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## Introduction

The theory of  $D$ -modules plays a key role in algebraic analysis. For the purposes of this text, by “algebraic analysis,” we mean analysis using algebraic methods, such as ring theory and homological algebra. In addition to the contributions by French mathematicians, J. Bernstein, and others, this area of research has been extensively developed since the 1960s by Japanese mathematicians, notably in the important contributions of M. Sato, T. Kawai, and M. Kashiwara of the Kyoto school.

To this day, there continue to be outstanding results and significant theories coming from the Kyoto school, including Sato’s hyperfunctions, microlocal analysis,  $D$ -modules and their applications to representation theory and mathematical physics. In particular, the theory of regular holonomic  $D$ -modules and their solution complexes (e.g., the theory of the Riemann–Hilbert correspondence which gave a sophisticated answer to Hilbert’s 21st problem) was a most important and influential result. Indeed, it provided the germ for the theory of perverse sheaves, which was a natural development from intersection cohomologies. Moreover, M. Saito used this result effectively to construct his theory of Hodge modules, which largely extended the scope of Hodge theory. In representation theory, this result opened totally new perspectives, such as the resolution of the Kazhdan–Lusztig conjecture.

As stated above, in addition to the strong impact on analysis which was the initial main motivation, the theory of algebraic analysis, especially that of  $D$ -modules, continues to play a central role in various fields of contemporary mathematics. In fact,  $D$ -module theory is a source for creating new research areas from which new theories emerge. This striking feature of  $D$ -module theory has stimulated mathematicians in various other fields to become interested in the subject.

Our aim is to give a comprehensive introduction to  $D$ -modules. Until recently, in order to really learn it, we had to read and become familiar with many articles, which took long time and considerable effort. However, as we mentioned in the preface, thanks to some textbooks and monographs, the theory has become much more accessible nowadays, especially for those who have some basic knowledge of complex analysis or algebraic geometry. Still, to understand and appreciate the real significance of the subject on a deep level, it would be better to learn both the theory and its typical applications.

In Part I of this book we introduce  $D$ -modules principally in the context of presenting the theory of the Riemann–Hilbert correspondence. Part II is devoted to explaining applications to representation theory, especially to the solution to the Kazhdan–Lusztig conjecture. Since we mainly treat the theory of algebraic  $D$ -modules on smooth algebraic varieties rather than the (original) analytic theory on complex manifolds, we shall follow the unpublished notes [Ber3] of Bernstein (the book [Bor3] is also written along this line). The topics treated in Part II reveal how useful  $D$ -module theory is in other branches of mathematics. Among other things, the essential usefulness of this theory contributed heavily to resolving the Kazhdan–Lusztig conjecture, which was of course a great breakthrough in representation theory.

As we started Part II by giving a brief introduction to some basic notions of Lie algebras and algebraic groups using concrete examples, we expect that researchers in other fields can also read Part II without much difficulty.

Let us give a brief overview of the topics developed in this text. First, we explain how  $D$ -modules are related to systems of linear partial differential equations. Let  $X$  be an open subset of  $\mathbb{C}^n$  and denote by  $\mathcal{O}$  the commutative ring of complex analytic functions globally defined on  $X$ . We denote by  $D$  the set of linear partial differential operators with coefficients in  $\mathcal{O}$ . Namely, the set  $D$  consists of the operators of the form

$$\sum_{i_1, i_2, \dots, i_n}^{\infty} f_{i_1, i_2, \dots, i_n} \left( \frac{\partial}{\partial x_1} \right)^{i_1} \left( \frac{\partial}{\partial x_2} \right)^{i_2} \cdots \left( \frac{\partial}{\partial x_n} \right)^{i_n} \quad (f_{i_1, i_2, \dots, i_n} \in \mathcal{O})$$

(each sum is a finite sum), where  $(x_1, x_2, \dots, x_n)$  is a coordinate system of  $\mathbb{C}^n$ . Note that  $D$  is a non-commutative ring by the composition of differential operators. Since the ring  $D$  acts on  $\mathcal{O}$  by differentiation,  $\mathcal{O}$  is a left  $D$ -module. Now, for  $P \in D$ , let us consider the differential equation

$$Pu = 0 \tag{0.0.1}$$

for an unknown function  $u$ . According to Sato, we associate to this equation the left  $D$ -module  $M = D/DP$ . In this setting, if we consider the set  $\text{Hom}_D(M, \mathcal{O})$  of  $D$ -linear homomorphisms from  $M$  to  $\mathcal{O}$ , we get the isomorphism

$$\begin{aligned} \text{Hom}_D(M, \mathcal{O}) &= \text{Hom}_D(D/DP, \mathcal{O}) \\ &\simeq \{\varphi \in \text{Hom}_D(D, \mathcal{O}) \mid \varphi(P) = 0\}. \end{aligned}$$

Hence we see by  $\text{Hom}_D(D, \mathcal{O}) \simeq \mathcal{O}$  ( $\varphi \mapsto \varphi(1)$ ) that

$$\text{Hom}_D(M, \mathcal{O}) \simeq \{f \in \mathcal{O} \mid Pf = 0\}$$

( $Pf = P\varphi(1) = \varphi(P1) = \varphi(P) = 0$ ). In other words, the (additive) group of the holomorphic solutions to the equation (0.0.1) is naturally isomorphic to  $\text{Hom}_D(M, \mathcal{O})$ . If we replace  $\mathcal{O}$  with another function space  $\mathcal{F}$  admitting a natural action of  $D$  (for example, the space of  $C^\infty$ -functions, Schwartz distributions,

Sato's hyperfunctions, etc.), then  $\text{Hom}_D(M, \mathcal{F})$  is the set of solutions to (0.0.1) in that function space.

More generally, a system of linear partial differential equations of  $l$ -unknown functions  $u_1, u_2, \dots, u_l$  can be written in the form

$$\sum_{j=1}^l P_{ij} u_j = 0 \quad (i = 1, 2, \dots, k) \quad (0.0.2)$$

by using some  $P_{ij} \in D$  ( $1 \leq i \leq k, 1 \leq j \leq l$ ). In this situation we have also a similar description of the space of solutions. Indeed if we define a left  $D$ -module  $M$  by the exact sequence

$$D^k \xrightarrow{\varphi} D^l \longrightarrow M \longrightarrow 0 \quad (0.0.3)$$

$$\varphi(Q_1, Q_2, \dots, Q_k) = \left( \sum_{i=1}^k Q_i P_{i1}, \sum_{i=1}^k Q_i P_{i2}, \dots, \sum_{i=1}^k Q_i P_{il} \right),$$

then the space of the holomorphic solutions to (0.0.2) is isomorphic to  $\text{Hom}_D(M, \mathcal{O})$ . Therefore, systems of linear partial differential equations can be identified with the  $D$ -modules having some finite presentations like (0.0.3), and the purpose of the theory of linear PDEs is to study the solution space  $\text{Hom}_D(M, \mathcal{O})$ . Since the space  $\text{Hom}_D(M, \mathcal{O})$  does not depend on the concrete descriptions (0.0.2) and (0.0.3) of  $M$  (it depends only on the  $D$ -linear isomorphism class of  $M$ ), we can study these analytical problems through left  $D$ -modules admitting finite presentations. In the language of categories, the theory of linear PDEs is nothing but the investigation of the contravariant functor  $\text{Hom}_D(\bullet, \mathcal{O})$  from the category  $M(D)$  of  $D$ -modules admitting finite presentations to the category  $M(\mathbb{C})$  of  $\mathbb{C}$ -modules.

In order to develop this basic idea, we need to introduce sheaf theory and homological algebra. First, let us explain why sheaf theory is indispensable. It is sometimes important to consider solutions locally, rather than globally on  $X$ . For example, in the case of ordinary differential equations (or more generally, the case of integrable systems), the space of local solutions is always finite dimensional; however, it may happen that the analytic continuations (after turning around a closed path) of a solution are different from the original one. This phenomenon is called monodromy. Hence we also have to take into account how local solutions are connected to each other globally.

Sheaf theory is the most appropriate language for treating such problems. Therefore, sheafifying  $\mathcal{O}, D$ , let us now consider the sheaf  $\mathcal{O}_X$  of holomorphic functions and the sheaf  $D_X$  (of rings) of differential operators with holomorphic coefficients. We also consider sheaves of  $D_X$ -modules (in what follows, we simply call them  $D_X$ -modules) instead of  $D$ -modules. In this setting, the main objects to be studied are left  $D_X$ -modules admitting locally finite presentations (i.e., coherent  $D_X$ -modules). Sheafifying also the solution space, we get the sheaf  $\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$  of the holomorphic solutions to a  $D_X$ -module  $M$ . It follows that what we should investigate is the contravariant functor  $\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X)$  from the category  $\text{Mod}_c(D_X)$  of coherent  $D_X$ -modules to the category  $\text{Mod}(\mathbb{C}_X)$  of (sheaves of)  $\mathbb{C}_X$ -modules.

Let us next explain the need for homological algebra. Although both  $\text{Mod}_c(D_X)$  and  $\text{Mod}(\mathbb{C}_X)$  are abelian categories,  $\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X)$  is not an exact functor. Indeed, for a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \quad (0.0.4)$$

in the category  $\text{Mod}_c(D_X)$  the sequence

$$0 \rightarrow \mathcal{H}om_{D_X}(M_3, \mathcal{O}_X) \rightarrow \mathcal{H}om_{D_X}(M_2, \mathcal{O}_X) \rightarrow \mathcal{H}om_{D_X}(M_1, \mathcal{O}_X) \quad (0.0.5)$$

associated to it is also exact; however, the final arrow  $\mathcal{H}om_{D_X}(M_2, \mathcal{O}_X) \rightarrow \mathcal{H}om_{D_X}(M_1, \mathcal{O}_X)$  is not necessarily surjective. Hence we cannot recover information about the solutions of  $M_2$  from those of  $M_1, M_3$ . A remedy for this is to consider also the “higher solutions”  $\mathcal{E}xt_{D_X}^i(M, \mathcal{O}_X)$  ( $i = 0, 1, 2, \dots$ ) by introducing techniques in homological algebra. We have  $\mathcal{E}xt_{D_X}^0(M, \mathcal{O}_X) = \mathcal{H}om_{D_X}(M, \mathcal{O}_X)$  and the exact sequence (0.0.5) is naturally extended to the long exact sequence

$$\begin{aligned} \cdots \rightarrow \mathcal{E}xt_{D_X}^i(M_3, \mathcal{O}_X) &\rightarrow \mathcal{E}xt_{D_X}^i(M_2, \mathcal{O}_X) \rightarrow \mathcal{E}xt_{D_X}^i(M_1, \mathcal{O}_X) \\ &\rightarrow \mathcal{E}xt_{D_X}^{i+1}(M_3, \mathcal{O}_X) \rightarrow \mathcal{E}xt_{D_X}^{i+1}(M_2, \mathcal{O}_X) \rightarrow \mathcal{E}xt_{D_X}^{i+1}(M_1, \mathcal{O}_X) \rightarrow \cdots \end{aligned}$$

Hence the theory will be developed more smoothly by considering all higher solutions together.

Furthermore, in order to apply the methods of homological algebra in full generality, it is even more effective to consider the object  $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$  in the derived category (it is a certain complex of sheaves of  $\mathbb{C}_X$ -modules whose  $i$ -th cohomology sheaf is  $\mathcal{E}xt_{D_X}^i(M, \mathcal{O}_X)$ ) instead of treating the sheaves  $\mathcal{E}xt_{D_X}^i(M, \mathcal{O}_X)$  separately for various  $i$ 's. Among the many other advantages for introducing the methods of homological algebra, we point out here the fact that the sheaf of a hyperfunction solution can be obtained by taking the local cohomology of the complex  $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$  of holomorphic solutions. This is quite natural since hyperfunctions are determined by the boundary values (local cohomologies) of holomorphic functions.

Although we have assumed so far that  $X$  is an open subset of  $\mathbb{C}^n$ , we may replace it with an arbitrary complex manifold. Moreover, also in the framework of smooth algebraic varieties over algebraically closed fields  $k$  of characteristic zero, almost all arguments remain valid except when considering the solution complex  $R\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X)$ , in which case we need to assume again that  $k = \mathbb{C}$  and return to the classical topology (not the Zariski topology) as a complex manifold. In this book we shall mainly treat  $D$ -modules on smooth algebraic varieties over  $\mathbb{C}$ ; however, in this introduction, we will continue to explain everything on complex manifolds. Hence  $X$  denotes a complex manifold in what follows.

There were some tentative approaches to  $D$ -modules by D. Quillen, Malgrange, and others in the 1960s; however, the real intensive investigation leading to later development was started by Kashiwara in his master thesis [Kas1] (we also note that this important contribution to  $D$ -module theory was also made independently by Bernstein [Ber1],[Ber2] around the same period). After this groundbreaking work, in collaboration with Kawai, Kashiwara developed the theory of (regular) holonomic

$D$ -modules [KK3], which is a main theme in Part I of this book. Let us discuss this subject.

It is well known that the space of the holomorphic solutions to every ordinary differential equation is finite dimensional. However, when  $X$  is higher dimensional, the dimensions of the spaces of holomorphic solutions can be infinite. This is because, in such cases, the solution contains parameters given by arbitrary functions unless the number of given equations is sufficiently large. Hence our task is to look for a suitable class of  $D_X$ -modules whose solution spaces are finite dimensional. That is, we want to find a generalization of the notion of ordinary differential equations in higher-dimensional cases.

For this purpose we consider the characteristic variety  $\text{Ch}(M)$  for a coherent  $D_X$ -module  $M$ , which is a closed analytic subset of the cotangent bundle  $T^*X$  of  $X$  (we sometimes call this the singular support of  $M$  and denote it by  $\text{SS}(M)$ ). We know by a fundamental theorem of algebraic analysis due to Sato–Kawai–Kashiwara [SKK] that  $\text{Ch}(M)$  is an involutive subvariety in  $T^*X$  with respect to the canonical symplectic structure of  $T^*X$ . In particular, we have  $\dim \text{Ch}(M) \geq \dim X$  for any coherent  $D_X$ -module  $M \neq 0$ .

Now we say that a coherent  $D_X$ -module  $M$  is holonomic (a maximally overdetermined system) if it satisfies the equality  $\dim \text{Ch}(M) = \dim X$ . Let us give the definition of characteristic varieties only in the simple case of  $D_X$ -modules

$$M = D_X/I, \quad I = D_X P_1 + D_X P_2 + \cdots + D_X P_k$$

associated to the systems

$$P_1 u = P_2 u = \cdots = P_k u = 0 \quad (P_i \in D_X) \quad (0.0.6)$$

for a single unknown function  $u$ . In this case, the characteristic variety  $\text{Ch}(M)$  of  $M$  is the common zero set of the principal symbols  $\sigma(Q)$  ( $Q \in I$ ) (recall that for  $Q \in D_X$  its principal symbol  $\sigma(Q)$  is a holomorphic function on  $T^*X$ ). In many cases  $\text{Ch}(M)$  coincides with the common zero set of  $\sigma(P_1), \sigma(P_2), \dots, \sigma(P_k)$ , but it sometimes happens to be smaller (we also see from this observation that the abstract  $D_X$ -module  $M$  itself is more essential than its concrete expression (0.0.6)).

To make the solution space as small (finite dimensional) as possible we should consider as many equations as possible. That is, we should take the ideal  $I \subset D_X$  as large as possible. This corresponds to making the ideal generated by the principal symbols  $\sigma(P)$  ( $P \in I$ ) (in the ring of functions on  $T^*X$ ) as large as possible, for which we have to take the characteristic variety  $\text{Ch}(M)$ , i.e., the zero set of the  $\sigma(P)$ 's, as small as possible. On the other hand, a non-zero coherent  $D_X$ -module is holonomic if the dimension of its characteristic variety takes the smallest possible value  $\dim X$ . This philosophical observation suggests a possible connection between the holonomicity and the finite dimensionality of the solution spaces. Indeed such connections were established by Kashiwara as we explain below.

Let us point out here that the introduction of the notion of characteristic varieties is motivated by the ideas of microlocal analysis. In microlocal analysis, the sheaf  $\mathcal{E}_X$  of microdifferential operators is employed instead of the sheaf  $D_X$  of differential

operators. This is a sheaf of rings on the cotangent bundle  $T^*X$  containing  $\pi^{-1}D_X$  ( $\pi : T^*X \rightarrow X$ ) as a subring. Originally, the characteristic variety  $\text{Ch}(M)$  of a coherent  $D_X$ -module  $M$  was defined to be the support  $\text{supp}(\mathcal{E}_X \otimes_{\pi^{-1}D_X} \pi^{-1}M)$  of the corresponding coherent  $\mathcal{E}_X$ -module  $\mathcal{E}_X \otimes_{\pi^{-1}D_X} \pi^{-1}M$ . A guiding principle of Sato–Kawai–Kashiwara [SKK] was to develop the theory in the category of  $\mathcal{E}_X$ -modules even if one wants results for  $D_X$ -modules. In this process, they almost completely classified coherent  $\mathcal{E}_X$ -modules and proved the involutivity of  $\text{Ch}(M)$ .

Let us return to holonomic  $D$ -modules. In his Ph.D. thesis [Kas3], Kashiwara proved for any holonomic  $D_X$ -module  $M$  that all of its higher solution sheaves  $\mathcal{E}xt_{D_X}^i(M, \mathcal{O}_X)$  are constructible sheaves (i.e., all its stalks are finite-dimensional vector spaces and for a stratification  $X = \bigsqcup X_i$  of  $X$  its restriction to each  $X_i$  is a locally constant sheaf on  $X_i$ ). From this result we can conclude that the notion of holonomic  $D_X$ -module is a natural generalization of that of linear ordinary differential equations to the case of higher-dimensional complex manifolds. We note that it is also proved in [Kas3] that the solution complex  $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$  satisfies the conditions of perversity (in language introduced later). The theory of perverse sheaves [BBD] must have been motivated (at least partially) by this result.

In the theory of linear ordinary differential equations, we have a good class of equations called equations with regular singularities, that is, equations admitting only mild singularities. We also have a successful generalization of this class to higher dimensions, that is, to regular holonomic  $D_X$ -modules. There are roughly two methods to define this class; the first (traditional) one will be to use higher-dimensional analogues of the properties characterizing ordinary differential equations with regular singularities, and the second (rather tactical) will be to define a holonomic  $D_X$ -module to be regular if its restriction to any algebraic curve is an ordinary differential equation with regular singularities. The two methods are known to be equivalent. We adopt here the latter as the definition. Moreover, we note that there is a conceptual difference between the complex analytic case and the algebraic case for the global meaning of regularity.

Next, let us explain the Riemann–Hilbert correspondence. By the monodromy of a linear differential equation we get a representation of the fundamental group of the base space. The original 21st problem of Hilbert asks for its converse: that is, for any representation of the fundamental group, is there an ordinary differential equation (with regular singularities) whose monodromy representation coincides with the given one? (there exist several points of view in formulating this problem more precisely, but we do not discuss them here. For example, see [AB], and others).

Let us consider the generalization in higher dimensions of this problem. A satisfactory answer in the case of integrable connections with regular singularities was given by P. Deligne [De1]. In this book, we deal with the problem for regular holonomic  $D_X$ -modules. As we have already seen, for any holonomic  $D_X$ -module  $M$ , its solutions  $\mathcal{E}xt_{D_X}^i(M, \mathcal{O}_X)$  are constructible sheaves. Hence, if we denote by  $D_c^b(\mathbb{C}_X)$  the derived category consisting of bounded complexes of  $\mathbb{C}_X$ -modules whose cohomology sheaves are constructible, the holomorphic solution complex  $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$  is an object of  $D_c^b(\mathbb{C}_X)$ . Therefore, denoting by  $D_{rh}^b(D_X)$  the

derived category consisting of bounded complexes of  $D_X$ -modules whose cohomology sheaves are regular holonomic  $D_X$ -modules, we can define the contravariant functor

$$R\mathcal{H}om_{D_X}(\bullet, \mathcal{O}_X) : D_{rh}^b(D_X) \longrightarrow D_c^b(\mathbb{C}_X). \quad (0.0.7)$$

One of the most important results in the theory of  $D$ -modules is the (contravariant) equivalence of categories  $D_{rh}^b(D_X) \simeq D_c^b(\mathbb{C}_X)$  via this functor. The crucial point of this equivalence (the Riemann–Hilbert correspondence, which we noted is the most sophisticated solution to Hilbert’s 21st problem) lies in the concept of regularity and this problem was properly settled by Kashiwara–Kawai [KK3]. The correct formulation of the above equivalence of categories was already conjectured by Kashiwara in the middle 1970s and the proof was completed around 1980 (see [Kas6]). The full proof was published in [Kas10]. For this purpose, Kashiwara constructed the inverse functor of the correspondence (0.0.7). We should note that another proof of this correspondence was also obtained by Mebkhout [Me4]. For the more detailed historical comments, compare the foreword by Schapira in the English translation [Kas16] of Kashiwara’s master thesis [Kas1]. As mentioned earlier we will mainly deal with algebraic  $D$ -modules in this book, and hence what we really consider is a version of the Riemann–Hilbert correspondence for algebraic  $D$ -modules. After the appearance of the theory of regular holonomic  $D$ -modules and the Riemann–Hilbert correspondence for analytic  $D$ -modules, A. Beilinson and J. Bernstein developed the corresponding theory for algebraic  $D$ -modules based on much simpler arguments. Some part of this book relies on their results.

The content of Part I is as follows. In Chapters 1–3 we develop the basic theory of algebraic  $D$ -modules. In Chapter 4 we give a survey of the theory of analytic  $D$ -modules and present some properties of the solution and the de Rham functors. Chapter 5 is concerned with results on regular meromorphic connections due to Deligne [De1]. As for the content of Chapter 5, we follow the notes of Malgrange in [Bor3], which will be a basis of the general theory of regular holonomic  $D$ -modules described in Chapters 6 and 7. In Chapter 6 we define the notion of regular holonomic algebraic  $D$ -modules and show its stability under various functors. In Chapter 7 we present a proof of an algebraic version the Riemann–Hilbert correspondence. The results in Chapters 6 and 7 are totally due to the unpublished notes of Bernstein [Ber3] explaining his work with Beilinson. In Chapter 8 we give a relatively self-contained account of the theory of intersection cohomology groups and perverse sheaves (M. Goresky–R. MacPherson [GM1], Beilinson–Bernstein–Deligne [BBD]) assuming basic facts about constructible sheaves. This part is independent of other parts of the book. We also include a brief survey of the theory of Hodge modules due to Mirohiko Saito [Sa1], [Sa2] without proofs.

We finally note that the readers of this book who are only interested in algebraic  $D$ -module theory (and not in the analytic one) can skip Sections 4.4 and 4.6, and need not become involved with symplectic geometry.

In the rest of the introduction we shall give a brief account of the content of Part II which deals with applications of  $D$ -module theory to representation theory.



The history of Lie groups and Lie algebras dates back to the 19th century, the period of S. Lie and F. Klein. Fundamental results about semisimple Lie groups such as those concerning structure theorems, classification, and finite-dimensional representation theory were obtained by W. Killing, E. Cartan, H. Weyl, and others until the 1930s. Afterwards, the theory of infinite-dimensional (unitary) representations was initiated during the period of World War II by E. P. Wigner, V. Bargmann, I. M. Gelfand, M. A. Naimark, and others, and partly motivated by problems in physics. Since then and until today the subject has been intensively investigated from various points of view. Besides functional analysis, which was the main method at the first stage, various theories from differential equations, differential geometry, algebraic geometry, algebraic analysis, etc. were applied to the theory of infinite-dimensional representations. The theory of automorphic forms also exerted a significant influence. Nowadays infinite-dimensional representation theory is a place where many branches of mathematics come together. As contributors representing the development until the 1970s, we mention the names of Harish-Chandra, B. Kostant, R. P. Langlands.

On the other hand, the theory of algebraic groups was started by the fundamental works of C. Chevalley, A. Borel, and others [Ch] and became recognized widely by the textbook of Borel [Bor1]. Algebraic groups are obtained by replacing the underlying complex or real manifolds of Lie groups with algebraic varieties. Over the fields of complex or real numbers algebraic groups form only a special class of Lie groups; however, various new classes of groups are produced by taking other fields as the base field. In this book we will only be concerned with semisimple groups over the field of complex numbers, for which Lie groups and algebraic groups provide the same class of groups. We regard them as algebraic groups since we basically employ the language of algebraic geometry.

The application of algebraic analysis to representation theory was started by the resolution of the Helgason conjecture [six] due to Kashiwara, A. Kowata, K. Mineura, K. Okamoto, T. Oshima, and M. Tanaka. In this book, we focus however on the resolution of the Kazhdan–Lusztig conjecture which was the first achievement in representation theory obtained by applying  $D$ -module theory.

Let us explain the problem. It is well known that all finite-dimensional irreducible representations of complex semisimple Lie algebras are highest weight modules with dominant integral highest weights. For such representations the characters are described by Weyl’s character formula. Inspired by the works of Harish-Chandra on infinite-dimensional representations of semisimple Lie groups, D. N. Verma proposed in the late 1960s the problem of determining the characters of (infinite-dimensional) irreducible highest weight modules with not necessarily dominant integral highest weights. Important contributions to this problem by a purely algebraic approach were made in the 1970s by Bernstein, I. M. Gelfand, S. I. Gelfand, and J. C. Jantzen, although the original problem was not solved.

A breakthrough using totally new methods was made around 1980. D. Kazhdan and G. Lusztig introduced a family of special polynomials (the Kazhdan–Lusztig polynomials) using Hecke algebras and proposed a conjecture giving the explicit form



of the characters of irreducible highest weight modules in terms of these polynomials [KL1]. They also gave a geometric meaning for Kazhdan–Lusztig polynomials using the intersection cohomology groups of Schubert varieties. Promptly responding to this, Beilinson–Bernstein [BB] and J.-L. Brylinski–Kashiwara independently solved the conjecture by establishing a correspondence between highest weight modules and the intersection cohomology complexes of Schubert varieties via  $D$ -modules on the flag manifold. This successful achievement, i.e., employing theories and methods, from other fields, was quite astonishing for the specialists who had been studying the problem using purely algebraic means. Since then  $D$ -module theory has brought numerous new developments in representation theory.

Let us explain more precisely the methods used to solve the Kazhdan–Lusztig conjecture. Let  $G$  be an algebraic group (or a Lie group),  $\mathfrak{g}$  its Lie algebra and  $U(\mathfrak{g})$  the universal enveloping algebra of  $\mathfrak{g}$ . If  $X$  is a smooth  $G$ -variety and  $\mathcal{V}$  is a  $G$ -equivariant vector bundle on  $X$ , the set  $\Gamma(X, \mathcal{V})$  of global sections of  $\mathcal{V}$  naturally has a  $\mathfrak{g}$ -module structure. The construction of the representation of  $\mathfrak{g}$  (or of  $G$ ) in this manner is a fundamental technique in representation theory.

Let us now try to generalize this construction. Denote by  $D_X^\mathcal{V} \subset \text{End}_{\mathbb{C}}(\mathcal{V})$  the sheaf of rings of differential operators acting on the sections of  $\mathcal{V}$ . Then  $D_X^\mathcal{V}$  is isomorphic to  $\mathcal{V} \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{V}^*$  which coincides with the usual  $D_X$  when  $\mathcal{V} = \mathcal{O}_X$ . In terms of  $D_X^\mathcal{V}$  the  $\mathfrak{g}$ -module structure on  $\Gamma(X, \mathcal{V})$  can be described as follows. Note that we have a canonical ring homomorphism  $U(\mathfrak{g}) \rightarrow \Gamma(X, D_X^\mathcal{V})$  induced by the  $G$ -action on  $\mathcal{V}$ . Since  $\mathcal{V}$  is a  $D_X^\mathcal{V}$ -module,  $\Gamma(X, \mathcal{V})$  is a  $\Gamma(X, D_X^\mathcal{V})$ -module, and hence a  $\mathfrak{g}$ -module through the ring homomorphism  $U(\mathfrak{g}) \rightarrow \Gamma(X, D_X^\mathcal{V})$ . From this observation, we see that we can replace  $\mathcal{V}$  with other  $D_X^\mathcal{V}$ -modules. That is, for any  $D_X^\mathcal{V}$ -module  $M$  the  $\mathbb{C}$ -vector space  $\Gamma(X, M)$  is endowed with a  $\mathfrak{g}$ -module structure.

Let us give an example. Let  $G = SL_2(\mathbb{C})$ . Since  $G$  acts on  $X = \mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$  by the linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x) = \left( \frac{ax + b}{cx + d} \right) \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, (x) \in X \right),$$

it follows from the above arguments that  $\Gamma(X, M)$  is a  $\mathfrak{g}$ -module for any  $D_X$ -module  $M$ . Let us consider the  $D_X$ -module  $M = D_X \delta$  given by Dirac's delta function  $\delta$  at the point  $x = \infty$ . In the coordinate  $z = \frac{1}{x}$  in a neighborhood of  $x = \infty$ , the equation satisfied by Dirac's delta function  $\delta$  is

$$z\delta = 0,$$

so we get

$$M = D_X / D_X z$$

in a neighborhood of  $x = \infty$ . Set  $\delta_n = \left(\frac{d}{dz}\right)^n \delta$ . Then  $\{\delta_n\}_{n=0}^\infty$  is the basis of  $\Gamma(X, M)$  and we have  $\frac{d}{dz} \delta_n = \delta_{n+1}$ ,  $z\delta_n = -n\delta_{n-1}$ .

Let us describe the action of  $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$  on  $\Gamma(X, M)$ . For this purpose consider the following elements in  $\mathfrak{g}$ :

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(these elements  $h, e, f$  form a basis of  $\mathfrak{g}$ ). Then the ring homomorphism  $U(\mathfrak{g}) \rightarrow \Gamma(X, D_X)$  is given by

$$h \mapsto 2z \frac{d}{dz}, \quad e \mapsto z^2 \frac{d}{dz}, \quad f \mapsto -\frac{d}{dz}.$$

For example, since

$$\exp(-te) \cdot \left( \frac{1}{z} \right) = \left( \frac{1}{z/(1-tz)} \right),$$

for  $\varphi(z) \in \mathcal{O}_X$  we get

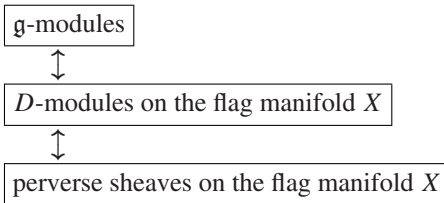
$$(e \cdot \varphi)(z) = \frac{d}{dt} \varphi \left( \frac{z}{1-tz} \right) \Big|_{t=0} = \left( z^2 \frac{d}{dz} \varphi \right) (z)$$

and  $e \mapsto z^2 \frac{d}{dz}$ . Therefore we obtain

$$h \cdot \delta_n = -2(n+1)\delta_n, \quad e \cdot \delta_n = n(n+1)\delta_{n-1}, \quad f \cdot \delta_n = -\delta_{n+1},$$

from which we see that  $\Gamma(X, M)$  is the infinite-dimensional irreducible highest weight module with highest weight  $-2$ .

For the proof of the Kazhdan–Lusztig conjecture, we need to consider the case when  $G$  is a semisimple algebraic group over the field of complex numbers and the  $G$ -variety  $X$  is the flag variety of  $G$ . For each Schubert variety  $Y$  in  $X$  we consider a  $D_X$ -module  $M$  satisfied by the delta function supported on  $Y$ . In our previous example, i.e., in the case of  $G = SL_2(\mathbb{C})$ , the flag variety is  $X = \mathbb{P}^1$  and  $Y = \{\infty\}$  is a Schubert variety. Since Schubert varieties  $Y \subset X$  may have singularities for general algebraic groups  $G$ , we take the regular holonomic  $D_X$ -module  $M$  characterized by the condition of having no subquotient whose support is contained in the boundary of  $Y$ . For this choice of  $M$ ,  $\Gamma(X, M)$  is an irreducible highest weight  $\mathfrak{g}$ -module and  $R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$  is the intersection cohomology complex of  $Y$ . A link between highest weight  $\mathfrak{g}$ -modules and the intersection cohomology complexes of Schubert varieties  $Y \subset X$  (perverse sheaves on the flag manifold  $X$ ) is given in this manner. Diagrammatically the strategy of the proof of the Kazhdan–Lusztig conjecture can be explained as follows:



Here the first arrow is what we have briefly explained above, and the second one is the Riemann–Hilbert correspondence, a general theory of  $D$ -modules. The first arrow is called the Beilinson–Bernstein correspondence, which asserts that the category of  $U(\mathfrak{g})$ -modules with the trivial central character and that of  $D_X$ -modules are equivalent. By this correspondence, for a  $D_X$ -module  $M$  on the flag manifold  $X$ , we associate to it the  $U(\mathfrak{g})$ -module  $\Gamma(X, M)$ . As a result, we can translate various problems for  $\mathfrak{g}$ -modules into those for regular holonomic  $D$ -modules (or through the Riemann–Hilbert correspondence, those for constructible sheaves).

The content of Part II is as follows. We review some preliminary results on algebraic groups in Chapters 9 and 10. In Chapters 11 and 12 we will explain how the Kazhdan–Lusztig conjecture was solved. Finally, in Chapter 13, a realization of Hecke algebras will be given by the theory of Hodge modules, and the relation between the intersection cohomology groups of Schubert varieties and Hecke algebras will be explained.

Let us briefly mention some developments of the theory, which could not be treated in this book. We can also formulate conjectures, similar to the Kazhdan–Lusztig conjecture, for Kac–Moody Lie algebras, i.e., natural generalizations of semisimple Lie algebras. In this case, we have to study two cases separately: (a) the case when the highest weight is conjugate to a dominant weight by the Weyl group, (b) the case when the highest weight is conjugate to an anti-dominant weight by the Weyl group. Moreover, Lusztig proposed certain Kazhdan–Lusztig type conjectures also for the following objects: (c) the representations of reductive algebraic groups in positive characteristics, (d) the representations of quantum groups in the case when the parameter  $q$  is a root of unity. The conjecture of the case (a) was solved by Kashiwara (and Tanisaki) [Kas15], [KT2] and L. Casian [Ca1]. Following the so-called Lusztig program, the other conjectures were also solved:

- (A) the equivalence of (c) and (d): H. H. Andersen, J. C. Jantzen, W. Soergel [AJS].
- (B) the equivalence of (b) and (d) for affine Lie algebras: Kazhdan–Lusztig [KL3].
- (C) the proof of (b) for affine Lie algebras: Kashiwara–Tanisaki [KT3] and Casian [Ca2].



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