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(*****************************************************************)
(*                                                               *)
(* Generalized Collocation Methods:Solutions to Nonlinear Problems *)
(* Bellomo,N.,Lods,B.,Revelli,R.,Ridolfi,L.                      *)
(* A Birkhäuser book                                             *)
(* ISBN:978-0-8176-4525-0                                        *)
(*                                                               *)
(* Program KdVIII                                                *)
(*                                                               *)
(*****************************************************************)

$TextStyle = {FontFamily -> "Times", FontSize → 12};
Off[General::"spell", General::"spell1"]

KdVIII[InitBoundData_, Nodes_, μ_, mreact_, Type_] := Module[{},
  (*** INITIAL CONDITION ***)
  φ[x_] := InitBoundData[[2]];
  hh = 1/(Nodes - 1);
  Which[Type === Sinc, Subscript[x1, i_] := (i - 1) * hh,
   Type === Lagrange, Subscript[x1, i_] := -1/2. (Cos[(i - 1) * π/(Nodes - 1)] - 1)];

  If[Type === Lagrange, Lagr[j_, x_] := Product[(If[p ≠ j, (x - Subscript[x1, p])/(Subscript[x1, j] - Subscript[x1, p]), 1]), {p, 1, Nodes}];

   InitValue[x_] := Sum[(φ[x] /. x → Subscript[x1, k]) * Lagr[k, x], {k, 1, Nodes}]];

  If[Type === Sinc, Sinc[j_, x_] := Which[0 ≤ x ≤ 1 && x ≠ (j - 1) * hh,
     Sin[(π * (x - (j - 1) * hh))/hh] / ((π * (x - (j - 1) * hh))/hh), x == (j - 1) * hh, 1];

   InitValue[x_] := Sum[(φ[x] /. x -> Subscript[x1, k]) * Sinc[k, x], {k, 1, Nodes}]];

  Print["Initial condition interpolation:"];
  Plot[Evaluate[{φ[x], InitValue[x]}], {x, 0, 1},
   PlotRange → All, PlotStyle → {GrayLevel[0], Dashing[{0.15, 0.05}]},
   AxesLabel → {x, u[x, 0]}, AxesOrigin → {0, 0}];

  Print["Don't worry! I'm working"];

  (*** DERIVATIVE MATRICES ***)

  If[Type === Lagrange,

   FDM[j_, i_] := Which[i == j, Sum[If[k == i, 0, 1/(Subscript[x1, i] - Subscript[x1, k])], {k, 1, Nodes}],

     i != j, Product[(If[p == i, 1, Subscript[x1, i] - Subscript[x1, p]]), {p, 1, Nodes}] /
       ((Subscript[x1, i] - Subscript[x1, j]) Product[(If[p == j, 1, Subscript[x1, j] - Subscript[x1, p]]), {p, 1, Nodes}])];

   FirstDer = Table[FDM[j, i], {j, 1, Nodes}, {i, 1, Nodes}];
   SecondDer =
    Table[If[i ≠ j, 2 (FirstDer[[j, i]] * FirstDer[[i, i]] - FirstDer[[j, i]]/(Subscript[x1, i] - Subscript[x1, j])), 0],
     {j, 1, Nodes}, {i, 1, Nodes}];
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  Do[SecondDer[[i, i]] = - Sum[SecondDer[[k, i]], {k, 1, Nodes}], {i, 1, Nodes}];
  ThirdDer =
   Table[If[i ≠ j, 3 (FirstDer[[j, i]] * SecondDer[[i, i]] - SecondDer[[j, i]] / (x1_i - x1_j)), 0],
    {j, 1, Nodes}, {i, 1, Nodes}];
  Do[ThirdDer[[i, i]] = - Sum[ThirdDer[[k, i]], {k, 1, Nodes}], {i, 1, Nodes}]];

 If[Type === Sinc,
  FDM[j_, i_] := Which[i != j, (-1)^(i-j) / (hh * (i - j)), i == j, 0];
  SDM[j_, i_] := Which[i != j, (2 * (-1)^(i-j+1)) / (hh^2 * (i - j)^2), i == j, - π^2 / (3 * hh^2)];
  TDM[j_, i_] := Which[i != j, (-1)^(i-j) / hh^3 (6 / (i - j)^3 - π^2 / (i - j)), i == j, 0];
  FirstDer = Table[FDM[j, i], {j, 1, Nodes}, {i, 1, Nodes}];
  SecondDer = Table[SDM[j, i], {j, 1, Nodes}, {i, 1, Nodes}];
  ThirdDer = Table[TDM[j, i], {j, 1, Nodes}, {i, 1, Nodes}]];

 (*** Nodal Equations ***)

 NdEq[i_] := C_i '[t] == ((( - (T1 - T0) / (X1 - X0)) * (Sum[FirstDer[[k, i]] * C_k[t], {k, 1, Nodes}]) * C_i[t]^mreact +
     ( - μ (T1 - T0) / (X1 - X0)^3) * (Sum[ThirdDer[[k, i]] * C_k[t], {k, 1, Nodes}])) /. x -> x1_i;

 EqsSys = Flatten[Table[{NdEq[i], C_i[0] == (φ[x] /. x → x1_i)}, {i, 2, Nodes - 1}]];
 NdEqsSys = Join[EqsSys, {C_1 '[t] == D[InitBoundData[[1]], t], C_1[0] == (φ[x] /. x → x1_1),
    C_Nodes '[t] == D[InitBoundData[[3]], t], C_Nodes[0] == (φ[x] /. x → x1_Nodes)}];
 UnKnown = Join[Table[C_i, {i, 1, Nodes}]];
 NumSol = NDSolve[NdEqsSys, UnKnown, {t, 0, 1}, MaxSteps → 10000000] // Flatten;

 (*** Solution ***)

 Which[Type === Sinc, TotSol[x_, t_] := Sum[NumSol[[k, 2]][t] * Sinc[k, x], {k, 1, Nodes}],

  Type === Lagrange, TotSol[x_, t_] := Sum[NumSol[[k, 2]][t] * Lagr[k, x], {k, 1, Nodes}]];

 Print["Solution:"];

 Plot3D[Evaluate[TotSol[x, t]], {x, 0, 1}, {t, 0, 1},
  PlotPoints → 81, Boxed → True, AxesLabel → {x, t, u}, PlotRange → All];
];
```

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(* One Soliton Solution *)

{T0, T1, X0, X1} = {0, 40, -30, 30};
{Nodes, m, μ, k, x0} = {51, 1, 1., 0.3, 0};
{A, ω} = {2 (m + 1) (m + 2) m^-2 μ k^2, 4 μ k^3 m^-2};
InitCond = (A Sech[k ((X1 - X0) x + X0) - ω (T1 - T0) t]^2 - x0)^(1/m) //. t → 0;
Bound0Cond = (A Sech[k ((X1 - X0) x + X0) - ω (T1 - T0) t]^2 - x0)^(1/m) //. x → 0;
Bound1Cond = (A Sech[k ((X1 - X0) x + X0) - ω (T1 - T0) t]^2 - x0)^(1/m) //. x → 1;
IBData = {Bound0Cond, InitCond, Bound1Cond};

KdVIII[IBData, Nodes, μ, m, Sinc];

(*** Invariants Calculus (Only for Sinc function) ***)

tt = {0, 0.25, 0.5, 0.75, 1};
```

$$\text{Inv1} = (\text{X1} - \text{X0}) \sum_{i=1}^{\text{Nodes}} \left(\text{NumSol}[[i, 2]][\text{tt}] * \sum_{j=1}^{\text{Nodes}} \text{hh} * \text{Sinc}[i, \text{x1}_j]\right)$$

$$\text{Inv2} = \frac{(\text{X1} - \text{X0})\ \text{hh}}{2} \sum_{j=1}^{\text{Nodes}} \left(\sum_{i=1}^{\text{Nodes}} (\text{NumSol}[[i, 2]][\text{tt}] * \text{Sinc}[i, \text{x1}_j])\right)^2$$

$$\text{Inv3} = (\text{X1} - \text{X0})\ \text{hh} \sum_{j=1}^{\text{Nodes}} \left(\left(\sum_{i=1}^{\text{Nodes}} (\text{NumSol}[[i, 2]][\text{tt}] * \text{Sinc}[i, \text{x1}_j])\right)^3 + \frac{1}{2} \left(\sum_{i=1}^{\text{Nodes}} \left(\text{NumSol}[[i, 2]][\text{tt}] * \frac{\text{FirstDer}[[i, j]]}{(\text{X1} - \text{X0})}\right)\right)^2\right)$$

$$\text{Inv4} = (\text{X1} - \text{X0})\ \text{hh} \sum_{j=1}^{\text{Nodes}} \left(5 \left(\sum_{i=1}^{\text{Nodes}} (\text{NumSol}[[i, 2]][\text{tt}] * \text{Sinc}[i, \text{x1}_j])\right)^4 + 10 \left(\sum_{i=1}^{\text{Nodes}} (\text{NumSol}[[i, 2]][\text{tt}] * \text{Sinc}[i, \text{x1}_j])\right) \left(\sum_{i=1}^{\text{Nodes}} \left(\text{NumSol}[[i, 2]][\text{tt}] * \frac{\text{FirstDer}[[i, j]]}{(\text{X1} - \text{X0})}\right)\right)^2 + \left(\sum_{i=1}^{\text{Nodes}} \left(\text{NumSol}[[i, 2]][\text{tt}] * \frac{\text{SecondDer}[[i, j]]}{(\text{X1} - \text{X0})^2}\right)\right)^2\right)$$

$$\text{Inv5} = (\text{X1} - \text{X0})\ \text{hh} \sum_{j=1}^{\text{Nodes}} \left(21 \left(\sum_{i=1}^{\text{Nodes}} (\text{NumSol}[[i, 2]][\text{tt}] * \text{Sinc}[i, \text{x1}_j]) \right)^5 + 105 \left(\sum_{i=1}^{\text{Nodes}} (\text{NumSol}[[i, 2]][\text{tt}] * \text{Sinc}[i, \text{x1}_j]) \right)^2 \left(\sum_{i=1}^{\text{Nodes}} \left(\text{NumSol}[[i, 2]][\text{tt}] * \frac{\text{FirstDer}[[i, j]]}{(\text{X1} - \text{X0})} \right) \right)^2 + 21 \left(\sum_{i=1}^{\text{Nodes}} (\text{NumSol}[[i, 2]][\text{tt}] * \text{Sinc}[i, \text{x1}_j]) \right) \left(\sum_{i=1}^{\text{Nodes}} \left(\text{NumSol}[[i, 2]][\text{tt}] * \frac{\text{SecondDer}[[i, j]]}{(\text{X1} - \text{X0})^2} \right) \right)^2 + \left(\sum_{i=1}^{\text{Nodes}} \left(\text{NumSol}[[i, 2]][\text{tt}] * \frac{\text{ThirdDer}[[i, j]]}{(\text{X1} - \text{X0})^3} \right) \right)^2 \right)$$

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(* Two Soliton Solution *)
{T0, T1, X0, X1} = {0, 360, -70, 70};
{Nodes, m, μ, k1, k2, x1, x2} = {81, 2, 1., 0.3, 0.2, -2, 3};
{A1, A2, ω1, ω2} =
  {2 (m + 1) (m + 2) m^-2 μ k1^2, A2 = 2 (m + 1) (m + 2) m^-2 μ k2^2, 4 μ k1^3 m^-2, 4 μ k2^3 m^-2};
InitCond = (A1 Sech[k1 ((X1 - X0) x + X0) - ω1 (T1 - T0) t - x1]^2)^(1/m) +
    (A2 Sech[k2 ((X1 - X0) x + X0) - ω2 (T1 - T0) t - x2]^2)^(1/m) //. t → 0;
Bound0Cond == 0;
Bound1Cond = 0;

IBData = {Bound0Cond, InitCond, Bound1Cond};

KdVIII[IBData, Nodes, μ, m, Sinc];

(*** Invariants Calculus (Only for Sinc function) ***)

tt = {0, 0.25, 0.5, 0.75, 1};
```

$$\text{Inv1} = (\text{X1} - \text{X0}) \sum_{i=1}^{\text{Nodes}} \left(\text{NumSol}[[i, 2]][\text{tt}] * \sum_{j=1}^{\text{Nodes}} \text{hh} * \text{Sinc}[i, \text{x1}_j] \right)$$

$$\text{Inv2} = \frac{(\text{X1} - \text{X0})\ \text{hh}}{2} \sum_{j=1}^{\text{Nodes}} \left(\sum_{i=1}^{\text{Nodes}} (\text{NumSol}[[i, 2]][\text{tt}] * \text{Sinc}[i, \text{x1}_j]) \right)^2$$