Preface

The aim of this book is to present the central parts of the theory for bases and frames. The content can naturally be split into two parts: Chapters 1–5 describe the theory on an abstract level, and Chapters 7–11 deal with explicit constructions in L^2-spaces. The link between these two parts is formed by Chapter 6, which introduces B-splines and their main properties.

Some years ago, I published the book *An Introduction to Frames and Riesz Bases* [10], which also appeared in the ANHA series. So, what are the reasons for another book on the topic? I will give some answers to this question.

Books written by mathematicians are usually focused on characterizations of various properties and the search for sufficient conditions for a desired conclusion to hold. Concrete constructions often play a minor role. The book [10] is no exception. During the past few years, frames have become increasingly popular, and several explicit constructions of frames of various types have been presented. Most of these constructions were based on quite direct methods rather than the classical sufficient conditions for obtaining a frame. With this in mind, it seems that there is a need for an updated version of the book [10], which moves the focus from the classical approach to a more constructive one.

Frame theory is developed in constant dialogue between mathematicians and engineers. Again, compared with [10], this is reflected in the current book by several new sections on applications and connections to engineering. The hope is that these sections will help the mathematically oriented readers to see where frames are used in practice — and the engineers to find the chapter containing the mathematical background for applications in their field.

The third main change compared with [10] is that the current book is meant to be a textbook, which should be directly suitable for use in a graduate course. We focus on the basic topics, without too many side-remarks; in contrast, [10] tried to cover the entire area, including the research aspects. The chapters from [10] dealing with research topics have been removed (or reduced: for example, parts of Chapter 15 about perturbation results now appear in Section 5.6). We frequently mention the names of the people who first proved a given result, but for the parts of the theory that can be considered classical, we do not state a reference to the original source. A professional reader might miss all the hints to more advanced literature and open problems; however, the hope is that the more streamlined writing makes it easier for students to follow the presentation.

For use in a graduate course, a number of exercises is included; they appear at the end of each chapter. Some of the removed material from [10] now appears in the exercises.

Let us describe the chapters in more detail. Chapter 1 gives an introduction to frames in finite-dimensional vector spaces with an inner product. This enables a reader with a basic knowledge of linear algebra to understand the idea behind frames without the technical complications in infinite-dimensional spaces. Many of the topics from the rest of the book are presented here, so Chapter 1 can also serve as an introduction to the later chapters.

Chapter 2 collects some definitions and conventions concerning infinite-dimensional vector spaces. Some standard results needed later in the book are also stated here. Special attention is given to the Hilbert space $L^2(\mathbb{R})$ and operators hereon. We expect the reader to be familiar with this material, so most of the results appear without proof. The exceptions are the sections about pseudo-inverse operators and some special operators on $L^2(\mathbb{R})$, which play a key role in Gabor theory and wavelet analysis; these subjects are not treated in classical analysis courses and are therefore described in detail.

Chapter 3 deals with the theory for bases in Hilbert spaces and Banach spaces. The most important part of the chapter is formed by a detailed discussion of Bessel sequences and Riesz bases. The chapter also contains sections on Fourier analysis and wavelet theory, which motivate the constructions in Chapters 7–11.

Chapter 4 highlights some of the limitations on the properties one can obtain from bases. Hereby, the reader is provided with motivation for considering the generalizations of bases studied in the rest of the book.

Chapter 5 contains the core material about frames in general Hilbert spaces. It gives a detailed description of frames with full proofs, relates frames and Riesz bases, and provides various ways of constructing frames.

Chapter 6 introduces B-splines and their main properties. We do not aim at a complete description of splines but concentrate on the properties that play a role in the current context.

Chapters 7–11 deal with frames having a special structure. A central part concerns theoretical conditions for obtaining dual pairs of frames and explicit constructions hereof. The most fundamental frames, namely frames consisting of translates of a single function in $L^2(\mathbb{R})$, are discussed in Chapter 7. In Chapter 8, these considerations are extended to frames generated by translations of a collection of functions rather than a single function. These frames naturally lead to Gabor frames in $L^2(\mathbb{R})$, which is the subject of Chapter 9. We provide characterizations of such frames, as well as explicit constructions of frames and some of their dual frames. The discrete counterpart in $\ell^2(\mathbb{Z})$ is treated in Chapter 10; in particular, it is shown how one can obtain Gabor frames in $\ell^2(\mathbb{Z})$ by sampling of Gabor frames in $L^2(\mathbb{R})$. Wavelet frames are introduced in Chapter 11. The main part of the chapter is formed by explicit constructions via multiscale methods, but the chapter also contains a section about general wavelet frames.

Most readers of the second part of the book will mainly be interested in either Gabor systems or wavelet systems. For this reason, Chapters 7–11 are to a large extent independent of each other. The most notable exception from that rule is that some of the fundamental results in Gabor analysis are based on results derived in the chapter about shift-invariant systems. In general, careful cross-references (and, if necessary, repetitions) between Chapters 7–11 are provided.

Depending on the level and specific interests of the students, a graduate course based on the book can proceed in various ways:

- Readers with a limited background in functional analysis (and readers who just want to get an idea about the topic) are encouraged to read Chapter 1. It will provide the reader with a good understanding for the topic, without all the technical complications in infinite–dimensional vector spaces.

- A short course on frames and Riesz bases in Hilbert spaces can be based on Sections 3.1–3.3 and Sections 5.1–5.2; these sections will make the reader able to proceed with most of the other parts of the book and with a large part of the research literature concerning abstract frame theory.

- A theoretical graduate course on bases and frames could be based on Chapter 2, Chapter 3, and Chapter 5. It would be natural to continue with one or more chapters on concrete frame constructions in $L^2(\mathbb{R})$.

- For a course focusing on either Gabor analysis or wavelets, the detailed analysis of frames in Chapter 5 is not necessary. It is enough to read Chapter 2, Section 3.5 (or Section 3.6), Chapter 4, Section 5.1, and parts of Chapter 6 before continuing with the relevant specialized chapters.

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