 0$. Therefore, by (2.32),

$$(2.33) \quad G(\varphi(24, u)) = G(u) + \int_0^{24} dG(\varphi(s, u)) \leq d^* - 2\bar{\varepsilon}.$$

By combining the above arguments, we see that for any $u \in A_1 \setminus \mathcal{D}^*$, there exists a $T_u > 0$ such that $\varphi(T_u, u) \in G^{d^* - \bar{\varepsilon}/2} \cup \mathcal{D}^*$. By continuity, there exists a neighborhood U_u such that $\varphi(T_u, U_u) \subset G^{d^* - \bar{\varepsilon}/3} \cup \mathcal{D}^*$. Because $A_1 \setminus \mathcal{D}^*$ is compact, we get a $T_1 > 0$ such that $\varphi(T_1, A_1 \setminus \mathcal{D}^*) \subset G^{d^* - \bar{\varepsilon}/4} \cup \mathcal{D}^*$. Then

$$(2.34) \quad \varphi(T_1, A_1) \subset G^{d^* - \bar{\varepsilon}/4} \cup \mathcal{D}^*.$$

Case 2. If $\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0] = \emptyset$, then (2.26) holds with $(\mathcal{K}[d^* - \varepsilon_0, d^* + \varepsilon_0])_{\delta_0/2} = \emptyset$ and $\beta(u) \equiv 1$. Then, trivially, (2.32)–(2.34) are still true.

Now we define

$$\Gamma^*(s, u) = \begin{cases} \varphi(2T_1 s, u), & s \in [0, 1/2], \\ \varphi(T_1, \Gamma(2s - 1, u)), & s \in [1/2, 1]. \end{cases}$$

Then, $\Gamma^* \in \Phi^*$. If $s \in [0, 1/2]$, we have that

$$\Gamma^*(s, A) \cap \mathcal{S} \subset \varphi(2T_1 s, A) \cap \mathcal{S} \subset G^{a_0} \cap \mathcal{S} \subset G^{d^* - \bar{\varepsilon}/4}.$$

If $s \in [1/2, 1]$, then

$$\begin{aligned} \Gamma^*(s, A) \cap \mathcal{S} &\subset \varphi(T_1, \Gamma(2s - 1, A)) \cap \mathcal{S} \\ &\subset \varphi(T_1, A_1) \cap \mathcal{S} \\ &\subset (G^{d^* - \bar{\varepsilon}/4} \cup \mathcal{D}^*) \cap \mathcal{S} \\ &\subset G^{d^* - \bar{\varepsilon}/4} \cap \mathcal{S} \\ &\subset G^{d^* - \bar{\varepsilon}/4}. \end{aligned}$$

It follows that $G(\Gamma^*([0, 1], A) \cap \mathcal{S}) \leq d^* - \bar{\varepsilon}/4$, a contradiction.

Next we consider the case of $d^* = b_0$. Here we have to construct a different vector field and need a careful analysis of the flow. We prove that $\mathcal{K}[d^*, d^*] \cap B \neq \emptyset$. If it were not true, there would exist numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that

$$(2.35) \quad \frac{(1 + \|u\|)^2 \|G'(u)\|^2}{1 + (1 + \|u\|)^2 \|G'(u)\|^2} \geq \varepsilon_1$$

for $|G(u) - d^*| < \varepsilon_2$ and $\text{dist}(u, B) \leq \varepsilon_3$. By decreasing ε_2 , we may assume that $\varepsilon_2 < \varepsilon_1 \varepsilon_3 / 16$. Let

$$\begin{aligned} \Omega_3 &:= \{u \in E : \text{dist}(u, B) \leq \varepsilon_3/2, |G(u) - d^*| \leq \varepsilon_2/2\}, \\ \Omega_4 &:= \{u \in E : \text{dist}(u, B) \leq \varepsilon_3/3, |G(u) - d^*| \leq \varepsilon_2/3\}. \end{aligned}$$

Then $\mathcal{K} \subset E \setminus \Omega_3$. Choose $\Gamma \in \Phi^*$ such that

$$(2.36) \quad \sup_{\Gamma([0, 1], A) \cap \mathcal{S}} G(u) \leq d^* + \varepsilon_2/3.$$

We can find a $u_0 \in \Gamma([0, 1], A) \cap B \cap \mathcal{S} \neq \emptyset$ because A links B and $B \subset \mathcal{S}$. This implies that

$$(2.37) \quad b_0 \leq G(u_0) \leq \sup_{\Gamma([0, 1], A) \cap \mathcal{S}} G(u) \leq d^* + \varepsilon_2/3;$$

that is, $u_0 \in \Omega_4 \subset \Omega_3$. Let

$$\vartheta_1(u) = \frac{\text{dist}(u, E \setminus \Omega_3)}{\text{dist}(u, E \setminus \Omega_3) + \text{dist}(u, \Omega_4)},$$

and consider the following Cauchy initial value problem,

$$\begin{cases} \frac{d\varphi_1(t, u)}{dt} = -\vartheta_1(\varphi_1(t, u))W(\varphi_1(t, u)), \\ \varphi_1(0, u) = u \in E, \end{cases}$$

which has a unique continuous solution $\varphi_1(t, u)$ in E . Obviously, by (2.35),

$$(2.38) \quad \frac{dG(\varphi_1(t, u))}{dt} \leq -\frac{\varepsilon_1}{8}\vartheta_1(\varphi_1(t, u)).$$

If $u \in G^{d^* + \varepsilon_2/3}$, then

$$G(\varphi_1(t, u)) \leq G(u) \leq d^* + \varepsilon_2/3$$

for all $t \geq 0$. If there is a $t_1 \leq \varepsilon_3/4$ such that $\varphi_1(t_1, u) \notin \Omega_4$, then either $G(\varphi_1(t_1, u)) < d^* - \varepsilon_2/3$ or $\text{dist}(\varphi_1(t_1, u), B) > \varepsilon_3/3$. For the latter case, we observe that $\text{dist}(\varphi_1(t, u), B) \geq \varepsilon_3/12$, and hence, $\varphi_1(t, u) \notin B$ for all $t \in [0, \varepsilon_3/4]$. If $\varphi_1(t, u) \in \Omega_4$ for all $t \in [0, \varepsilon_3/4]$, then

$$\begin{aligned} G\left(\varphi_1\left(\frac{\varepsilon_3}{4}, u\right)\right) &= G(u) + \int_0^{\varepsilon_3/4} dG(\varphi_1(t, u)) \\ &\leq d^* + \frac{\varepsilon_2}{3} - \frac{\varepsilon_3\varepsilon_1}{32} \\ &\leq d^* - \frac{\varepsilon_2}{6}. \end{aligned}$$

That is, either

$$G(\varphi_1(\varepsilon_3/4, u)) < d^* - \varepsilon_2/6 = b_0 - \varepsilon_2/6$$

or $\varphi_1(t, u) \notin B$ for all $t \in [0, \varepsilon_3/4]$ and each $u \in G^{d^* + \varepsilon_2/3}$. It follows that $\varphi_1(\varepsilon_3/4, u) \notin B$ for any $u \in G^{d^* + \varepsilon_2/3}$. Next we prove that for all $u \in A, t \in [0, \varepsilon_3/4]$, we must have $\varphi_1(t, u) \notin B$. Note that if $u \in A, u \notin \mathcal{S}$, then $u \in \mathcal{D}^*$. Following an argument similar to that of the proof of the first case, we see that $\varphi_1(t, u) \in \mathcal{D}^*$. Hence $\varphi_1(t, u) \notin B \subset \mathcal{S}$ for all $t \geq 0$. Therefore, we may only consider the case $u \in A \cap \mathcal{S}$. Evidently, $\varphi_1(\varepsilon_3/4, u) \notin B$. Furthermore, by (2.38), we see that

$$\begin{aligned} G(\varphi_1(t, u)) &\leq G(u) - \frac{\varepsilon_1}{8} \int_0^t \vartheta_1(\varphi_1(s, u)) ds \\ &\leq d^* - \frac{\varepsilon_1}{8} \int_0^t \vartheta_1(\varphi_1(s, u)) ds. \end{aligned}$$

If $\varphi_1(t, u) \in B$, then $G(\varphi_1(t, u)) \geq b_0 = d^*$, and we must have $\vartheta_1(\varphi_1(s, u)) \equiv 0$ for $s \in [0, t]$. This implies that $\varphi_1(s, u) \notin \Omega_4$ and either $G(\varphi_1(s, u)) < d^* - \varepsilon_2/3$ or $\text{dist}(\varphi_1(s, u), B) > \varepsilon_3/3$ for all $s \in [0, t]$. Both cases imply $\varphi_1(t, u) \notin B$. This proves that $\varphi_1([0, \varepsilon_3/4], A) \cap B = \emptyset$. Let

$$\Gamma_1(t, u) = \begin{cases} \varphi_1(2t\varepsilon_3/4, u), & 0 \leq t \leq 1/2, \\ \varphi_1(\varepsilon_3/4, \Gamma(2t-1, u)), & 1/2 \leq t \leq 1. \end{cases}$$

Then it is easy to check that $\Gamma_1 \in \Phi^*$. But by the above arguments,

$$\Gamma_1([0, 1], A) \cap B = \emptyset,$$

which contradicts the fact that A links B . $\square$

Theorem 2.14. *Suppose that (2.16) and (A_1) hold and that $\Theta_G : E \rightarrow E$ is a compact operator. Assume that $E = Y \oplus M$, $1 < \dim Y < \infty$, and that*

- (1) $G(v) \leq \alpha$ for all $v \in Y$, where α is a positive constant,
- (2) $G(w) \geq \alpha$ for all $w \in B := \{w : w \in M, \|w\| = \rho\} \subset \mathcal{S}$, where ρ is a positive constant,
- (3) $G(sw_0 + v) \leq T_0$ for all $s \geq 0, v \in Y; w_0 \in M \setminus \{0\}$ is a fixed element, and T_0 is a constant.

If G satisfies the $(w\text{-PS})_c$ condition for all $c > 0$, then there exists a sequence $\{u_n\} \subset E \setminus (-\mathcal{P} \cup \mathcal{P})$ such that

$$G'(u_n) \rightarrow 0, \quad G'(u_n) = \frac{T_n}{n}u_n, \quad G(u_n) \rightarrow c,$$

where $\{T_n\}$ is a bounded sequence and $c \in [\alpha/2, 2T_0]$.

Proof. Define $\psi \in \mathbf{C}^\infty(\mathbf{R})$ such that $\psi = 0$ in $(-\infty, 1/2)$ and $\psi = 1$ in $(1, \infty)$, $0 \leq \psi \leq 1$. We may assume that $\|w_0\| = 1$. Write $u \in E$ as $u = v + w, v \in Y, w \in M$. Let

$$G_n(u) = G(u) - \left(T_0 + \frac{1}{n}\right) \psi \left(\frac{\|u\|^2}{n}\right), \quad n = 1, 2, \dots$$

Then

$$G'(u) - G'_n(u) = 2 \left(T_0 + \frac{1}{n}\right) \psi' \left(\frac{\|u\|^2}{n}\right) \frac{u}{n},$$

$$\|G'(u) - G'_n(u)\| \leq T_1 n^{-1/2}.$$

We claim that G_n satisfies (w-PS) for each n sufficiently large if G does. In fact, assume that $\{u_k\}$ is a (w-PS) sequence: $G_n(u_k) \rightarrow c$ and $(1 + \|u_k\|)G'_n(u_k) \rightarrow 0$ as $k \rightarrow \infty$. If, for a renamed subsequence, $\|u_k\|^2/n > 1$, then $\psi'(\|u_k\|^2/n) = 0$ and

$$(1 + \|u_k\|)G'_n(u_k) = (1 + \|u_k\|)G'(u_k) \rightarrow 0.$$

Then $\{u_k\}$ has a convergent subsequence. If $\|u_k\|^2/n \leq 1$, then $\{u_k\}$ is bounded and (w-PS) follows immediately. To see this, note that

$$G'(u_k) - 2 \left(T_0 + \frac{1}{n} \right) \psi' \left(\frac{\|u_k\|^2}{n} \right) \frac{u_k}{n} \rightarrow 0.$$

Take n so large that

$$b(u) = \kappa(u) - \left[2 \left(T_0 + \frac{1}{n} \right) \psi' \left(\frac{\|u\|^2}{n} \right) / n \right]$$

is bounded and bounded away from 0. Then

$$b(u_k)u_k - \Theta_G u_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Because the $\{u_k\}$ is bounded, there is a renamed subsequence such that both $b(u_k)$ and $\Theta_G u_k$ converge. Hence, this subsequence converges as well. Thus, in both cases, the (w-PS) condition is satisfied. Moreover, $G_n(v) \leq \alpha$ for all $v \in Y$. For any $w \in M$, if $\|w\| = \rho$, then $\psi(\|w\|^2/n) = 0$ for $n > 2\rho^2$ and consequently $G_n(w) = G(w) \geq \alpha$. Choose $\|sw_0 + v\| := n^{1/2} := R_n$. Then $R_n > \rho$ if n large enough, and

$$G_n(sw_0 + v) = G(sw_0 + v) - (T_0 + 1/n)\psi \left(\frac{\|sw_0 + v\|^2}{n} \right) \leq -\frac{1}{n}.$$

Let

$$B := \{w \in M : \|w\| = \rho\},$$

and

$$A_n := \{v \in Y : \|v\| \leq R_n\} \cup \{sw_0 + v : s \geq 0, v \in Y, \|sw_0 + v\| = R_n\}.$$

Then A_n links B , and G_n satisfies all the conditions of Theorem 2.13. Hence, there exists a $u_n \in E \setminus (-\mathcal{P} \cup \mathcal{P})$ such that

$$G'_n(u_n) = 0, \quad G_n(u_n) \in \left[\alpha/2, \sup_{(t,u) \in [0,1] \times A_n} G_n((1-t)u) \right].$$

Evidently,

$$\|G'(u_n) - G'_n(u_n)\| = \|G'(u_n)\| \leq T_1 n^{-1/2} \rightarrow 0,$$

$$\alpha/2 \leq G_n(u_n) \leq G(u_n) \leq G_n(u_n) + T_0 + 1/n,$$

$$\sup_{(t,u) \in [0,1] \times A_n} G_n((1-t)u) \leq T_0.$$

Therefore, $G(u_n) \rightarrow c \in [\alpha/2, 2T_0]$. Finally,

$$G'(u_n) = G'(u_n) - G'_n(u_n) = 2 \left(T_0 + \frac{1}{n} \right) \psi' \left(\frac{\|u_n\|}{n} \right) \frac{u_n}{n} = \frac{T_n}{n} u_n,$$

where $\{T_n\}$ is a bounded sequence. $\square$

The statement $G'(u_n) = (T_n/n)u_n$ in Theorem 2.14 is quite helpful for getting a sign-changing limit of the sequence $\{u_n\}$.

We now assume that there is another norm $\|\cdot\|_\star$ of E such that $\|u\|_\star \leq C_\star \|u\|$ for all $u \in E$; here $C_\star > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_\star \rightarrow 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. In the sequel, all properties are with respect to the norm $\|\cdot\|$ if without specific indication. Write $E = M \oplus Y$, where $Y, M := Y^\perp$ are closed subspaces with $\dim Y < \infty$ and $(M \setminus \{0\}) \cap (-\mathcal{P} \cup \mathcal{P}) = \emptyset$; that is, the nontrivial elements of M are sign-changing. Let $y_0 \in M \setminus \{0\}$ with $\|y_0\| = 1$ and $0 < \rho < R$ with

$$R^{p-2} \|y_0\|_\star^p + \frac{R \|y_0\|_\star}{1 + D_\star \|y_0\|_\star} > \rho, \quad D_\star > 0, p > 2 \text{ are constants.}$$

Let

$$A := \{u = v + sy_0 : v \in Y, s \geq 0, \|u\| = R\} \cup [Y \cap \bar{B}_R],$$

$$B := \left\{ u \in M : \frac{\|u\|_\star^p}{\|u\|^2} + \frac{\|u\| \|u\|_\star}{\|u\| + D_\star \|u\|_\star} = \rho \right\}.$$

Then by Proposition 2.10, A links B in the sense of Definition 2.1. Choose

$$(2.39) \quad a_\star > \sup_{[0,1] \times A} G((1-t)u) + 2.$$

Define

$$(2.40) \quad B^\star := B \cap G^{a_\star}.$$

Choose $\Gamma(t, u) = (1-t)u \in \Phi^\star$; then $\Gamma(t, a) \in B$ for some $(t, a) \in [0, 1] \times A$. Moreover, $\Gamma(t, a) \in G^{a_\star}$, hence, $B^\star := B \cap G^{a_\star} \neq \emptyset$. Set

$$(2.41) \quad \Phi^{\star\star} := \{\Gamma \in \Phi^\star : \Gamma([0, 1], A) \subset G^{a_\star}\}.$$

Then $\Gamma(t, u) = (1-t)u \in \Phi^\star \cap \Phi^{\star\star}$.

Lemma 2.15. $\|u\|_\star \leq c_1, \quad \forall u \in B$; here c_1 is a constant.

(A₃) Assume that for any $a, b > 0$, there is a $c = c(a, b) > 0$ such that

$$G(u) \leq a \quad \text{and} \quad \|u\|_\star \leq b \Rightarrow \|u\| \leq c.$$

Lemma 2.16. *Assume (A_3) . Then we have that*

$$\text{dist}(B^* := B \cap G^{a_*}, \mathcal{P}) := \delta_1 > 0.$$

Proof. By negation, we assume that $\text{dist}(B^*, \mathcal{P}) = 0$. Then we find $\{u_n\} \subset B^*$, $\{p_n\} \subset \mathcal{P}$ such that $\|u_n - p_n\| \rightarrow 0$. Then $\{u_n\}$, hence $\{p_n\}$, is bounded in both $(E, \|\cdot\|)$ and $(E, \|\cdot\|_*)$. We assume that $u_n \rightharpoonup u^* \in E$; $p_n \rightharpoonup p^* \in \mathcal{P}$ weakly in $(E, \|\cdot\|)$; $u_n \rightarrow u^*$ strongly in $(E, \|\cdot\|_*)$. Then we observe that $u^* \in M$. Because

$$\frac{\|u_n\|_*^p}{\|u_n\|^2} + \frac{\|u_n\| \|u_n\|_*}{\|u_n\| + D_* \|u_n\|_*} = \rho$$

and $\|u_n - u^*\|_* \rightarrow 0$, then $u^* \neq 0$. However, because $u^* = p^*$, we get a contradiction in as much as all nonzero elements of M are sign-changing. $\square$

In view of Lemma 2.16, we may assume that $B^* \subset \mathcal{S}$ as long as the δ of Condition (A_1) is small enough; this is indeed true in our applications.

Theorem 2.17. *Suppose that (2.16), (A_1) , and (A_3) hold. Assume*

$$a_0 := \sup_A G \leq b_0^* := \inf_{B^*} G.$$

Define

$$d^* := \inf_{\Gamma \in \Phi^{**}} \sup_{\Gamma([0,1], A) \cap \mathcal{S}} G(u);$$

then

$$d^* \in [b_0^*, \sup_{(t,u) \in [0,1] \times A} G((1-t)u)].$$

Furthermore, if G satisfies the $(w\text{-PS})_c$ condition for any c with

$$c \in [b_0^*, \sup_{(t,u) \in [0,1] \times A} G((1-t)u)],$$

then

$$\mathcal{K}[d^* - \varepsilon, d^* + \varepsilon] \cap (E \setminus (-\mathcal{P} \cup \mathcal{P})) \neq \emptyset$$

for all $\varepsilon > 0$ small. Moreover, if $d^ = b_0^*$, then $\mathcal{K}[d^*, d^*] \subset B^*$.*

Proof. It suffices to note that any flow ϕ considered in the proof of Theorem 2.13 is nonincreasing in the sense that $G(\phi(t, u))$ is nonincreasing in t . Then the proof of this theorem is the same as that of Theorem 2.13 where B is replaced by B^* . $\square$

Theorem 2.18. *Suppose that (A_3) holds. Theorem 2.14 is still true if we replace B by B^* and $a_* := T_0 + 2$.*

Notes and Comments. The ideas of the proofs for Lemmas 2.11 and 2.12 first come from Sun [316] (see also Guo [163] and a paper by Liu and Sun [211]). In [316], $\mathcal{D}^*$ itself is a convex set. In [211], it is assumed that $\Theta_G(\partial \mathcal{D}(\delta)) \subset \mathcal{D}(\delta)$

and that $G' = \mathbf{id} - \Theta_G$. Lemma 2.11 and Theorems 2.13 and 2.14 are proved in Schechter and Zou [288].

Condition (A_1) is applied in Conti et al. [107]. In particular, [107] is the pioneering paper where the neighborhood of a cone is introduced which satisfies the type of condition (A_1) . By using the invariant sets of flows and lower (upper) solutions, Conti et al. obtained the existence of multiple solutions with ordering relations. Similar ideas are also used in Conti et al. [108] for the existence of many solutions for superlinear elliptic systems. Later, this idea of the neighborhood of a cone is used by Bartsch et al., Liu and Wang, Schechter and Zou, and Zou, among others.

In Theorem 2.14, T_0 is an arbitrary constant that is not necessarily equal to α . This novelty makes it powerful in applications, especially in dealing with asymptotically linear equations. Note that T_0 must be equal to α in classical linking (cf. Benci and Rabinowitz [55], Brézis and Nirenberg [70], Li and Liu [197], Li and Willem [200], Silva [295], and Tintarev [328]).

2.3 Jumping Dirichlet Equations

Consider the sign-changing solutions to the following Dirichlet boundary value problem

$$(2.42) \quad \begin{cases} -\Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with the smooth boundary $\partial\Omega$ and finite measure $\text{meas } \Omega := |\Omega|$.

Let $E := H_0^1(\Omega)$ be the usual Sobolev space endowed with the inner product

$$\langle u, v \rangle := \int_{\Omega} (\nabla u \nabla v) dx$$

for $u, v \in E$ and the norm $\|u\| := \langle u, u \rangle^{1/2}$. Let

$$0 < \lambda_1 < \cdots < \lambda_k < \cdots$$

denote the distinct Dirichlet eigenvalues of $-\Delta$ on Ω with zero boundary value. Then each λ_k has finite multiplicity. The principal eigenvalue λ_1 is simple with a positive eigenfunction φ_1 , and the eigenfunctions φ_k corresponding to λ_k ($k \geq 2$) are sign-changing. Let N_k denote the eigenspace of λ_k . Then $\dim N_k < \infty$. We fix k and let $E_k := N_1 \oplus \cdots \oplus N_k$. In this section, we consider the case of

$$(2.43) \quad \lim_{t \rightarrow +\infty} \frac{f(x, t)}{t} = \beta_+(x), \quad \lim_{t \rightarrow -\infty} \frac{f(x, t)}{t} = \beta_-(x)$$

uniformly for $x \in \Omega$. In particular,

$$\lambda_k < \beta_{\pm}(x) < c, \quad \text{where } c > 0 \text{ is a fixed constant.}$$

Throughout this section, we assume

(B₁) f is a Carathéodory function and $f(x, t)t \geq 0$ for $(x, t) \in \Omega \times \mathbf{R}$;
 $\lim_{t \rightarrow 0} (f(x, t))/t = 0$ uniformly for $x \in \Omega$.

(B₂) $2F(x, t) \geq \lambda_{k-1}t^2 - c_0$ for all $x \in \Omega, t \in \mathbf{R}$, where $F(x, t) = \int_0^t f(x, s)ds$; $c_0 > 0$ is a constant.

By the above assumptions, we may find a $C_F > 0$ such that

$$(2.44) \quad F(x, t) \leq \frac{1}{4}\lambda_1|t|^2 + C_F|t|^p, \quad \forall x \in \Omega, \quad t \in \mathbf{R};$$

here $2 < p < 2^*$. Also, we can get another constant $\Lambda_0 > 0$ such that

$$(2.45) \quad 2F(x, t) \leq \Lambda_0 t^2 \quad \text{for all } x \in \Omega, \quad t \in \mathbf{R}.$$

Recall the Gagliardo–Nirenberg inequality,

$$(2.46) \quad \|u\|_p \leq c_p \|u\|^\alpha \|u\|_2^{(1-\alpha)}, \quad u \in E,$$

where $\alpha \in (0, 1)$ is defined by

$$(2.47) \quad \frac{1}{p} = \alpha \left(\frac{1}{2} - \frac{1}{N} \right) + (1 - \alpha) \frac{1}{2}.$$

On the other hand, we have a constant $\Lambda_p > 0$ such that

$$(2.48) \quad \|u\|_p \leq \Lambda_p \|u\|, \quad u \in E.$$

Without loss of generality, we assume that $\Lambda_p > 1$ and $c_p > 1$. Set

$$(2.49) \quad \Lambda_p^* := \min \left\{ \frac{1}{4\Lambda_p^2 c_p^{(p-2)}}, (4\Lambda_p^2 c_p^{(p-2)})^{-(1/(p-2))} \right\},$$

$$(2.50) \quad T_1 := \min \{ \lambda_k^{(1-\alpha)(p-2)}, \lambda_k^{(1-\alpha)} \},$$

$$(2.51) \quad T_2 := \min \left\{ \frac{1}{64C_F^2}, (8C_F)^{-(1/(p-2))} \right\}.$$

(B₃) Assume that

$$c_0 \leq \frac{1}{4|\Omega|} (\Lambda_p^*)^2 T_1 T_2.$$

The first result deals with the case of a jump not crossing eigenvalues: $\lambda_k < \beta_{\pm}(x) \leq \lambda_{k+1}$. Resonance may occur at λ_{k+1} .

Theorem 2.19. *Assume that (B_1) – (B_3) and (2.43) hold with $\lambda_k < \beta_{\pm}(x) \leq \lambda_{k+1}$. If either $\beta_+(x) < \lambda_{k+1}$ for $x \in \Omega$ or $\beta_-(x) < \lambda_{k+1}$ for $x \in \Omega$, then Equation (2.42) has a sign-changing solution.*

If we strengthen the condition on f , we have the following theorem where the jump is allowed to cross an arbitrarily finite number of eigen-values.

Theorem 2.20. *Assume that (B_1) – (B_3) and (2.43) hold. If $\lambda_k < \beta_{\pm}(x)$ for $x \in \Omega$, and*

(B₄) *there exists a $C_0(x) \in L^1(\Omega)$ such that*

- (i) $f(x, t)t - 2F(x, t) \geq C_0(x)$, for $(x, t) \in \Omega \times \mathbf{R}$,
- (ii) $\lim_{|t| \rightarrow \infty} (f(x, t)t - 2F(x, t)) = \infty$ for $x \in \Omega$.

then Equation (2.42) has a sign-changing solution.

Theorem 2.20 permits $\beta_{\pm}(x)$ to be arbitrary bounded functions greater than λ_k and to cross an arbitrarily finite number of eigenvalues of $-\Delta$ with zero boundary value condition. Therefore, the jump has much more freedom.

For the Dirichlet boundary value problem (2.42), it is usually called jumping nonlinearity at $\pm\infty$ if

$$\begin{cases} f(x, t)/t \rightarrow a & \text{a.e. } x \in \Omega \text{ as } t \rightarrow -\infty, \\ f(x, t)/t \rightarrow b & \text{a.e. } x \in \Omega \text{ as } t \rightarrow \infty. \end{cases}$$

The existence of solutions of (2.42) is closely related to the equation

$$-\Delta u = bu^+ - au^-, \quad \text{where } u^{\pm} = \max\{\pm u, 0\}.$$

Conventionally, the set

$$\Sigma := \{(a, b) \in \mathbf{R}^2 : -\Delta u = bu^+ - au^- \text{ has nontrivial solutions}\}$$

is called the Fučík spectrum of $-\Delta$ (see Dancer [127], Fučík [151], and Schechter [269]). It plays a key role in most results of this aspect. However, so far no complete description of Σ has been found. If

$$0 < \lambda_1 < \cdots < \lambda_k < \cdots$$

are the distinct Dirichlet eigenvalues of $-\Delta$ on Ω with zero boundary value, it was shown in Schechter [269] that in the square $(\lambda_{l-1}, \lambda_{l+1})^2$ there are decreasing curves C_{l1}, C_{l2} (which may or may not coincide) passing through the point (λ_l, λ_l) such that all points above or below both curves in the square (the so-called type (I) region) are not in Σ , whereas points on the curves are in Σ . Usually, the status of points between the curves (referred to as the type (II) region if the curves do not coincide) is unknown. However, it was shown in Gallouët and Kavian [154] that when λ is a simple eigenvalue, then

points of the type (II) region are not in Σ . On the other hand, Margulies and Margulies [223] have shown that there are boundary value problems for which many curves in Σ emanate from a point (λ_l, λ_l) when λ_l is a multiple eigenvalue. Certainly, these curves are contained in region (II).

Next we proceed to prove the above theorems. Define

$$G(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(x, u)dx, \quad u \in E.$$

Then $G \in C^1(E, \mathbf{R})$ and $G'(u) = u - \Theta_G(u)$, $u \in E$, where $\Theta_G : E \rightarrow E$ is a compact operator. Actually, $\Theta_G(u) = (-\Delta)^{-1}(f(x, u))$.

Lemma 2.21. *Under the assumptions of Theorems 2.19–2.20, $G(u) \rightarrow -\infty$ for $u \in E_k$ as $\|u\| \rightarrow \infty$.*

Proof. Rewrite G as

$$G(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} \left(\frac{1}{2}\beta_+(x)(u^+)^2 + \frac{1}{2}\beta_-(x)(u^-)^2 + H(x, u) \right) dx, \quad u \in E,$$

where $H(x, u) := \int_0^u h(x, t)dt$;

$$h(x, t) = f(x, t) - (\beta_+(x)t^+ - \beta_-(x)t^-); \quad t^{\pm} = \max\{\pm t, 0\}.$$

Therefore,

$$\begin{aligned} G(u) &= \frac{1}{2}\|u\|^2 - \int_{\Omega} H(x, u)dx \\ &\quad - \frac{1}{2} \left(\int_{\beta_-(x) \geq \beta_+(x)} + \int_{\beta_-(x) < \beta_+(x)} \right) (\beta_+(x)(u^+)^2 + \beta_-(x)(u^-)^2)dx \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{2} \int_{\beta_-(x) \geq \beta_+(x)} \beta_+(x)u^2 dx \\ &\quad - \frac{1}{2} \int_{\beta_-(x) \geq \beta_+(x)} (\beta_-(x) - \beta_+(x))(u^-)^2 dx \\ &\quad - \frac{1}{2} \int_{\beta_-(x) < \beta_+(x)} \beta_-(x)u^2 dx \\ &\quad - \frac{1}{2} \int_{\beta_-(x) < \beta_+(x)} (\beta_+(x) - \beta_-(x))(u^+)^2 dx - \int_{\Omega} H(x, u)dx. \end{aligned}$$

Note $\min\{\beta_+(x), \beta_-(x)\} > \lambda_k$ and recall the variational characterization of eigenvalues $\{\lambda_k\}$; we have the following estimates for any $u \in E_k$.

$$\begin{aligned}
G(u) &\leq \frac{1}{2}\|u\|^2 - \frac{1}{2} \int_{\beta_-(x) \geq \beta_+(x)} \beta_+(x) u^2 dx \\
&\quad - \frac{1}{2} \int_{\beta_-(x) < \beta_+(x)} \beta_-(x) u^2 dx - \int_{\Omega} H(x, u) dx \\
&\leq \frac{1}{2}\|u\|^2 - \frac{1}{2} \int_{\Omega} \min\{\beta_+(x), \beta_-(x)\} u^2 dx - \int_{\Omega} H(x, u) dx \\
&\leq -\delta\|u\|^2 - \int_{\Omega} H(x, u) dx,
\end{aligned}$$

where $\delta > 0$ is a constant. The last inequality is due to the finite dimension of E_k and the Schechter–Simon Theorem 1.62. Therefore,

$$\lim_{\|u\| \rightarrow \infty, u \in E_k} \frac{G(u)}{\|u\|^2} \leq -\delta$$

because

$$\lim_{t \rightarrow \infty} \frac{h(x, t)}{t} = 0$$

and $\dim E_k < \infty$. □

Lemma 2.22. *Assume (B_2) . Then*

$$G(u) \leq \frac{c_0|\Omega|}{2}, \quad \forall u \in E_{k-1}.$$

Proof. For $u \in E_{k-1}$,

$$\begin{aligned}
G(u) &= \frac{1}{2}\|u\|^2 - \int_{\Omega} F(x, u) dx \\
&\leq \frac{1}{2}\|u\|^2 - \frac{1}{2} \int_{\Omega} \lambda_{k-1} u^2 dx + \frac{1}{2} \int_{\Omega} c_0 dx \\
&\leq \frac{c_0|\Omega|}{2}.
\end{aligned}$$
□

For $p > 2$ given in (2.44), we let

$$(2.52) \quad \xi_0(u) := \begin{cases} \frac{\|u\|_p^p}{\|u\|^2} + \frac{\|u\| \|u\|_p}{\|u\| + \lambda_k^\beta \|u\|_p}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0, \end{cases}$$

where $\beta = (1 - \alpha)(p - 2)$. Then $\xi_0 : E \rightarrow E$ is continuous. Set

$$(2.53) \quad S_0 := \{u \in E_{k-1}^\perp : \xi_0(u) = \rho\}, \quad \rho := \frac{1}{8C_F} > 0.$$

For $u \in S_0$, by (2.48) we have

$$\begin{aligned} \rho &= \frac{\|u\|_p^p}{\|u\|^2} + \frac{\|u\|\|u\|_p}{\|u\| + \lambda_k^\beta \|u\|_p} \\ &\leq \frac{\|u\|\|u\|_p}{2(\|u\|\lambda_k^\beta \|u\|_p)^{1/2}} + \frac{\|u\|_p^2}{\|u\|^2} \|u\|_p^{p-2} \\ &\leq \frac{(\|u\|\|u\|_p)^{1/2}}{2(\lambda_k^\beta)^{1/2}} + A_p^2 \|u\|_p^{p-2} \\ &\leq \frac{(A_p)^{1/2} \|u\|}{2(\lambda_k^\beta)^{1/2}} + A_p^2 \|u\|_p^{p-2}. \end{aligned}$$

By the Gagliardo–Nirenberg inequality in (2.46) and (2.47),

$$(2.54) \quad \|u\|_p^{p-2} \leq c_p^{p-2} \|u\|^{\alpha(p-2)} \|u\|_2^{(1-\alpha)(p-2)}.$$

But $u \in E_{k-1}^\perp$; we see that

$$\lambda_k \|u\|_2^2 \leq \|u\|^2 \quad \text{and} \quad \|u\|_2 \leq \frac{1}{\lambda_k^{1/2}} \|u\|.$$

Hence,

$$(2.55) \quad \|u\|_p^{p-2} \leq c_p^{p-2} \|u\|^{p-2} \lambda_k^{-((1-\alpha)(p-2))/2}.$$

Therefore,

$$\begin{aligned} \rho &\leq \frac{(A_p)^{1/2} \|u\|}{2(\lambda_k^\beta)^{1/2}} + (A_p)^2 c_p^{p-2} \|u\|^{p-2} \lambda_k^{-((1-\alpha)(p-2))/2} \\ &\leq \left(\frac{1}{(\lambda_k^\beta)^{1/2}} + \frac{1}{\lambda_k^{((1-\alpha)(p-2))/2}} \right) (2A_p^2 c_p^{p-2}) \max\{\|u\|, \|u\|^{p-2}\}. \end{aligned}$$

Then we have that

$$(2.56) \quad \frac{\lambda_k^{((1-\alpha)(p-2))/2}}{(4A_p^2 c_p^{p-2})} \rho \leq \max\{\|u\|, \|u\|^{p-2}\}.$$

Lemma 2.23. *For all $u \in S_0$,*

$$\|u\| \geq \Lambda_p^* \min\{\lambda_k^{((1-\alpha)(p-2))/2}, \lambda_k^{(1-\alpha)/2}\} \min\{\rho, \rho^{1/(p-2)}\}.$$

Lemma 2.24. $\|u\|_p^p / \|u\|^2 \leq \rho, \quad \forall u \in S_0.$

Lemma 2.25. $\|u\|_p \leq c_1, \quad \forall u \in S_0.$

Proof. If $\|u\|_p \rightarrow \infty$, then so does $\|u\| \rightarrow \infty$; hence

$$\frac{\|u\| \|u\|_p}{\|u\| + \lambda_k^\beta \|u\|_p} \rightarrow \infty,$$

a contradiction. □

By (2.44), we know that $F(x, u) \leq \lambda_1/4 |u|^2 + C_F |u|^p, \quad \forall x \in \Omega, u \in \mathbf{R}.$
Consider the functional

$$G(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega} F(x, u) dx, \quad u \in H_0^1(\Omega).$$

Then

$$\begin{aligned} G(u) &\geq \frac{1}{2} \|u\|^2 - \frac{\lambda_1}{4} \|u\|_2^2 - C_F \|u\|_p^p \\ &\geq \frac{1}{4} \|u\|^2 - C_F \|u\|_p^p \\ &\geq \|u\|^2 \left(\frac{1}{4} - C_F \frac{\|u\|_p^p}{\|u\|^2} \right). \end{aligned}$$

Combining Lemma 2.23 and Lemma 2.24, we have the following.

Lemma 2.26. *For any $u \in S_0$, we have that*

$$G(u) \geq \frac{1}{8} (\Lambda_p^*)^2 T_1 T_2 \geq \frac{1}{2} |\Omega| c_0.$$

Lemma 2.27. *Under the assumptions of Theorem 2.19, G satisfies the (w-PS) condition.*

Proof. Assume that $\{u_n\}$ is a (w-PS) sequence:

$$G(u_n) \rightarrow c, \quad (1 + \|u_n\|) G'(u_n) \rightarrow 0.$$

By negation, we assume that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Let $w_n = u_n / \|u_n\|$. Then $\|w_n\| = 1$ and there is a renamed subsequence such that $w_n \rightarrow w$ weakly in E , strongly in $L^2(\Omega)$, and a.e. in Ω . Moreover,

$$\langle G'(u_n), v \rangle = \langle u_n, v \rangle - \int_{\Omega} f(x, u_n) v dx \rightarrow 0$$

and

$$\langle w_n, v \rangle - \int_{\Omega} \frac{f(x, u_n) v}{\|u_n\|} dx \rightarrow 0.$$

By (2.43), we see that

$$-\Delta w = \beta_+ w^+ - \beta_- w^-.$$

Because

$$G(u_n)/\|u_n\|^2 = 1/2 - \int_{\Omega} F(x, u_n) dx / \|u_n\|^2 \rightarrow 0,$$

we see that $\int_{\Omega} (\beta_+ (w^+)^2 + \beta_- (w^-)^2) dx = 1$. It implies that $w \neq 0$. Let $w := w_- + w_+$ with $w_- \in E_k$, $w_+ \in E_k^{\perp}$, $\tilde{w} := w_+ - w_-$. Let $q(x) = \beta_+(x)$ when $w(x) \geq 0$; $q(x) = \beta_-(x)$ when $w(x) < 0$. Then we have that $-\Delta w = q(x)w$ and hence

$$\|w_+\|^2 - \|w_-\|^2 = \int_{\Omega} q(x)(w_+)^2 dx - \int_{\Omega} q(x)(w_-)^2 dx.$$

It follows that

$$\begin{aligned} 0 &\leq \|w_+\|^2 - \lambda_{k+1} \|w_+\|_2^2 \leq \|w_+\|^2 - \int_{\Omega} q(x)(w_+)^2 dx \\ &= \|w_-\|^2 - \int_{\Omega} q(x)(w_-)^2 dx \leq \|w_-\|^2 - \lambda_k \int_{\Omega} (w_-)^2 dx \leq 0. \end{aligned}$$

That is, $\|w_{\pm}\|^2 = \int_{\Omega} q(x)(w_{\pm})^2 dx$. The only way this can happen is $q(x) = \lambda_k$ when $w_-(x) \neq 0$ and $q(x) = \lambda_{k+1}$ when $w_+(x) \neq 0$ and therefore, either w_- is an eigenfunction of λ_k or w_+ is an eigenfunction of λ_{k+1} . But the first case cannot occur because $\beta_{\pm} > \lambda_k$. If w_+ is an eigenfunction of λ_{k+1} , then w_+ is sign-changing. Because $-\Delta w_+ = \beta_+(x)w_+^+ - \beta_-(x)w_+^-$, we have $\beta_- = \lambda_{k+1}$ on a subset of Ω of positive measure and $\beta_+ = \lambda_{k+1}$ on another subset of Ω of positive measure. This contradicts the assumption of the theorem. $\square$

Lemma 2.28. *Under the assumptions of Theorem 2.20, G satisfies the (w-PS) condition.*

Proof. Assume that $\{u_n\}$ is a (w-PS) sequence: $(1 + \|u_n\|)\|G'(u_n)\| \rightarrow 0$ and $\{G(u_n)\}$ is bounded. Then

$$(2.57) \quad G(u_n) - \frac{1}{2} \langle G'(u_n), u_n \rangle = \int_{\Omega} \left(\frac{1}{2} f(x, u_n) u_n - F(x, u_n) \right) dx < c$$

and

$$\frac{1}{2}\|u_n\|^2 \leq c + \int_{\Omega} F(x, u_n) dx \leq c + \int_{\Omega} A_0 u_n^2 dx.$$

If $\{\|u_n\|\}$ is unbounded, then, for a renamed subsequence,

$$1 \leq 2A_0 \lim_{n \rightarrow \infty} \int_{\Omega} \frac{u_n^2}{\|u_n\|^2} dx.$$

It follows that $\lim_{n \rightarrow \infty} |u_n|^2 = \infty$ on a subset of Ω with a positive measure. Combining this with (B_4) , we have $\int_{\Omega} (\frac{1}{2}f(x, u_n)u_n - F(x, u_n)) dx \rightarrow \infty$, which contradicts (2.57). $\square$

To prove Theorems 2.19 and 2.20, we apply Theorem 2.17. First, we let

$$\mathcal{P} := \{u \in E : u(x) \geq 0 \text{ for a.e. } x \in \Omega\}.$$

Then $\mathcal{P}$ ($-\mathcal{P}$) is the positive (negative) cone of E and $\pm\mathcal{P}$ has an empty interior. Let

$$A := \{u = v + sy_0 : v \in E_{k-1}, s \geq 0, \|u\| = R\} \cup (E_{k-1} \cap B_R(0)),$$

where $y_0 \in E_{k-1}^\perp$, $\|y_0\| = 1$ and R large enough. Let ρ be defined in (2.53). By Proposition 2.10, A links S_0 . Choose

$$(2.58) \quad a_* > \sup_{[0,1] \times A} G((1-t)u) + 2.$$

Define

$$(2.59) \quad B^* := S_0 \cap G^{a_*}.$$

In Lemmas 2.15 and 2.16, we chose $\|\cdot\|_* = \|\cdot\|_p$. Then by Lemma 2.16,

$$\text{dist}(B^* := S_0 \cap G^{a_*}, \mathcal{P}) := \delta_1 > 0.$$

We define

$$\mathcal{D}(\mu_0) := \{u \in E : \text{dist}(u, \mathcal{P}) < \mu_0\}.$$

Lemma 2.29. *Under the assumptions of (B_1) , there exists a $\mu_0 \in (0, \delta_1)$ such that $\Theta_G(\pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu_0/2)$.*

Proof. Write $u^\pm = \max\{\pm u, 0\}$. For any $u \in E$,

$$\|u^+\|_2 = \min_{w \in (-\mathcal{P})} \|u - w\|_2$$

$$\begin{aligned}
&\leq \frac{1}{\lambda_1^{1/2}} \min_{w \in (-\mathcal{P})} \|u - w\| \\
(2.60) \quad &= \frac{1}{\lambda_1^{1/2}} \text{dist}(u, -\mathcal{P})
\end{aligned}$$

and, for each $s \in (2, 2^*]$, there exists a $C_s > 0$ such that

$$(2.61) \quad \|u^\pm\|_s \leq C_s \text{dist}(u, \mp \mathcal{P}).$$

By assumption (B_1) , for each $\varepsilon' > 0$ small enough, there exists a $C_{\varepsilon'} > 0$ such that

$$(2.62) \quad f(x, t)t \leq \varepsilon' t^2 + C_{\varepsilon'} |t|^p, \quad x \in \Omega, t \in \mathbf{R},$$

where $p > 2$ is a constant. Let $v = \Theta_G(u)$. Then by (2.60)–(2.62),

$$\begin{aligned}
&\text{dist}(v, -\mathcal{P}) \|v^+\| \\
&\leq \|v^+\|^2 \\
&= \langle v, v^+ \rangle \\
&= \int_{\Omega} f(x, u^+) v^+ dx \\
&\leq \int_{\Omega} (\varepsilon' |u^+| + C_{\varepsilon'} |u^+|^{p-1}) |v^+| dx \\
&\leq \left(\frac{2}{5} \text{dist}(u, -\mathcal{P}) + C (\text{dist}(u, -\mathcal{P}))^{p-1} \right) \|v^+\|.
\end{aligned}$$

That is,

$$\text{dist}(\Theta_G(u), -\mathcal{P}) \leq \left(\frac{2}{5} \right) \text{dist}(u, -\mathcal{P}) + C (\text{dist}(u, -\mathcal{P}))^{p-1}.$$

So, there exists a $\mu_0 < \delta_1$ such that $\text{dist}(\Theta_G(u), -\mathcal{P}) < \frac{1}{2} \mu_0$ for every $u \in -\mathcal{D}(\mu_0)$. Similarly, $\text{dist}(\Theta_G(u), \mathcal{P}) < \frac{1}{2} \mu_0$ for every $u \in \mathcal{D}(\mu_0)$. The conclusion follows. $\square$

Proofs of Theorems 2.19–2.20. By Theorem 2.17, there exists a $u \in E \setminus (-\mathcal{P} \cup \mathcal{P})$ (sign-changing critical point) such that

$$G'(u) = 0, \quad G(u) \in \left[b_0^* - \bar{\varepsilon}, \sup_{(t,u) \in [0,1] \times A} G((1-t)u) + \bar{\varepsilon} \right],$$

where $b_0^* = c_0 |\Omega|/2$; $\bar{\varepsilon}$ is small enough. $\square$

Notes and Comments. The study of the Fučík spectrum began with Ambrosetti and Prodi [17], Dancer [125], and Fučík [151]. They first realized

that the set Σ is an important factor in the study of semilinear elliptic boundary value problems with jumping nonlinearities. There are many papers on the existence of solutions to Dirichlet elliptic boundary value problems with jumping nonlinearities; see Căc [75], Dancer [127, 128], Giannoni and Micheletti [159], Hirano and Nishimura [169], Lazer and McKenna [191, 192], Liu and Wu [209], Margulies and Margulies [223], Marino and Saccon [220], Perera and Schechter [244–246], Schechter [269, 272, 273, 275, 276], and their references cited therein. They were mainly concerned with the existence results without sign-changingness of the solution. Dancer and Du [130] (jumping at zero), Dancer and Zhang [132], Li and Zhang [201], and Schechter et al. [279] got sign-changing solutions for Dirichlet zero-boundary value problems where the Fučík spectrum of Dirichlet boundary value problems is essential to their arguments. Note that in Schechter et al. [279], the authors first proved that the sign-changing solutions of Dirichlet boundary value problems are independent of the Fučík spectrum. The Fučík spectrum for Schrödinger equations is an open question. Lemma 2.29 is due to Bartsch, Liu and Weth [37], whose earlier ideas can be found in Conti, Merizzi and Terracini [107, 108].

2.4 Oscillating Dirichlet Equations

In this section, we consider the following case,

$$(2.63) \quad \liminf_{t \rightarrow \pm\infty} \frac{f(x, t)}{t} := \theta_{\pm}(x); \quad \limsup_{t \rightarrow \pm\infty} \frac{f(x, t)}{t} := \vartheta_{\pm}(x),$$

where $\theta_{\pm}, \vartheta_{\pm} \in L^{\infty}(\Omega)$. Assumption (2.63) implies that the nonlinearities are jumping and oscillating. Assume

$$(B_5) \quad 2F(x, t) \geq \max\{\lambda_{k-1}t^2, \theta_+(x)(t^+)^2 + \theta_-(x)(t^-)^2\} - c_0 \text{ for } x \in \Omega, t \in \mathbf{R};$$

$c_0 > 0$ is a constant.

Theorem 2.30. *Assume (B_1) , (B_3) , and (B_5) . For each pair of numbers α_+, β_- in the interval $(\lambda_k, \lambda_{k+1})$ there are numbers $\alpha_- < \lambda_k$ and $\beta_+ > \lambda_{k+1}$ such that*

$$\alpha_{\pm} \leq \theta_{\pm}(x) \leq \vartheta_{\pm}(x) \leq \beta_{\pm}, \quad x \in \Omega.$$

Then Equation (2.42) has a sign-changing solution.

Theorem 2.31. *Assume (B_1) , (B_3) , and (B_5) . For each pair of numbers α_-, β_+ in the interval $(\lambda_k, \lambda_{k+1})$ there are numbers $\alpha_+ < \lambda_k$ and $\beta_- > \lambda_{k+1}$ such that*

$$\alpha_{\pm} \leq \theta_{\pm}(x) \leq \vartheta_{\pm}(x) \leq \beta_{\pm}, \quad x \in \Omega.$$

Then Equation (2.42) has a sign-changing solution.

Theorem 2.32. Assume (B_1) , (B_3) , and (B_5) . Suppose that

$$\|v\|^2 \leq \int_{\mathbf{R}^N} (\theta_+(v^+)^2 + \theta_-(v^-)^2) dx, \quad \forall v \in E_k; \quad \vartheta_{\pm}(x) \leq \lambda_{k+1}, \quad x \in \Omega$$

and that no eigenfunction corresponding to λ_{k+1} satisfies

$$-\Delta u + V(x)u = \vartheta_+ u^+ - \vartheta_- u^-,$$

and no function in $E_k \setminus \{0\}$ satisfies $-\Delta u + V(x)u = \theta_+ u^+ - \theta_- u^-$. Then Equation (2.42) has a sign-changing solution.

Theorem 2.33. Assume (B_1) , (B_3) , and (B_5) . Suppose that

$$\lambda_k \leq \theta_{\pm}(x) \leq \vartheta_{\pm}(x) \leq \lambda_{k+1}, \quad x \in \Omega$$

and that no eigenfunction corresponding to λ_k satisfies

$$-\Delta u = \theta_+ u^+ - \theta_- u^-$$

and that no eigenfunction corresponding to λ_{k+1} satisfies $-\Delta u = \vartheta_+ u^+ - \vartheta_- u^-$. Then Equation (2.42) has a sign-changing solution.

Some lemmas are necessary for proving the above theorems.

Lemma 2.34. For each pair of numbers $\alpha_+, \beta_- \in (\lambda_k, \lambda_{k+1})$, there are numbers $\alpha_- < \lambda_k, \beta_+ > \lambda_{k+1}$ such that

$$(2.64) \quad \|u\|^2 < \int_{\Omega} (\alpha_+(u^+)^2 + \alpha_-(u^-)^2) dx, \quad \forall u \in E_k \setminus \{0\};$$

$$(2.65) \quad \|u\|^2 > \int_{\mathbf{R}^N} (\beta_+(u^+)^2 + \beta_-(u^-)^2) dx, \quad \forall u \in E_k^{\perp} \setminus \{0\}.$$

Proof. To prove (2.64), we define

$$\bar{c}_0 := \max_{u \in E_k, \|u\|_2=1} \left(\|u\|^2 - \int_{\Omega} \alpha_+(u^+)^2 dx - \int_{\Omega} \lambda_k (u^-)^2 dx \right).$$

Because $\dim E_k < \infty$, $\bar{c}_0$ exists and is attained at a point $u_0 \in E_k$ with $\|u_0\|_2 = 1$. Then,

$$\bar{c}_0 = \left(\|u_0\|^2 - \int_{\Omega} \lambda_k u_0^2 dx \right) + \int_{\Omega} (\lambda_k - \alpha_+) (u_0^+)^2 dx \leq 0.$$

Note that both terms in the middle above are less than or equal to zero. If $\|u_0\|^2 - \int_{\Omega} \lambda_k u_0^2 dx = 0$, then $u_0 \in N_k$ is an eigenfunction of λ_k . Hence, $u_0^+ \not\equiv 0$ because the eigenfunction is sign-changing. Thus, the second term, hence $\bar{c}_0$, is less than zero. Therefore, for all $u \in E_k \setminus \{0\}$,

$$\|u\|^2 - \int_{\Omega} (\alpha_+(u^+)^2 + \alpha_-(u^-)^2) dx \leq (\bar{c}_0 + \lambda_k - \alpha_-) \int_{\Omega} u^2 dx < 0$$

for an appropriate $\alpha_- < \lambda_k$. To prove (2.65), we define

$$d_0 = \inf_{u \in E_k^\perp, \|u\|_2=1} \left(\|u\|^2 - \int_{\Omega} \lambda_{k+1}(u^+)^2 dx - \int_{\Omega} \beta_-(u^-)^2 dx \right).$$

Note that $\int_{\Omega} \lambda_{k+1}(u)^2 dx \leq \|u\|^2$ for all $u \in E_k^\perp$; we see that

$$\begin{aligned} (2.66) \quad & \|u\|^2 - \int_{\Omega} \lambda_{k+1}(u^+)^2 dx - \int_{\Omega} \beta_-(u^-)^2 dx \\ & \geq \int_{\Omega} (\lambda_{k+1} - \beta_-)(u^-)^2 dx \\ & \geq 0. \end{aligned}$$

It follows that $d_0 \geq 0$. It suffices to show that $d_0 > 0$. But, if this were not true, we would have a sequence $\{u_n\} \subset E_k^\perp, \|u_n\|_2 = 1$ such that

$$d_n := \|u_n\|^2 - \int_{\Omega} \lambda_{k+1}(u_n)^2 dx - \int_{\Omega} (\lambda_{k+1} - \beta_-)(u_n^-)^2 dx \rightarrow 0$$

as $n \rightarrow \infty$. It follows that

$$\|u_n\|^2 \leq \lambda_{k+1} + d_n.$$

We may assume that $u_n \rightarrow u_*$ weakly in E and strongly in $L^2(\Omega)$, hence, $\|u_*\|_2 = 1$. Therefore,

$$\|u_*\|^2 - \int_{\mathbf{R}^N} \lambda_{k+1}(u_*)^2 dx - \int_{\Omega} (\lambda_{k+1} - \beta_-)(u_*^-)^2 dx \leq \lim_{n \rightarrow \infty} d_n = 0.$$

This implies that $u_*^- = 0$ and

$$\|u_*\|^2 = \int_{\Omega} \lambda_{k+1}(u_*)^2 dx.$$

All these mean that u_* is a positive eigenfunction of λ_{k+1} . This contradiction completes the proof of the lemma. $\square$

Lemma 2.35. *Under the assumptions of Theorem 2.30, G satisfies the (PS) condition.*

Proof. Lemma 2.34 and the conditions of Theorem 2.30 imply that

$$(2.67) \quad \|u\|^2 < \int_{\mathbf{R}^N} (\theta_+(u^+)^2 + \theta_-(u^-)^2) dx, \quad \forall u \in E_k \setminus \{0\};$$

$$(2.68) \quad \|u\|^2 > \int_{\Omega} (\vartheta_+(u^+)^2 + \vartheta_-(u^-)^2) dx, \quad \forall u \in E_k^\perp \setminus \{0\}.$$

Now let $\{u_n\}$ be a (PS) sequence: $\|G'(u_n)\| \rightarrow 0$ and $\{G(u_n)\}$ is bounded. We just have to show that $\{u_n\}$ is bounded. To show this, assume that $\|u_n\| \rightarrow \infty$. Let $\bar{u}_n = u_n/\|u_n\|$. Then $\bar{u}_n \rightarrow \bar{u}$ weakly in E , strongly in $L^2(\Omega)$, and a.e. in Ω . Because $|f(x, u_n)|/\|u_n\| \leq F_0|\bar{u}_n|$, we may assume that $(f(x, u_n))/\|u_n\|$ converges strongly in $L^2(\Omega)$ to a function $h(x)$. Observe that

$$\liminf_{n \rightarrow \infty} \frac{f(x, u_n)}{\|u_n\|} \geq \bar{u}(x) \liminf_{t \rightarrow \infty} \frac{f(x, t)}{t} = \bar{u}(x)\theta_+(x), \quad \text{if } \bar{u}(x) > 0.$$

In a similar way, we can show that

$$\bar{u}(x)\theta_+(x) \leq \liminf_{n \rightarrow \infty} \frac{f(x, u_n)}{\|u_n\|} \leq \limsup_{n \rightarrow \infty} \frac{f(x, u_n)}{\|u_n\|} \leq \bar{u}(x)\vartheta_+(x), \quad \text{if } \bar{u}(x) > 0;$$

$$\bar{u}(x)\vartheta_-(x) \leq \liminf_{n \rightarrow \infty} \frac{f(x, u_n)}{\|u_n\|} \leq \limsup_{n \rightarrow \infty} \frac{f(x, u_n)}{\|u_n\|} \leq \bar{u}(x)\theta_-(x), \quad \text{if } \bar{u}(x) < 0.$$

This gives

$$\bar{u}(x)\theta_+(x) \leq h(x) \leq \bar{u}(x)\vartheta_+(x), \quad \text{if } \bar{u}(x) > 0;$$

$$\bar{u}(x)\vartheta_-(x) \leq h(x) \leq \bar{u}(x)\theta_-(x), \quad \text{if } \bar{u}(x) < 0.$$

Let $q(x) = h(x)/\bar{u}(x)$ if $\bar{u}(x) \neq 0$; otherwise, $q(x) = 0$. Then

$$(2.69) \quad \theta_+(x) \leq q(x) \leq \vartheta_+(x), \quad \text{if } \bar{u}(x) > 0;$$

$$(2.70) \quad \theta_-(x) \leq q(x) \leq \vartheta_-(x), \quad \text{if } \bar{u}(x) < 0.$$

On the other hand, $G'(u_n) \rightarrow 0$ implies that

$$(2.71) \quad \langle \bar{u}(x), v \rangle - \int_{\Omega} h(x)v dx = \langle \bar{u}(x), v \rangle - \int_{\Omega} q(x)\bar{u}v dx = 0.$$

Let $\bar{u} = \bar{v} + \bar{w}$ with $\bar{v} \in E_k$, $\bar{w} \in E_k^\perp$, and $\tilde{u} = \bar{w} - \bar{v}$. Therefore, by (2.71),

$$(2.72) \quad \|\bar{w}\|^2 - \|\bar{v}\|^2 = \int_{\Omega} q(x)(\bar{w})^2 dx - \int_{\Omega} q(x)(\bar{v})^2 dx.$$

Recalling (2.67)–(2.69) and (2.72), we have

$$\begin{aligned} 0 &\leq \int_{\Omega} (\theta_+(\bar{v}^+)^2 + \theta_-(\bar{v}^-)^2) dx - \|\bar{v}\|^2 \\ &\leq \int_{\Omega} q(x)(\bar{v})^2 dx - \|\bar{v}\|^2 \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} q(x)(\bar{w})^2 dx - \|\bar{w}\|^2 \\
&\leq \int_{\Omega} (\vartheta_+(\bar{w}^+)^2 + \vartheta_-(\bar{w}^-)^2) dx - \|\bar{w}\|^2 \\
&\leq 0.
\end{aligned}$$

It follows that

$$(2.73) \quad \int_{\Omega} (\theta_+(\bar{v}^+)^2 + \theta_-(\bar{v}^-)^2) dx = \|\bar{v}\|^2;$$

$$(2.74) \quad \int_{\Omega} (\vartheta_+(\bar{w}^+)^2 + \vartheta_-(\bar{w}^-)^2) dx = \|\bar{w}\|^2.$$

Using (2.67) and (2.68) once again, we see that $\bar{v} = \bar{w} = \bar{u} = 0$. Hence,

$$\left\langle G'(u_n), \frac{u_n}{\|u_n\|^2} \right\rangle = 1 - \int_{\mathbf{R}^N} \frac{f(x, u_n)}{\|u_n\|} \bar{u}_n(x) dx \rightarrow 1,$$

providing a contradiction. $\square$

By a similar argument, we can prove the following.

Lemma 2.36. *Under the assumptions of Theorem 2.31, G satisfies the (PS) condition.*

Lemma 2.37. *Under the assumptions of Theorem 2.32, G satisfies the (PS) condition.*

Proof. By the assumptions of the Theorem 2.32, we have

$$\begin{aligned}
\|u\|^2 &\leq \int_{\Omega} (\theta_+(u^+)^2 + \theta_-(u^-)^2) dx, \quad u \in E_k; \\
\|u\|^2 &\geq \lambda_{k+1} \|u\|_2^2 \geq \int_{\Omega} (\vartheta_+(u^+)^2 + \vartheta_-(u^-)^2) dx, \quad u \in E_k^{\perp}.
\end{aligned}$$

Then (2.74) still holds. Hence

$$\int_{\Omega} (\lambda_{k+1} - \vartheta_+)(\bar{w}^+)^2 dx + \int_{\Omega} (\lambda_{k+1} - \vartheta_-)(\bar{w}^-)^2 dx = 0.$$

It follows that $\vartheta_+ = \lambda_{k+1}$ if $\bar{w} > 0$, and $\vartheta_- = \lambda_{k+1}$ if $\bar{w} < 0$ and $\bar{w}$ is an eigenfunction of λ_{k+1} . Therefore,

$$-\Delta \bar{w} + V(x)\bar{w} = \lambda_{k+1}\bar{w} = \vartheta_+\bar{w}^+ - \vartheta_-\bar{w}^-,$$

which implies that $\bar{w} = 0$. Furthermore,

$$\int_{\Omega} (\theta_+(\bar{v}^+)^2 + \theta_-(\bar{v}^-)^2) dx = \int_{\Omega} q(x)(\bar{v})^2 dx.$$

Thus, $q(x) = \theta_+(x)$ if $\bar{u} > 0$; $q(x) = \theta_-(x)$ if $\bar{u} < 0$ and

$$-\Delta \bar{u} + V(x)\bar{u} = \theta_+ \bar{u}^+ - \theta_- \bar{u}^-.$$

It follows that $\bar{u} = \bar{v} = 0$. Using an argument similar to that used in proving Lemma 2.35, we get a contradiction if the (PS) sequence is unbounded. $\square$

Similarly, we have

Lemma 2.38. *Under the assumptions of Theorem 2.33, G satisfies the (PS) condition.*

Proofs of Theorems 2.30–2.33. By Lemmas 2.22–2.26, we see that $G(u) \leq c_0|\Omega|/2$ for all $u \in E_{k-1} := Y$ and $G(u) \geq c_0|\Omega|/2$ for all $u \in S_0$. By (B_5) , (2.63), and (2.67), similar to the proof of Lemma 2.22, we have $G(u) \leq T_0$ for all $u \in E_k$; here T_0 is a constant. Then, G satisfies all the conditions of Theorem 2.18. Therefore, there exists a sequence $\{u_k\} \in E \setminus (-\mathcal{P} \cup \mathcal{P})$ such that

$$G'(u_k) \rightarrow 0, \quad G'(u_k) = C_k u_k / k, G(u_k) \in \left[\frac{1}{4} c_0 |\Omega|, 2T_0 \right],$$

as $k \rightarrow \infty$, where the sequence $\{C_k\}$ is bounded. By Lemmas 2.35–2.38, $u_k \rightarrow u$, where u satisfies

$$G'(u) = 0, \quad G(u) \in \left[\frac{1}{4} c_0 |\Omega|, 2T_0 \right].$$

We now show that u is sign-changing. In fact, because $G'(u_k) - C_k u_k / k = 0$, we have

$$\|u_k^\pm\|^2 - \frac{C_k}{k} \|u_k^\pm\|^2 = \int_{\mathbf{R}^N} f(x, u_k^\pm) u_k^\pm dx \leq \frac{1}{3} \|u_k^\pm\|^2 + C \|u_k^\pm\|_p^p.$$

It follows that $\|u_k^\pm\| \geq s_0 > 0$, where s_0 is a constant independent of k . This implies that the limit u is sign-changing and $G'(u) = 0$, $G(u) \in [\frac{1}{4} c_0 |\Omega|, 2T_0]$. $\square$

Notes and Comments. The existence results of Theorems 2.30–2.33 are essentially known (cf. C ac [75], Berestycki and de Figueiredo [58], Furtado et al. [152, 153], Habets et al. [165], and Schechter [269]). But in those papers the signs of the solutions cannot be decided. Theorems 2.30–2.33 are neither consequences of the usual linking theorems nor straightforward results of the methods developed in Bartsch [30], Li and Wang [199], and Bartsch et al. [37]. A similar result to that of Lemma 2.34 can be found in C ac [75], Lazer and McKenna [191], and Schechter [269].

2.5 Double Resonant Cases

Consider the following case,

$$(2.75) \quad \lambda_k \leq \Psi_1(x) := \liminf_{|t| \rightarrow \infty} \frac{f(x, t)}{t} \leq \limsup_{|t| \rightarrow \infty} \frac{f(x, t)}{t} := \Psi_2(x) \leq \lambda_{k+1}$$

uniformly for $x \in \Omega$. We have the following.

Theorem 2.39. *Assume that (B_1) – (B_4) and (2.75) hold with $\Psi_1(x) \not\equiv \lambda_k$. Then Equation (2.42) has a sign-changing solution.*

Lemma 2.40. *Under the assumptions of Theorem 2.39, $G(u) \rightarrow -\infty$ for $u \in E_k$ and $\|u\| \rightarrow \infty$.*

Proof. Because $\Psi_1(x) \geq \lambda_k$, $\Psi_1(x) \not\equiv \lambda_k$, and $\dim E_k < \infty$, by the variational characterization of the eigenvalues $\{\lambda_k\}$, there is a $\rho > 0$ such that

$$(2.76) \quad \|u\|^2 - \int_{\Omega} \Psi_1(x) u^2 dx \leq -\rho \|u\|^2 \quad \text{for all } u \in E_k.$$

In fact, this is an immediate consequence of the Schechter–Simon Theorem 1.62. Furthermore, by (2.75), for $\varepsilon > 0$ small enough, there exists a $C_\varepsilon > 0$ such that

$$\frac{1}{2} \Psi_1(x) t^2 - F(x, t) \leq \frac{1}{2} \varepsilon t^2 + C_\varepsilon$$

for all $x \in \Omega, t \in \mathbf{R}$. Therefore, combining (2.76),

$$\begin{aligned} G(u) &= \frac{1}{2} \|u\|^2 - \frac{1}{2} \int_{\Omega} \Psi_1(x) u^2 dx + \int_{\Omega} \left(\frac{1}{2} \Psi_1(x) u^2 - F(x, u) \right) dx \\ &\leq -\frac{3\rho}{8} \|u\|^2 + \int_{\Omega} \left(\frac{1}{2} \varepsilon u^2 + C_\varepsilon \right) dx \\ &\leq -\frac{\rho}{4} \|u\|^2 + \int_{\Omega} C_\varepsilon dx. \end{aligned}$$

The lemma follows immediately. $\square$

Proof of Theorem 2.39. Similar to Lemma 2.28, G satisfies the (w-PS) condition. The remainder is analogous to the proof of Theorem 2.20. We leave the details to the readers. $\square$

Notes and Comments. To study the sign-changing solutions, several authors developed some methods. In Bartsch [30], the author established an abstract critical theory in partially ordered Hilbert spaces by virtue of critical groups and studied superlinear problems. In Li and Wang [199], a Ljusternik–Schnirelman theory was established for studying the sign-changing solutions

of an even functional. Some linking-type theorems were also obtained in partially ordered Hilbert spaces. The methods and abstract critical point theory of Bartsch [30], Bartsch and Weth [45], and Li and Wang [199, 198] involved the dense Banach space $\mathbf{C}(\Omega)$ of continuous functions in the Hilbert space $H_0^1(\Omega)$, where the cone has a nonempty interior. This plays a crucial role. To fit that framework, much stronger hypotheses (e.g., boundedness of the domain and stronger smoothness of the nonlinearities) were imposed. In [37], the method of dealing with superlinear non-odd f was based on Liu and Sun [211] by using arguments of invariant sets. In [37], this idea of the neighborhood of a cone due to Conti et al. [107] was applied and modified by the authors to construct the invariant set. Their idea also can be traced back to Bartsch [29].

Under the Ambrosetti–Rabinowitz’s super-quadratic (ARS, for short), Wang [331] obtained the existence result of three solutions (one is positive, another one is negative) on a superlinear Dirichlet elliptic equation and later in Bartsch and Wang [40], the authors proved for semilinear Dirichlet problems that the third solution is sign-changing. This result was generalized to nonlinear Schrödinger equations in Bartsch and Wang [41] where the (ARS) condition plays an important role. Recall the papers of Coti Zelati and Rabinowitz [121, 122], where $V(x)$ and $f(x, t)$ were periodic for each x variable and infinitely many sign-changing solutions were obtained by a totally different theory.

In Bartsch and Wang [44], the existence of sign-changing entire solutions defined on $\mathbf{R}^N$ was studied. They constructed a series of Dirichlet problems on the ball and then expanded the ball to whole space.

Other papers on sign-changing solutions include Bartsch et al. [31], Castro et al. [80, 81], Castro and Finan [82], Dancer and Du [129], Dancer and Yan [131], and Schechter et al. [279]. Other variants of the linking theorem can be found in Schechter [273, 267, 270, 274, 271]. Finally, we mention other papers on resonant problems. In Arcoya and Costa [18], Bartolo et al. [27], Bartsch and Li [36], Hirano et al. [171], and Hirano and Nishimura [169], the strong resonant elliptic equation was studied. In Schechter [276] and Zou and Liu [348], general resonant problems were considered.



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