Preface

It seems to have been decided that undergraduate mathematics today rests on two foundations: calculus and linear algebra. These may not be the best foundations for, say, number theory or combinatorics, but they serve quite well for undergraduate analysis and several varieties of undergraduate algebra and geometry. The really *perfect* sequel to calculus and linear algebra, however, would be a blend of the two—a subject in which calculus throws light on linear algebra and vice versa. Look no further! This perfect blend of calculus and linear algebra is Lie theory (named to honor the Norwegian mathematician Sophus Lie—pronounced “Lee ”). So why is Lie theory not a standard undergraduate topic?

The problem is that, until recently, Lie theory was a subject for mature mathematicians or else a tool for chemists and physicists. There was no Lie theory for novice mathematicians. Only in the last few years have there been serious attempts to write Lie theory books for undergraduates. These books broke through to the undergraduate level by making some sensible compromises with generality; they stick to matrix groups and mainly to the classical ones, such as rotation groups of n-dimensional space.

In this book I stick to similar subject matter. The classical groups are introduced via a study of rotations in two, three, and four dimensions, which is also an appropriate place to bring in complex numbers and quaternions. From there it is only a short step to studying rotations in real, complex, and quaternion spaces of any dimension. In so doing, one has introduced the classical simple Lie groups, in their most geometric form, using only basic linear algebra. Then calculus intervenes to find the tangent spaces of the classical groups—their Lie algebras—and to move back and forth between the group and its algebra via the log and exponential functions. Again, the basics suffice: single-variable differentiation and the Taylor series for e^x and $\log(1+x)$.

Where my book diverges from the others is at the next level, the miraculous level where one discovers that the (curved) structure of a Lie group is almost completely captured by the structure of its (flat) Lie algebra. At this level, the other books retain many traces of the sophisticated approach to Lie theory. For example, they rely on deep ideas from outside Lie theory, such as the inverse function theorem, existence theorems for ODEs, and representation theory. Even inside Lie theory, they depend on the Killing form and the whole root system machine to prove simplicity of the classical Lie algebras, and they use everything under the sun to prove the Campbell–Baker–Hausdorff theorem that lifts structure from the Lie algebra to the Lie group. But actually, proving simplicity of the classical Lie algebras can be done by basic matrix arithmetic, and there is an amazing elementary proof of Campbell–Baker–Hausdorff due to Eichler [1968].

The existence of these little-known elementary proofs convinced me that a naive approach to Lie theory is possible and desirable. The aim of this book is to carry it out—developing the central concepts and results of Lie theory by the simplest possible methods, mainly from single-variable calculus and linear algebra. Familiarity with elementary group theory is also desirable, but I provide a crash course on the basics of group theory in Sections 2.1 and 2.2.

The naive approach to Lie theory is due to von Neumann [1929], and it is now possible to streamline it by using standard results of undergraduate mathematics, particularly the results of linear algebra. Of course, there is a downside to naiveté. It is probably not powerful enough to prove some of the results for which Lie theory is famous, such as the classification of the simple Lie algebras and the discovery of the five exceptional algebras.[1] To compensate for this lack of technical power, the end-of-chapter discussions introduce important results beyond those proved in the book, as part of an informal sketch of Lie theory and its history. It is also true that the naive methods do not afford the same insights as more sophisticated methods. But they offer another insight that is often undervalued—some important theorems are not as difficult as they look! I think that all mathematics students appreciate this kind of insight.

In any case, my approach is not entirely naive. A certain amount of topology is essential, even in basic Lie theory, and in Chapter 8 I take

[1] I say so from painful experience, having entered Lie theory with the aim of understanding the exceptional groups. My opinion now is that the Lie theory that precedes the classification is a book in itself.

the opportunity to develop all the appropriate concepts from scratch. This includes everything from open and closed sets to simple connectedness, so the book contains in effect a minicourse on topology, with the rich class of multidimensional examples that Lie theory provides. Readers already familiar with topology can probably skip this chapter, or simply skim it to see how Lie theory influences the subject. (Also, if time does not permit covering the whole book, then the end of Chapter 7 is a good place to stop.)

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Finally, a word about my title. Readers of a certain age will remember the book *Naive Set Theory* by Paul Halmos—a lean and lively volume covering the parts of set theory that all mathematicians ought to know. Paul Halmos (1916–2006) was my mentor in mathematical writing, and I dedicate this book to his memory. While not attempting to emulate his style (which is inimitable), I hope that *Naive Lie Theory* can serve as a similar introduction to Lie groups and Lie algebras. Lie theory today has become the subject that all mathematicians ought to know something about, so I believe the time has come for a naive, but mathematical, approach.

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