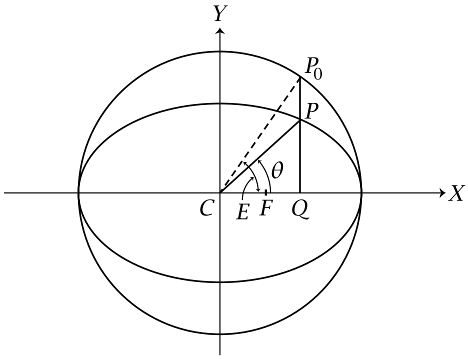
2

The Kepler Problem

For the Newtonian $1/r^2$ force law, a miracle occurs — all of the solutions are periodic instead of just quasi-periodic. To put it another way, the two-dimensional tori are further decomposed into invariant circles. This highly degenerate situation seems unbelievable from the point of view of general theory, yet it is the most interesting feature of the problem.

— *Richard Moeckel,* Bull. AMS., **41**:1 (2003), pp. 121–2. Review of *Classical and Celestial Mechanics, the Recife Lectures,* Cabral and Diacu (eds.), Princeton University Press, 2002.

A necessary preliminary to a full understanding of the Kepler problem is a full familiarity with the geometric and analytic features of the conics — particularly those of the ellipse.



1 Features of the Ellipse: Geometry and Analysis

Placing the origin at the center C, with X- and Y-coordinate axes coinciding respectively with the major and minor axes of the ellipse, then in terms of these Cartesian coordinates, the equation of the ellipse reads

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \tag{1.1}$$

where a and b measure the semimajor and semiminor axes, respectively. The equation can be characterized parametrically in the form

D. Ó Mathúna, *Integrable Systems in Celestial Mechanics*, doi: 10.1007/978-0-8176-4595-3_2,
© Birkhäuser Boston, a part of Springer Science + Business Media, LLC 2008

$$X = a \cos E, \quad Y = b \sin E. \tag{1.2}$$

The line PQ normal to the major axis through an arbitrary point $P(X, Y)$ meets the circumscribed circle at $P_0(X, Y_0)$. With E denoting the angle subtended at the center C between CP_0 and CQ, the interpretation of (1.2) is clear, and furthermore we see that

$$CQ = a \cos E, \quad P_0 Q = a \sin E, \quad PQ = b \sin E. \tag{1.3}$$

For the radius vector $CP = R$ from the center to the arbitrary point $P(X, Y)$ of the ellipse, we have

$$R^2 = X^2 + Y^2 = a^2 - (a^2 - b^2) \sin^2 E. \tag{1.4}$$

The eccentricity e of the ellipse may be defined by

$$b^2 = a^2(1 - e^2) \tag{1.5}$$

so that for (1.4), we may write

$$R^2 = a^2[1 - e^2 \sin^2 E] \tag{1.6a}$$

or

$$R = a[1 - e^2 \sin^2 E]^{1/2} \tag{1.6b}$$

as the equation for the ellipse in terms of the "eccentric angle" E.

For the corresponding equation in terms of center-based polar coordinates (R, θ), we note

$$X = R \cos \theta, \quad Y = R \sin \theta \tag{1.7}$$

and equation (1.1) becomes

$$R^2 \left[\frac{b^2}{a^2} \cos^2 \theta + \sin^2 \theta \right] = b^2 \tag{1.8}$$

which, on the introduction of (1.5) yields

$$R^2[1 - e^2 \cos^2 \theta] = a^2(1 - e^2) \tag{1.9a}$$

$$R = \frac{a\sqrt{1 - e^2}}{[1 - e^2 \cos^2 \theta]^{1/2}} \tag{1.9b}$$

as the required equation.

The point $F(ae, 0)$ is a focus of the ellipse. Moving the origin to the focus through the translation

$$x = X - ae, \quad y = Y \tag{1.10}$$

the Cartesian equation (1.1) becomes

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1 . \tag{1.11}$$

Substituting from (1.5) into (1.11) and rearranging yields

$$x^2 + y^2 = [a(1 - e^2) - ex]^2 . \tag{1.12}$$

We now introduce polar coordinates (r, f) centered at the focus so that

$$x = r \cos f, \quad y = r \sin f \tag{1.13}$$

and relation (1.12) may be written

$$r = a(1 - e^2) - er \cos f = e\left[\frac{a(1 - e^2)}{e} - r \cos f\right] . \tag{1.14}$$

If we consider the line $x = a(1 - e^2)/e$ (parallel to the y-axis), which we call the directrix, then the factor in square brackets on the right of (1.14) measures the distance from an arbitrary point on the ellipse to the directrix. Hence equation (1.14) merely states that for an arbitrary point on the curve, the ratio of the distance from the focus to the distance from the directrix is given by the eccentricity e. This, in fact, can be taken as the general definition of a conic, which for $e < 1$ is an ellipse, whereas for $e > 1$ it is a hyperbola. Returning to (1.14), we note that it can be put in the neater — and possibly more recognizable — form

$$r = \frac{a(1 - e^2)}{1 + e \cos f} \tag{1.15}$$

which, with $e < 1$, we take as the standard equation for the ellipse.

For the corresponding relation in terms of the "eccentric angle" E,

$$\begin{aligned} r^2 = x^2 + y^2 &= (X - ae)^2 + Y^2 \\ &= X^2 + Y^2 - 2aeX + a^2e^2 \\ &= R^2 - 2a^2e \cos E + a^2e^2 . \end{aligned} \tag{1.16}$$

Introducing R from (1.6) into (1.16) yields

$$r^2 = a^2[1 - e^2 \sin^2 E - 2e \cos E + e^2] = a^2[1 - e \cos E]^2 \tag{1.17}$$

so that

$$r = a[1 - e \cos E] \tag{1.18}$$

as the sought-for relation.

For the ellipse, therefore, we note the following:

1. Equation (1.6) relates the center-based radius vector R at the point P to the angle-parameter E, being the angle subtended at the center between the major axis and the radius to the point where the normal to the major axis through P meets the circumscribed circle.

2. Equation (1.9) gives the equation of the ellipse in terms of the center-based polar coordinates (R, θ).
3. Equation (1.15) gives the equation of the ellipse in terms of the focus-based polar coordinates (r, f).
4. Equation (1.18) relates the radial coordinate r of the focus-based system (r, f) of item 3 above to the angle parameter E referred to in item 1 above. The attractive simplicity of (1.18) must be balanced against its mixed nature, involving coordinate systems of different origins.

A straightforward exercise yields the relation between the angles E and f. Since

$$x = r\cos f = r\left[2\cos^2\frac{f}{2} - 1\right] = r\left[1 - 2\sin^2\frac{f}{2}\right] \tag{1.19}$$

we have

$$\begin{aligned}(1)\qquad 2r\cos^2\frac{f}{2} &= r + x = r + X - ae = a(1 - e\cos E) + a\cos E - ae\\ &= a(1-e)[1+\cos E] = 2a(1-e)\cos^2\frac{E}{2}\end{aligned} \tag{1.20}$$

and hence

$$r\cos^2\frac{f}{2} = a(1-e)\cos^2\frac{E}{2}. \tag{1.21}$$

$$\begin{aligned}(2)\qquad 2r\sin^2\frac{f}{2} &= r - x = r - X + ae = a(1 - e\cos E) - a\cos E + ae\\ &= a(1+e)[1-\cos E] = 2a(1+e)\sin^2\frac{E}{2}\end{aligned}$$

and hence

$$r\sin^2\frac{f}{2} = a(1+e)\sin^2\frac{E}{2}. \tag{1.22}$$

Dividing (1.22) by (1.21) yields

$$\tan^2\frac{f}{2} = \frac{1+e}{1-e}\tan^2\frac{E}{2}, \qquad \tan^2\frac{E}{2} = \frac{1-e}{1+e}\tan^2\frac{f}{2}. \tag{1.23a,b}$$

This latter relation can now be used to derive the equation for R in terms of f, but its algebraic complexity limits its utility.

Returning to the standard equation (1.15), we see that (with prime denoting differentiation with respect to f)

$$r' = \frac{\mathrm{d}r}{\mathrm{d}f} = \frac{ae(1-e^2)\sin f}{(1+e\cos f)^2}. \tag{1.24}$$

Hence $r' = 0$ for $f = 0, \pm\pi, \ldots, \pm n\pi$. It can be easily checked that $f = 0$ is a minimum point for r (as also are $f = \pm 2n\pi$) while $f = \pi$ (as well as

$f = \pm(2n+1)\pi$) is the maximum point for r. The point $f = 0$, at which $r = a(1-e)$, we shall call the *pericenter*; the point $f = \pi$, at which $r = a(1+e)$, we shall call the *apocenter*.

At the extremity of the semiminor axis, we have

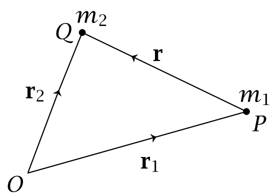
$$x = -ae, \quad y = b = a\sqrt{1-e^2} \tag{1.25}$$

from which it follows that, at that extremity,

$$r = a, \quad \cos f = -e \tag{1.26}$$

and hence we have that $[a, \arccos e]$ are the focus-based polar coordinates of the extremity of the positive semiminor axis.

2 The Two-Body Problem



We consider the motion of two bodies moving under the influence of their mutual attraction. Denoting the masses of the two bodies by m_1 and m_2, with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, referred to the origin at 0, we write

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \,. \tag{2.1}$$

In accordance with the inverse square law governing the gravitational attraction of m_1 and m_2, the equations of motion for m_1 and m_2 are given respectively by

$$m_1\ddot{\mathbf{r}}_1 = \frac{Gm_1m_2}{r^2}\mathbf{e}_r = \frac{Gm_1m_2}{r^3}\mathbf{r}, \qquad \text{and hence} \quad \ddot{\mathbf{r}}_1 = \frac{Gm_2}{r^3}\mathbf{r} \tag{2.2a}$$

$$m_2\ddot{\mathbf{r}}_2 = -\frac{Gm_1m_2}{r^2}\mathbf{e}_r = -\frac{Gm_1m_2}{r^3}\mathbf{r}, \qquad \text{and hence} \quad \ddot{\mathbf{r}}_2 = -\frac{Gm_1}{r^3}\mathbf{r} \tag{2.2b}$$

where we have used the “dot” to denote differentiation with respect to time t, and where the unit vector $\mathbf{e}_r$ is defined by $\mathbf{r} = |\mathbf{r}|\,\mathbf{e}_r = r\mathbf{e}_r$. Subtracting (2.2a) from (2.2b), we have

$$\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = \ddot{\mathbf{r}} = -\frac{G(m_1+m_2)}{r^3}\mathbf{r} \tag{2.3}$$

and as the equation is unaltered by the replacement of $\mathbf{r}$ by $-\mathbf{r}$, or by the interchange of m_1 and m_2, equation (2.3) describes the motion of either body relative to the other. Moreover, equation (2.3) shows that the problem has been reduced to that of the motion of a particle of unit mass in the gravitational field of a body of mass m, situated at the origin, where

$$m = m_1 + m_2 \tag{2.4}$$

and if we set

$$\mu = G(m_1 + m_2) = Gm \tag{2.5}$$

then equation (2.3) reads

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} = -\frac{\mu}{r^2}\mathbf{e}_r \tag{2.6}$$

which is the standard form.

In the case of planetary motion, one may think of m_1 as the Sun and m_2 as the planet. In that case, we may write

$$m = m_1 + m_2 = m_1\left(1 + \frac{m_2}{m_1}\right) \tag{2.7}$$

and (2.6) describes the motion of the planet in the heliocentric coordinate system. We may also note that the dominance of the mass of the Sun would permit the approximation

$$m \approx m_1, \quad \mu \approx Gm_1 \tag{2.8}$$

when such an approximation is appropriate.

At this point, we introduce the gravitational potential. At an arbitrary point P in the gravitational fields of a mass m at Q, the function U defined by

$$U = \frac{Gm}{|PQ|} = \frac{Gm}{r} = \frac{\mu}{r} \tag{2.9}$$

is the potential per unit mass: it has the feature that the force defined by the gradient of this function U is in fact the Newtonian gravitational force acting on a particle of unit mass, namely

$$F = \nabla U = -\frac{Gm}{r^2}\mathbf{e}_r = -\frac{Gm}{r^3}\mathbf{r} = -\frac{\mu}{r^3}\mathbf{r} \tag{2.10}$$

so that, for the equation of motion of a particle P of unit mass, we have

$$\ddot{\mathbf{r}} = -\frac{Gm}{r^3}\mathbf{r} = -\frac{\mu}{r^3}\mathbf{r} \tag{2.11}$$

identical with (2.6).

In case of several masses m_i, $i = 1, \ldots, n$, situated respectively at Q_i, $i = 1, \ldots, n$, the potential function per unit mass at P is given by

$$U = \sum_{i=1}^{n} \frac{Gm_i}{|PQ_i|} \tag{2.12}$$

to which we shall have occasion to refer later.

In the next section when we encounter the conservation of energy, we shall see that the potential energy V per unit mass for a particle in the gravitational field of a mass m is given by

$$V = -\frac{Gm}{r} = -\frac{\mu}{r} = -U \tag{2.13}$$

so that the potential function is the negative of the potential energy.

The problem defined by the differential equations (2.6) with μ given by (2.5) is known as the *Kepler problem*.

3 The Kepler Problem: Vectorial Treatment

In the class of problems in Celestial Mechanics, the Kepler problem is distinguished by several features: it has every possible "degeneracy" — the "frequencies" associated with all three coordinates coincide so that all bound orbits are periodic (except for collision orbits); but more relevant at this point is the fact that the motion is always planar. This means that it admits a vectorial treatment to which other problems are not amenable.

In terms of a (heliocentric) spherical coordinate system (r, θ, φ) with unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\varphi$, it follows from

$$\mathbf{r} = r\mathbf{e}_r \tag{3.1}$$

that the velocity vector $\mathbf{v}$ is given by

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{r}}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\sin\theta \cdot \dot{\varphi}\mathbf{e}_\varphi \tag{3.2}$$

where again the dot denotes differentiation with respect to time; there follows

$$\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r} \tag{3.3a}$$

$$v^2 = \mathbf{v} \cdot \mathbf{v} = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\,. \tag{3.3b}$$

We note that the fundamental equation (2.6) admits an immediate first integral — which we shall recognize as the energy integral. Taking the scalar product of (2.6) with the velocity vector $\dot{\mathbf{r}}$, we find

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \cdot \dot{\mathbf{r}} = -\frac{\mu}{2r^3}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \cdot \mathbf{r}) = -\frac{\mu}{2r^3}\frac{\mathrm{d}}{\mathrm{d}t}(r^2) = -\frac{\mu}{r^2}\dot{r} \tag{3.4}$$

and so

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{r}\right) \tag{3.5}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}v^2 - \frac{\mu}{r}\right] = 0\,. \tag{3.6}$$

Letting $\mathcal{E}$ denote the constant of integration, we therefore have the energy integral in the form

$$\tfrac{1}{2}v^2 - \frac{\mu}{r} = \mathcal{E}\,. \tag{3.7}$$

For a particle of unit mass, the first term is clearly the kinetic energy and the second term is the potential energy; accordingly, if we use T to denote the kinetic and V the potential energy, then

$$T = \tfrac{1}{2}v^2, \qquad V = -\frac{\mu}{r}, \qquad T + V = \mathcal{E} \tag{3.8a,b,c}$$

and the definition of V is consistent with (2.13).

Rewriting (3.7) in the form

$$\tfrac{1}{2}v^2 = \mathcal{E} + \frac{\mu}{r} \tag{3.9}$$

and noting that the left-hand side is always positive, then if $\mathcal{E}$ is negative, relation (3.9) sets the lower limit on μ/r: if we exhibit the case of negative energy by writing

$$\mathcal{E} = -\alpha^2 \tag{3.10}$$

and define a length scale a by setting

$$a = \frac{\mu}{2\alpha^2} \tag{3.11}$$

then we have that

$$\frac{\mu}{r} - \alpha^2 \geq 0 \quad \text{implying} \quad \frac{\mu}{r} \geq \alpha^2 \tag{3.12}$$

and hence

$$r \leq \frac{\mu}{\alpha^2} = 2a \tag{3.13}$$

giving the corresponding upper limit on r: negative energy implies bound orbits, and these shall be the main focus of our attention.

Returning to relations (3.1) and (3.2) we form the angular momentum vector $\mathbf{C}$ by taking the cross product of $\mathbf{r}$ and $\mathbf{v}$, to find

$$\mathbf{C} = \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = -r^2 \sin\theta\,\dot{\varphi}\,\mathbf{e}_\theta + r^2\dot{\theta}\,\mathbf{e}_\varphi \tag{3.14}$$

and we further note that

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0 - \mathbf{r} \times \frac{\mu}{r^3}\mathbf{r} = 0\,. \tag{3.15}$$

Hence in the central gravitational field, the angular momentum vector $\mathbf{C}$ is constant. At this point, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{e}_r) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{r\dot{\mathbf{r}} - \dot{r}\mathbf{r}}{r^2} = \frac{r^2\dot{\mathbf{r}} - r\dot{r}\mathbf{r}}{r^3}$$
$$= \frac{(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}}{r^3} = \frac{\mathbf{r}\times(\dot{\mathbf{r}}\times\mathbf{r})}{r^3} = -\frac{\mathbf{r}\times\mathbf{C}}{r^3} = \frac{\mathbf{C}\times\mathbf{r}}{r^3}. \quad (3.16)$$

When $\mathbf{C} = 0$, the above relation implies that, in that case, the unit vector $\mathbf{e}_r$ is constant — hence the motion is rectilinear along the radius vector toward the origin, leading to collision. When $\mathbf{C} \neq 0$, it follows from (3.14) that

$$\mathbf{r}\cdot\mathbf{C} = \mathbf{r}\cdot(\mathbf{r}\times\dot{\mathbf{r}}) = 0 \quad (3.17)$$

so that $\mathbf{r}$ remains normal to the fixed vector $\mathbf{C}$; hence the motion takes place in the plane defined by the fixed (constant) vector $\mathbf{C}$.

It further follows from (3.14) that

$$C^2 = \mathbf{C}\cdot\mathbf{C} = (\mathbf{r}\times\dot{\mathbf{r}})\cdot(\mathbf{r}\times\dot{\mathbf{r}}) = r^4\sin^2\theta\dot{\varphi}^2 + r^4\dot{\theta}^2$$
$$= r^2[r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2] = r^2[v^2 - \dot{r}^2] \quad (3.18)$$

and we have a second integral, this one involving the magnitude of the angular momentum vector $\mathbf{C}$, namely

$$r^2[v^2 - \dot{r}^2] = C^2\,. \quad (3.19)$$

Moreover, rewriting the latter as an expression for v^2, and recalling the energy integral (3.7), we have

$$\tfrac{1}{2}v^2 = \tfrac{1}{2}\left[\frac{C^2}{r^2} + \dot{r}^2\right] = \mathcal{E} + \frac{\mu}{r} \quad (3.20)$$

giving the relation between the constants C and $\mathcal{E}$.

Returning to (3.16) and again applying the gravitational equation (2.6) and also noting that $\dot{\mathbf{C}} = 0$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{e}_r) = \frac{\mathbf{C}\times\mathbf{r}}{r^3} = -\frac{\mathbf{C}\times\ddot{\mathbf{r}}}{\mu} = -\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{C}\times\dot{\mathbf{r}}) = \frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\times\mathbf{C})\,. \quad (3.21)$$

If we let $\mathbf{e}$ denote the arbitrary constant vector introduced by the integration of this latter vector differential equation, we have

$$\mu(\mathbf{e}_r + \mathbf{e}) = \mathbf{v}\times\mathbf{C} = \dot{\mathbf{r}}\times(\mathbf{r}\times\dot{\mathbf{r}}) = (\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\mathbf{r} - (\mathbf{r}\cdot\dot{\mathbf{r}})\dot{\mathbf{r}} = v^2\mathbf{r} - r\dot{r}\dot{\mathbf{r}}\,. \quad (3.22)$$

Again, we note in passing that if $\mathbf{C} = 0$, then $\mathbf{e} = -\mathbf{e}_r$, so that $\mathbf{e}$ is the unit vector along the radius vector toward the origin. For $\mathbf{C} \neq 0$, we take the scalar product with $\mathbf{C}$ across (3.22), and noting that $\mathbf{C}$ is normal to both $\mathbf{r}$ and $\dot{\mathbf{r}}$, we find

$$\mathbf{e}\cdot\mathbf{C} = 0 \quad (3.23)$$

which implies that the vector $\mathbf{e}$ lies in the plane of the motion.

Taking the scalar product with $\mathbf{r}$ across (3.22) gives

$$\mu(r + \mathbf{e} \cdot \mathbf{r}) = v^2 r^2 - r^2 \dot{r}^2 = r^2(v^2 - \dot{r}^2) = C^2 \tag{3.24}$$

wherein we have introduced (3.18); we now rewrite (3.24) in the form

$$\mathbf{e} \cdot \mathbf{e}_r = \frac{C^2}{\mu r} - 1\,. \tag{3.25}$$

If, in the plane of the motion, we let the vector **e**, whose magnitude we denote by e, define a base axis and if we let f denote the angle in this plane between this base vector and the radius vector **r**, then (r, f) constitute a polar coordinate basis in the plane of the motion, and equation (3.25) can be written in the form

$$r[1 + e\cos f] = \frac{C^2}{\mu}\,. \tag{3.26}$$

For $e = 0$, the motion is circular. For $e \neq 0$, we rewrite (3.26) [in accord with (1.14)] as

$$r = e\left[\frac{C^2}{e\mu} - r\cos f\right] \tag{3.27}$$

which [referring to equation (1.14) and the subsequent paragraph] defines a conic with a directrix at a distance $C^2/\mu e$ from the origin and with eccentricity e. And for $e < 1$, this conic is an ellipse, and the vector **e** is the vector based at the focus (origin) directed at the pericenter and with magnitude e.

The vector **e** is known as the *Runge–Lenz vector* and also the *eccentric axis vector*.

There is one more exercise to be performed on relation (3.22). We recall that since **C** is normal to **v**, there follows that

$$|\mathbf{v} \times \mathbf{C}| = vC, \quad (\mathbf{v} \times \mathbf{C})^2 = v^2 C^2\,. \tag{3.28}$$

Accordingly, if we square both sides of (3.22), then on reversing the order we find

$$\begin{aligned} v^2C^2 &= \mu^2(\mathbf{e} + \mathbf{e}_r)^2 = \mu^2[1 + e^2 + 2\mathbf{e} \cdot \mathbf{e}_r] \\ &= \mu^2\left[1 + e^2 + 2\left(\frac{C^2}{\mu r} - 1\right)\right] = \mu^2(e^2 - 1) + 2\mu\frac{C^2}{r} \end{aligned} \tag{3.29}$$

in which we have introduced (3.25) and rearranged. Hence

$$\mu^2(1 - e^2) = -2C^2\left[\tfrac{1}{2}v^2 - \frac{\mu}{r}\right] = -2C^2\mathcal{E} \tag{3.30}$$

from which it is immediately evident that

$$e \lesseqgtr 1 \quad \text{corresponds to} \quad \mathcal{E} \lesseqgtr 0 \tag{3.31}$$

i.e., negative/positive energy corresponds to elliptic/hyperbolic orbits — as anticipated earlier.

Restricting our attention to bound orbits (negative energy), we introduce (3.10) and (3.11) into (3.30), to obtain

$$1 - e^2 = \frac{2C^2}{\mu^2}\alpha^2 = \frac{C^2}{\mu}\Big/\frac{\mu}{2\alpha^2} = \frac{C^2}{\mu}\cdot\frac{1}{a} \tag{3.32}$$

and hence

$$\frac{C^2}{\mu} = a(1 - e^2) = p \tag{3.33}$$

where we introduce the symbol p to denote the semi–latus rectum — the value of r at $f = \pi/2$. In terms of these length parameters, equation (3.26) reads

$$r = \frac{p}{1 + e\cos f} = \frac{a(1 - e^2)}{1 + e\cos f} \tag{3.34}$$

as an alternate form for the equation of the orbit, and we write

$$b = a\sqrt{1 - e^2} \tag{3.35}$$

as the length parameter of the semiminor axis.

The polar coordinates (r, f) in the orbit plane together with the axis normal to the plane constitute a cylindrical polar coordinate system. With base unit vectors $\mathbf{e}_r$ and $\mathbf{e}_f$ in the orbit plane together with the axial unit vector $\mathbf{e}_k$, we may write

$$\mathbf{r} = r\mathbf{e}_r \tag{3.36a}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{f}\mathbf{e}_f \tag{3.36b}$$

and, for the angular momentum, we have

$$\mathbf{C} = \mathbf{r}\times\mathbf{v} = r^2\dot{f}\mathbf{e}_k\,. \tag{3.37}$$

It follows that, for the magnitude of the angular momentum, we have

$$r^2\dot{f} = C = \sqrt{\mu p} = \sqrt{\mu a(1 - e^2)} = \sqrt{\mu a}\sqrt{1 - e^2} \tag{3.38}$$

wherein we have introduced (3.33). If we let τ denote the time for a complete orbit and if we also introduce the mean motion n, measuring the frequency, by the relation

$$n = \frac{2\pi}{\tau} \tag{3.39}$$

and note that the area traced out in one orbit is πab, we have that the mean areal velocity over an orbit is given by

$$\frac{\pi ab}{\tau} = \pi ab\cdot\frac{n}{2\pi} = \tfrac{1}{2}nab = \tfrac{1}{2}na^2\sqrt{1 - e^2}\,. \tag{3.40}$$

However, the areal velocity is, in fact, given by one-half the angular momentum of (3.38). Identifying the quantity in (3.38) with twice the quantity in (3.40) gives, after cancellation of the common $\sqrt{1-e^2}$ factor,

$$na^2 = \sqrt{\mu a} \tag{3.41}$$

and hence the important relation

$$n^2 a^3 = \mu = Gm = G(m_1 + m_2) \tag{3.42}$$

whence we have substituted for μ from (2.5).

We are now in a position to make some observations:

1. The motion takes place in a plane defined by the angular momentum vector, and for negative energy the orbit is the ellipse (3.34); this is Kepler's First Law.
2. The constancy of the angular momentum (3.38) implies a constant mean areal velocity; this is Kepler's Second Law.
3. If the approximation (2.8) were to be introduced into (3.42), we would have $n^2a^3 = Gm$, a constant for all planets; this is Kepler's Third Law, more usually stated as the square of the orbit period is proportional to the cube of the semimajor axis.

Recalling equation (3.20) for the case of negative energy so that $\mathcal{E} = -\alpha^2$, we rearrange to obtain

$$\begin{aligned} r^2\dot{r}^2 &= -[2\alpha^2 r^2 - 2\mu r + C^2] \\ &= -2\alpha^2\left[r^2 - \frac{\mu r}{\alpha^2} + \frac{C^2}{2\alpha^2}\right]. \end{aligned} \tag{3.43}$$

The singularity at $r = 0$ in this differential equation can be regularized by means of a regularizing transformation whereby a new independent variable E is introduced through the defining relation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\sqrt{2\alpha^2}}{r} \quad \text{so that} \quad r\frac{\mathrm{d}}{\mathrm{d}t} = \sqrt{2\alpha^2}\frac{\mathrm{d}}{\mathrm{d}E} \tag{3.44}$$

and, on the introduction of (3.44) and some rearrangement, equation (3.43) becomes

$$\begin{aligned} \left(\frac{\mathrm{d}r}{\mathrm{d}E}\right)^2 &= -\left[r^2 - \frac{\mu}{\alpha^2}r + \frac{C^2}{\mu}\cdot\frac{\mu}{2\alpha^2}\right] \\ &= -[r^2 - 2ar + a^2(1-e^2)] \\ &= -[(a-r)^2 - a^2e^2] \end{aligned} \tag{3.45}$$

where we have introduced (3.11) and (3.33). By means of the substitution $a - r = aeZ$, this immediately integrates, and we find

$$r = a[1 - e\cos E] \tag{3.46}$$

satisfying the condition that $E = 0$ when $r = a(1-e)$. Recalling relation (1.18), it is evident that E can be identified with the eccentric angle introduced in (1.2).

It remains to determine the relation between the angle E and the time t. From the defining relation (3.44), we have

$$\sqrt{2\alpha^2}\frac{\mathrm{d}t}{\mathrm{d}E} = r = a[1 - e\cos E] \tag{3.47}$$

so that, on integration

$$\sqrt{2\alpha^2}(t - t_0) = a[E - e\sin E] \tag{3.48}$$

satisfying the requirement that $E = 0$ when $t = t_0$. From (3.41), we note that

$$n^2a^2 = \frac{\mu}{a} = 2\alpha^2 \tag{3.49}$$

and hence

$$M = n(t - t_0) = E - e\sin E, \tag{3.50}$$

known as *Kepler's equation*. The eccentric angle E defined by (3.44) is, in Celestial Mechanics, called the *eccentric anomaly*, and the quantity $M = n(t - t_0)$ is called the *mean anomaly*. The angle f, introduced in equation (3.26), is called the *true anomaly*. We postpone to the next section the full treatment of the true anomaly.

The vectorial treatment gives a full account of the Kepler orbit in its plane. The fuller picture of the motion in space, including the orientation of the orbit plane, is more clearly seen in the Lagrangian analysis, which is the subject of the next section.

4 The Kepler Problem: Lagrangian Analysis

In terms of spherical coordinates (r, θ, φ) (of the heliocentric system), the three Cartesian coordinates can be expressed as

$$x = r\sin\theta\cos\varphi \tag{4.1a}$$

$$y = r\sin\theta\sin\varphi \tag{4.1b}$$

$$z = r\cos\theta \tag{4.1c}$$

from which it can readily be deduced that the metric coefficients g_{ij} are given by

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta, \quad g_{ij} = 0, \quad i \neq j\,. \tag{4.2}$$

Then for the kinetic and potential energies per unit mass, we have, respectively,

$$T = \tfrac{1}{2}v^2 = \tfrac{1}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2], \quad V = -\frac{\mu}{r} \tag{4.3}$$

and the Hamiltonian, reflecting the total energy, is

$$H = T + V = \tfrac{1}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\cdot\dot{\varphi}^2] - \frac{\mu}{r} \tag{4.4}$$

while, for the Lagrangian, we have

$$L = T - V = \tfrac{1}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2] + \frac{\mu}{r}. \tag{4.5}$$

From the latter there follows the system of Lagrangian equations, which takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[\dot{r}] = r\dot{\theta}^2 + r\sin^2\theta\cdot\dot{\varphi}^2 - \frac{\mu}{r^2} \tag{4.6a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[r^2\dot{\theta}] = r^2\sin\theta\cos\theta\cdot\dot{\varphi}^2 \tag{4.6b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[r^2\sin^2\theta\cdot\dot{\varphi}] = 0. \tag{4.6c}$$

As the coordinate φ does not appear explicitly in the Lagrangian (4.5), it is an ignorable coordinate, and the procedure outlined in Chapter 1 may be followed; or we may proceed directly.

From (4.6c) there follows an immediate integration yielding

$$r^2\sin^2\theta\cdot\dot{\varphi} = C_3, \quad \text{or} \quad \dot{\varphi} = \frac{C_3}{r^2\sin^2\theta} \tag{4.7a,b}$$

where C_3 is the constant of integration and represents the polar component of angular momentum. The introduction of (4.7) into (4.6a,b) yields, respectively

$$\frac{\mathrm{d}}{\mathrm{d}t}[\dot{r}] = r\dot{\theta}^2 - \frac{\mu}{r^2} + \frac{C_3^2}{r^3\sin^2\theta} \tag{4.8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[r^2\dot{\theta}] = C_3^2\frac{\cos\theta}{r^2\sin^3\theta}. \tag{4.8b}$$

Considering (4.8b), we multiply across by $r^2\dot{\theta}$ to obtain

$$r^2\dot{\theta}\frac{\mathrm{d}}{\mathrm{d}t}[r^2\dot{\theta}] = C_3^2\frac{\cos\theta\cdot\dot{\theta}}{\sin^3\theta} \tag{4.9}$$

which may be rearranged as

$$\frac{\mathrm{d}}{\mathrm{d}t}[r^2\dot{\theta}]^2 = -C_3^2\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\sin^2\theta}\right] \tag{4.10}$$

or alternatively

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(r^2\dot{\theta})^2 + \frac{C_3^2}{\sin^2\theta}\right] = 0. \tag{4.11}$$

This implies that the expression in square brackets is constant; however, if we substitute for C_3 in terms of $\dot{\varphi}$ from (4.7a), the expression becomes

$$r^2[r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2] = r^2(v^2 - \dot{r}^2) \tag{4.12}$$

and if we recall (3.18), we see that this constant is the square of the angular momentum, namely C^2. Accordingly, the integral of (4.11) may be written

$$r^4\dot{\theta}^2 + \frac{C_3^2}{\sin^2\theta} = C^2 \tag{4.13}$$

or alternatively

$$r\dot{\theta}^2 = \frac{1}{r^3}\left[C^2 - \frac{C_3^2}{\sin^2\theta}\right] \tag{4.14}$$

as the form appropriate for the reduction of (4.8a), which we effect prior to the integration of (4.13).

If we substitute for $r\dot{\theta}^2$ from (4.14) and for $\dot{\varphi}$ from (4.7b) into equation (4.8a), we see that the terms with C_3^2 cancel and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\dot{r}] = \frac{C^2}{r^3} - \frac{\mu}{r^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mu}{r} - \frac{1}{2}\frac{C^2}{r^2}\right]. \tag{4.15}$$

If we multiply across by $\dot{r}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{2}\dot{r}^2] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mu}{r} - \frac{1}{2}\frac{C^2}{r^2}\right] \tag{4.16}$$

or on rearrangement

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}\dot{r}^2 - \frac{\mu}{r} + \frac{1}{2}\frac{C^2}{r^2}\right] = 0 \tag{4.17}$$

and so the expression in square brackets must be constant. Again, recalling (3.19) we see that the expression

$$\tfrac{1}{2}\dot{r}^2 - \frac{\mu}{r} + \frac{1}{2}\frac{C^2}{r^2} = \tfrac{1}{2}\dot{r}^2 - \frac{\mu}{r} + \tfrac{1}{2}(v^2 - \dot{r}^2) = \tfrac{1}{2}v^2 - \frac{\mu}{r} \tag{4.18}$$

is in fact the energy integral, whose constant has already been designated as $\mathcal{E}$ (3.7) and for negative energy has been identified by $-\alpha^2$ (3.10). Accordingly, the integrated relation reads

$$\tfrac{1}{2}\dot{r}^2 - \frac{\mu}{r} + \frac{1}{2}\frac{C^2}{r^2} = -\alpha^2 \tag{4.19}$$

or alternatively

$$r^2\dot{r}^2 = -2\alpha^2\left[r^2 - \frac{\mu}{\alpha^2}r + \frac{C^2}{2\alpha^2}\right] \tag{4.20}$$

identical with the previously derived (3.43). The subsequent analysis leading to the solution (3.46)

$$r = a[1 - e\cos E] \tag{4.21}$$

follows an identical pattern.

In item 4 below relation (1.18), we have noted a shortcoming of this simple form of the equation of the ellipse: for the dynamic problem a second shortcoming is now coming into view. If one were to apply the transformation (3.44) to equation (4.13) for $\dot{\theta}$, we would still have a coupled equation, and if one were to substitute for r from (4.21), one has a differential equation that is not readily integrable.

In fact, an inspection of equation (4.13) suggests the form of the alternative regularizing transformation that will effect the uncoupling of equations (4.13) and (4.17), whose uncoupled form admits a ready integration in the case of each equation.

The singularity in the differential equation (4.13) can be regularized by means of the [regularizing] transformation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{C}{r^2}, \qquad C\frac{\mathrm{d}}{\mathrm{d}f} = r^2\frac{\mathrm{d}}{\mathrm{d}t} \tag{4.22a,b}$$

and with f as the new independent variable, and with prime denoting differentiation with respect to f, equation (4.13) becomes

$$C^2\theta'^2 + \frac{C_3^2}{\sin^2\theta} = C^2\,. \tag{4.23}$$

If we now introduce a new parameter ν, representing the inclination of the orbit plane, and defined by

$$\nu = \frac{C_3}{C} \tag{4.24}$$

then equation (4.23) may be written

$$\sin^2\theta\cdot\theta'^2 = (1-\nu^2) - \cos^2\theta \tag{4.25}$$

which, as we shall see, admits a straightforward integration.

Returning to equation (4.20), we multiply by a further r^2-factor to obtain

$$r^4\dot{r}^2 = -r^2[2\alpha^2r^2 - 2\mu r + C^2]\,. \tag{4.26}$$

If we utilize the transformation (4.22) to introduce the new independent variable f, then after dividing across by C^2 we have

$$r'^2 = -r^2\left[1 - \frac{2\mu}{C^2}r + \frac{2\alpha^2}{C^2}\right] \tag{4.27a}$$

$$= -r^2\left[1 - \frac{2}{p}r + \frac{1}{ap}r^2\right] \tag{4.27b}$$

where we have introduced the length scales from (3.11) and (3.33) into (4.27a) to obtain (4.27b).

The integration of (4.27b) is facilitated by the introduction of an auxiliary dependent variable u, defined by

$$u = \frac{1}{r}, \quad r = \frac{1}{u}, \quad r' = -\frac{1}{u^2}u' \tag{4.28}$$

and, after a little manipulation, equation (4.27b) becomes

$$u'^2 = -\left[\left(u - \frac{1}{p}\right)^2 + \frac{1}{ap} - \frac{1}{p^2}\right] = \left[\frac{e^2}{p^2} - \left(u - \frac{1}{p}\right)^2\right]. \tag{4.29}$$

By setting

$$u - \frac{1}{p} = \frac{e}{p}w \tag{4.30}$$

the differential equation for w reads

$$w'^2 = 1 - w^2 \tag{4.31}$$

with solution

$$w = \cos(f + \omega_0) \tag{4.32}$$

where ω_0 is the constant of integration. It follows from (4.30) that

$$u = \frac{1}{p}[1 + e\cos(f + \omega_0)] \tag{4.33}$$

and hence, noting (4.28), we have

$$r = \frac{p}{1 + e\cos(f + \omega_0)}\,. \tag{4.34a}$$

Except for the factor ω_0, this is identical with (1.15) for the ellipse, so the variable f has the obvious angular interpretation; moreover, if the angle is measured from the pericenter so that

$$f = 0 \quad \text{corresponds to} \quad r = a(1 - e) \tag{4.34b}$$

then clearly $\omega_0 = 0$ and we have

$$r = \frac{p}{1 + e\cos f} \tag{4.35}$$

as the solution for r, identical with (1.15).

Returning to equation (4.25), we note that the integration can be facilitated by setting

$$\cos\theta = \sqrt{1 - \nu^2}\,S \tag{4.36}$$

so that equation (4.25) becomes

$$S'^2 = 1 - S^2 \tag{4.37}$$

with solution

$$S = \sin(f + \omega) \tag{4.38}$$

where ω is the constant of integration. The point where the orbit crosses the z-plane is called the *node* and the line joining it to the focus is called the *nodal line*. The crossing of the z-plane corresponds to $\theta = \pi/2$, and so noting (4.36) and (4.38), this must correspond to $f = -\omega$; hence ω measures the angle in the orbit plane subtended at the focus between the major axis and the nodal line. And we may write

$$\cos\theta = \sqrt{1 - \nu^2}\,\sin(f + \omega) \tag{4.39}$$

as the complete solution for the θ-coordinate.

It remains to integrate equation (4.7) for the third coordinate φ. Writing (4.7) in the form

$$r^2\dot{\varphi} = \frac{C_3}{\sin^2\theta} \tag{4.40}$$

we introduce the regularizing transformation (4.22) replacing t as the independent variable by f. We then have

$$C\varphi' = \frac{C_3}{\sin^2\theta} \tag{4.41}$$

and if we divide across by C and note the defining relation (4.24) for ν, we obtain

$$\varphi' = \frac{\nu}{\sin^2\theta} = \frac{\nu}{1 - \cos^2\theta}\,. \tag{4.42}$$

If we introduce $\cos\theta$ in terms of f from (4.39), we obtain

$$\begin{aligned}\varphi' &= \frac{\nu}{1 - (1 - \nu^2)\sin^2(f + \omega)} \\ &= \frac{\nu}{\cos^2(f + \omega) + \nu^2\sin^2(f + \omega)} = \frac{\nu\sec^2(f + \omega)}{1 + \nu^2\tan^2(f + \omega)}\,.\end{aligned} \tag{4.43}$$

The integration of equation (4.43) is facilitated by the substitution

$$\tan\Phi = \nu\tan(f + \omega) \tag{4.44}$$

from which we have

$$\sec^2\Phi\cdot\Phi' = \nu\sec^2(f + \omega), \quad \sec^2\Phi = 1 + \nu^2\tan^2(f + \omega) \tag{4.45a,b}$$

and from (4.43) there follows

$$\varphi' = \Phi' \quad \text{implying} \quad \Phi = \varphi + \varphi_0 \tag{4.45c}$$

where φ_0 is the constant of integration. Hence (4.44) implies that

$$\tan(\varphi + \varphi_0) = \nu \tan(f + \omega)\,. \tag{4.46}$$

We have already noted that at the nodal crossing, $f = -\omega$; if we now let Ω denote the longitude at this nodal line, then from (4.46) there follows

$$\tan(\Omega + \varphi_0) = 0, \quad \text{implying} \quad \varphi_0 = -\Omega \tag{4.47}$$

and hence, from (4.46), we have

$$\tan(\varphi - \Omega) = \nu \tan(f + \omega) \tag{4.48}$$

as the solution for the third coordinate φ.

The completion of the solution requires the determination of the time-angle relation connecting the time with the true anomaly f. For this we introduce the expression (4.35) into the inverted form of the defining relation (4.22a), and if we substitute for C from (3.33), we find

$$\frac{\mathrm{d}t}{\mathrm{d}f} = \frac{r^2}{C} = \frac{1}{\sqrt{\mu a}\sqrt{1-e^2}}\,\frac{a^2(1-e^2)^2}{(1+e\cos f)^2}\,. \tag{4.49}$$

If we recall from (3.41) that $\sqrt{\mu a} = na^2$, it follows that

$$n\frac{\mathrm{d}t}{\mathrm{d}f} = \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2}\,. \tag{4.50}$$

For the integration of this expression we first note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}f}\left[\frac{e\sin f}{1+e\cos f}\right] &= \frac{(1+e\cos f)e\cos f + e^2\sin^2 f}{(1+e\cos f)^2} = \frac{e^2 + e\cos f}{(1+e\cos f)^2} \\ &= \frac{(1+e\cos f)-(1-e^2)}{(1+e\cos f)^2} \\ &= \frac{1}{1+e\cos f} - \frac{(1-e^2)}{(1+e\cos f)^2}\end{aligned} \tag{4.51}$$

and hence, on multiplying by $\sqrt{1-e^2}$ and rearranging, we have

$$\frac{(1-e^2)^{3/2}}{(1+e\cos f)^2} = \frac{\sqrt{1-e^2}}{1+e\cos f} - \frac{\mathrm{d}}{\mathrm{d}f}\left[\frac{e\sqrt{1-e^2}\sin f}{1+e\cos f}\right]. \tag{4.52}$$

For the integration of the first term on the right we note that if we set

$$\tan\chi = \frac{\sqrt{1-e^2}\sin f}{e+\cos f} \tag{4.53}$$

there follows

$$\sec^2 \chi = \frac{(1 + e\cos f)^2}{(e + \cos f)^2}, \quad \cos \chi = \frac{e + \cos f}{1 + e\cos f}, \tag{4.54a,b}$$

$$\sin \chi = \frac{\sqrt{1 - e^2}\sin f}{1 + e\cos f}\,. \tag{4.54c}$$

Taking the derivative of (4.53), we find

$$\begin{aligned}
\sec^2 \chi \cdot \chi' &= \frac{(1 + e\cos f)^2}{(e + \cos f)^2}\chi' \\
&= \frac{\sqrt{1 - e^2}(e + \cos f)\cos f + \sqrt{1 - e^2}\sin^2 f}{(e + \cos f)^2} \\
&= \sqrt{1 - e^2}\frac{(1 + e\cos f)}{(e + \cos f)^2}
\end{aligned} \tag{4.55}$$

which, with (4.54a), yields

$$\chi' = \frac{\sqrt{1 - e^2}}{1 + e\cos f} \tag{4.56}$$

and hence, noting (4.53), we have

$$\int \frac{\sqrt{1 - e^2}}{1 + e\cos f}\,\mathrm{d}f = \chi = \arctan\left[\frac{\sqrt{1 - e^2}\sin f}{e + \cos f}\right]. \tag{4.57}$$

Accordingly, the integration of (4.50) is accomplished by combining (4.52) and (4.57) to yield

$$M = n(t - t_0) = \arctan\left[\frac{\sqrt{1 - e^2}\sin f}{e + \cos f}\right] - \frac{e\sqrt{1 - e^2}\sin f}{1 + e\cos f} \tag{4.58}$$

where t_0, reflecting the constant introduced by the integration, is the time of the pericenter passage, i.e., $t = t_0$ corresponds to $f = 0$.

http://www.springer.com/978-0-8176-4096-5

Integrable Systems in Celestial Mechanics
Ó'Mathúna, D.
2008, X, 234 p. 24 illus., Hardcover
ISBN: 978-0-8176-4096-5
A product of Birkhäuser Basel