2

Infinite-dimensional Vector Spaces and Sequences

After the introduction to frames in finite-dimensional vector spaces in Chapter 1, the rest of the book will deal with expansions in infinite-dimensional vector spaces. Here great care is needed: we need to replace finite sequences $\{f_k\}_{k=1}^n$ by infinite sequences $\{f_k\}_{k=1}^\infty$, and suddenly the question of convergence properties becomes a central issue. The vector space itself might also cause problems, e.g., in the sense that Cauchy sequences might not be convergent. We expect the reader to have a basic knowledge about these problems and the way to circumvent them, but for completeness we repeat the central themes in Sections 2.1–2.4. Section 2.5 deals with pseudo-inverse operators; this subject is not expected to be known and is treated in more detail. Section 2.6 introduces the so-called moment problems in Hilbert spaces. In Sections 2.7–2.9, we discuss the Hilbert space $L^2(\mathbb{R})$ consisting of the square integrable functions on $\mathbb{R}$ and three classes of operators hereon, as well as the Fourier transform. The material in those sections is not needed for the study of frames and bases on abstract Hilbert spaces in Chapter 3 (except Section 3.5 and Section 3.6) and Chapter 5, but it forms the basis for all the constructions in Chapters 7–11.

2.1 Normed vector spaces and sequences

A central theme in this book is to find conditions on a sequence $\{f_k\}$ in a vector space X such that every $f \in X$ has a representation as a

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superposition of the vectors f_k. In most spaces appearing in functional analysis, this cannot be done with a finite sequence $\{f_k\}$. We are therefore forced to work with infinite sequences, say, $\{f_k\}_{k=1}^{\infty}$, and the representation of f in terms of $\{f_k\}_{k=1}^{\infty}$ will be via an infinite series. For this reason, the starting point must be a discussion of convergence of infinite series. We collect the basic definitions here together with some conventions.

Throughout the section, we let X denote a complex vector space. A *norm* on X is a function $||\cdot|| : X \to [0, \infty[$ satisfying the following three conditions:

(i) $||x|| = 0 \Leftrightarrow x = 0$;

(ii) $||\alpha x|| = |\alpha| \; ||x||, \; \forall x \in X, \; \alpha \in \mathbb{C}$;

(iii) $||x + y|| \leq ||x|| + ||y||, \; \forall x, y \in X$.

In situations where more than one vector space appear, we will frequently denote the norm on X by $||\cdot||_X$. If X is equipped with a norm, we say that X is a *normed vector space*. The *opposite triangle inequality* is satisfied in any normed vector space:

$$||x - y|| \geq \big| \, ||x|| - ||y|| \, \big|, \; x, y \in X. \tag{2.1}$$

We say that a sequence $\{x_k\}_{k=1}^{\infty}$ in X

(i) converges to $x \in X$ if

$$||x - x_k|| \to 0 \text{ for } k \to \infty;$$

(ii) is a *Cauchy sequence* if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$||x_k - x_l|| \leq \epsilon \text{ whenever } k, l \geq N.$$

A convergent sequence is automatically a Cauchy sequence, but the opposite is not true in general. There are, however, normed vector spaces in which a sequence is convergent if and only if it is a Cauchy sequence; a space X with this property is called a *Banach space.*

Imitating the finite-dimensional setting described in Chapter 1, we want to study sequences $\{f_k\}_{k=1}^{\infty}$ in X with the property that each $f \in X$ has a representation $f = \sum_{k=1}^{\infty} c_k f_k$ for some coefficients $c_k \in \mathbb{C}$. In order to do so, we have to explain exactly what we mean by convergence of an infinite series. There are, in fact, at least three different options; we will now discuss these options.

First, the notation $\{f_k\}_{k=1}^{\infty}$ indicates that we have chosen some ordering of the vectors f_k,

$$f_1, f_2, f_3, \ldots, f_k, f_{k+1}, \cdots \quad .$$

We say that an *infinite series* $\sum_{k=1}^{\infty} c_k f_k$ is convergent with sum $f \in X$ if

$$\left|\left| f - \sum_{k=1}^{n} c_k f_k \right|\right| \to 0 \text{ as } n \to \infty.$$

If this condition is satisfied, we write

$$f = \sum_{k=1}^{\infty} c_k f_k. \tag{2.2}$$

Thus, the definition of a convergent infinite series corresponds exactly to our definition of a convergent sequence with $x_n = \sum_{k=1}^{n} c_k f_k$.

Above we insisted on a fixed ordering of the sequence $\{f_k\}_{k=1}^{\infty}$. It is very important to notice that convergence properties of $\sum_{k=1}^{\infty} c_k f_k$ not only depend on the sequence $\{f_k\}_{k=1}^{\infty}$ and the coefficients $\{c_k\}_{k=1}^{\infty}$ but also on the ordering. Even if we consider a sequence in the simplest possible Banach space, i.e., a sequence $\{a_k\}_{k=1}^{\infty}$ in $\mathbb{C}$, it can happen that $\sum_{k=1}^{\infty} a_k$ is convergent but that $\sum_{k=1}^{\infty} a_{\sigma(k)}$ is divergent for a certain permutation σ of the natural numbers (Exercise 2.1). This observation leads to the second definition of convergence. If $\{f_k\}_{k=1}^{\infty}$ is a sequence in X and $\sum_{k=1}^{\infty} f_{\sigma(k)}$ is convergent for all permutations σ, we say that $\sum_{k=1}^{\infty} f_k$ is *unconditionally convergent.* In that case, the limit is the same regardless of the order of summation.

Finally, an infinite series $\sum_{k=1}^{\infty} f_k$ is said to be *absolutely convergent* if

$$\sum_{k=1}^{\infty} ||f_k|| < \infty.$$

In any Banach space, absolute convergence of $\sum_{k=1}^{\infty} f_k$ implies that the series converges unconditionally (Exercise 2.2), but the opposite does not hold in infinite-dimensional spaces. In finite-dimensional spaces, the two types of convergence are identical.

A subset $Z \subseteq X$ (countable or not) is said to be *dense* in X if for each $f \in X$ and each $\epsilon > 0$ there exists $g \in Z$ such that

$$||f - g|| \leq \epsilon.$$

In words, this means that elements in X can be approximated arbitrarily well by elements in Z.

For a given sequence $\{f_k\}_{k=1}^{\infty}$ in X, we let $\text{span}\{f_k\}_{k=1}^{\infty}$ denote the vector space consisting of all *finite* linear combinations of vectors f_k. The definition of convergence shows that if each $f \in X$ has a representation of the type (2.2), then each $f \in X$ can be approximated arbitrarily well in norm by elements in $\text{span}\{f_k\}_{k=1}^{\infty}$, i.e.,

$$\overline{\text{span}}\{f_k\}_{k=1}^{\infty} = X. \tag{2.3}$$

A sequence $\{f_k\}_{k=1}^{\infty}$ having the property (2.3) is said to be *complete* or *total.* We note that there exist normed spaces where no sequence $\{f_k\}_{k=1}^{\infty}$ is complete. A normed vector space, in which a countable and dense family exists, is said to be *separable.*

When we speak about a *finite sequence,* we mean a sequence $\{c_k\}_{k=1}^{\infty}$ where at most finitely many entries c_k are non-zero.

2.2 Operators on Banach spaces

Let X and Y denote Banach spaces. An linear map $U : X \to Y$ is called an *operator*, and U is *bounded* or *continuous* if there exists a constant $K > 0$ such that

$$||Ux||_Y \leq K\,||x||_X, \ \forall x \in X. \tag{2.4}$$

Usually, it will be clear from the context which norm we use, so we will write $||\cdot||$ for both $||\cdot||_X$ and $||\cdot||_Y$. The *norm* of the operator U, denoted by $||U||$, is the smallest constant K that can be used in (2.4). Alternatively,

$$||U|| = \sup\{||Ux|| \ : \ x \in X, ||x|| = 1\}.$$

If U_1 and U_2 are operators for which the range of U_2 is contained in the domain of U_1, we can consider the composed operator U_1U_2; if U_1 and U_2 are bounded, then also U_1U_2 is bounded, and

$$||U_1U_2|| \leq ||U_1||\ ||U_2||. \tag{2.5}$$

Now consider a sequence of operators $U_n : X \to Y, \ n \in \mathbb{N}$, which converges pointwise to a mapping $U : X \to Y$, i.e.,

$$U_nx \to Ux, \text{ as } n \to \infty, \ \forall x \in X.$$

We say that U_n converges to U in the *strong operator topology.* The *Banach–Steinhaus Theorem*, also known as the *uniform boundedness principle,* states the following:

Theorem 2.2.1 *Let $U_n : X \to Y, \ n \in \mathbb{N}$, be a sequence of bounded operators, which converges pointwise to a mapping $U : X \to Y$. Then U is linear and bounded. Furthermore, the sequence of norms $||U_n||$ is bounded, and $||U|| \leq \liminf ||U_n||$.*

An operator $U : X \to Y$ is *invertible* if U is surjective and injective. For a bounded, invertible operator, the inverse operator is bounded:

Theorem 2.2.2 *A bounded bijective operator between Banach spaces has a bounded inverse.*

In case $X = Y$, it makes sense to speak about the identity operator I on X. The *Neumann Theorem* states that an operator $U : X \to X$ is invertible if it is close enough to the identity operator:

Theorem 2.2.3 *If $U : X \to X$ is bounded and $||I - U|| < 1$, then U is invertible, and*

$$U^{-1} = \sum_{k=0}^{\infty}(I - U)^k. \tag{2.6}$$

Furthermore,

$$||U^{-1}|| \le \frac{1}{1 - ||I - U||}.$$

Note that (2.6) should be interpreted in the sense of the operator norm, i.e., as

$$\left|\left|U^{-1} - \sum_{k=0}^{N} (I-U)^k\right|\right| \to 0 \text{ as } N \to \infty.$$

2.3 Hilbert spaces

A special class of normed vector spaces is formed by *inner product spaces.* Recall that an inner product on a complex vector space X is a mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ for which

(i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle, \ \forall x, y, z \in X, \ \alpha, \beta \in \mathbb{C};$

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}, \ \forall x, y \in X;$

(iii) $\langle x, x \rangle \ge 0, \ \forall x \in X$, and $\langle x, x \rangle = 0 \Leftrightarrow x = 0.$

Note that we have chosen to let the inner product be linear in the first entry. It implies that the inner product is conjugated linear in the second entry. Frequently, the opposite convention is used in the literature.

A vector space X with an inner product $\langle \cdot, \cdot \rangle$ can be equipped with the norm

$$||x|| := \sqrt{\langle x, x \rangle}, \ x \in X.$$

If X is a Banach space with respect to this norm, then X is called a *Hilbert space*. We reserve the letter $\mathcal{H}$ for these spaces. We will always assume that $\mathcal{H}$ is *non-trivial*, i.e., that $\mathcal{H} \neq \{0\}$. The standard examples are the spaces $L^2(\mathbb{R})$ and $\ell^2(\mathbb{N})$ discussed in Section 2.7.

In any Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$, *Cauchy–Schwarz' inequality* holds: it states that

$$|\langle x, y \rangle| \le ||x|| \, ||y||, \quad \forall x, y \in \mathcal{H}.$$

Two elements $x, y \in \mathcal{H}$ are *orthogonal* if $\langle x, y \rangle = 0$; and the *orthogonal complement* of a subspace U of $\mathcal{H}$ is

$$U^{\perp} = \{x \in \mathcal{H} : \ \langle x, y \rangle = 0, \ \forall y \in U\}.$$

The above definitions and results are valid whether $\mathcal{H}$ is finite-dimensional or infinite-dimensional. Also note that norms and inner products are defined in a similar way on real vector spaces (just replace the scalars $\mathbb{C}$ by the real scalars $\mathbb{R}$).

We state a few elementary results concerning Hilbert spaces that will be used repeatedly during the book. The proof of the first is left to the reader as Exercise 2.3.

Lemma 2.3.1 *For a sequence $\{x_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ the following are equivalent:*

(i) $\{x_k\}_{k=1}^{\infty}$ *is complete.*

(ii) If $\langle x, x_k \rangle = 0$ *for all* $k \in \mathbb{N}$, *then* $x = 0$.

Among the linear operators on a Hilbert space, a special role is played by the continuous linear operators $U : \mathcal{H} \to \mathbb{C}$. They are called *functionals* and are characterized in *Riesz' Representation Theorem*:

Theorem 2.3.2 *Let* $U : \mathcal{H} \to \mathbb{C}$ *be a continuous linear mapping. Then there exists a unique* $y \in \mathcal{H}$ *such that* $Ux = \langle x, y \rangle$ *for all* $x \in \mathcal{H}$.

The uniqueness of the element $y \in \mathcal{H}$ associated with a given functional has the following important consequence.

Corollary 2.3.3 *Let $\mathcal{H}$ be a Hilbert space. Assume that $x, y \in \mathcal{H}$ satisfy that*

$$\langle x, z \rangle = \langle y, z \rangle, \ \forall z \in \mathcal{H}.$$

Then $x = y$.

Finally, we note that the norm of an arbitrary element $x \in \mathcal{H}$ can be recovered based on the inner product between x and the elements in the unit sphere in $\mathcal{H}$:

Lemma 2.3.4 *For any* $x \in \mathcal{H}$,

$$||x|| = \sup_{||y||=1} |\langle x, y \rangle|.$$

2.4 Operators on Hilbert spaces

Let U be a bounded operator from the Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$ into the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$. The *adjoint* operator is defined as the unique operator $U^* : \mathcal{H} \to \mathcal{K}$ satisfying that

$$\langle x, Uy \rangle_{\mathcal{H}} = \langle U^* x, y \rangle_{\mathcal{K}}, \ \forall x \in \mathcal{H}, y \in \mathcal{K}.$$

Usually, we will write $\langle \cdot, \cdot \rangle$ for both inner products; it will always be clear from the context in which space the inner product is taken.

We collect some relationships between U and U^*; the proofs can be found in, e.g., Theorem 4.14 and Theorem 4.15 in [60].

Lemma 2.4.1 *Let $U : \mathcal{K} \to \mathcal{H}$ be a bounded operator. Then the following holds:*

(i) $||U|| = ||U^*||$, *and* $||UU^*|| = ||U||^2$.

(ii) $\mathcal{R}_U$ *is closed in* $\mathcal{H}$ *if and only if* $\mathcal{R}_{U^*}$ *is closed in* $\mathcal{K}$.

(iii) U *is surjective if and only if there exists a constant* $C > 0$ *such that*

$$||U^*y|| \geq C\ ||y||, \quad \forall y \in \mathcal{H}.$$

In the rest of this section, we consider the case $\mathcal{K} = \mathcal{H}$. A bounded operator $U : \mathcal{H} \to \mathcal{H}$ is *unitary* if $UU^* = U^*U = I$. If U is unitary, then

$$\langle Ux, Uy \rangle = \langle x, y \rangle, \ \forall x, y \in \mathcal{H}.$$

A bounded operator $U : \mathcal{H} \to \mathcal{H}$ is *self-adjoint* if $U = U^*$. When U is self-adjoint,

$$||U|| = \sup_{||x||=1} |\langle Ux, x \rangle|\,. \tag{2.7}$$

For a self-adjoint operator U, the inner product $\langle Ux, x \rangle$ is real for all $x \in \mathcal{H}$. One can introduce a partial order on the set of self-adjoint operators by

$$U_1 \leq U_2 \Leftrightarrow \langle U_1 x, x \rangle \leq \langle U_2 x, x \rangle, \ \forall x \in \mathcal{H}.$$

Using this order, one can work with self-adjoint operators almost as with real numbers. For example, under certain conditions it is possible to "multiply" an operator inequality with a bounded operator. The precise statement below can be found in [43]:

Theorem 2.4.2 *Let U_1, U_2, U_3 be self-adjoint operators. If $U_1 \leq U_2$, $U_3 \geq 0$, and U_3 commutes with U_1 and U_2, then $U_1U_3 \leq U_2U_3$.*

An important class of self-adjoint operators consists of the *orthogonal projections.* Given a closed subspace V of $\mathcal{H}$, the orthogonal projection of $\mathcal{H}$ onto V is the operator $P : \mathcal{H} \to \mathcal{H}$ for which

$$Px = x, \ x \in V, \ Px = 0, \ x \in V^{\perp}.$$

If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for V, the operator P is given explicitly by

$$Px = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k, \ x \in \mathcal{H}.$$

In case $\mathcal{H}$ is a complex Hilbert space and U is a bounded operator on $\mathcal{H}$, a direct calculation gives that

$$\begin{aligned} 4\langle Ux, y \rangle &= \langle U(x+y), x+y \rangle - \langle U(x-y), x-y \rangle \\ &\quad + i\langle U(x+iy), x+iy \rangle - i\langle U(x-iy), x-iy \rangle. \end{aligned} \tag{2.8}$$

In particular, we can recover the inner product in $\mathcal{H}$ from the norm by

$$4\langle x,y\rangle = ||x+y||^2 - ||x-y||^2 + i\ ||x+iy||^2 - i\ ||x-iy||^2,\ x,y\in\mathcal{H},$$

a result that is known as the *polarization identity.*

Lemma 2.4.3 *Let $U:\mathcal{H}\to\mathcal{H}$ be a bounded operator, and assume that $\langle Ux,x\rangle=0$ for all $x\in\mathcal{H}$. Then the following holds:*

(i) If $\mathcal{H}$ is a complex Hilbert space, then $U=0$.

(ii) If $\mathcal{H}$ is a real Hilbert space and U is self-adjoint, then $U=0$.

Proof. If $\mathcal{H}$ is complex, we can use (2.8); thus, if $\langle Ux,x\rangle=0$ for all $x\in\mathcal{H}$, then $\langle Ux,y\rangle=0$ for all $x,y\in\mathcal{H}$, and therefore $U=0$.

In case $\mathcal{H}$ is a real Hilbert space, we must use a different approach. Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. Then, for arbitrary $j,k\in\mathbb{N}$,

$$\begin{aligned} 0 &= \langle U(e_k+e_j), e_k+e_j\rangle \\ &= \langle Ue_k,e_k\rangle + \langle Ue_j,e_j\rangle + \langle Ue_k,e_j\rangle + \langle Ue_j,e_k\rangle \\ &= \langle Ue_k,e_j\rangle + \langle e_j,Ue_k\rangle \\ &= 2\,\langle Ue_j,e_k\rangle; \end{aligned}$$

therefore $U=0$. □

Note that without the assumption $U=U^*$, the second part of the lemma would fail; to see that, let U be a rotation of 90° in $\mathbb{R}^2$.

A bounded operator $U:\mathcal{H}\to\mathcal{H}$ is *positive* if $\langle Ux,x\rangle\geq 0,\ \forall x\in\mathcal{H}$. On a complex Hilbert space, every bounded positive operator is self-adjoint. For a positive operator U, we will often use the following result about the existence of a *square root*, i.e., a bounded operator W such that $W^2=U$:

Lemma 2.4.4 *Every bounded and positive operator $U:\mathcal{H}\to\mathcal{H}$ has a unique bounded and positive square root W. The operator W has the following properties:*

(i) If U is self-adjoint, then W is self-adjoint.

(ii) If U is invertible, then W is also invertible.

(iii) W can be expressed as a limit (in the strong operator topology) of a sequence of polynomials in U, and commutes with U.

2.5 The pseudo-inverse operator

It is well-known that not all bounded operators U on a Hilbert space $\mathcal{H}$ are invertible: an operator U needs to be injective and surjective in order to be invertible. We will now prove that if an operator U has closed range, there exists a "*right-inverse operator*" $U^\dagger$ in the following sense:

Lemma 2.5.1 *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, and suppose that $U : \mathcal{K} \to \mathcal{H}$ is a bounded operator with closed range $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger : \mathcal{H} \to \mathcal{K}$ for which*

$$UU^\dagger x = x, \ \forall x \in \mathcal{R}_U. \tag{2.9}$$

Proof. Consider the restriction of U to an operator on the orthogonal complement of the kernel of U, i.e., let

$$\tilde{U} := U_{|\mathcal{N}_U^\perp} : \mathcal{N}_U^\perp \to \mathcal{H}.$$

Clearly, $\tilde{U}$ is linear and bounded. $\tilde{U}$ is also injective: if $\tilde{U}x = 0$, it follows that $x \in \mathcal{N}_U^\perp \cap \mathcal{N}_U = \{0\}$. We now prove that the range of $\tilde{U}$ equals the range of U. Given $y \in \mathcal{R}_U$, there exists $x \in \mathcal{K}$ such that $Ux = y$. By writing $x = x_1 + x_2$, where $x_1 \in \mathcal{N}_U^\perp, \ x_2 \in \mathcal{N}_U$, we obtain that

$$\tilde{U}x_1 = Ux_1 = U(x_1 + x_2) = Ux = y.$$

It follows from Theorem 2.2.2 that $\tilde{U}$ has a bounded inverse

$$\tilde{U}^{-1} : \mathcal{R}_U \to \mathcal{N}_U^\perp.$$

Extending $\tilde{U}^{-1}$ by zero on the orthogonal complement of $\mathcal{R}_U$ we obtain a bounded operator $U^\dagger : \mathcal{H} \to \mathcal{K}$ for which $UU^\dagger x = x$ for all $x \in \mathcal{R}_U$. $\square$

The operator $U^\dagger$ constructed in the proof of Lemma 2.5.1 is called the *pseudo-inverse* of U. In the literature, one will often see the pseudo-inverse of an operator U with closed range defined as the unique operator $U^\dagger$ satisfying that

$$\mathcal{N}_{U^\dagger} = \mathcal{R}_U^\perp, \ \ \mathcal{R}_{U^\dagger} = \mathcal{N}_U^\perp, \text{ and } UU^\dagger x = x, x \in \mathcal{R}_U; \tag{2.10}$$

this definition is equivalent to the above construction (Exercise 2.4). We collect some properties of $U^\dagger$ and its relationship to U.

Lemma 2.5.2 *Let $U : \mathcal{K} \to \mathcal{H}$ be a bounded operator with closed range. Then the following holds:*

(i) The orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}_U$ is given by $UU^\dagger$.

(ii) The orthogonal projection of $\mathcal{K}$ onto $\mathcal{R}_{U^\dagger}$ is given by $U^\dagger U$.

(iii) U^* has closed range, and $(U^*)^\dagger = (U^\dagger)^*$.

(iv) On $\mathcal{R}_U$, the operator $U^\dagger$ is given explicitly by

$$U^\dagger = U^*(UU^*)^{-1}. \tag{2.11}$$

Proof. All statements follow from the characterization of $U^\dagger$ in (2.10). For example, it shows that

$$UU^\dagger = I \text{ on } \mathcal{R}_U \text{ and that } UU^\dagger = 0 \text{ on } \mathcal{N}_{U^\dagger} = \mathcal{R}_U^\perp;$$

this gives (i) by the definition of an orthogonal projection. The proof of (ii) is similar. That $\mathcal{R}_{U^*}$ is closed was stated already in Lemma 2.4.1; thus $(U^*)^\dagger$ is well defined. That $(U^*)^\dagger$ equals $(U^\dagger)^*$ follows by verifying that $(U^\dagger)^*$ satisfies (2.10) with U replaced by U^*. Finally, UU^* is invertible as an operator on $\mathcal{R}_U$, and the operator given by

$$U^*(UU^*)^{-1} \text{ on } \mathcal{R}_U, \quad \text{and} \quad 0 \text{ on } \mathcal{R}_U^\perp$$

satisfies the conditions (2.10) characterizing $U^\dagger$. □

The pseudo-inverse gives the solution to an important optimization problem:

Theorem 2.5.3 *Let $U : \mathcal{K} \to \mathcal{H}$ be a bounded surjective operator. Given $y \in \mathcal{H}$, the equation $Ux = y$ has a unique solution of minimal norm, namely $x = U^\dagger y$.*

The proof is identical with the proof of Theorem 1.4.2.

2.6 A moment problem

Before we leave the discussion of abstract Hilbert spaces, we mention a special class of equations, known as moment problems. For the purpose of the current book, they are only needed in Section 7.4.

The general version of a *moment problem* is as follows: given a collection of elements $\{x_k\}_{k=1}^\infty$ in a Hilbert space $\mathcal{H}$ and a sequence $\{a_k\}_{k=1}^\infty$ of complex numbers, can we find an element $x \in \mathcal{H}$ such that

$$\langle x, x_k \rangle = a_k, \text{ for all } k \in \mathbb{N}?$$

We will only need a special moment problem:

Lemma 2.6.1 *Let $\{x_k\}_{k=1}^N$ be a collection of vectors in $\mathcal{H}$ and consider the moment problem*

$$\langle x, x_k \rangle = \begin{cases} 1 \text{ if } k = 1, \\ 0 \text{ if } k = 2, \dots, N. \end{cases} \tag{2.12}$$

Then the following are equivalent:

(i) The moment problem (2.12) *has a solution x.*

(ii) If $\sum_{k=1}^N c_k x_k = 0$ for some scalar coefficients c_k, then $c_1 = 0$.

(iii) $x_1 \notin \text{span}\{x_k\}_{k=2}^N$.

In case the moment problem (2.12) *has a solution, it can be chosen of the form $x = \sum_{k=1}^N d_k x_k$ for some scalar coefficients d_k.*

Proof. Assume first that (i) is satisfied, i.e., (2.12) has a solution x. Then, if $\sum_{k=1}^N c_k x_k = 0$ for some coefficients $\{c_k\}_{k=1}^N$, we have that

$$0 = \langle x, \sum_{k=1}^N c_k x_k \rangle = \sum_{k=1}^N c_k \langle x, x_k \rangle = c_1,$$

i.e., (ii) holds. Now assume that (ii) is satisfied. Then $x_1 \notin \text{span}\{x_k\}_{k=2}^N$. Let P denote the orthogonal projection of $\mathcal{H}$ onto $\text{span}\{x_k\}_{k=2}^N$, and put $\varphi = x_1 - Px_1$. Then

$$\langle \varphi, x_1 \rangle = \langle x_1 - Px_1, x_1 - Px_1 \rangle + \langle x_1 - Px_1, Px_1 \rangle = ||x_1 - Px_1||^2 \neq 0,$$

and $\langle \varphi, x_k \rangle = 0$ for $k = 2, \dots, N$. Thus, the element

$$x := \frac{\varphi}{||x_1 - Px_1||^2} \tag{2.13}$$

solves the moment problem (2.12), i.e., (i) is satisfied. The equivalence of (ii) and (iii) is clear. In case the equivalent conditions are satisfied, the construction of x in (2.13) shows that $x \in \text{span}\{x_k\}_{k=1}^N$. $\square$

2.7 The spaces $L^p(\mathbb{R})$, $L^2(\mathbb{R})$, and $\ell^2(\mathbb{N})$

The most important class of Banach spaces is formed by the L^p-spaces, $1 \leq p \leq \infty$. Before we define these spaces, we will remind the reader about some basic facts from the theory of integration. The proofs and further results can be found in any standard book on the subject, e.g., [59].

We begin with *Fatou's Lemma*. For our purpose, it is enough to consider the case of the Lebesgue measure on the real axis $\mathbb{R}$, equipped with the (Borel-) measurable sets:

Lemma 2.7.1 *Let $f_n : \mathbb{R} \to [0, \infty]$, $n \in \mathbb{N}$ be a sequence of measurable functions. Then the function $\liminf_{n\to\infty} f_n$ is measurable, and*

$$\int_{-\infty}^{\infty} \liminf_{n\to\infty} f_n(x)\, dx \leq \liminf_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx.$$

Lebesgue's Dominated Convergence Theorem is the main tool to interchange limits and integrals:

Theorem 2.7.2 *Suppose that $f_n : \mathbb{R} \to \mathbb{C}$, $n \in \mathbb{N}$ is a sequence of measurable functions, that $f_n(x) \to f(x)$ pointwise as $n \to \infty$, and that there exists a positive, measurable function g such that $|f_n| \leq g$ for all $n \in \mathbb{N}$ and $\int_{-\infty}^{\infty} g(x)\, dx < \infty$. Then f is integrable, and*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

A *null set* is a measurable set with measure zero. A condition holds *almost everywhere* (abbreviated a.e.) if it holds except on a null set.

We are now ready to define the Banach spaces $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$. First, we define $L^\infty(\mathbb{R})$ as the space of essentially bounded measurable functions $f : \mathbb{R} \to \mathbb{C}$, equipped with the essential supremums-norm. For $1 \leq p < \infty$, $L^p(\mathbb{R})$ is the space of functions f for which $|f|^p$ is integrable with respect to the Lebesgue measure:

$$L^p(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ is measurable and } \int_{-\infty}^{\infty} |f(x)|^p \, dx < \infty \right\}.$$

The norm on $L^p(\mathbb{R})$ is

$$||f|| = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{1/p}.$$

To be more precise, $L^p(\mathbb{R})$ consists of equivalence classes of functions that are equal almost everywhere, and for which a representative (and hence all) for the equivalence class satisfies the integrability condition. In order not to be too tedious, we adopt the standard terminology and speak about functions in $L^p(\mathbb{R})$ rather than equivalence classes.

The case $p = 2$ plays a special role: in fact, the space

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ is measurable and } \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \right\}$$

is the only one of the $L^p(\mathbb{R})$-spaces that can be equipped with an inner product. Actually, $L^2(\mathbb{R})$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \, dx, \;\; f, g \in L^2(\mathbb{R}).$$

In $L^2(\mathbb{R})$, Cauchy–Schwarz' inequality states that for all $f, g \in L^2(\mathbb{R})$,

$$\left| \int_{-\infty}^{\infty} f(x)g(x) \, dx \right| \leq \left(\int_{-\infty}^{\infty} |f(x)|^2 \, dx \right)^{1/2} \left(\int_{-\infty}^{\infty} |g(x)|^2 \, dx \right)^{1/2}.$$

The spaces $L^2(\Omega)$, where Ω is an open subset of $\mathbb{R}$, are defined similarly. According to the general definition, a sequence of functions $\{g_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$ converges to $g \in L^2(\Omega)$ if

$$||g - g_k|| = \left(\int_{\Omega} |g(x) - g_k(x)|^2 \, dx \right)^{1/2} \to 0 \text{ as } k \to \infty.$$

Convergence in L^2 is very different from pointwise convergence. As a positive result, we have *Riesz' Subsequence Theorem:*

Theorem 2.7.3 *Let $\Omega \subseteq \mathbb{R}$ be an open set, and let $\{g_k\}$ be a sequence in $L^2(\Omega)$ that converges to $g \in L^2(\Omega)$. Then $\{g_k\}$ has a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ such that*

$$g(x) = \lim_{k\to\infty} g_{n_k}(x)$$

for a.e. $x \in \Omega$.

The result holds no matter how we choose the representatives for the equivalence classes. This is typical for this book, where we rarely deal with a specific representative for a given class. There are, however, a few important exceptions. When we speak about a continuous function, it is clear that we have chosen a specific representative, and the same is the case when we discuss *Lebesgue points.* By definition, a point $y \in \mathbb{R}$ is a Lebesgue point for a function f if

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{y-\frac{1}{2}\epsilon}^{y+\frac{1}{2}\epsilon} |f(y) - f(x)|\, dx = 0.$$

If f is continuous in y, then y is a Lebesgue point (Exercise 2.5). More generally, one can prove that if $f \in L^1(\mathbb{R})$, then almost every $y \in \mathbb{R}$ is a Lebesgue point.

It is clear from the definition that different representatives for the same equivalence class will have different Lebesgue points. For example, every $y \in \mathbb{R}$ is a Lebesgue point for the function $f = 0$; changing the definition of f in a single point y will not change the equivalence class, but y will no longer be a Lebesgue point. See Exercise 2.5 for some related observations.

The discrete analogue of $L^2(\mathbb{R})$ is $\ell^2(I)$, the space of square sumable scalar sequences with a countable index set I:

$$\ell^2(I) := \left\{ \{x_k\}_{k\in I} \mid x_k \in \mathbb{C},\ \sum_{k\in I} |x_k|^2 < \infty \right\}.$$

The definition of the space $\ell^2(I)$ corresponds to our definition of $L^2(\mathbb{R})$ with the set $\mathbb{R}$ replaced by I and the Lebesgue measure replaced by the counting measure. $\ell^2(I)$ is a Hilbert space with respect to the inner product

$$\langle \{x_k\}, \{y_k\} \rangle = \sum_{k\in I} x_k \overline{y_k};$$

in this case, Cauchy–Schwarz' inequality gives that

$$\left| \sum_{k\in I} x_k \overline{y_k} \right|^2 \leq \sum_{k\in I} |x_k|^2 \sum_{k\in I} |y_k|^2,\ \{x_k\}_{k\in I}, \{y_k\}_{k\in I} \in \ell^2(I).$$

We will frequently use the discrete version of Fatou's lemma:

Lemma 2.7.4 *Let I be a countable index set and $f_n : I \to [0, \infty],\ n \in \mathbb{N}$, a sequence of functions. Then*

$$\sum_{k \in I} \liminf_{n \to \infty} f_n(k) \le \liminf_{n \to \infty} \sum_{k \in I} f_n(k).$$

2.8 The Fourier transform and convolution

For $f \in L^1(\mathbb{R})$, the *Fourier transform* $\hat{f} : \mathbb{R} \to \mathbb{C}$ is defined by

$$\hat{f}(\gamma) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \gamma}\, dx,\ \gamma \in \mathbb{R}.$$

Frequently, we will also denote the Fourier transform of f by $\mathcal{F}f$.

If $(L^1 \cap L^2)(\mathbb{R})$ is equipped with the $L^2(\mathbb{R})$-norm, the Fourier transform is an isometry from $(L^1 \cap L^2)(\mathbb{R})$ into $L^2(\mathbb{R})$. If $f \in L^2(\mathbb{R})$ and $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions in $(L^1 \cap L^2)(\mathbb{R})$ that converges to f in L^2-sense, then the sequence $\{\hat{f}_k\}_{k=1}^{\infty}$ is also convergent in $L^2(\mathbb{R})$, with a limit that is independent of the choice of $\{f_k\}_{k=1}^{\infty}$. Defining

$$\hat{f} := \lim_{k \to \infty} \hat{f}_k$$

we can extend the Fourier transform to a unitary mapping of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. We will use the same notation to denote this extension. In particular, we have *Plancherel's equation:*

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle,\ \forall f, g \in L^2(\mathbb{R}),\ \text{and } ||\hat{f}|| = ||f||. \tag{2.14}$$

If $f \in L^1(\mathbb{R})$, then $\hat{f}$ is continuous. If the function f as well as $\hat{f}$ belong to $L^1(\mathbb{R})$, the *inversion formula* describes how to come back to f from the function values $\hat{f}(\gamma)$, see [2]:

Theorem 2.8.1 *Assume that $f, \hat{f} \in L^1(\mathbb{R})$. Then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{2\pi i x \gamma} d\gamma,\ \text{a.e. } x \in \mathbb{R}. \tag{2.15}$$

If f is continuous, the pointwise formula (2.15) *holds for all $x \in \mathbb{R}$. In general, it holds at least for all Lebesgue points for f.*

Given two functions $f, g \in L^1(\mathbb{R})$, the *convolution* $f * g : \mathbb{R} \to \mathbb{C}$ is defined by

$$f * g(y) = \int_{-\infty}^{\infty} f(y - x) g(x)\, dx,\ y \in \mathbb{R}.$$

The function $f * g$ is well defined for all $y \in \mathbb{R}$ and belongs to $L^1(\mathbb{R})$. If $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$, the convolution $f * g(y)$ is well defined for a.e. $y \in \mathbb{R}$ and defines a function in $L^2(\mathbb{R})$.

The Fourier transform and convolution are related by the following important result.

Theorem 2.8.2 *If $f, g \in L^1(\mathbb{R})$, then $\widehat{f * g}(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma)$ for all $\gamma \in \mathbb{R}$; if $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$, the formula holds for a.e. $\gamma \in \mathbb{R}$.*

2.9 Operators on $L^2(\mathbb{R})$

In this section, we consider three classes of operators on $L^2(\mathbb{R})$ that will play a key role in our analysis of Gabor frames and wavelets. Their definitions are as follows:

$$\textit{Translation by } a \in \mathbb{R},\ T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (T_a f)(x) = f(x-a); \quad (2.16)$$

$$\textit{Modulation by } b \in \mathbb{R},\ E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (E_b f)(x) = e^{2\pi i b x} f(x); \quad (2.17)$$

$$\textit{Dilation by } a \neq 0,\ D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (D_a f)(x) = \frac{1}{\sqrt{|a|}} f\left(\frac{x}{a}\right). \quad (2.18)$$

A comment about notation: we will usually skip the parentheses and simply write $T_a f(x)$, and similarly for the other operators. Frequently, we will also let E_b denote the function $x \mapsto e^{2\pi i b x}$. We collect some of the most important properties for the operators in (2.16)–(2.18):

Lemma 2.9.1 *The translation operators satisfy the following:*

(i) T_a is unitary for all $a \in \mathbb{R}$.

(ii) For each $f \in L^2(\mathbb{R})$, the mapping $y \mapsto T_y f$ is continuous from $\mathbb{R}$ to $L^2(\mathbb{R})$.

Similar statements hold for $E_b, b \in \mathbb{R}$, and $D_a,\ a \neq 0$.

Proof. Let us prove that the operators T_a are unitary. Since

$$\begin{aligned}\langle T_a f, g\rangle = \int_{-\infty}^{\infty} f(x-a)\overline{g(x)}\,dx &= \int_{-\infty}^{\infty} f(x)\overline{g(x+a)}\,dx \\ &= \langle f, T_{-a} g\rangle,\ \forall f, g \in L^2(\mathbb{R}),\end{aligned}$$

we see that $T_a^* = T_{-a}$. On the other hand, T_a is clearly an invertible operator with $T_a^{-1} = T_{-a}$, so we conclude that $T_a^{-1} = T_a^*$.

To prove the continuity of the mapping $y \mapsto T_y f$, we first assume that the function f is continuous and has compact support, say, contained in the bounded interval $[c, d]$. For notational convenience, we prove the continuity in $y_0 = 0$. First, for $y \in]-\frac{1}{2}, \frac{1}{2}[$ the function

$$\phi(x) = T_y f(x) - T_{y_0} f(x) = f(x-y) - f(x)$$

has support in the interval $[-\frac{1}{2}+c, d+\frac{1}{2}]$. Since f is uniformly continuous, we can for any given $\epsilon > 0$ find $\delta > 0$ such that

$$|f(x-y) - f(x)| \leq \epsilon \ \text{ for all } x \in \mathbb{R} \text{ whenever } |y| \leq \delta;$$

with this choice of δ, we thus obtain that

$$\begin{aligned} ||T_y f - T_{y_0} f|| &= \left(\int_{-\frac{1}{2}+c}^{\frac{1}{2}+d} |f(x-y) - f(x)|^2 \, dx \right)^{1/2} \\ &\leq \epsilon \sqrt{d-c+1}. \end{aligned}$$

This proves the continuity in the considered special case. The case of an arbitrary function $f \in L^2(\mathbb{R})$ follows by an approximation argument, using that the continuous functions with compact support are dense in $L^2(\mathbb{R})$ (Exercise 2.6). The proofs of the statements for E_b and D_a are left to the reader (Exercise 2.7). □

Chapters 9–11 will deal with Gabor systems and wavelet systems in $L^2(\mathbb{R})$; both classes consist of functions in $L^2(\mathbb{R})$ that are defined by compositions of some of the operators T_a, E_b, and D_a. For this reason, the following *commutator relations* are important:

$$T_a E_b f(x) = e^{-2\pi iba} E_b T_a f(x) = e^{2\pi ib(x-a)} f(x-a), \tag{2.19}$$

$$T_b D_a f(x) = D_a T_{b/a} f(x) = \frac{1}{\sqrt{|a|}} f(\frac{x}{a} - \frac{b}{a}), \tag{2.20}$$

$$D_a E_b f(x) = \frac{1}{\sqrt{|a|}} e^{2\pi ixb/a} f(\frac{x}{a}) = E_{\frac{b}{a}} D_a f(x). \tag{2.21}$$

In wavelet analysis, the dilation operator $D_{1/2}$ plays a special role, and we simply write

$$Df(x) := 2^{1/2} f(2x).$$

With this notation, the commutator relation (2.20) in particular implies that

$$T_k D^j = D^j T_{2^j k} \text{ and } D^j T_k = T_{2^{-j} k} D^j, \ j, k \in \mathbb{Z}. \tag{2.22}$$

We will often use the Fourier transformation in connection with Gabor systems and wavelet systems. In this context, we need the commutator relations

$$\mathcal{F} T_a = E_{-a} \mathcal{F}, \ \ \mathcal{F} E_a = T_a \mathcal{F}, \ \ \mathcal{F} D_a = D_{1/a} \mathcal{F}, \ \ \mathcal{F} D = D^{-1} \mathcal{F}. \tag{2.23}$$

2.10 Exercises

2.1 Find a sequence $\{a_k\}_{k=1}^{\infty}$ of real numbers for which $\sum_{k=1}^{\infty} a_k$ is convergent but not unconditionally convergent.

2.2 Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in a Banach space. Prove that absolute convergence of $\sum_{k=1}^{\infty} f_k$ implies unconditional convergence.

2.3 Prove Lemma 2.3.1.

2.4 Prove that the conditions in (2.10) are equivalent to the construction of the pseudo-inverse in Lemma 2.5.1.

2.5 Here we ask the reader to prove some results concerning Lebesgue points.

(i) Assume that $f : \mathbb{R} \to \mathbb{C}$ is continuous. Prove that every $y \in \mathbb{R}$ is a Lebesgue point.

(ii) Prove that $x = 0$ is not a Lebesgue point for the function $\chi_{[0,1]}$.

(iii) Let $f = \chi_{\mathbb{Q}}$. Prove that every $y \notin \mathbb{Q}$ is a Lebesgue point and that the rational numbers are not Lebesgue points.

2.6 Complete the proof of Lemma 2.9.1 by showing the continuity of the mapping $y \mapsto T_y f$ for $f \in L^2(\mathbb{R})$.

2.7 Prove the statements about E_b and D_a in Lemma 2.9.1.

2.8 Prove the commutator relations (2.23).

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