What Are the Chances? Assigning Probabilities

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Chance, too, which seems to rush along with slack reins, is bridled and governed by laws.
Boethius, *The Consolation of Philosophy*

Misunderstanding of probability may be the greatest of all impediments to scientific literacy.
Stephen Jay Gould

Introduction

Probability plays a central role in quantifying chance with the probability of an event being a measure of the event's uncertainty. When people talk about the probability of an event, say, the probability of rain tomorrow, they may have in their mind a scale of expressions that distinguish the different degrees of probability in some way, for example, 'it is certain not to rain', 'it is very unlikely to rain', 'it is unlikely to rain', 'it is as likely to rain as not', 'it is likely to rain', 'it is very likely to rain', 'it is certain to rain'. But these expressions, apart from the first, fourth and last are vague and different people may interpret them differently. The vagueness is of little consequence when discussing the weather with a friend but might be of concern in conversations with your bookmaker, insurance salesman or doctor. We need a more precise, *numerical* scale for probability.

By convention, probabilities are measured on a scale from zero to one. To say an event with probability of one fifth (1/5 or 0.2) implies that there is a 1 in 5 or 20% chance of the event happening. The zero point on the probability scale corresponds to an event which is impossible—the author now ever running a marathon in less than two and a half hours suggests itself. (Come to think of it, that was always pretty unlikely!) The highest point on the probability scale, one, corresponds to an event which is certain to happen—the author's death within the next 35 years unfortunately comes to mind. But how do we assign probabilities to events that are

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uncertain but possible, rather than certain or impossible? Well, there are a number of possibilities.

Using Relative Frequencies as Probabilities

Suppose 1,000,000 children are born in a particular year with 501,000 of them being boys. The relative frequency of boys is simply 501,000/1,000,000 = 0.501. Now suppose that in a small village somewhere, 10 children are born in the year, of whom 7 are boys. The relative frequency of boys in this case is 7/10 = 0.7. As the rabbis of over two millennia realized, the relative frequency of an event relates to the event's chance or probability of occurrence. In fact, this concept is now used to provide an intuitive definition of the probability of an event as the relative frequency of the event in a *large* number of trials. Thus in the examples given, the observation of 501,000 male births amongst a million births suggests we can claim that the probability of a male birth is 0.501, but it would be unconvincing to use the counts of births from the small village to suggest that the probability of a male child is 0.7.

The obvious question that needs to be asked here is, 'How large is large enough?' Sadly there is no simple answer, which may appear to imply that the relative frequency approach to assigning probabilities is not perfect. But before jumping to such a conclusion we might ponder for a moment that a perfect model of the solar system was not needed to successfully send probes to photograph the outer planets. And relative frequency probabilities remain very useful despite being somewhat less than perfect.

The Classical Definition of Probability

Suppose we are interested in assigning a probability to the event that a fair coin, when tossed, comes down heads. We could use the relative frequency approach, by simply tossing the coin a large number of times and then dividing the number of heads obtained by the total number of tosses. Undertaking this task myself I tossed a 50-pence (50p) coin 500 times and obtained 238 heads, leading to a relative-frequency-based value for the probability of a head of 0.48. But unlike this frequency approach, the so-called classical definition allows probabilities to be assigned to single events. We might, for example, be interested in finding the probability of getting a two on a single roll of a die. Calling the case of two 'favourable', the numerical value of its probability is found by dividing the number of favourable possible cases by the total number of all equally possible cases. For a true die there are six outcomes, all of which are equally possible, but only one of which is favourable; consequently the probability of throwing a two is 1/6.

In essence, this approach to assigning probabilities assumes that without further information about the physical process that, say, flips a coin or rolls a die, a sensible observer would view all outcomes as being equally likely; hence one half for the probability of a head and one sixth for the probability of each side of a die.

We shall look more at tossing coins in Chapter 4 and at rolling dice in Chapter 5; here we will consider a different example of assigning probabilities using the classical approach.

Let's look at the following three problems involving leap years;

1. What is the chance of a year, which is not a leap year, having 53 Sundays?
2. What is the chance that a leap year, will contain 53 Sundays?
3. What is the chance that a leap year which is known not to be the last year in the century should be a leap year?

To get the answer to each question we need to count the 'favourable' outcomes in each case and divide the resulting number by the total number of equally possible outcomes. In this way we are led to the following solutions;

1. A non-leap year of 365 days consists of 52 complete weeks, and one day over. This odd day may be one of the 7 days of the week, and there is nothing to make one more likely than another. Only one will lead to the result that the year will have 53 Sundays; consequently the probability that the year has 53 Sundays is simply 1/7.
2. A leap year of 366 days consists of 52 complete weeks, and 2 days over. These days may be

   Sunday and Monday,
   Monday and Tuesday,
   Tuesday and Wednesday,
   Wednesday and Thursday,
   Thursday and Friday,
   Friday and Saturday,
   Saturday and Sunday.

   And all seven possibilities are equally likely. Two of them (the first and the last) will produce the required result so the chance of a leap year containing 53 Sundays is 2/7, twice the probability for a non-leap year.

3. The year may be any of the remaining 99 of any century, and all these are equally likely; 24 of the years are leap years so the chance that the year in question is a leap year is 24/99.

Subjective Probability

The classical and relative frequency methods of assigning probabilities cannot be used in all circumstances where we are interested in making probability statements. What, for example, is the probability that life exists on a planet circling the nearest star, Proxima Centauri? For that matter, what is the probability that there even *is* a planet circling Proxima Centauri? The classical definition of probability provides no answer for either question, and there are no relative frequency data available that might help. In such circumstances we are forced to express a subjective opinion about the required probabilities—to assign a subjective probability. We might use experience, intuition, or a hunch to arrive at a value, but the assigned probability will be subjective, and different people can be expected to assign different probabilities to the same event. Subjective probabilities express a person's degree of belief in the event and so are often referred to as personal probabilities.

A useful short summary of subjective probability has been provided by Peter Lenk of the University of Michigan:

> In the frequency interpretation, probability is an innate characteristic; flipped coins come up heads 50% of the time because that is the property of coins. In subjective probability, flipped coins come up heads 50% of the time because the observer does not have information that would lead him or her to believe that one side is preferred over the other. The locus of uncertainty is not the coin and its environment, but resides within the observer. If the observer knew more about the physical laws of coin flips and the initial conditions, then he or she could refine his or her probability assessment.

Although we shall not be concerned with subjective probabilities in the remainder of this book one word of advice may be appropriate: Never let the values of the subjective probabilities you assign be zero or one. For example, my own subjective probabilities for flying saucers and the Loch Ness monster are very near to zero, but there no doubt exist people whose subjective probabilities for these two phenomena are close to one. And my own current very low subjective probability for flying saucers, say, would increase dramatically if such a machine landed in my garden.

For scientists in particular, complete disbelief (a subjective probability of zero) or complete certainty (a subjective probability of one), are equally undesirable. Scientists (and others) need to avoid the paralysis that can be caused by the pursuit of certainty. However successful and reliable a theory, say, may be up to any point in time, further observations may come along and show a need for adjustment of the theory, while at the other extreme, however little confidence one has in a theory, new information may change the situation. But scientists and non-scientists both, court the danger of descending into an emotional pit of pride and prejudice once their subjective probabilities for any event or phenomenon take either the value one or zero. In the words of that well known Chinese philosopher, Chan: 'Human mind like parachute: work best when open.'

Rules for Combining Probabilities

Suppose we roll a fair die and want to know the probability of getting either a six or a one. Since a six and a one cannot occur together they are usually called *mutually exclusive events*, and the probability of either of them occurring is simply the sum of the individual probabilities, in this case $1/6+1/6=1/3$. This is a general rule for finding the probability of mutually exclusive events; if there are two such events A and B then

$$\Pr(\text{A or B}) = \Pr(\text{A}) + \Pr(\text{B}),$$

where Pr stands for 'the probability of' and the equation is translated as 'the probability of event A or event B is equal to the probability of event A plus the probability of event B'. So the die example is described by

$$\Pr(6 \text{ or } 1) = \Pr(6) + \Pr(1).$$

For three mutually exclusive events, A, B and C,

$$\Pr(\text{A or B or C}) = \Pr(\text{A}) + \Pr(\text{B}) + \Pr(\text{C}).$$

For example, tossing an even number with a fair die,

$$\Pr(2 \text{ or } 4 \text{ or } 6) = \Pr(2) + \Pr(4) + \Pr(6) = 1/6 + 1/6 + 1/6 = 1/2,$$

and so on for more than three mutually exclusive events.

Suppose now we roll two dice separately and want the probability of getting a double six. Since what happens to each die is not affected by what happens to the other, the two events, six with one die and six with the other, are said to be *independent*. The probability that both occur is then simply obtained by multiplying together the probabilities of each event, i.e., $1/6 \times 1/6 = 1/36$. This is a general rule for finding the probability of independent events; if there are two such events, A and B then,

$$\Pr(\text{A and B}) = \Pr(\text{A}) \times \Pr(\text{B}).$$

For three independent events, A, B and C,

$$\Pr(\text{A and B and C}) = \Pr(\text{A}) \times \Pr(\text{B}) \times \Pr(\text{C}).$$

For example getting a triple six when rolling three dice,

$$\Pr(\text{triple six}) = \Pr(6) \times \Pr(6) \times \Pr(6) = 1/6 \times 1/6 \times 1/6 = 1/216$$

and so on for more than three independent events.

By combining the *addition rule* and the *multiplication rule*, probabilities of more complex events can be determined, as we shall find in later chapters.

Odds

Gamblers usually prefer to quantify their uncertainty about an event in terms of *odds* rather than probabilities, although the two are actually completely synonymous. An event with a probability of 1/5 would be said by an experienced gambler to have odds against of 4 to 1. This simply expresses the fact that the probability of the event not occurring is four times that of it occurring. Similarly, an event with probability 4/5 would be said by the gambler, to be 4 to 1 on—here the probability of the event occurring is four times that of it not occurring. The exact relation between odds and probability can be defined mathematically:

- An event with odds of 'F to 1 in its favour (odds-on)' has probability $F/(F+1)$.
- An event with odds of 'A to 1 against' has probability $1/(A+1)$.
- An event with probability P implies that the odds in favour are $P/(1-P)$ to 1, whereas the odds against are $1/P-1$.

Odds and probabilities are completely equivalent ways of expressing the uncertainty of an event but gamblers generally deal in the former and scientists in the latter, although it has to be said that many medical statisticians are, perhaps due to a misspent youth, also very adept with using odds.

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