Chapter II

Introduction to kinetics (statics and impulse theory)

§1. Contrast between continuously acting forces and impact forces; the impulse for a single free mass particle

While in the preceding kinematic chapter we had no occasion to speak of the principles of mechanics, we will have to draw upon these principles in some form for the kinetic considerations that now follow. This can be done in various ways.

The original setting of the principles of mechanics by Newton assumes the *concept of force* as something immediately acknowledged and understood. More recent presentations often seek to eliminate this concept from the foundations, and introduce it only later as a convenient abbreviated designation in mechanics. The suitability of one or the other procedure depends essentially on the objective that one pursues. If one sets out merely to construct a consistent conceptual system, as Hertz does in his beautiful work on the "Principles of Mechanics," then one can well dispense with the concept of force. If one wishes, however, to attain a lively comprehension of mechanical phenomena and a rapid orientation in specific questions, as is our intention in these lectures, then the concept of force appears particularly valuable on psychological grounds; namely, this concept is immediately associated with the activities of man, who has in his muscles the possibility of doing work. Such a performance of work is bound in our perception with the feeling of *exertion.* We involuntarily carry over this perception to the external predecessors of motion as well. The root of the concept of force lies, without doubt, in this anthropomorphic interpretation of external events. We will not, therefore, suppress this interpretation, but rather proceed with this point of view directly in the foreground.[46]

R.J. Nagem, G. Sandri, *The Theory of the Top*, DOI: 10.1007/978-0-8176-4721-6_II,

We determine whether a force acts in a certain direction on a mass particle, which is thought to be fixed somehow in space, by displacing the particle slightly in the opposite direction. If we must perform work (exert our muscles) in doing so, then a force is present; otherwise, the particle is free of force in the given direction. The same procedure can also serve for the measurement of force: *we measure a force acting on a particle in a specified direction by the ratio of the work that we must perform in a displacement of the particle in the opposite direction to this displacement* $\left(P = \frac{A}{l}\right)$. If the units of length and work are prescribed, then the unit of force is determined as well.

We have already defined *the unit of length* on page 11 when we accepted the "absolute system of measurement." *The unit of work* is defined in this measurement system, as is well known, by first establishing *the unit of mass* as the gram, and then referring to experience with freely falling bodies (the so-called law of falling bodies). The unit of work is determined in this way as the 980.60^{th} part of the work that is performed in the elevation of one gram by one centimeter at the 45^{th} degree of latitude.[47] Work thus receives the dimension $\frac{ml^2}{t^2}$, and force the dimension $\frac{ml}{t^2}$. We must certainly declare this definition of the unit of work to be rather indirect from the above psychological point of view. Since force and work lie more immediately in our perception than mass, we could demand that the unit of work be established before the unit of mass, and could advocate the introduction of a general measurement system (which, moreover, is suggested occasionally from other points of view) in which work is used in addition to space and time as the third fundamental dimension. Whether such a measurement system would be recommended practically, we leave here completely undecided.

One traditionally distinguishes, furthermore, two kinds of forces: *continuously acting forces* and *impact* or *instantaneous forces.* The rule given above for the measurement of forces refers, as we must explicitly emphasize, to continuously acting forces. To include the measurement of impact forces, we say that *an impact force is equivalent to an extraordinarily large continuous force that acts for only an extraordinarily short time.*

We then measure the magnitude of an impact force[48] by the *product of the magnitude of that continuously applied force and the length of the interval of application* $\left([P] = \dfrac{ml}{t}\right)$.

We now recall briefly the empirical facts (or axioms) that constitute the statics of forces (both continuous forces and impact forces) for a single mass particle. They can be summarized in one statement:

The most general system of forces on a single freely moving mass particle has the character of a vector emanating from the particle.

In this statement we comprise the theorem of the parallelogram of forces as well as the decomposition of a force into components; *forces add exactly as vectors.*

The vector representation of a force is so familiar to us that we can easily hold our axiom as self-evident and undervalue its significance. It is, therefore, perhaps not superfluous to consider the most immediate consequences of our axiom. A force has, just as a vector, a specified magnitude and a designated direction. We now introduce a mass particle that may be subjected to arbitrary external forces. The direction of the resulting force is opposite to the direction in which we must perform the maximum work for a displacement. If we displace the particle, on the other hand, in a direction deviating from this designated direction, then, according to our fundamental theorem, the work thus performed is equal to the previous maximum work multiplied by the cosine of the angle that the current displacement encloses with the previous. In particular, we will, whenever we must expend a certain amount of work in a specified displacement, acquire the same quantity of work in an oppositely directed displacement.

We next recall the axiom that lies at the foundation of the kinetics of a single mass particle. It is well known: *a continuously applied force causes an acceleration of the particle whose direction coincides with the direction of the force and whose magnitude, multiplied by the mass of the particle (using the previously accepted absolute system of measurement), is equal to the magnitude of the force;* further, *an instantaneous force causes an instantaneous change of velocity, whose direction coincides with the direction of the force and whose magnitude, multiplied by the mass of the particle, equals the magnitude of the impact force.*

Since the velocity and the acceleration of a particle have, just as the force, the character of a vector, the equality of the vector components follows immediately from our axiom that states the equality of the vectors. If X and $[X]$ denote the x-component of a continuous and an impact force, respectively, then we have

$$(1) \qquad X = m\frac{d^2x}{dt^2} \text{ etc.}, \qquad\qquad (2) \qquad [X] = m\Delta\left(\frac{dx}{dt}\right) \text{ etc.}$$

Here $\Delta\left(\frac{dx}{dt}\right)$ simply denotes the change of $\left(\frac{dx}{dt}\right)$.

Moreover, impact forces can be attributed to continuously acting forces and conversely, as is demonstrated by well-known examples from physics. In the kinetic theory of gases, for example, the continuously applied pressure of a gas against the wall of a vessel is thought to be produced by the collisions of the gas molecules. The continuous pressure is thus resolved into a series of very small and very rapidly successive individual impacts. One proceeds conversely in the theory of elasticity if one wishes to follow, in detail, the impact of two spheres. The apparently instantaneous impact force is replaced by a very short but still continuously increasing and decreasing force that is applied from the impacting to the impacted sphere.

One recognizes immediately that equations (1) and (2) are transformed into each other by one or the other interpretation. If we imagine, as in the example from the theory of gases, a series of individual impacts $[X_1]$, $[X_2]$, …, $[X_n]$ that follow each other by the very short time interval Δt, and we assume that the ratio $[X_i] : \Delta t$ tends to a finite limit X with decreasing Δt, then we obtain equation (1) from equation (2) through the passage to the limit $\Delta t = 0$. If we begin, on the other hand, from a continuously applied force X that differs from zero only in the time interval Δt, but that has very significant intensity, so much so that the "time integral"

$$\int_{t_0}^{t_0+\Delta t} X\, dt$$

retains a finite value $[X]$ if we let Δt decrease to zero, then we obtain equation (2) from equation (1) by integration with respect to t.

Since impact forces and continuous forces are thus, in a certain sense, mathematically equivalent, it is possible to base mechanics equally well on one or the other. Each of the two methods corresponds to a particular point of view of natural philosophy. Whoever is of the opinion that no discontinuous processes can occur in nature will prefer continuous forces. But whoever holds that continuity in nature is only apparent, and that our imperfect sensory organs yield an indistinct picture of the world, will generally wish to return to impact forces*). For the mathematician as such these questions do not come into consideration; the mathematician will evaluate the advantages of the two methods according to their greater or lesser mathematical usefulness and convenience.

From this point of view, we cannot refrain from giving preference to the method of impact forces over the now usual method of continuous forces for the general introduction of mechanics. We will also make use of this method when it is not at all a matter of treating sudden or rapidly successive changes in the state of motion. We thus take up again a manner of representation that was generally accepted by the actual founders of mechanics.

Fundamentally, moreover, it is only a matter of grasping the meaning of the differential equations of mechanics, or of differential equations in general, independently of formulas, according to their inner conceptual content.

We illustrate our intentions first with the example of a single freely moving mass particle.

We consider our particle at any position on its path, and always imagine *that impact force which is able to transform the particle instantaneously, and without change of position, from the state of rest into the actual state of motion.* If suddenly stopped at the considered place in its motion, the particle would be able to exert this same impact force against the obstacle. We call this impact force the *i m p u l s e of the particle*; our primary interest is now, to a certain extent, to follow not the motion of the particle, but rather the change in its impulse.

On the basis of the general axiom of statics we can say:

The impulse of a single free mass particle is a vector.

On the basis of the kinetic axiom we can, in addition, immediately give the magnitude and direction of this vector. According to equation (2), it is, namely, an impact force that changes the velocity 0 into the

*) Cf. here, for example, L. B o l t z m a n n: Über die Unentbehrlichkeit der Atomistik in der Naturwissenschaft, Berichte der Wiener Akademie 1896, as well as Wied. Ann. 1897.[49]

velocity v, equal in magnitude to mv; the direction coincides with the direction of v. We will therefore say:

The magnitude of the impulse of a single mass particle equals the so-called quantity of motion;[50] *the direction of the impulse coincides with the actual direction of the advancement.*

Using the concept of the impulse, the motion of a single particle can now be described in a particularly simple way.

We have first

Theorem I. *If no forces act on the particle, the impulse remains constant in magnitude and direction in space.* (Galileo's law of inertia = Newton's lex prima).

If, on the contrary, the particle is subjected to external influences, these can consist of instantaneous impacts $[P_i]$ or a continuous force P. In the first case, the impacts $[P_i]$ are composed with the already present impulse according to the parallelogram of forces. In the second case, we construct the single impact force that is equivalent to the continuous force during the time element Δt; we thus form

$$[P] = \int_t^{t+\Delta t} P \,.\, dt = P \,.\, \Delta t.$$

This "infinitesimal impact" is again composed with the already present impulse according to the parallelogram of forces.

We thus formulate

Theorem II. *If external forces act on our particle, the impulse is altered so that its change at each moment Δt is equal in magnitude and direction to the (finite or infinitesimal) impact produced during this moment of time by the external forces* (Newton's lex secunda).

These fundamental theorems carry over word for word, as we will later see, to the case of the top, and mutatis mutandis to arbitrary mechanical systems. —

We next ask for *the work that the impulse or any continuous force must perform for the generation of the instantaneous state of motion; that is, the work that is necessary to transform the particle from the state of rest to the actually present state of motion.* This same work may be supplied by the particle in reverse if we arrest its motion suddenly or gradually. This explains the designation of our quantum of work as the *vis viva*[51] *of the particle.*

The work that a *continuous force* (X, Y, Z) performs on a mass particle in a displacement (dx, dy, dz) is evidently

$$dA = X\,dx + Y\,dy + Z\,dz = (Xx' + Yy' + Zz')\,dt. \tag{3}$$

This equation, in fact, is nothing more than the analytic expression for our original definition of the concept of force.

We substitute here for X, Y, Z the values from equation (1), in which we suppose that (X, Y, Z) is the force that has generated the actual motion of the particle beginning from rest. Then the element of work is

$$dA = m(x''x' + y''y' + z''z')\,dt.$$

The total work, which amounts exactly to the *vis viva* of the particle, follows through integration with respect to time; we thus obtain the well-known formula

$$T = \frac{m}{2}(x'^2 + y'^2 + z'^2). \tag{4}$$

The *vis viva* of the particle is thus determined by the momentary state of motion of the particle alone; it is independent of how we imagine this state to be produced, or in what way we let x', y', z' or X, Y, Z vary during the generation of the motion. It is therefore also valid, in particular, if we imagine the motion to be generated instantaneously by an impulse, as we now wish to do.

If we take up the calculation of the *vis viva* again under this special assumption, then the matter will become particularly clear. We can first suppose, for this purpose, that a force (X, Y, Z) with a constant and very large magnitude is applied during the very small time interval Δt, so that the velocity (x', y', z') increases *uniformly* during this interval. From our above expression for the work there then follows, if we denote the duration of the force by Δt,

$$T = \int_0^{\Delta t} dA = X\int_0^{\Delta t} x'\,dt + Y\int_0^{\Delta t} y'\,dt + Z\int_0^{\Delta t} z'\,dt.$$

At the beginning of the interval Δt, the velocity of the particle is equal to zero, and at the end it has become (x', y', z'). Since, moreover, it increases from its initial value to its final value uniformly, the time integrals on the right-hand side have the values $\frac{1}{2}x'\Delta t$, $\frac{1}{2}y'\Delta t$, $\frac{1}{2}z'\Delta t$. We thus have

$$T = \frac{1}{2}(Xx' + Yy' + Zz')\,\Delta t.$$

We now let Δt decrease to zero. Then the products $X\Delta t$, $Y\Delta t$, $Z\Delta t$ go over into the components of the impulse, which we denote by $[X]$, $[Y]$, $[Z]$:

$$[X]=\int_0^{\Delta t} X\,dt = X\Delta t,\ [Y]=\int_0^{\Delta t} Y\,dt = Y\Delta t,\ [Z]=\int_0^{\Delta t} Z\,dt = Z\Delta t.$$

As a result, we can write the latter expression for the *vis viva* as

$$T = \frac{1}{2}([X]x' + [Y]y' + [Z]z'). \tag{5}$$

The vis viva *generated by the impulse appears here as half the product of the magnitude of the impulse and the length of the velocity vector.*

The expression (5) is, naturally, identical with (4). In fact, it follows immediately from our definition of the impulse given above that

$$[X] = mx', \qquad [Y] = my', \qquad [Z] = mz'. \tag{6}$$

These equations can also be written in the noteworthy form

$$[X] = \frac{\partial T}{\partial x'}, \qquad [Y] = \frac{\partial T}{\partial y'}, \qquad [Z] = \frac{\partial T}{\partial z'}. \tag{7}$$

In the execution of the partial differentiation, T is to be in the form (4); that is, written as a function of the velocity components.

We can, on the other hand, also conceive T as a function of the components of the impulse. Namely, there follows from equation (4) and (6)

$$T = \frac{1}{2m}([X]^2 + [Y]^2 + [Z]^2). \tag{4$'$}$$

As a result, we can also give equations (7) the form

$$x' = \frac{\partial T}{\partial [X]}, \qquad y' = \frac{\partial T}{\partial [Y]}, \qquad z' = \frac{\partial T}{\partial [Z]}. \tag{7$'$}$$

In the formation of these equations, T is naturally defined by equations (4′); that is, T is conceived as a function of the impulse components. In so far as a misunderstanding is precluded by the context, as here, we will not specifically indicate by the notation whether we assume one or the other conception of the *vis viva*.

In addition to equations (7) and (7′), we present as analytic expressions for the force components X, Y, Z the equations

$$X = \frac{\partial A}{\partial x}, \qquad Y = \frac{\partial A}{\partial y}, \qquad Z = \frac{\partial A}{\partial z}, \tag{8}$$

which result, respectively, from our original introduction of the concept of force in equation (3)*).

It is an immediate consequence of our impulse theorems I) and II) that the *vis viva* remains constant ($dT = 0$) for force-free motion of the particle, and that the change in the *vis viva* for motion influenced by external forces is equal to the work performed by these forces ($dT = dA$).

Equations (7), or the equivalent equations (7′), give us relations between the velocity vector, the impulse vector, and the expression for the *vis viva*, which we bring together in words in the following way:

The impulse (velocity) components are the partial differential quotients of the vis viva *with respect to the velocity (impulse) components, where we think of the* vis viva *given as a function of the velocity (impulse) components.*

In addition to equations (7) or (7′), there is a second triplet of equations that indicate how the impulse is changed by external influences. This triplet of equations is only the analytic expression of the law stated in Theorem II. If a continuous force (X, Y, Z) acts on our particle, then we obviously have, according to II, with the use of rectangular coordinates,

$$\frac{d[X]}{dt} = X, \qquad \frac{d[Y]}{dt} = Y, \qquad \frac{d[Z]}{dt} = Z. \tag{9}$$

Formulas (7), (8), *and* (9) *are the very well known fundamental equations of particle mechanics in rectangular coordinates.* —

We now wish to ask how these equations change if we introduce, instead of *rectangular coordinates*, an arbitrary set of *generalized coordinates*. There is, to be sure, no reason for departing from rectangular coordinates for a single freely moving mass particle. The following considerations, however, should serve as a preparation for more difficult cases in which we cannot manage with rectangular coordinates.

We wish to think of the position of a particle in space as given not by three mutually perpendicular planes, but rather by three arbitrary surfaces, and to consider as coordinates not the quantities x, y, z, but rather any three functions $\varphi = \varphi(x, y, z)$, $\psi = \psi(x, y, z)$, $\vartheta = \vartheta(x, y, z)$. Instead of the usual velocity coordinates x', y', z', we introduce the

*) It is noted that the meaning of the differential symbols in (8) deviates from the usual meaning, in so far as they are indeed "differential quotients," but are not to be used as "derivatives" of a function of the coordinates, since the expression (3) does not, in general, represent a perfect differential.

corresponding "generalized velocity coordinates" φ', ψ', ϑ'; that is, the derivatives of our quantities φ, ψ, ϑ with respect to time.

One recognizes immediately, if one differentiates the defining equations for φ, ψ, ϑ with respect to time, that *the new velocity coordinates are linear functions of the old, and vice versa.* In particular, we wish to denote the coefficients of φ', ψ', ϑ' in the linear expressions for x', y', z' by a_{ik}, so that

$$(10) \qquad \begin{cases} x' = a_{11}\varphi' + a_{12}\psi' + a_{13}\vartheta', \\ y' = a_{21}\varphi' + a_{22}\psi' + a_{23}\vartheta', \\ z' = a_{31}\varphi' + a_{32}\psi' + a_{33}\vartheta'; \end{cases}$$

the meaning of the coefficients a_{ik} is evidently

$$(11) \qquad a_{11} = \frac{\partial x}{\partial \varphi}, \quad a_{12} = \frac{\partial x}{\partial \psi}, \;\cdots\; a_{21} = \frac{\partial y}{\partial \varphi}, \;\cdots\; .$$

It will now be further necessary, however, to consider also *the corresponding generalized coordinates of the force and the impulse.* To define these coordinates, we return to our original definition of force. We ask for the work that we must perform in an infinitesimal change of the coordinate φ, with fixed values of ψ and ϑ. We define the φ coordinate of the force as the ratio of this work to the change in φ. If we denote this coordinate of force by Φ, then we have

$$\Phi = \frac{\partial A}{\partial \varphi}$$

(where the meaning of this differentiation symbol is defined by the conditions $\psi = \text{const.}$, $\vartheta = \text{const.}$); the force components Ψ and Θ have corresponding meanings. The expression for the work in an arbitrarily small displacement $(d\varphi, d\psi, d\vartheta)$ of our mass particle thus becomes

$$(12) \qquad dA = \Phi\, d\varphi + \Psi\, d\psi + \Theta\, d\vartheta = (\Phi\varphi' + \Psi\psi' + \Theta\vartheta')\, dt.$$

Our definition of the generalized force components thus implies that the expression for the work retains exactly the earlier form (3) *with the introduction of generalized coordinates.*

We can, therefore, easily express $\Phi\,\Psi\,\Theta$ in terms of XYZ. Namely, if we replace x', y', z' in (3) by their values in terms of φ', ψ', ϑ' given in equations (10) and order according to the latter quantities, then Φ, Ψ, Θ become the coefficients of φ', ψ', ϑ', respectively. We thus obtain

$$
(13) \qquad \begin{cases} \Phi = a_{11}X + a_{21}Y + a_{31}Z, \\ \Psi = a_{12}X + a_{22}Y + a_{32}Z, \\ \Theta = a_{13}X + a_{23}Y + a_{33}Z. \end{cases}
$$

The expressions for the new force coordinates in terms of the old are thus entirely analogous to the expressions for the old velocity coordinates in terms of the new; namely, the coefficients of the force transformation result from the coefficients of the velocity transformation by the interchange of the horizontal and vertical ranks. To state this concisely, we say:

The force coordinates are contragredient to the velocity coordinates.

The coordinates of an impulse, which we can indeed conceive as the limiting case of a continuous force, behave just like the coordinates of a continuous force; further, the expression for the *vis viva*, which we have defined as a certain finite quantity of work, transforms just like the expression for the infinitesimal work.

We thus obtain

$$
(14) \qquad \begin{cases} [\Phi] = a_{11}[X] + a_{21}[Y] + a_{31}[Z], \\ [\Psi] = a_{12}[X] + a_{22}[Y] + a_{32}[Z], \\ [\Theta] = a_{13}[X] + a_{23}[Y] + a_{33}[Z] \end{cases}
$$

and

$$
(15) \qquad T = \frac{1}{2}\left([\Phi]\varphi' + [\Psi]\psi' + [\Theta]\vartheta'\right).
$$

On the basis of equations (14), we verify without difficulty the relations

$$
(16) \qquad [\Phi] = \frac{\partial T}{\partial \varphi'}, \qquad [\Psi] = \frac{\partial T}{\partial \psi'}, \qquad [\Theta] = \frac{\partial T}{\partial \vartheta'},
$$

in which we suppose T to be expressed as a function of the velocity coordinates. In fact, it follows from $T = \frac{m}{2}(x'^2 + y'^2 + z'^2)$, for example, that

$$
\frac{\partial T}{\partial \varphi'} = mx'\frac{\partial x'}{\partial \varphi'} + my'\frac{\partial y'}{\partial \varphi'} + mz'\frac{\partial z'}{\partial \varphi'} = [X]a_{11} + [Y]a_{21} + [Z]a_{31} = [\Phi].
$$

Equations (16) are precisely analogous to our earlier relations (7). *The relations* (7) *are completely unchanged in form by the introduction of generalized coordinates.*

The same holds also for equations (7′); we assure ourselves of this briefly in the following way.

Solved for $\varphi', \ldots$, equations (10) give

$$(10') \qquad \begin{cases} \varphi' = A_{11}x' + A_{21}y' + A_{31}z', \\ \psi' = A_{12}x' + A_{22}y' + A_{32}z', \\ \vartheta' = A_{13}x' + A_{23}y' + A_{33}z', \end{cases}$$

where the A_{ik} generally denote the subdeterminants of the determinant a_{ik} divided by this determinant. In the same way, there result from equations (14)

$$(14') \qquad \begin{cases} [X] = A_{11}[\Phi] + A_{12}[\Psi] + A_{13}[\Theta], \\ [Y] = A_{21}[\Phi] + A_{22}[\Psi] + A_{23}[\Theta], \\ [Z] = A_{31}[\Phi] + A_{32}[\Psi] + A_{33}[\Theta]. \end{cases}$$

We now imagine that T is expressed in terms of $[\Phi]$, $[\Psi]$, $[\Theta]$ by substituting the values of $[X]$, $[Y]$, $[Z]$ from (14′) into equation (4′). We then form $\dfrac{\partial T}{\partial[\Phi]}$, in which we hold $[\Psi]$ and $[\Theta]$ as well as φ, ψ, ϑ fixed. We thus have, with consideration of (7′) and (14′),

$$\frac{\partial T}{\partial[\Phi]} = \frac{\partial T}{\partial[X]} \cdot \frac{\partial[X]}{\partial[\Phi]} + \frac{\partial T}{\partial[Y]} \cdot \frac{\partial[Y]}{\partial[\Phi]} + \frac{\partial T}{\partial[Z]} \cdot \frac{\partial[Z]}{\partial[\Phi]} = A_{11}x' + A_{21}y' + A_{31}z'.$$

But from this it follows, according to (10′), if we adjoin at the same time the analogous equations,

$$(16') \qquad \varphi' = \frac{\partial T}{\partial[\Phi]}, \quad \psi' = \frac{\partial T}{\partial[\Psi]}, \quad \vartheta' = \frac{\partial T}{\partial[\Theta]}.$$

If we wish to formulate equations (16) and (16′) as theorems, then the manner of expression of page 77 can serve us word for word.

We then have in analogy to equations (8), according to (12),

$$(17) \qquad \Phi = \frac{\partial A}{\partial\varphi}, \quad \Psi = \frac{\partial A}{\partial\psi}, \quad \Theta = \frac{\partial A}{\partial\vartheta}.$$

For the meaning of these differential symbols, cf. the footnote on p. 77.

It will also be well to rewrite equations (9) in the generalized coordinates φ, ψ, ϑ. For this purpose, we first multiply equations (9) sequentially by a_{11}, a_{21}, a_{31} and add them. Then there results on the right-hand side, according to (13), the component Φ of the external force. We write the left side as

$$\frac{d}{dt}(a_{11}[X] + a_{21}[Y] + a_{31}[Z]) - \left([X]\frac{da_{11}}{dt} + [Y]\frac{da_{21}}{dt} + [Z]\frac{da_{31}}{dt}\right).$$

The first term here is simply the differential quotient of the impulse component $[\Phi]$ with respect to time; the second term becomes equal, with consideration of (6) and (11), to

$$m\left(x'\frac{\partial x'}{\partial\varphi} + y'\frac{\partial y'}{\partial\varphi} + z'\frac{\partial z'}{\partial\varphi}\right);$$

this is, however, nothing other than the partial differential quotient of the *vis viva* taken with respect to φ, where we think of the *vis viva* expressed in terms of the velocity coordinates φ', ψ', ϑ'. We thus arrive at the following law for the change of the φ-component of the impulse:

$$\frac{d[\Phi]}{dt} - \frac{\partial T}{\partial \varphi} = \Phi. \tag{18}$$

In precisely the same way there result

$$\begin{cases} \dfrac{d[\Psi]}{dt} - \dfrac{\partial T}{\partial \psi} = \Psi, \\[2ex] \dfrac{d[\Theta]}{dt} - \dfrac{\partial T}{\partial \vartheta} = \Theta. \end{cases} \tag{18}$$

Equations (18) state nothing other, according to their derivation, than equations (9); the simplicity of the general law II is only somewhat veiled here by the introduction of the coordinates φ, ψ, ϑ.

Equations (7), (8), and (9) or equations (16), (17), and (18) represent, taken together, the equations of motion for a single mass particle. One has, in the notation we have chosen, the simplest case of the so-called *Lagrange equations of the second kind.*[52] We will often return to these equations, and already remark here that we will always succeed, with help of an impulse concept analogous to the above, in interpreting them in a manner similar to the equations of motion of a single mass particle.

We have taken the term "impulse" from the work of Thomson and Tait, in which our concept plays an important role. Maxwell applies the same term in the attempt to use energy as the basis of the general equations of mechanics. The somewhat colorless word *momentum* is usually used in English books instead of impulse;[53] the components of the impulse are then called "the moments of momentum"(!). Hertz, on the other hand, uses the word *moment* as synonymous with our impulse. The otherwise very common designation "quantity of motion" (quantité de mouvement) expresses only the length but not the direction of the impulse vector, and therefore appears to us inappropriate.

§2. The elementary statics of rigid bodies

Before we can attack the kinetics of the top, we must orient ourselves with respect to the composition and decomposition of a system of forces applied to our body. One groups, as is well known, all those investigations that treat merely of forces, without regard to the resulting motion, under the rather inappropriate name of *statics*, in which the question of the composition of a given system of forces is reduced to the search for such forces which, added to the given system of forces, would produce equilibrium. The word *dynamics* would be more suitable; that word is usually applied, however, to the part of mechanics that we designate as kinetics.

The treatment of statics for the present case of the rigid body can be accomplished according to two essentially different methods, which are characterized by the names of P o i n s o t and L a g r a n g e. We wish to refer briefly to both methods.

The statics of rigid bodies in the geometric treatment of Poinsot is based on a series of *axioms* that we recognize, in part, from the previous case of a single mass particle. We said that the force on a single mass particle has the character of a vector, and that multiple forces applied at the same point add as vectors. For rigid bodies the following axiom is added: *the point of application of a force can be displaced arbitrarily in the direction of the force.* This axiom is obviously independent of the previous, since its validity is bound essentially to the nature of a rigid body; it can be regarded directly as the definition of the latter.[54] For actual bodies, which are always elastic to a certain degree, the axiom is naturally fulfilled only approximately. Moreover, N e w t o n's lex tertia, which states the equality of actio and reactio, is interpreted so that it comprises our axiom, which appears somewhat artificial indeed.

With the help of this axiom, one now investigates the composition of forces which, somehow given, are distributed on the body. One first sees immediately that two forces that act in parallel directions can always be replaced by a single force whose direction is parallel to the directions of the original forces. The determination of the point of application and the magnitude of this single force forms the content of the so-called "law of the lever." If the forces are equal in magnitude and opposite in direction, a remarkable singularity results. Namely, the point of application then moves to infinity, while, at the same time, the magnitude become infinitely small.[55] *A force-pair* (that is, a pair of oppositely directed equal and parallel forces) *is thus equivalent to an infinitely small force that acts on an infinitely long lever arm.*

To keep the presentation elementary, however, the consideration of infinitely small forces and infinitely large lever arms is usually avoided. As a result, one is obliged to regard the force-pair as an irreducible element of the statics of rigid bodies. It becomes necessary, further, to discuss the composition and decomposition of force-pairs in a manner similar to the composition and decomposition of forces. New axioms are not needed for this purpose, since the very definition of the force-pair allows the question of the equilibrium of pairs to be reduced to the question of the equilibrium of forces.

We summarize the results of these investigations in the following way: we represent the pair by a *vector*, which we lay down perpendicular to the plane lying through the forces, on that side from which the forces appear to act in the clockwise sense. The length of the vector (in the centimeter scale chosen once and for all) is equal to the "moment of the pair"; that is, equal to the product of the magnitude of the forces and their shortest distance of separation. The initial point of our vector can be taken arbitrarily in the plane of the pair, or also arbitrarily in space. Thus obtains the theorem: *two force-pairs are composed in such a way that the corresponding vectors add geometrically. Composition of multiple pairs always results again in a pair.*

Force-pairs thus have, just as forces, the character of vectors. We must, however, emphasize the following difference. The vector of a force applied to a rigid body may be displaced only in its direction, while the vector of a pair, in contrast, may be transported parallel to itself arbitrarily in space. *The vector of a force is* (in the manner of expression of Mr. B u d d e) *a sliding*[56] *vector, while that of a pair is a free vector; that is, a vector with a completely arbitrary point of attachment.*

It is evident from the preceding that a specified force-pair can be replaced, with regard to its static effect, in a great variety of ways by another force-pair. In fact, two force-pairs that give the same vector by the given construction are completely equivalent. It is thus advisable to abandon the particular manifestation of the force-pair and retain only the representing vector. We wish to accommodate this circumstance in the designation as well, and prefer to speak of a *turning-force* (or a *turning-moment*) instead of a force-pair. We also designate the direction of the turning-force, according to its definition as a vector, as the "axis of the turning-force"; we mark the sense of the vector with an arrow that surrounds the axis in the sense that the forces of the pair appear to act as seen from the axis. The dimension of the turning-force is force times lever arm $\left(D = m\frac{l^2}{t^2}\right)$. In contrast to the expression turning-force (force-pair), we will temporarily use the word *pushing-force* (single-force).

We must beware, however, of associating the expression turning-force with the perception that the turning-force would tend to turn about a determined straight line. The turning-force occurs here in a purely static way. Its kinetic effect can be discussed only later, when we have treated of the kinetic effects of forces in general, and have made certain assumptions about the mass distribution of the body. The designation pushing-force should, naturally, imply equally little that the kinetic effect of a pushing-force necessarily consists in a parallel displacement.

We now enter into the general problem of statics, and therefore assume that forces are spatially distributed on the rigid body in an arbitrary manner. We proceed, as usual, in the following way: we adopt an arbitrary point O as the *reference point*, and place vectors through this point with the same and opposite senses as each of the given vectors. We group each of the given forces with the oppositely directed force through O into a force-pair, and replace the force-pair, according to the rule above, with the vector of a turning-force, where we conveniently choose the reference point O as the initial point of the vector. We thus obtain as many turning-forces as the number of original forces. We compose all these turning-forces into a resultant turning-force D, whose axis may pass though O. There then remain the forces (pushing-forces) through O in the same direction as the given forces. We compose these also into a resultant S. Thus the theorem:

An arbitrary system of forces applied to our rigid body may be replaced by a combination of a pushing-force and a turning-force emanating from an arbitrary point O.

We note the general rule for the calculation of S and D. Let P_i be one of the forces applied to our body, and P_i^x, P_i^y, P_i^z the projections of the vector P_i on the axes of a rectangular coordinate system whose origin coincides with the reference point. Further, let X_i, Y_i, Z_i be the coordinates of the application point of P_i, and S^x, S^y, S^z and D^x, D^y, D^z the components of S and D, respectively. We then have

$$(1)\quad \begin{cases} S^x = \Sigma\, P_i^x, & S^y = \Sigma\, P_i^y, \qquad\qquad S^z = \Sigma\, P_i^z, \\ D^x = \Sigma\,(P_i^z Y_i - P_i^y Z_i), & D^y = \Sigma\,(P_i^x Z_i - P_i^z X_i),\ D^z = \Sigma\,(P_i^y X_i - P_i^x Y_i). \end{cases}$$

These very well known formulas are the immediate analytic expression of the geometric construction for S and D described above. One concisely calls S^x, S^y, S^z, D^x, D^y, D^z *the coordinates of the force system* (applied to the rigid body).

In general, the direction of S and the axis of D will form an angle that depends on the choice of the reference point and the nature of the given force system. But we can always choose the point O so that the vectors S and D coincide exactly in their directions, and O can still be chosen arbitrarily on a certain line. We will call this simplest equivalent of a general system of forces—a pushing-force combined with a turning-force that has the direction of the pushing-force as its axis—a *screw* (or, more precisely, a *force-screw*). We can then designate the quantities $S^x, \ldots, D^x, \ldots$ as *the coordinates of the force-screw* and can make the preceding theorem more precise in the following way:

An arbitrary force system applied to our rigid body may always be conceived as a screw whose coordinates are determined by (1).

If one places the reference point on the axis of the screw, then the components of S become proportional to those of D. However, one will often forgo this simplification in the formulas, and will prefer to give the reference point a position distinguished by the nature of the problem; this will, naturally, not hinder us from still representing the force system as a screw (although a screw not passing through O). Thus one prefers to choose the reference point for the freely moving rigid body as the center of gravity; we will later make this choice when we treat of the top moving on a plane. On the other hand, it is generally advisable to place the reference point at the fixed support point for the case of the (generalized or symmetric) top. If we construct the turning-force and the pushing-force at this point O, we will then have, in the following, to consider only the turning-force.

In fact, whatever the details of the circumstances that effect the fixed position of the support may be, they must in any case provide, in opposition to the applied pushing-force at O, an equally large oppositely directed resistance force. This resistance force is called the *reaction force of the support point.* It is equal in magnitude to the resultant S found above, and is opposite in direction. If we add this reaction force to our system of forces, then the pushing-force S is directly canceled, while the turning-force remains exactly the previous quantity D. We can thus completely disregard the appearance the pushing-force from the start, and will have to return to it later only occasionally, if we would calculate the force that the foundation of the top must bear due to its motion.

At the same time, our general theorems above simplify in the present case. We can say:

The most general system of forces applied to our top can be replaced, with consideration of the fixed position of the support point, by a single turning-force. —

One observes the beautiful analogy between our static theorems and the kinematic theorems developed at the beginning of these lectures. The analogy for a free body is such that turning-forces must be compared with (infinitesimal) parallel displacements, and pushing-forces with (infinitesimal) rotations. The same geometric figure, the screw, appears once as a motion-screw, and once again as a force-screw. For the top, turning forces and (infinitesimal) rotations about O appear directly in parallel. Both are represented by vectors.

It is remarked, further, that the top is not inferior in simplicity, with respect to statics, to a single mass particle. The possibility of the later elementary geometric development for the theory of the top rests essentially on this circumstance.

As an example, we discuss the particularly simple case in which the original force system is provided by gravity. On each element dm of the top, the gravitational force $g\,dm$ acts vertically downward.

We take an X, Y, Z coordinate system with the origin at O and the Z-axis vertically upward. From equations (1) there follow immediately, if m is the total mass of the top and ξ, η, ζ are the coordinates of the center of gravity,

$$S^x = 0, \quad S^y = 0, \quad S^z = -g \int dm = -mg,$$

$$D^x = -g \int Y\, dm = -mg\eta, \quad D^y = g \int X\, dm = mg\xi, \quad D^z = 0.$$

We thus arrive at the very well known fact that *the effect of gravity is the same as a single force of magnitude mg applied perpendicularly downward at the center of gravity.*

For the symmetric top, the center of gravity obviously lies either on the figure axis or on its extension through point O. Let E be the

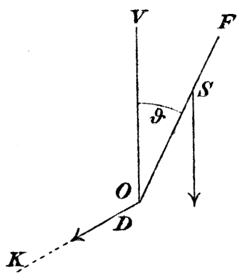

Fig. 11.

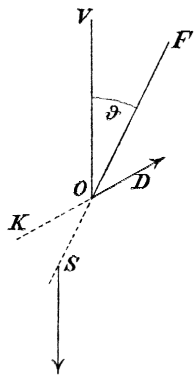

Fig. 12.

distance of the center of gravity from O, and ϑ, as earlier, the angle between the figure axis and the vertical. Then the vector D has length

$$mgE \sin\vartheta;$$

its direction is perpendicular to the vertical as well as to the figure axis. If we recall, further, the definition of the line of nodes on page 17, then we can say that the vector D lies on the line of nodes or on its extension through O, according to whether S lies on the figure axis or on its extension. We can also express this by saying that the vector D always lies on the line of nodes, and has the magnitude

$$D = P \sin\vartheta, \quad P = \pm mgE,$$

where the upper or lower sign is chosen according to whether the center of gravity lies above or below the support point (sc. for a vertically erected figure axis). The latter manner of expression, which we will later accept, has the advantage that we first treat of the two different cases uniformly, and can separate them from one another afterward through the simple conditions $P > 0$ and $P < 0$.

As was the case in the latter example, it is usually assumed implicitly that the forces which are treated in statics are continuously acting forces. But one sees immediately that our entire development remains valid for impact forces, in so far as the collected impacts are applied only at the same time. In fact, each statement about continuous forces carries over immediately to impact forces (cf. the definition of the latter on page 70). In analogy to the concept of a turning-force, we will introduce the concept of a *turning-impact*; that is, the embodiment of a pair of equally large, oppositely directed parallel *pushing-impacts*. The magnitude, axis, and sense of the turning-impact are determined, just as for the turning-force, by the moment and the position of the constitutive single forces. The dimension of the turning-impact is $[D] = m\frac{l^2}{t}$. We therefore state the general theorems:

The most general system of impact forces applied to a freely moving rigid body may always be replaced by a single screw (more precisely said, an impact-screw).

and

The most general system of impact forces that is applied in any way to the points of our top may always be conceived as a single turning-impact, and may therefore be represented by a single vector emanating from O.

As mentioned previously, statics was founded in the form considered thus far by Poinsot (in whose work, however, the word "screw" does not appear). His fundamental work *éléments de statique* first appeared in the year 1803; an extraordinarily large number of editions has since followed. The proofs of the previously given theorems can be reviewed there. Statics was placed in relation with projective geometry, and especially with the linear geometry of Plücker, by Moebius.[57] In addition to the previously referenced theory of screws by Ball, we cite in particular, among more recent presentations, the two-volume textbook of Routh*), *Analytical Statics*, which may be especially recommended for precision and richness of content. —

We now enter into the previously mentioned second method of establishing the fundamental principles of statics, which is, in essence, due to Lagrange. It has, in contrast to the previously discussed

*) Cambridge, 2nd edition 1896.

presentation of Poinsot, the advantage of greater capacity for generalization; it may, on the other hand, appear less elementary. In this method, we derive the *composition of forces* from the *composition of the quantities of work* that the forces perform in an infinitesimal displacement of the rigid body.

We imagine again that an arbitrary system of forces P_i with arbitrary application points acts on our body. We compose the total work dA of our force system from the work elements dA_i of the individual forces P_i. We use the general fundamental theorem that *the work is independent of the choice of coordinate system; it is a scalar quantity; furthermore, quantities of work compose, as do scalar quantities, by addition in the algebraic sense.*

According to page 75, the work that the force P_i performs in an infinitesimal displacement of its point of application is

$$dA_i = (P_i^x x_i' + P_i^y y_i' + P_i^z z_i')\, dt,$$

where P_i^x, P_i^y, P_i^z, are the components of P_i and x_i, y_i, z_i are the coordinates of its point of application with respect to a coordinate system (x, y, z) fixed in space. Thus the total work is

$$dA = \Sigma\, dA_i = \Sigma\,(P_i^x x_i' + P_i^y y_i' + P_i^z z_i')\, dt.$$

We now recall the results of the first chapter, according to which every infinitesimal change of position of a rigid body consists of a parallel displacement and a rotation, and can be represented analytically by the six velocity coordinates x', y', z', p, q, r (cf. page 47). Using these coordinates, we first express the velocity (x_i', y_i', z_i') of the application point of P_i. As a result of the parallel displacement, the application point of P_i receives (just as every point of the body) the velocity (x', y', z'); as a result of the rotation, it acquires (see eqn. $(3')$ of page 41) the velocity

$$(-Y_i r + Z_i q,\ -Z_i p + X_i r,\ -X_i q + Y_i p).$$

Thus the resultant velocity of the application point of P_i is

$$\begin{aligned} x_i' &= x' - Y_i r\ + Z_i q, \\ y_i' &= y' - Z_i p + X_i r, \\ z_i' &= z' - X_i q + Y_i p. \end{aligned}$$

We enter these values into our above expression for the total work. This work is then written as

$$dA = (S^x x' + S^y y' + S^z z' + D^x p + D^y q + D^z r)\, dt, \tag{2}$$

where

$$\begin{cases} S^x = \Sigma\, P_i^x, \quad S^y = \Sigma\, P_i^y, \quad S^z = \Sigma\, P_i^z, \\ D^x = \Sigma\,(P_i^z Y_i - P_i^y Z_i), \quad D^y = \Sigma\,(P_i^x Z_i - P_i^z X_i), \\ D^z = \Sigma\,(P_i^y X_i - P_i^x Y_i). \end{cases} \tag{3}$$

These are precisely the quantities that appeared above in equations (1).

We wish to clarify the meaning of these quantities independently of what has been said earlier, and thus arrive at a simple new definition of them. According to equation (2), S^x is the ratio of the work that our system performs in a displacement of the body in the direction of the x-axis to the magnitude of this displacement. In the same way, D^x equals the ratio of the work that our system of forces performs in a pure rotation of the body about the X-axis, or, which is the same, the x-axis, to the magnitude of the rotation angle. The quantities S^x, …, D^x, … therefore have meanings that are completely analogous to the meanings of the components P^x, … of the force P applied to a single mass particle, which indeed, for their part, were originally defined (cf. page 70) as the ratio of a certain quantity of work to a certain infinitesimal motion. As a result, the extension of the concept of force from the single mass particle to our rigid body is immediate. We are able to speak briefly of a *total force applied to the rigid body, which is equivalent to the given system of individual forces. This total force is decomposed into a pushing-force S and a turning-force D, each of which can be resolved into three components with respect to the coordinate axes.* We will again designate the quantities S^x, S^y, S^z, D^x, D^y, D^z as the coordinates of our total force, as we have named the quantities x', y', z', p, q, r the coordinates of the instantaneous velocity. We can then say in brief:

The coordinates of the force are, according to their definition, nothing other than the factors that multiply the coordinates of the velocity in the expression for the work.

If, in particular, we treat of a body with a fixed support point, which we will take, as previously, to be the reference point, then we have for the pushing velocity $x' = y' = z' = 0$, and we can abstract the pushing-force S^x, S^y, S^z. The turning-force, in contrast, will again be defined precisely by equation (3). The total work that this turning-force performs in the infinitesimal displacement $(p, q, r)\,dt$ is therefore

$$dA = (D^x p + D^y q + D^z r)\,dt. \tag{2'}$$

The present definitions of the turning and pushing-force are, in many respects, preferable to the earlier ones, since they relate immediately to the concept of force for a single mass particle; they relieve us, in particular, from the introduction of the concept of the force-pair, which we earlier encountered by necessity. That the new and old definitions come to the same result is shown by the comparison of the expressions (1) and (3).

We could now develop anew the collected lessons of elementary statics; in particular, the facts that pushing-forces and turning-forces are sliding and free vectors, respectively, would appear as immediate consequences of the corresponding composition law of velocities and our fundamental theorem above, according to which quantities of work add as scalars.

Statics was established in this manner by Lagrange in his famous *mécanique analytique.* If we place the expression for work before the composition of forces, this is the same, in essence, as if we evaluate the equilibrium of a system on the basis of the *principle of virtual displacements*, according to the precedent of Lagrange. In fact, this principle states, as is well known, that an arbitrary system of given forces is in equilibrium if the work performed by the system vanishes for any possible infinitesimal displacement, or, somewhat more generally stated, that two different force systems are equivalent if the work performed by each of the two systems is equal for any possible displacement. In conformity with this principle, we have replaced the given force system of the P_i by a combination of a pushing-force S and a turning-force D. The means of expression was merely somewhat different than for Lagrange, in whose time the concept and the designation of work were not yet prevalent.[58] —

We wish, finally, to use the expression for the work to establish a convention for what we mean by the "*generalized coordinates of a force system*," just as was done for a single mass particle. We will proceed in complete analogy to page 78 for the single mass particle.

We can obviously establish the state of motion of the rigid body by many sets of six parameters other than the quantities x', y', z', p, q, r.

The nearest modification would be to select a different reference point, and to furthermore vary the positions of the xyz system in space and the XYZ system in the body. If we then decompose a specific force system into a pushing-force and a turning-force according to the above rule, we will find values for the components of these forces that differ from the previous. The coordinates of an infinitesimal change of position will change just as well. It is clear, on the other hand, that the work which a specific force system performs in a specific infinitesimal change of position must retain exactly the previous value. For a fixed choice of the units of length, time, and mass, the work has a fixed numerical value that is independent of the choice of coordinate system; it is (with respect to a change of the coordinate system) an *absolute invariant*, as we can say.

But we will consider still further changes in the coordinate specification of the instantaneous state of motion. We will establish, for example, the instantaneous rotation by the rates of change φ', ψ', ϑ' of the Euler angles instead of the quantities p, q, r, and we can further specify (as, for example, in the previous section) the position and velocity of the reference point in terms of the magnitudes and magnitude changes of three curvilinear coordinates ξ, η, ζ. The most general assumption is that we set x', y', z', p, q, r equal to arbitrary linear functions of six arbitrary velocity parameters ξ', η', ζ', φ', ψ', ϑ', with coefficients that depend on the values of ξ, η, ζ, φ, ψ, ϑ themselves. It is then asked how the coordinates of the force system are changed, or, more correctly, what we should understand by the words "coordinates of the force system." We establish in this respect the following convention.

We introduce the values of $x', \ldots, r$ in terms of $\xi', \ldots, \vartheta'$ into the expression (2) for the work, and order the expression according to the latter quantities. *We then define the quantities that appear as the factors of ξ', η', ζ', φ', ψ', ϑ' to be the coordinates of the force system corresponding to the velocity coordinates ξ', η', ζ', φ', ψ', ϑ', respectively.* This definition of the force components stands in exact analogy to our definition of those words for a single mass particle. The force coordinate corresponding to ξ, for example, is the ratio of the work that our force system performs in the displacement $d\xi$ to this displacement.

The new force coordinates will obviously be linear functions of the old, and the transformation equations that lead from the latter to the former appear entirely analogous to the transformation equations that express the old velocity coordinates in terms of the new; the coefficients of the horizontal and vertical ranks are merely interchanged with one another. We express this fact concisely when we say:

On the basis of our definition, the generalized force coordinates are always contragredient to the velocity coordinates.

We have already stated on page 79 the corresponding theorem, or, more correctly, the corresponding convention, for a single mass particle.

§3. The concept of the impulse for the generalized top. Relation between the impulse vector and the rotation vector. Connection to the expression for the *vis viva*

We now make the passage from statics to kinetics, and thus ask for the relation between the motion and the forces that cause the motion. The mass distribution of the body is now of decisive importance, so that we will from now on treat of the generalized and the symmetric top separately.

As in the case of the single mass particle, we place the *c o n c e p t o f t h e i m p u l s e* at the summit of kinetics. We illustrate this concept first for a freely moving rigid body, and proceed at once to a body fixed at one of its points.

The definition of the impulse is the following:

We consider the rigid body in an arbitrary state of motion, and ask for any system of impact forces that is capable of transforming the body instantaneously in its instantaneous position from rest to the considered state of motion. *This or any equivalent system of impact forces is called the impulse of the body.*

With consideration of the investigations of the preceding sections, we can immediately state the following theorems:

The impulse of a freely moving rigid body is a combination of a pushing-impact and a turning-impact; it can be conceived concisely as a screw.

and

The impulse of a rigid body with a fixed support point O is a single turning-impact; we can visualize it by the simple image of a vector emanating from O.

Remaining in the latter case, we place the kinematic vector of the rotational velocity into consideration alongside the static vector of the impulse.

A first exercise in the kinetics of the top will be to establish the mutual dependence of these two vectors.

To this end, we consider both the velocities and the impulses of all the individual mass elements of which the top is made.

We take the axis of the rotational velocity as the first coordinate axis of a rectangular coordinate frame XYZ, which has an unchanging position with respect to the body, and whose origin coincides with the support point. We denote the components of the impulse vector with respect to the coordinate axes by L, M, N, and those of the rotation vector, as earlier, by p, q, r. By assumption, the only nonzero value among the latter is p.

We now consider any element P of the body with mass dm. By virtue of the rotation along the X-axis, our element has a linear velocity

$$v = p\sqrt{Y^2 + Z^2}.$$

The impact force that is required to produce this velocity instantaneously has the magnitude $v\,dm$; its components with respect to the coordinate axes are, as one easily recognizes,

$$0, \quad -pZ\,dm, \quad pY\,dm.$$

An impact force of this form acts on our body for each distinguishable element P. The turning-force that corresponds to the system of these impact forces is then our impulse. Its components are calculated, according to the analytic rule of page 85, as

$$(1) \quad L = p\int (Y^2 + Z^2)\,dm, \quad M = -p\int YX\,dm, \quad N = -p\int ZX\,dm,$$

where the integral extends over the total mass of the body. The given expressions show immediately that the impulse vector generally differs in direction from the vector of the rotational velocity; while, according to assumption, the vector of the rotational velocity falls on the X-axis, the vector of the impulse also has components in the directions of the Y- and Z-axes.

If we assume that the instantaneous rotation occurs about the Y- or Z-axis, then we obviously have, by a cyclical interchange in a completely corresponding way, the following values for the components of the associated impulses:

$$(1') \quad L = -q\int XY\,dm, \quad M = q\int (Z^2+X^2)\,dm, \quad N = -q\int ZY\,dm,$$

and

$$(1'') \quad L = -r\int XZ\,dm, \quad M = -r\int YZ\,dm, \quad N = r\int (X^2+Y^2)\,dm.$$

But the impulse for a general position of the rotation vector now follows immediately from the preceding equations. As we know, rotational velocities, as well as turning-forces (impulses), compose as vectors; that is, their components simply add. To a rotation (p, q, r) about the axis $p:q:r$ there thus corresponds an impulse whose components are equal to the respective sums of the impulse components calculated in equations (1), (1′), and (1″). The associated impulse is thus

$$(2) \quad \begin{cases} L \;=\; p\displaystyle\int (Y^2+Z^2)\,dm - q\int XY\,dm - r\int XZ\,dm, \\ M = -\,p\displaystyle\int YX\,dm + q\int (Z^2+X^2)\,dm - r\int YZ\,dm, \\ N \;= -\,p\displaystyle\int ZX\,dm - q\int ZY\,dm + r\int (X^2+Y^2)\,dm. \end{cases}$$

The preceding equations immediately assume a very clear form if we introduce the quadratic form of the velocity components

$$(3) \quad T = \frac{1}{2}\left\{p^2\int (Y^2+Z^2)\,dm + q^2\int (Z^2+X^2)\,dm + r^2\int (X^2+Y^2)\,dm \right.$$
$$\left. - 2qr\int YZ\,dm - 2rp\int ZX\,dm - 2pq\int XY\,dm\right\}.$$

Namely, there then follow simply

$$(4) \qquad L = \frac{\partial T}{\partial p}, \quad M = \frac{\partial T}{\partial q}, \quad N = \frac{\partial T}{\partial r}.$$

We ask for the mechanical meaning of our quadratic form T. It appears that T *is the* vis viva *of the top; that is, the work which the impulse performs in the generation of the instantaneous state of motion.*

In fact, we calculate this work if we first compose the individual work that the individual applied impulses perform on the mass elements of the body. According to page 75, the work that is performed on the mass element dm in the generation of the velocity (x', y', z') is

$$dA = \frac{1}{2}(x'^2 + y'^2 + z'^2)\,dm.$$

The total work is calculated from this by integration over the total mass of the body; it becomes

$$\frac{1}{2}\int (x'^2 + y'^2 + z'^2)\, dm.$$

We reformulate this expression by introducing the rotational velocity p, q, r. We have produced the necessary expressions for x', y', z' on page 41. We thus obtain

$$\frac{1}{2}\int \{(-Zq + Yr)^2 + (-Xr + Zp)^2 + (-Yp + Xq)^2\}\, dm.$$

But the working out of this expression directly yields the right-hand side of equation (3). We will therefore say:

The vis viva *of the top is a homogeneous quadratic function of the components of the rotation vector, with constant coefficients that depend only on the mass distribution of the body.*

After we have recognized the meaning of T, we can declare equations (4) to be the exact analogue of equations (7) on page 76 for a single mass particle.

We will rewrite the expression for the *vis viva* in a series of other interesting forms. We first remark that according to a well-known theorem for homogeneous functions,

$$T = \frac{1}{2}\left(p\frac{\partial T}{\partial p} + q\frac{\partial T}{\partial q} + r\frac{\partial T}{\partial r}\right).$$

With consideration of equations (4), we can write instead

$$T = \frac{1}{2}(pL + qM + rN). \tag{5}$$

We express this formula in words in the following way:

The vis viva *is equal to half the product of the magnitude of the impulse vector and the projection of the rotation vector onto the impulse vector (or also equal to half the product of the magnitude of the rotation vector and the projection of the impulse vector onto the rotation vector).*

In the language of vector analysis (cf. page 62) we can also say concisely:

The vis viva *is equal to half the scalar product of the impulse vector and the rotation vector.*

We may also develop the last formula directly from the consideration of the total system, without entering into the synthesis of the rigid body from its individual mass elements, if we carry over a consideration given on page 75 for a single mass particle directly to our case.

We begin from the work that an arbitrary continuous turning-force (D^x, D^y, D^z) performs on our top in a displacement $p\,dt$, $q\,dt$, $r\,dt$. This work, according to equation $(2')$ of page 91, is

$$dA = (D^x p + D^y q + D^z r)\,dt. \tag{6}$$

From this we derive the expression for the finite work that our turning-impacts L, M, N perform in the *generation* of the rotation p, q, r (that is, exactly the expression for the *vis viva*) by integration with respect to time in the following manner.

We can conceive our turning-impact L, M, N as a continuous turning-force of very large constant magnitude and very small application duration Δt. We can therefore set

$$L = \int_0^{\Delta t} D^x\,dt = D^x \Delta t,\ M = \int_0^{\Delta t} D^y\,dt = D^y \Delta t,\ N = \int_0^{\Delta t} D^z\,dt = D^z \Delta t. \tag{7}$$

At the beginning of the interval Δt, the rotational velocity of the body equals zero, and at the end of the interval Δt it equals (p, q, r). We must now assume that in the intermediate time the velocity increases *uniformly*, so that

$$\int_0^{\Delta t} p\,dt = \frac{1}{2} p \Delta t,\quad \int_0^{\Delta t} q\,dt = \frac{1}{2} q \Delta t,\quad \int_0^{\Delta t} r\,dt = \frac{1}{2} r \Delta t. \tag{8}$$

If we integrate the expression (6) for the work between $t = 0$ and $t = \Delta t$, then we obtain, with consideration of (7) and (8),

$$\begin{cases} T = \displaystyle\int_0^{\Delta t} dA = D^x \int_0^{\Delta t} p\,dt + D^y \int_0^{\Delta t} q\,dt + D^z \int_0^{\Delta t} r\,dt \\[2ex] \qquad\qquad = \dfrac{1}{2} (D^x p + D^y q + D^z r)\,\Delta t \\[2ex] \qquad\qquad = \dfrac{1}{2} (Lp + Mq + Nr). \end{cases} \tag{9}$$

We thus return directly to equation (5).

In equations (4), we have assumed T to be a function of the velocity coordinates p, q, r. But we can also calculate T as a function of the impulse coordinates. It is enough, for this purpose, to solve equations (2) for p, q r and enter the resulting values of the latter quantities into (5). From (2) there first result

$$\begin{cases} p = A_{11} L + A_{21} M + A_{31} N, \\ q = A_{12} L + A_{22} M + A_{32} N, \\ r = A_{13} L + A_{23} M + A_{33} N, \end{cases} \tag{2'}$$

where the A_{ik} denote the values of the subdeterminants of the coefficient schema in (2) divided by the determinant, and where $A_{ik} = A_{ki}$. With consideration of (5), we now obtain for T the expression

$$T = \frac{1}{2}(A_{11}L^2 + 2A_{12}LM + \cdots + A_{33}N^2). \tag{3'}$$

Conceived as a function of the impulse coordinates, T is again a homogeneous quadratic form with constant coefficients.

We will, further, form the partial differential quotients of this function with respect to L, M, N. These will obviously be equal to the right-hand sides of equations (2′), so that we find the relations

$$p = \frac{\partial T}{\partial L}, \quad q = \frac{\partial T}{\partial M}, \quad r = \frac{\partial T}{\partial N}. \tag{4'}$$

The equations (4′) *represent the solution of equations* (4) *written in a characteristically symmetric form.* It is emphasized that T is expressed above as a function of p, q, r, and now as a function of L, M, N.

Equations (4), or the equivalent equations (4′), yield the desired relation between the impulse vector and the rotation vector in the most general form. They represent the first and most important equations in the kinetics of the top. They have, moreover, exactly the same form as the analogous equations for a single mass particle (cf. page 76). We can, collecting both triplets of equations, repeat the statement of page 77:

The impulse (velocity) components are the partial differential coefficients of the vis viva *taken with respect to the velocity (impulse) components, where we must think of the* vis viva *expressed as a function of the velocity (impulse) components.*

We next bring the expression for the *vis viva* into relation with the *concept of the moment of inertia.* The coefficients of $\frac{1}{2}p^2$, $\frac{1}{2}q^2$, $\frac{1}{2}r^2$ in expression (3) are designated, as is well known, as the moments of inertia of the body about the axis of X, Y, Z, respectively. On the other hand, the coefficients of $-pq$, $-qr$, $-rp$ in the same expression are occasionally called the "products of inertia" (or also the "centrifugal moments"). Further, the moment of inertia M of the body about an arbitrary axis is defined by the equation

$$M = \int R^2 \, dm,$$

where R denotes the distance of the element dm from the considered axis, and where the integral extends over the entire mass of the body.

But we arrive at the same integrals in the expression for the *vis viva*. We note that the linear velocity of an element dm of our body is equal to the product of the angular velocity of the body about the instantaneous rotation axis and the distance of the element from this axis; if we denote the former by Ω and the latter by R, then

$$x'^2 + y'^2 + z'^2 = \Omega^2 R^2.$$

Thus

$$(10) \qquad T = \frac{1}{2}\int (x'^2 + y'^2 + z'^2)\, dm = \frac{\Omega^2}{2}\int R^2\, dm = \frac{M}{2}\,\Omega^2.$$

We compare the expression $T = \dfrac{M}{2}\Omega^2$ for the *vis viva* of a rigid body with the formula $T = \dfrac{m}{2}v^2$ for an individual mass particle. We can then say:

The vis viva *of the top is calculated from the angular velocity and the moment of inertia corresponding to the instantaneous axis of rotation in exactly the same manner as the* vis viva *of a single particle is calculated from the velocity and the mass.*

We may use equation (10), further, to establish a general expression for M. If we denote the direction cosines of the instantaneous rotation axis $p\!:\!q\!:\!r$ with respect to the coordinate frame XYZ by α, β, γ, so that

$$\alpha = \frac{p}{\Omega}, \quad \beta = \frac{q}{\Omega}, \quad \gamma = \frac{r}{\Omega},$$

then there results from (10) and (3)

$$(11)\begin{cases} M = \alpha^2\displaystyle\int (Y^2+Z^2)\, dm + \beta^2\int (Z^2+X^2)\, dm + \gamma^2\int (X^2+Y^2)\, dm \\ \qquad\qquad -2\beta\gamma\displaystyle\int YZ\, dm - 2\gamma\alpha\int ZX\, dm - 2\alpha\beta\int XY\, dm. \end{cases}$$

The moment of inertia about an arbitrary axis (α, β, γ) is therefore a homogeneous quadratic function of the direction cosines α, β, γ, and depends on these quantities in completely the same manner as $2T$ depends on the velocity components p, q, r.

We introduce next the concept of the *ellipsoid of inertia*, customary since Poinsot, by first laying off on the axis (α, β, γ) the segment $\varrho = \sqrt{1/M}$ as a radius vector. The endpoint of this segment has the coordinates

$$\xi = \alpha\varrho, \quad \eta = \beta\varrho, \quad \zeta = \gamma\varrho.$$

If we make the same construction for all possible axes (α, β, γ), then there results a surface of the second degree, and, in fact, an ellipsoid that has the equation

$$1 = \xi^2 \int (Y^2 + Z^2)\, dm + \eta^2 \int (Z^2 + X^2)\, dm + \zeta^2 \int (X^2 + Y^2)\, dm$$
$$-2\eta\zeta \int YZ\, dm - 2\zeta\xi \int ZX\, dm - 2\xi\eta \int XY\, dm.$$

The three principal axes of this ellipsoid are the so-called *principal inertial axes.* If we imagine the coordinates X, Y, Z shifted to the principal inertial axes, then the products $\eta\zeta$, $\zeta\xi$, $\xi\eta$ in the equation for the ellipsoid of inertia must vanish. *The principal axes are therefore distinguished in that the products of inertia are equal to zero if the principal axes are used as the coordinate axes.* The equation for the ellipsoid of inertia in these coordinates takes the form

$$1 = A\xi^2 + B\eta^2 + C\zeta^2, \tag{12}$$

where the quantities

$$A = \int (Y^2 + Z^2)\, dm, \quad B = \int (Z^2 + X^2)\, dm, \quad C = \int (X^2 + Y^2)\, dm$$

are called the *principal moments of inertia* with respect to the support point O.

Not every ellipsoid, moreover, can be an ellipsoid of inertia. Namely, one easily recognizes from the given expressions for A, B, C that these quantities satisfy the inequalities

$$A < B + C, \quad B < C + A, \quad C < A + B,$$

inequalities that we summarize in the simplest way by the statement that A, B, C are the sides of a possible straight-line triangle. *Thus the only ellipsoids that correspond to ellipsoids of inertia of actual bodies are those for which the squares of the reciprocals of the major axes can form a triangle.*

The expression (11) for the moment of inertia about an arbitrary axis goes over, in our present choice of coordinates, to

$$M = A\alpha^2 + B\beta^2 + C\gamma^2.$$

But the *vis viva* of the body transforms in just the same way as this expression. According to (10), we obtain for the *vis viva*

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2). \tag{13}$$

Finally, equations (2) also simplify substantially if we let the coordinate frame coincide with the "principal coordinate frame." *Namely, the impulse vector (L, M, N) that corresponds to an arbitrary rotation* *vector* (p, q, r) *is now determined through the fundamental equations*

$$L = Ap, \quad M = Bq, \quad N = Cr. \tag{14}$$

In consequence, there results from (13) the following expression for the *vis viva* in terms of the impulse coordinates:

$$T = \frac{1}{2}\left(\frac{L^2}{A} + \frac{M^2}{B} + \frac{N^2}{C}\right). \tag{13'}$$

We are able to conclude from equation (14), just as from the earlier equation (1), that the rotation vector and the impulse vector form, in general, a nonzero angle with each other. In fact, in so far as the principal moments of inertia are all different from one another, the proportion

$$L : M : N = p : q : r$$

is fulfilled only if two components of the rotation vector (impulse vector) vanish. *The rotation vector and the impulse vector thus coincide only if one of the two vectors (and therefore simultaneously the other) lies on one of the three principal axes.*

The relation between our two vectors may be described in geometric form, finally, by a simple construction that was already given, in essence, by Poinsot.

We begin from the ellipsoid of inertia

$$A\xi^2 + B\eta^2 + C\zeta^2 = 1$$

and place through the endpoint of the rotation vector p, q, r the ellipsoid similar and similarly situated with the ellipsoid of inertia. This ellipsoid will have the equation

$$A\xi^2 + B\eta^2 + C\zeta^2 = Ap^2 + Bq^2 + Cr^2 = 2T.$$

At the endpoint of the rotation vector we then place the tangent plane

$$Ap\xi + Bq\eta + Cr\zeta = 2T$$

to this ellipsoid. The perpendicular from O to this plane has the direction

$$Ap : Bq : Cr = L : M : N;$$

that is, the direction of the impulse vector. The length of the perpendicular is

$$\frac{2T}{G},$$

where

$$G^2 = L^2 + M^2 + N^2$$

denotes the square of the length of the impulse vector. If the rotation vector, and therefore also the magnitude T of the *vis viva*, are given to us, then the direction and the magnitude of the impulse are determined from our construction. In particular, we can state the theorem:

The direction of the impulse vector lies perpendicular to the plane that is conjugate to the rotation vector with respect to the ellipsoid of inertia.[59]

A completely analogous construction leads to the magnitude and direction of the rotation vector if the impulse vector is given. We place the plane normal to the impulse vector through its endpoint. The equation of this plane is

$$Ap\xi + Bq\eta + Cr\zeta = G^2.$$

Among the surfaces similar and similarly situated with the ellipsoid of inertia, there is one that is tangent to our plane. It is the ellipsoid

$$A\xi^2 + B\eta^2 + C\zeta^2 = \frac{G^4}{2T}.$$

The binding line of the tangent point with O then yields the direction of the rotation vector. We obtain the magnitude of the rotation vector if we compare any one linear dimension of the latter ellipsoid with the corresponding linear dimension of the ellipsoid of inertia. Two such lengths stand in the proportion $G^2 : \sqrt{2T}$. Since G now is given, the magnitude of T and thus also the length of the rotation vector are known. —

It would not be difficult to give the corresponding development more generally for the case of the freely moving rigid body. We can limit ourselves to writing the results for this case directly, since their derivation differs only slightly from the development above.

We denote the coordinates of the impulse screw by X, Y, Z, L, M, N and the coordinates of the motion screw, as earlier, by x', y', z', p, q, r. The former quantities are determined from the latter in the simplest way by means of the equations

$$\text{(16)} \qquad \begin{cases} X = \dfrac{\partial T}{\partial x'}, & Y = \dfrac{\partial T}{\partial y'}, \quad Z = \dfrac{\partial T}{\partial z'}, \\[2ex] L = \dfrac{\partial T}{\partial p}, & M = \dfrac{\partial T}{\partial q}, \quad N = \dfrac{\partial T}{\partial r}. \end{cases}$$

Here T is the expression for the *vis viva* written as a function of the velocity coordinates. To arrange this expression most conveniently, one places the reference point at the center of gravity, and lets the coordinate system fixed in the body coincide with the principal axes passing through the center of gravity. The *vis viva* T then becomes simply

$$T = \frac{m}{2}\,(x'^2 + y'^2 + z'^2) + \frac{1}{2}\,(Ap^2 + Bq^2 + Cr^2).$$

Written in terms of the impulse coordinates, T takes the form

$$T = \frac{1}{2m}\,(X^2 + Y^2 + Z^2) + \frac{1}{2}\Big(\frac{L^2}{A} + \frac{M^2}{B} + \frac{N^2}{C}\Big).$$

From this one recognizes that the inversion of equations (16) is

$$(16') \qquad \begin{cases} x' = \dfrac{\partial T}{\partial X}, & y' = \dfrac{\partial T}{\partial Y}, \quad z' = \dfrac{\partial T}{\partial Z}, \\[2mm] p = \dfrac{\partial T}{\partial L}, & q = \dfrac{\partial T}{\partial M}, \quad r = \dfrac{\partial T}{\partial N}, \end{cases}$$

where the expression for T in terms of the coordinates of the impulse screw is used for the differentiation. These equations are, to be sure, derived here for the sake of simplicity only for a special position of the reference point and for a special choice of the coordinate axes XYZ. They are, however, independent of this choice, and hold just as generally as equations (16).

The first parts of equations (16) and (16′) (referring to the motion of the center of gravity) are exactly identical to equations (7) and (7′) of the first section, and the second parts of equations (16) and (16′) (referring to the motion about the center of gravity) are exactly identical to equations (4) and (4′) of this section. They represent the first and most important determining equations in the kinetics of a free rigid body.

The expressions given above for the *vis viva* immediately result in the equation

$$T = \frac{1}{2}\,(x'X + y'Y + z'Z + pL + qM + rN),$$

which naturally can again be derived directly from the expression (2) on page 90 for the work performed in an infinitesimal displacement. The parenthesis on the right-hand of this equation naturally has a simple geometric meaning that depends only on the nature of the two screws and their relative positions, but not on their absolute position in space, and will be called the *moment of the two screws upon each other.* In terms of the pitches h and h' of the two screws, the shortest distance Δ, the angle of inclination φ of the two screw axes, the magnitude Ω of the rotational velocity, and the magnitude S of the pushing-impulse, the moment is expressed as*):

$$\Omega S\{2\pi\Delta\sin\varphi + (h + h')\cos\varphi\}.$$

*) Cf. F. Klein, Math. Ann. Bd. II, p. 368. Ball (l. c.) designates the expression in question as the “virtual coefficient” of the screws.

The concept of the impulse for the generalized top is completely developed by P o i n s o t in his repeatedly cited work. The designation of P o i n s o t is, somewhat ceremoniously, *couple d'impulsion* (translated in the German edition of S c h e l l b a c h as "the motion-stimulating force-pair" (!)).

§4. Transference of the preceding results to the special case of the symmetric top

We now proceed specifically to the symmetric top, and therefore assume that our body has rotational symmetry about the figure axis. The question now is the simplification of the preceding kinetic consideration that results from this assumption. The ellipsoid of inertia naturally has rotational symmetry about the figure axis in the same way as the mass distribution of the body. The ellipsoid of inertia thus becomes a surface of revolution. In addition to the figure axis, all axes in the equatorial plane of the top become principal axes of the ellipsoid and principal inertial axes of the body. All the principal moments of inertia corresponding to these principal axes, moreover, are equal to each other.

If we wish to refer the top to a principal inertial frame as a coordinate frame, then we need only place the Z-axis on the figure axis; the axes of X and Y then fall in the equatorial plane, and become principal inertial axes with equal principal moments of inertia. If we denote, as earlier, the principal moments of inertia by A, B, and C, then we have *the characteristic relation for the symmetric top*

$$A = B.$$

For the present choice of the coordinate system, the equation of the ellipsoid of inertia is thus

$$A(\xi^2 + \eta^2) + C\zeta^2 = 1;$$

the expression for the *vis viva* becomes

$$T = \frac{1}{2}\Big(A(p^2 + q^2) + Cr^2\Big),$$

and the relations between the components of the impulse vector and the rotation vector are written in their simplest form as

$$L = Ap, \quad M = Aq, \quad N = Cr.$$

We wish to return first to the generalization of the definition of the symmetric top that was mentioned already in the Introduction. *We* *will always call a rigid body with a fixed support point O a symmetric top (or simply a top) if two of the three principal moments of inertia through O are equal to each other and, moreover, the center of gravity lies on the axis of the third principal moment of inertia.* Such a body will behave, with respect to all questions concerning the rotation about the point O under the influence of gravity, exactly like a body that has the previously assumed *geometric* rotational symmetry about the figure axis. We will likewise carry over the designations "figure axis" and "equatorial plane of the top" to the figure axis and equatorial plane of the ellipsoid of inertia of our generalized body. The equatorial plane is thus distinguished by the fact that its collected axes are principal inertial axes with the same principal moment of inertia A. We can say of such a body that it has not a *geometric* rotational symmetry, but rather a *mechanical rotational symmetry about the figure axis.*

We will distinguish, moreover, three subclasses of symmetric tops, according to whether the ellipsoid of inertia is a prolate ellipsoid of revolution, an oblate ellipsoid of revolution, or, in particular, a sphere. We thus speak of a *prolate top*, an *oblate top*, or a *spherical top.* The spherical top is distinguished, in particular, by the fact that any axis passing through O represents a principal inertial axis of the body. Since the principal axes of the ellipsoid of inertia are the reciprocal values of $\sqrt{A}$ and $\sqrt{C}$, the ellipsoid of inertia will be prolate when $A > C$ and oblate when $A < C$. In the limiting case $A = C$, the ellipsoid of inertia goes over into a sphere. Thus the conditions for the three cases are

$$\begin{aligned} \textit{prolate top} &: A > C, \\ \textit{oblate}\ \text{''} &: A < C, \\ \textit{spherical}\ \text{''} &: A = C. \end{aligned}$$

As examples of the three types of tops with *geometric* rotational symmetry, we can always take an ellipsoid of revolution filled with homogeneous mass, which accordingly is prolate, oblate, or a sphere. It is also easy, however, to construct examples of tops with only *mechanical* rotational symmetry. In fact, four mass particles of equal mass that form the corners of a square, and are imagined to be bound to each other by rigid massless rods, represent a *symmetric top with an oblate ellipsoid of inertia* that has only *mechanical* rotational symmetry. If we fix a fifth mass particle on the figure axis of this top (that is, on the normal erected from the midpoint O of the square), then we obtain, depending on the distance of this particle from O and its mass, an oblate top or a spherical top with again only *mechanical* rotational symmetry. In particular, we emphasize here for later use that we can construct in this manner a *spherical top that has an arbitrarily given (positive or negative) gravitational turning-moment P, and whose center of gravity does not, therefore, coincide with O.* For this purpose, we can choose the masses of four formerly cited particles equal, for example, to 1 gr, and the side length of the square in whose corners the particles are fixed equal to 1 cm. For the fifth particle, we must then arrange that its mass m and its distance E from O are

$$m = \left(\frac{P}{g}\right)^2, \qquad E = \frac{g}{P},$$

where g is the acceleration of gravity.[60]

We now carry out the *P o i n s o t* construction, through which we clarify the relation between the impulse axis and the axis of rotation for the symmetric top. The simplification here, compared with the earlier case of the generalized top, is that we can carry out the construction in a plane; namely, the meridian plane passing through the instantaneous axis of rotation. A characteristic distinction between our three types of tops is thus made noticeable.

We imagine the rotation vector as somehow given. Through the axis OR of this vector we place the meridian plane FOR, which we will use in the following as the plane of the drawing. We draw the figure axis vertically upward. The tangent plane at the point R', the intersection of the rotation axis with the ellipsoid of inertia or one of the similar and similarly situated ellipsoids used on page 101, is perpendicular to the plane of the drawing, and thus the perpendicular from O to this plane lies in the plane of the drawing. Instead of the tangent plane, it is therefore enough to consider the tangent to the ellipsoid of inertia lying in our meridian plane. The situation in detail is as follows.

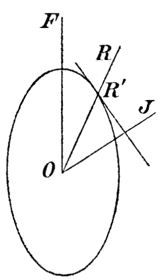

Fig. 13.

1. *The prolate top*, $A > C$. The perpendicular from O to the tangent at R' falls on the opposite side of the rotation axis as the figure axis (cf. Fig. 13). *For the prolate top, the rotation axis therefore lies between the impulse axis and the figure axis.*

2. *The oblate top*, $A < C$. The perpendicular from O to the tangent

at R' lies in the acute angle between the figure axis and the rotation axis (cf. Fig. 14). *For the oblate top, the impulse axis therefore lies between the rotation axis and the figure axis.*

3. *The spherical top*, $A = C$. Since the meridian intersection of the ellipsoid of inertia degenerates into a circle, the perpendicular from O

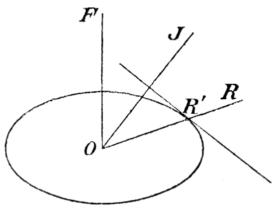

Fig. 14.

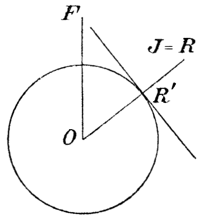

Fig. 15.

to the tangent through the contact point R' also passes through R' (cf. Fig. 15). *For the spherical top, the impulse axis and the rotation axis necessarily coincide.*

Only the spherical top behaves, so to speak, "*isotropically*"; that is, in the sense that the axis of the rotational motion coincides with the turning-force that generates the motion. The oblate and the prolate top exhibit the corresponding behavior only when the rotation is about the figure axis or about an axis of the equatorial plane, as is immediately evident from our construction. All these cases, moreover, are included in our general rule above, according to which the rotation vector and the impulse vector coincide in direction if and only if one of the two vectors lies along a principal inertial axis of the body. —

While we have thus far expressed the state of motion in terms of the components p, q, r, and have represented the impulse, correspondingly, in terms of the components L, M, N, we now wish to think of the instantaneous rotation of the symmetric top as given in terms of the changes in the Euler angles φ, ψ, ϑ, and ask for the "corresponding components of the impulse." The following considerations are also valid, moreover, for the generalized top. We give the development here first for the symmetric top only because the formulas for the generalized top become somewhat long.

We have already established on page 92 how we define, in general, the force coordinate corresponding to a velocity coordinate for a rigid body. (We refer also to the completely analogous consideration on page 78 for a single mass particle.) What was said for the force coordinates naturally holds equally well for the impulse coordinates; what was developed for the freely moving rigid body carries over in an immediately more understandable manner to the top.

According to the noted rule, we proceed from the equations that express the old velocity coordinates p, q, r in terms of the new coordinates φ', ψ', ϑ'. These are the "kinematic" equations

$$\text{(1)} \qquad \begin{cases} p = \qquad \psi' \sin\vartheta \sin\varphi + \vartheta' \cos\varphi, \\ q = \qquad \psi' \sin\vartheta \cos\varphi - \vartheta' \sin\varphi, \\ r = \varphi' + \psi' \cos\vartheta \end{cases}$$

of page 45. As a result, the new impulse components, which we denote by $[\Phi]$, $[\Psi]$, $[\Theta]$, depend on the old components in the following way:

$$\begin{aligned} [\Phi] &= N, \\ [\Psi] &= L \sin\vartheta \sin\varphi + M \sin\vartheta \cos\varphi + N \cos\vartheta, \\ [\Theta] &= L \cos\varphi - M \sin\varphi. \end{aligned}$$

Here we replace L, M, N by their expressions in terms of φ', ψ', ϑ', which are given by equations (1), if we multiply those equations by A, A, and C, respectively. Thus we obtain

$$\text{(2)} \qquad \begin{cases} [\Phi] = C(\varphi' + \psi' \cos\vartheta), \\ [\Psi] = C \cos\vartheta\, \varphi' + (C \cos^2\vartheta + A \sin^2\vartheta)\psi', \\ [\Theta] = A\vartheta'. \end{cases}$$

We refer once more to the relation that holds between the impulse and velocity coordinates and the partial differential quotients of the *vis viva*. It is self-evident from our definition of the impulse coordinates that this relationship, which we know to be valid for the velocity coordinates p, q, r, remains valid if we introduce new coordinates φ', ψ', ϑ' that depend linearly on the old coordinates. But we may nevertheless convince ourselves of this as follows. According to equations (1), the expression for the *vis viva*

$$T = \frac{1}{2}\left(A(p^2 + q^2) + Cr^2\right)$$

is written in terms of the coordinates φ', ψ', ϑ' as

$$\text{(3)} \qquad T = \frac{1}{2}\left(A(\vartheta'^2 + \sin^2\vartheta\,.\,\psi'^2) + C(\varphi' + \cos\vartheta\,.\,\psi')^2\right).$$

But from this there result immediately, in analogy to equations (4) of the previous section,

$$\text{(4)} \qquad [\Phi] = \frac{\partial T}{\partial\varphi'}, \quad [\Psi] = \frac{\partial T}{\partial\psi'}, \quad [\Theta] = \frac{\partial T}{\partial\vartheta'}.$$

The relations analogous to equations (4′) are easily verified in the same way.

Finally, we ask for the *geometric meaning of our impulse coordinates* $[\Phi]$, $[\Psi]$, $[\Theta]$; we infer this from the geometric meaning of the expression $(2')$ on page 91 for the work of an infinitesimal displacement of our top.

Just as we could conceive the expression for the *vis viva* on page 96 as half the product of the length of the rotation vector and the projection of the impulse vector onto the rotation vector, we will now say that *the expression*

$$dA = (D^x p + D^y q + D^z r)\,dt$$

for the work that would be performed by an arbitrary turning-force D applied to our top in an infinitesimal displacement $(p, q, r)\,dt$ is equal, up to the factor dt, to the product of the magnitude of the turning velocity and the projection of the turning-force onto the axis of the turning velocity. If we now introduce our velocity coordinates φ', ψ', ϑ' instead of p, q, r, then the expression for the work retains, as we know, its earlier form. The preceding equation is thus transformed, if we denote the corresponding components of D by Φ, Ψ, Θ, into

$$dA = (\Phi\varphi' + \Psi\psi' + \Theta\vartheta')\,dt.$$

We now consider, in particular, an infinitesimal rotation for which $\psi' = \vartheta' = 0$, so that the corresponding work becomes equal to $\Phi\varphi'\,dt$. In this case, the rotation vector lies along the figure axis, since φ' denotes a rotation about the figure axis. From the geometric meaning of dA, it now follows immediately *that Φ denotes the perpendicular projection of the vector D onto the figure axis.* Further, the rotation vectors ψ' and ϑ' fall, respectively, in the directions of the vertical and the line of nodes. Thus it follows in the same way *that Ψ and Θ represent the perpendicular projections of the vector D onto the vertical and the line of nodes.* Exactly the same geometric meaning naturally belongs to our impulse coordinates $[\Phi]$, $[\Psi]$, $[\Theta]$. *These quantities are equal, respectively, to the perpendicular projections of the impulse vector onto the figure axis, the vertical, and the line of nodes.*

We can, without further elaboration, generalize the result of the last consideration by saying: *if, on the basis of any three skew-angular axes, we decompose the rotation vector into components parallel to these axes, then we obtain the corresponding decomposition of the force or impulse vector if we project perpendicularly to those axes.*

§5. The two fundamental theorems on the behavior of the impulse vector in the course of the motion

While we have oriented ourselves thus far with respect to the impulse that corresponds to an instantaneous state of motion of a body, or, equivalently, to the instantaneous state of motion resulting from a given impulse, it will be our next exercise to investigate the course of the motion in time. The previously discussed relation between the impulse vector and the instantaneous rotation vector is completely independent of the external circumstances under which the motion proceeds; that is, of the continuous forces that act on the rigid body. Our further considerations, however, will be determined in an essential way by these external forces. In this respect, we assume, on the one hand, that our body is entirely free of external forces, including, in particular, the force of gravity. On the other hand, we will admit of arbitrary continuous forces. We argue first for the freely moving rigid body. In doing so, we place the *investigation of the impulse* before the *consideration of the state of motion.* We therefore ask, in the first place: *how is the impulse of our body changed for force-free motion?* We derive the change of the state of motion from the behavior of the impulse only later. The answer to our question is simply the following:

The impulse is completely unchanged; it remains constant in space during the motion.

We establish this fundamental theorem in the most elementary manner from the kinetics of individual mass particles as follows.

We begin with a single mass element P, which moves freely and is subjected to no external forces. As we know, the direction and the magnitude of the impulse of such an element remain constant in space. (Galileo's law of inertia.)

We now consider two mass elements P and P' that are rigidly bound and are free, moreover, of external forces. We replace the action of the rigid binding dynamically by forces; in particular, we have a force directed toward P' at the point P and a force of the same magnitude directed toward P at the point P'. (Newton's lex tertia.) The common magnitude of the two forces depends on the requirements of the rigid binding, and modifies, in magnitude and direction, the directly applicable individual impulses of P and P'. If we include these forces — we can call them reaction forces — we may then treat of our two mass particles as freely moving particles. Now the individual impulses of P and P'

change continuously as a result of the presence of the reaction forces; the impulses add geometrically with the infinitesimal impacts corresponding to the reaction forces (Newton's lex secunda). It is different with the total impulse of the system formed by the two mass particles. We construct this total impulse by composing the individual impulses of the two particles according to the rules of elementary statics. In this construction, however, the two oppositely directed equal reaction forces obviously cancel at each moment. The resulting impact-screw therefore behaves exactly as if our reaction forces did not exist and our mass particles were free. *The impulse thus remains constant for the entire duration of the motion.*

A system of three rigidly bound mass particles that are free of external forces behaves no differently. Here we must consider not one but rather three pairs of equal and opposite reaction forces that act on the sides of the triangle formed by our particles. The individual impulses of the system particles are again changed by these reaction forces. In the construction of the impact-screw corresponding to the entire system, however, the reaction forces do not come into consideration; this screw behaves exactly as if our three mass particles moved freely according to the Galilean law of inertia.

The same deliberation carries over immediately to the case of arbitrarily many bound mass particles, and, further, to a spatially continuous mass system that is free of external forces. It holds even for the more general case of a nonrigid system whose particles are subjected only to internal forces that satisfy the principle of action and reaction; the result is valid, for example, for an elastic body, for the planetary system, or for a fluid quantum.

For the case of a freely moving rigid body, we state the result here explicitly as the first of the theorems governing the course of the motion:

T h e o r e m I: *The impulse-screw of a rigid body remains constant in space for force-free motion.*

It will be useful to carry over this theorem into the language of ordinary analytic mechanics. To this end, we calculate the components of the impulse-screw according to the rule on page 85; for P_i^x, P_i^y, P_i^z, we must take the components of the individual impulses of all the mass particles that constitute the body; we therefore set

$$P_i^x = x_i' \Delta m_i, \quad P_i^y = y_i' \Delta m_i, \quad P_i^z = z_i' \Delta m_i$$

and must finally go over from summation to integration. We thus obtain

$$S^x = \int x' \, dm = c',$$

$$S^y = \int y' \, dm = c'',$$

$$S^z = \int z' \, dm = c'''.$$

The integrals appearing here are nothing other than the velocities of the center of gravity (multiplied by m). *Our theorem thus states that the velocity of the center of gravity is constant; the theorem is identical with the simplest case of the so-called center of gravity theorem.*

In the same manner, we calculate the turning-moments D^x, D^y, D^z of our screw about O. According to the same rule as above, there follow

$$D^x = \int (z'y - y'z) \, dm = c^{IV},$$
$$D^y = \int (x'z - z'x) \, dm = c^{V},$$
$$D^z = \int (y'x - x'y) \, dm = c^{VI}.$$

These equations are also well known to us from ordinary mechanics; they are simply the so-called area theorems.[61] *The second part of our theorem regarding the turning components is therefore identical with the area theorems,* which, as is well known, apply to the free motion of a rigid body.

One notes, in particular, that our geometric consideration implies certain simple integrations that one is forced to carry out in the analytic derivation. The equivalent of the integration process is the recognition that the infinitesimal additional impulses which originate from the reaction forces are mutually annihilated in the formation of the total impulse.

We could have defined the impulse of the body directly in terms of the constants of the center of gravity and the area theorems; the opposite path followed here, however, appears to us more instructive. Our derivation forced us to return to the true root of the theorems, the mechanical principles that are otherwise easily hidden behind formulas; it leaves, we believe, nothing more to be desired in transparency. It corresponds completely, moreover, to the tendencies of Poinsot, who gives, for his part, a much less simple proof*). In contrast, a consideration entirely analogous to the above is found in a beautiful work of R. B. Hayward**), to which we will make further reference.[62]

The transference of our theorem to a body with a fixed support is now given immediately. We must now decompose our impulse screw at the fixed point O into a pushing-impact S and a turning-impact D. The former will be canceled by the fastening of the body; that is, by the reaction force at the support point. There remains only the turning-impact, and this, for the top, is what we called the impulse. There follows again, in analogy to the above,

Theorem Ia: *The impulse vector of a body supported at one point remains constant in magnitude and direction in space for force-free motion.*

This theorem is identical with the statement that the *area theorems* of the previous case remain valid, while the *center of gravity theorems* obviously become invalid. At the same time, the theorem appears as the exact analogue of the Galilean law of inertia, if we give the latter the form of Theorem I of page 74. —

It is now assumed that *arbitrary continuously applied external forces* act at the points of our body; the body will again be taken first as freely moving. Its impulse will no longer remain constant, and the question will be to determine the manner in which it is changed.

We arrange this consideration as above. If the body consists of a single particle, then its impulse and the impulses of the infinitesimal impacts of the external forces compose successively at each moment according to the parallelogram rule (Newton's lex secunda). If the body consists of two particle at an invariable distance of separation, then the individual impulses of these two particles will be changed by the external forces as well as by the reaction forces acting between them, by which we replace the rigid binding. If we consider, however, the total impulse of the system formed by the two particles, then the reaction forces cancel. The impact-screw in question therefore consists of a constant part that corresponds to the original impulse of our two

*) Théorie nouvelle ..., Kap. II, §5.

**) On a direct method of estimating velocities with respect to axes moveable in space. Cambridge Phil. Transact. Vol. X, 1856.

particles, and a variable part, which is caused only by the additional impacts corresponding to the external forces in the considered time period. We can first compose these latter impacts at each moment into an infinitesimal pushing-impact dS and a turning-impact dD with respect to a reference point O, and then combine dS and dD with the corresponding components S and D of the current impulse-screw according to the parallelogram construction. This procedure justifies the following theorem, which we immediately generalize to a rigid system of arbitrarily many particles and, further, to a rigid body of continuous mass distribution (not to mention generalized systems with only internal forces):

T h e o r e m I I: *The impulse screw of a freely moving rigid body on which arbitrary external forces act is changed during the motion so that it is composed at each moment according to the rules of statics with the infinitesimal impact screw caused by the external forces.*

This theorem corresponds to the cessation of validity of the simple center of gravity and area theorems for motion influenced by external forces. For one of the simple area or center of gravity theorems to remain valid, the system of external forces must fulfill a special condition; it must, as one says according to the precedent of Lie, admit of a certain infinitesimal transformation, namely an infinitesimal rotation or translation. In this case, a turning or pushing component of the force-screw corresponding to the external forces would vanish; at the same time, our geometric construction would immediately imply that a component of the impulse-screw would remain constant during the motion.

One notices the characteristic inappropriateness and asymmetric manner of notation that is used in analytic mechanics with respect to the center of mass and area theorems. One speaks of the existence of an area theorem only when the turning-moment of the external forces about an axis is equal to zero; that is, only when the turning part of the impulse has an invariable component in time. In contrast, one speaks of the center of gravity theorem also when external forces are applied, and therefore if the pushing part of the impulse changes arbitrarily during the course of the motion. This discrepancy arises essentially because the concept of the impulse, which is bound organically to the area and center of mass theorems, is not considered in the usual presentations. The natural use of language would obviously be to conceive the words area theorem just as generally as the words center of gravity theorem, and therefore to understand the area theorem to mean that the turning-impact of the impulse adds geometrically with the successive turning-impacts of the exterior forces. In the case of constant turning- and pushing-impacts, one would then speak, as above, of the "simple" area and center of gravity theorems.

We make once more the passage to a body with a fixed support point, for which the constituent part S of the impulse will be canceled by the reaction force at O. Our deliberation above then leads to

Theorem IIa: *The impulse vector of a top on which arbitrary continuous external forces act is altered at each moment so that its change in direction and magnitude is equal to the infinitesimal turning-impact caused by the external forces.*

This theorem*) coincides in form precisely with the second Newtonian axiom, in so far as we state the latter as in Theorem II of page 74.

§6. The theorem of the *vis viva*

The preceding impulse theorems, together with our earlier relation between the impulse vector and the rotation vector, determine the motion of the top just as completely as the Newtonian axioms, to which they correspond exactly in form and content, govern the mechanics of a single mass particle. In fact, the successive changes of the impulse in space are established by our previous considerations. From these changes in impulse, however, follow the position of the rotation vector in the body, and therefore also the instantaneous motion, by virtue of the results of §3.

It will, therefore, no longer be necessary to return to the synthesis of the body from its individual mass elements and to follow (by means of the Newtonian axioms) the motion of the latter from the standpoint of particle mechanics. If we later do this on occasion, as, for example, at the end of this section, it is only on secondary and didactic grounds, since particle mechanics is especially familiar through general use.

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*) In a monograph on the top: *A. de Saint-Germain, Résumé de la théorie du movement d'un solide autor d'un point fixe*, Paris 1887, this theorem is attributed in error to Résal. The first volume of the *Traité de Mécanique général* by Résal, in which the theorem in question appears on page 247, first appeared in 1873, while our theorem (compare, for example, the year of the Hayward treatise) is certainly much older.

The preceding impulse theorems must also include, in so far as the top is concerned, the theorem of the *vis viva*. In fact, this theorem, as we will immediately show, is only a corollary of our impulse theorems.

We first assume that no external forces act on our body, naturally disregarding the reaction force at the support point and such forces that are canceled by this reaction force.

The expression for twice the *vis viva*

$$Lp + Mq + Nr \tag{1}$$

signifies geometrically, as mentioned on page 96, the scalar product of the impulse and rotation vectors, and has, as such, a value that depends only on the magnitudes and the relative positions of the two vectors, and not on their positions in space.

We temporarily consider a uniform rotation of the body about the axis $p:q:r$, which we imagine fixed in the body and therefore also in space, while at the same time the vector (L, M, N) will be imagined as fixed in space. The change in the above scalar product due to this motion, that is, the quantity

$$p\,dL + q\,dM + r\,dN,$$

is equal to zero, since the magnitudes and the relative positions of our two vectors are not changed.[63]

The actually occurring force-free motion can, however, in so far as the behavior of the impulse vector and the motion of the body are concerned, be identified at each moment, in the first approximation, with a motion of the assumed nature. Thus it is also valid for the actual motion that

$$p\,dL + q\,dM + r\,dN = 0. \tag{2}$$

We next remark that for the actual motion, according to equation (4′) of §3,

$$p = \frac{\partial T}{\partial L}, \quad q = \frac{\partial T}{\partial M}, \quad r = \frac{\partial T}{\partial N},$$

where T is understood as the expression for the *vis viva* written in terms of the impulse coordinates. Thus the left-hand side of equation (2) is transformed into the perfect differential of the *vis viva*. We therefore obtain the equation $dT = 0$, or integrated $T = h$, which yields the theorem:

For force-free motion of the top, the vis viva *of the body is not changed.*

The *conservation of the* vis viva thus follows immediately, as we see, from the *conservation of the impulse.*

Now let, on the other hand, arbitrary external forces act on our top. We compose these forces, with respect to the support point O, into a pushing-force and a turning-force. The former we can neglect, and we denote the components of the latter, in the coordinate system to which L, M, N are also referred, by $\mathsf{\Lambda}$, M, N. In the time element dt, the impulse vector in space now undergoes the displacement $\mathsf{\Lambda}\,dt$, $\mathsf{M}\,dt$, $\mathsf{N}\,dt$ (Theorem IIa of the preceding §). The impulse vector, therefore, does not retain its magnitude and position in space. We must, at each moment, make in reverse the displacement of the endpoint of the impulse vector caused by the external forces in order to obtain a point fixed in space. The displacement of this point relative to the body is, resolved into components,

$$dL - \mathsf{\Lambda}\,dt, \quad dM - \mathsf{M}\,dt, \quad dN - \mathsf{N}\,dt.$$

Its binding segment with O yields a vector that has the same magnitude and relative position with respect to the rotation vector at the end of the time interval dt as the impulse vector had at the beginning of the time interval.

The same thus holds for this vector as for the impulse vector itself in force-free motion. Equation (2) is, therefore, now to be replaced by the equation

$$(3) \qquad p\,dL + q\,dM + r\,dN = (\mathsf{\Lambda}p + \mathsf{M}q + \mathsf{N}r)\,dt.$$

The left side of this equation is again the perfect differential of the *vis viva* (dT); the right side represents (according to equation ($2'$) of page 91) the work performed during the time interval dt by the external forces. We thus have the theorem:

For motion of the top influenced by external forces, the vis viva *is altered at each moment so that its change is equal to the infinitesimal work performed by the external forces* ($dT = dA$).

It can occur, in particular, that the finite work which the external forces perform on our body, when we bring the body from a fixed initial position into any new position, depends only on this end position, and not on the intermediate positions passed through in the motion. As is well known, one calls the negative value of this work the *potential energy* U, and correspondingly designates T as the kinetic energy and $T + U$ as the total energy of the body. Then dA is the perfect differential of the function $-U$. As a result, the preceding theorem takes the simpler form $dT = -dU$, or $T + U = h$, and can be stated in the following way:

If the external forces that influence the motion have a "potential," the total energy of the body does not change during the motion.

This *theorem of the change of the kinetic energy or the conservation of the total energy* is therefore also, as we see, a simple consequence of our *theorem of the change of the impulse.*

We prove in a corresponding manner the theorem of the *vis viva* for the motion of a free rigid body.

Here we must replace the expression (1) by the expression

$$Xx' + Yy' + Zz' + Lp + Mq + Nr, \tag{4}$$

in which x', y', z', p, q, r denote the coordinates of the instantaneous motion-screw and X, Y, Z, L, M, N denote the coordinates of the impulse-screw. This expression, as emphasized on page 103, has a geometric meaning that is independent of the position of the two screws in space, and depends only on their pitches and their relative positions.

Force-free motion is treated first. We carry out the infinitesimal screw $(x', y', z', p, q, r)\,dt$ and consider the relative motion of the impulse screw with respect to the body. We imagine the motion-screw fixed in the body and therefore also in space; the impulse-screw, which according to the previous section is fixed in space, will thus perform a screw motion about the motion-screw, so that its relative position with respect to the motion-screw and its pitch are not changed. If dX, dY, dZ, dL, dM, dN denote the relative coordinate changes of the impulse, then there holds for the motion considered here, and also for the actual motion,

$$x'\,dX + y'\,dY + z'\,dZ + p\,dL + q\,dM + r\,dN = 0. \tag{5}$$

But the left side is, in consequence of equation $(16')$ of page 103, the perfect differential dT of the *vis viva*; we thus have $dT = 0$, or $T = h$.

The vis viva *again remains unchanged for force-free motion of a rigid body.*

For the case in which arbitrary external forces influence the motion of the rigid body, the consideration is generalized immediately in the following evident manner.

We first compose the exterior forces, with respect to the reference point, into a pushing-force $(\mathsf{\Xi}, \mathsf{H}, \mathsf{Z})$ and a turning-force $(\mathsf{\Lambda}, \mathsf{M}, \mathsf{N})$. The changes of the impulse coordinates relative to space during the time element dt are equal, according to Theorem II of the previous section, to $\Xi\,dt$, $\mathsf{H}\,dt$, $\mathsf{Z}\,dt$, $\Lambda\,dt$, $\mathsf{M}\,dt$, $\mathsf{N}\,dt$. The changes dX, dY, dZ, dL, dM, dN of the impulse coordinates relative to the body thus come only in part from the infinitesimal screw $(x', y', z', p, q, r)\,dt$; another part will be caused by the external forces. We must make the changes corresponding to the latter in reverse to obtain a screw that lies relative to the motion-screw at the end of the time interval dt just as the impulse-screw lay at the beginning of the infinitesimal motion. In other words, we must replace the quantities dX, …, dN in equation (5) by

$$dX - \Xi\,dt,\ dY - \mathsf{H}\,dt,\ dZ - \mathsf{Z}\,dt,\ dL - \Lambda\,dt,\ dM - \mathsf{M}\,dt,\ dN - \mathsf{N}\,dt.$$

Thus there follows

$$\begin{aligned} &x'\,dX + y'\,dY + z'\,dZ + p\,dL + q\,dM + r\,dN \\ &= (\Xi\,x' + \mathsf{H}\,y' + \mathsf{Z}\,z' + \Lambda\,p + \mathsf{M}\,q + \mathsf{N}\,r)\,dt. \end{aligned} \tag{6}$$

The left side, however, is the perfect differential of the *vis viva*, and the right side denotes, according to equation (2) of page 90, the work performed by the external forces. We thus have $dT = dA$:

The change of the vis viva *is equal at each moment to the work performed by the external forces.*

It is perhaps useful to carry out the proof of this theorem yet again according to the method of the previous section, in which we imagine our body resolved into its individual mass elements. It is enough, for this purpose, to consider a system of two rigidly bound mass elements.

We remark in advance that the change of the *vis viva* of a single mass particle is equal, on the basis of the formula

$$dT = x'd[X] + y'd[Y] + z'd[Z],$$

to the scalar product of the velocity vector (x', y', z') with the change of the impulse vector $([X], [Y], [Z])$.

At each of our two rigidly bound particles 1 and 2, we imagine the vectors of the individual impulses 1 and 2, which coincide in direction with the velocity vectors 1 and 2; we imagine as well the reaction forces 1 and 2, which replace the rigid binding of the points and act along the binding line of the points. External forces may not be present.

An evident consequence of the rigid binding is that the projection of the velocity vector 1 on the binding line is equal to that of the velocity vector 2. In place of this, we can say, on the basis of the Newtonian lex tertia:

The sum of the scalar products of the velocity vectors with the corresponding reaction forces is equal to zero.

In fact, the equality of the considered projections is transformed into the negative equality of the considered scalar products because of the opposing sense of the two reaction forces. Now the reaction forces, however, are proportional in magnitude and direction to the changes of the individual impulses (Newton's lex secunda). *Therefore the sum of the scalar products of the changes of the individual impulses with the individual velocity vectors will also equal zero.*

According to the prefatory remark, however, the two terms of this sum equal the respective changes of the *vis viva* of our two mass particles. The sum itself is therefore equal to the change of the *vis viva* of the system.

Thus the vis viva *of our system remains constant, just as for a single mass particle that moves according to the Galilean law of inertia.*

The generalization of our deliberation to the case in which external forces act, or to that of arbitrarily many particles bound to form a rigid system, as well as the specialization to the case of the top, is so simple that we can pass over it.

We remark, further, that the analytic proof which is usually given for the theorem of the *vis viva* runs precisely parallel to the preceding geometric proof. Indeed, the last consideration, in which we returned to the individual mass elements of the rigid body, corresponds in analytic mechanics to taking as a basis the differential equations in the form of the so-called *Lagrange equations of the first kind*, while the earlier consideration of the total system corresponds to the standpoint of the so-called *generalized Lagrange equations (the equations of the second kind).* We will enter more deeply into both systems of equations in the following chapter (cf. §3).

§7. Geometric treatment of force-free motion of the top

We will now use the preceding general theorems to provide a clear geometric image of the motion of the top in the simplest conceivable case. We assume that no external forces act on the top. To specifically eliminate the effect of gravity, we imagine the top supported at its center of gravity.

The geometric theory of this motion, which was first given by P o i n s o t, can now be written down immediately.

We consider, in the first place, that the impulse vector of the top remains constant in space for force-free motion. We imagine, once and for all, that this vector is erected vertically upward from O. The magnitude and direction of this vector are naturally given by the initial impact through which our body has been set in motion.

In the second place, we consider that the *vis viva* of the body also remains constant for force-free motion. We give this fact a twofold geometric expression.

On the one hand, the *vis viva* denotes one-half the product of the magnitude of the impulse vector with the projection of the rotation vector onto the impulse vector. From the constancy of the impulse and the constancy of the *vis viva* taken together, it follows that the projection of the rotation vector onto the impulse vector has an invariable length. The magnitude of this projection depends again on the nature of the original impulse. *We therefore have a plane e, fixed in space and perpendicular to the impulse axis, that yields a locus for the endpoint of the rotation vector with respect to its position in space.*

A further geometric meaning of the theorem of the *vis viva* results from the expression

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2).$$

The endpoint (p, q, r) of the rotation vector thus lies on an ellipsoid rigidly bound to the top, which is similar and similarly situated with the ellipsoid of inertia. The constancy of the *vis viva* now means that this ellipsoid continuously retains it dimensions during the motion. *We therefore have, in addition, an ellipsoid E, fixed in the body, that yields a locus for the endpoint of the rotation vector with respect to its position in the body.*

We consider, finally, the relation between the impulse vector and the rotation vector. This relation can be expressed by means of the Poinsot construction of page 101, so that the tangent plane of the ellipsoid E at the endpoint of the rotation vector stands perpendicular to the axis of the impulse. This tangent plane is thus, first of all, permanently parallel to our plane e; since the plane e, moreover, constantly passes through the endpoint of the rotation vector, the tangent plane coincides directly with the plane e. In other words:

Our ellipsoid E is constantly tangent to our plane e during the motion.

Now the polhode curve runs somehow in the ellipsoid E, and the herpolhode curve runs somehow in the plane e. Since the two curves roll on one another in the motion, our ellipsoid E also rolls without sliding on our plane e during the motion. *Thus we can emulate the completely kinetically determined motion in a purely kinematic way if we let an ellipsoid, described with O as its midpoint and fixed in the body, roll without sliding on a plane fixed in space.*[64]

We owe this beautiful and transparent image for the motion of the force-free top, as is well known, to the investigations of Poinsot. The motion in question will therefore be designated concisely as *Poinsot motion*. If, following the precedent of Poinsot, we thus visualize the motion by the rolling of an ellipsoid on a plane, we place ourselves, curiously, in a certain opposition to the general Poinsot theory of rotation. According to this general theory, we should, in the first place, make clear to ourselves the forms of the rolling cones and seek from these to acquire a representation of the motion. But the forms of these cones are rather complicated even in the preceding simple example, as we will show below for the herpolhode cone, and the differing construction above is much more transparent. In more difficult cases (with the addition of gravity), we can expect even less to manage with the discussion of the rolling cones alone.

To treat only of the successive positions of the body in space, our picture of the motion above is fully sufficient. If we wish, however, to give expression also to the velocity of the actual motion in our kinematic picture, then we must add a further specification regarding the manner of the rolling:

The velocity of rolling should be measured so that the rotation of the ellipsoid, which naturally takes place about the radius erected from O to the momentary tangent point of E and e, is equal to this radius.

The way in which this condition is also to be realized purely kinematically was first shown by S y l v e s t e r*). We cannot enter into this here.[65]

We now complete our representation of the Poinsot motion by studying, in detail, the behavior of the different geometric elements of the motion.

*) Cf. Sylvester: On the motion of a rigid body etc., London R. S. Phil. Transactions 1866.

The course of the motion with respect to the body is treated first. In this respect we wish to know, above all, *the curve that the endpoint of the impulse vector describes relative to the body.* We will occasionally call this curve, for want of a better expression, the "impulse curve."

In a single time element of the relative motion under discussion, the endpoint of the impulse vector always moves in a circular arc around the instantaneous rotation axis. Its relative displacement with respect to the body is therefore perpendicular to the instantaneous rotation axis. The direction of this displacement with respect to the XYZ frame is determined by the ratios of the coordinate changes $dL : dM : dN$, and the direction of the rotation vector is determined through the ratios $p:q:r$. Thus the equation

$$p\,dL + q\,dM + r\,dN = 0$$

is satisfied.

On the basis of the general relation between the impulse vector and the rotation vector, we can also write our equation as

$$\frac{L\,dL}{A} + \frac{M\,dM}{B} + \frac{N\,dN}{C} = 0.$$

Through integration there follows

$$\frac{1}{2}\left(\frac{L^2}{A} + \frac{M^2}{B} + \frac{N^2}{C}\right) = h. \tag{1}$$

Equation (1) is naturally equivalent to the equation of the *vis viva*, since our deliberation given in the previous section for the proof of the theorem of the *vis viva* is only repeated in a somewhat more special setting.

It is clear, further, that the endpoint of the impulse vector, because of its constant length G, must always lie on a sphere of radius G. We thus have

$$L^2 + M^2 + N^2 = G^2. \tag{2}$$

The desired curve is determined by equations (1) and (2). We can say:

The path that the endpoint of the impulse vector describes with respect to the body is a spherical curve; it is given by the intersection of the ellipsoid (1) *with the sphere* (2).

The form of the curve is sketched in Figures 18, 19, and 20 (see the following section) for a few characteristic cases.

At the same time, the curve that the endpoint of the rotation vector describes with respect to the body (that is, the polhode curve) is also determined. We obtain this curve from the just derived curve of the impulse through a simple deformation with respect to the principal axes of the body. It also lies simultaneously on two surfaces of the second degree, the first of which is already known to us as our ellipsoid E. Namely, it follows from (1) and (2), with consideration of the relation between the impulse vector and the rotation vector, that

$$\frac{1}{2}(Ap^2 + Bq^2 + Cr^2) = h \tag{3}$$

and

$$A^2p^2 + B^2q^2 + C^2r^2 = G^2. \tag{4}$$

The polhode curve is therefore the intersection of the two concentric ellipsoids (3) *and* (4).

In a corresponding manner, we treat of the curves that the endpoints of the impulse vector and the rotation vector describe relative to fixed space. The former curve naturally reduces to a point; we therefore consider the latter; that is, our *herpolhode curve.* We first know from the theorem of the *vis viva* that this curve lies in the fixed plane e.

The herpolhode curve is therefore a plane curve. The distance of its plane from O (that is, the projection of the rotation vector on the vertical impulse axis), which we denote in agreement with the previous by ϱ, results from the theorem of the *vis viva* as

$$\varrho = \frac{2h}{G}. \tag{5}$$

The more precise form of the herpolhode curve may not, as it was for the polhode curve, be defined by an elementary geometric construction, since it is, in general, of a transcendental nature. But it is certainly possible to discover it from our kinematic image of the motion.

We describe spheres about O at the greatest and least distances of the polhode curve from O; these spheres determine two circles in our plane e. The common midpoint of these circles is the point of intersection of the impulse axis with e. Between these two circles the herpolhode curve must evidently run to and fro in regular windings, in which it is alternatively tangent to these circles or, in special cases, touches one of these circles with a cusp. The herpolhode curve consists of an infinite series of congruent arcs that are twisted with respect to one another by the same amount. Each individual arc corresponds to a single unwinding of the polhode curve. In general, the curve does not close, but rather encircles the midpoint of the figure, the intersection point of the impulse axis with its plane, infinitely often. From this it already follows that the equation of the curve will be a transcendental equation. The corresponding result holds, naturally, for the herpolhode cone, which this curve projects from O. The form of the herpolhode is represented in the adjacent figure*) for the special case

$$A = \frac{1}{36}, \quad B = \frac{1}{25}, \quad C = \frac{1}{16},$$

$$h = 50, \quad G^2 = 5.$$

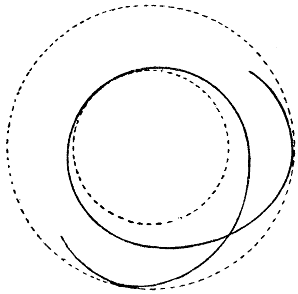

Fig. 16.

We will now see how these matters are modified in the case of the *symmetric top.* Here the ellipsoid of inertia, as well as all the previously used ellipsoids (1), (3), and (4), has rotational symmetry about the figure axis. But if we bring together for intersection an ellipsoid of revolution and a sphere (as for the construction of the impulse curve), or two ellipsoids of revolution with coinciding figure axes (as for the construction of the polhode), then the intersection curve is always a pair of diametrically situated circles in parallel planes.

Thus the polhode curve, as well as the curve that the impulse describes with respect to the body, goes over into a circle.

If we let, further, an ellipsoid of revolution with fixed midpoint O roll on a plane, then the locus of the contact point in this plane is obviously a circle. This shows, for example, that the collected points of the polhode curve have a constant distance (Ω) from O; the collected points of the herpolhode curve must also lie at the distance Ω from O. The latter curve is therefore the intersection of a plane with a sphere of radius Ω.

Thus the herpolhode curve for a symmetric top also goes over into a circle.

The character of the resulting motion may now be given in a word:

The most general motion of the force-free symmetric top is regular precession.

*) The figure is taken from the dissertation of Mr. W. H e s s: "Das Rollen einer Fläche zweiten Grades auf einer invariabeln Ebene, München 1880." Mr. H e s s shows that the herpolhode curve, because of the inequalities that hold between the principal moments of inertia A, B, C, can have no (real) inflection points; the figure originally given by P o i n s o t, Liouville's Journal sér. I, t. 16 was erroneous in this respect.[66]

In fact, we could have characterized regular precession in §6 of the preceding chapter by the condition that the curves and cones of the polhode and the herpolhode were circles and circular cones, respectively. The axis of precession is the impulse axis. We will reserve for later the more precise classification of this precessional motion in the sense of the distinctions of page 53.

In the special case of the symmetric top, the following deliberation, which is perhaps even simpler and more direct, also leads to the goal. We again take the impulse axis vertically upward and draw the figure axis, in its initial position, inclined by an arbitrary angle ϑ with respect to the impulse axis. The position of the instantaneous axis of rotation follows from the positions of the impulse axis and the figure axis. The rotation axis always lies, according to the construction given on page 106, in the same plane as the impulse and figure axes, and divides the angle ϑ, as we can say concisely, in a fixed ratio depending only on the mass distribution of the top (that is, on the values of A and C), and, in particular, is interior or exterior according to whether the top is prolate or oblate. For greater clarity we lay a unit sphere about O and mark its points of intersection with the impulse axis, the rotation axis, and the figure axis, which we denote, respectively, by J, R, and F. The point J, according to our fundamental principle, is a fixed point, the "north pole" of the sphere. The points R and F, in contrast, move; *we claim that they each describe a parallel circle about the north pole.*

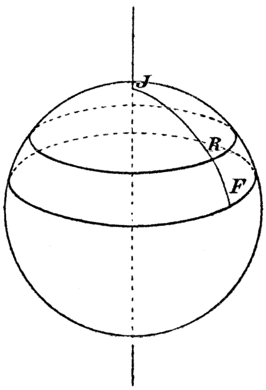

Fig. 17.

In fact (cf. Fig. 17), the instantaneous motion of the top consists of a rotation about OR. Since F and R lie on the same meridian of the sphere, the point F wanders at the first moment in the direction of the parallel circle through F, so that the angle ϑ will be initially unchanged. As a result, another line of the body that is contained in the meridian plane JOF must now take over the role of the turning axis. If we denote its intersection point with the unit sphere by R, then R lies on the meridian JF. Since this point now divides the arc JF in a fixed ratio and the arc JF must retain its initial length, the arc JR

must also have its original magnitude. Thus R likewise wanders, at the first moment, on the parallel circle through R, and indeed, as we said, so far that J, R, and F will be points on one and the same meridian. We are thus led back precisely to the initial conditions of the motion. In consequence, our deliberation also holds at each following time.

For force-free motion of the symmetric top, the figure axis and the rotation axis each describe a circular cone about the impulse axis.

From our earlier construction of the rotation vector, it follows, further, that the length of the rotation vector is likewise constant for constant length of the impulse vector and constant inclination angle ϑ. Now, however, the progressive velocity of the figure axis on its circular cone is proportional to the magnitude of the rotation vector. *The figure axis therefore traverses its circular cone with constant velocity.* Further, the rotational velocity of the top about the figure axis equals the projection of the rotation vector onto the figure axis. *The top therefore rotates relative to the figure axis with constant angular velocity.*

By these remarks, however, the motion is again characterized as *regular precession.* —

In conclusion, we remark that our treatment of the force-free top also finds application to the motion of a free rigid body in space that is subjected to no external forces or only such external forces that, by an appropriate choice of the reference point, may be composed into a pure pushing-force, as, for example, the force of gravity in the case where we choose the center of gravity as the reference point. Namely, we can then treat, according to the general impulse theorems of the fifth section, of the translation of the reference point on the one hand, and the rotation of the body about the reference point on the other hand, the latter according to the preceding theory of the force-free generalized top, and the former according to the laws of the mechanics of a single particle. Since for the case of the action of gravity, in particular, the trajectory of a single mass particle (the parabola) is sufficiently known, we now already command, with recourse to the preceding results, *the motion of a heavy rigid body moving freely in space.*

§8. Rotation of the top about a permanent turning axis and the so-called stability of the rotation axis of a rapidly rotating top

We have already emphasized many times the analogy between the motion of a single mass particle and the rotation of the top. Kinematically, both problems have three degrees of freedom; statically, the impulse in both cases can be conceived as a vector; a deep analogy also exists kinetically, in so far as we direct our attention to the behavior of the impulse vector (cf. §5). The situation is different if we compare the behavior of the velocity vector in the two cases. While the velocity vector (just as the impulse vector) retains its direction and magnitude in space for the single force-free mass particle, the vector of the turning velocity for force-free motion of the top continuously changes in magnitude and direction, both in space and in the body. We will pose the problem of determining the circumstances under which the velocity vector of the top also remains constant in magnitude and direction in space; or, in other words, *under what circumstances a uniform rotation of the top about a fixed spatial axis occurs.*

We know that for the *Poinsot motion* the rotation vector describes a conical shell that has the impulse vector in its interior. If the rotation axis is now stationary in space, the herpolhode cone reduces to a single line, so that this line must coincide with the direction of the impulse. According to page 101, however, the rotation axis and the impulse axis coincide only when their common direction is a principal axis of the body. Conversely, it follows from the construction of page 102 that the rotation axis then retains a fixed position in space and in the body, and that the rotational velocity is uniform. If we denote, as usual, an axis about which a continuous uniform rotation is possible as a "permanent axis," then we can say:

For the generalized top there are only three permanent axes, the principal axes of the body.

If the top rotates about one of these three principal axes, then the polhode, herpolhode, and the curve that the impulse describes in the body each reduce, obviously, to a single point.

The three principal axes present an interesting difference with respect to the *stability* of the considered uniform rotation, as was already noted by Poinsot.

The concept of the stability of a motion, which we encounter here for the first time, plays an important role in modern mechanics, and will be treated with a certain care. To determine whether we call a form of motion of our top stable or unstable, we will proceed in the following way (where the precise sense of the words chosen by us will become fully clear, perhaps, only in the course of the further development): we impart to the top, while it executes the motion in question, a small impact of an arbitrary nature. If the collected elements of the motion (for example, the successive positions of the top in space and the positions of the rotation vector and the impulse vector with respect to the top and in space) are always changed to a lesser extent as the applied impact is made smaller, then we will call the motion stable; in every other case it will be called unstable.

In this spirit, we first investigate the *rotation of the top about the axis of the greatest or least principal moment of inertia*, and consider, above all, the *curve that the endpoint of the impulse describes in the body.* The ellipsoid 1) and the sphere 2) of page 123, whose common points yield our impulse curve, must, obviously, now be tangent at two points that lie on the longest or shortest principal axis of inertia, respectively. In the former case, the ellipsoid will be completely enclosed by the sphere, and in the latter the sphere will be completely enclosed by the ellipsoid. If we now impart a small impact to the top, then we slightly change the constants h and G; that is, the sizes of our ellipsoid and our sphere. The tangent point dissolves, in both cases, into a small encircling curve, which lies, in its entire extent, always closer to the previous tangent point as the changes in h and G are made smaller. (For arbitrary changes in h and G, the tangent point can dissolve, to be sure, into an imaginary curve; the necessary values of our constants, however, are incompatible with the mechanical meaning of these quantities, so that we can therefore disregard them.)

The *polhode* behaves in a manner entirely similar to the impulse curve; indeed, the polhode curve can be derived from the impulse curve by a simple deformation of the principal axes of the body. The polhode curve, which for uniform rotation consists of a single point, is also transformed by the addition of a small external impact into a small oval that continually lies very near the previous single point. We thus conclude that in the rolling of the polhode on our fixed plane of page 121, there results a *herpolhode curve* whose dimensions are likewise always smaller as the disturbing impact is chosen smaller. In summary, it follows that:

Uniform rotation of the top about the greatest or least principal axis of the ellipsoid of inertia is a stable form of motion.

We next assume that the rotation occurs about the *intermediate principal axis of the ellipsoid of inertia.* Again, we first consider the impulse curve. This curve reduces, in our case, to one of the intersection points of the intermediate principal axis with the sphere of radius G. Our sphere 2) is tangent to the ellipsoid 1) at this point as well as at the diametrically opposed point. One sees immediately from the adjacent figure, however, that there are still infinitely many other points which are common to the two surfaces. Namely, the two surfaces must necessarily intersect; the ends of the greatest principal axes of the ellipsoid, for example, extend out of the sphere; the smallest principal axes lie entirely in the interior of the sphere. The complete intersection curve consists of two circles; namely, the intersection circles of the ellipsoid, which cross at the endpoints of the intermediate principal axis. It is easy to give the analytic condition that is required for the occurrence of the present case. If B is the moment of inertia corresponding to the intermediate principal axis, then we must choose the constants h and G so that equations (1) and (2) on page 123 give the same value of M^2 for $L = N = 0$; our condition is thus

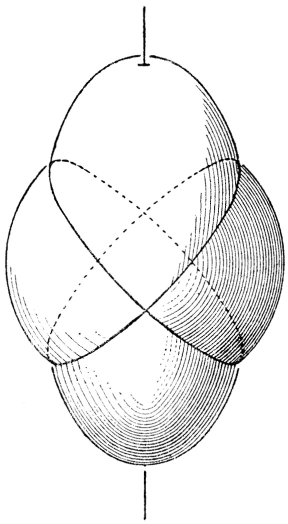

Fig. 18.

$$G^2 = 2hB.$$

If we now alter the dimensions of the sphere and ellipsoid slightly by the addition of an external impact, then we always obtain a curve that departs from the original tangent point by a finite distance. The two tangent points (together with the circles passing through them) dissolve into two ovals that surround the endpoints of the largest or smallest principal axis of the altered ellipsoid. This is seen in Figures 19 and 20. The external disturbance for these figures is specifically chosen so that the ellipsoid retains its size and only the sphere is changed; in particular, the sphere is made larger with respect to the original in Figure 19 ($G^2 > 2hB$) and smaller in Figure 20 ($G^2 < 2hB$).

On the basis of the depicted behavior of the impulse curve, we can already state the interesting theorem:

Uniform rotation of the generalized top about the axis of intermediate principal moment of inertia is an unstable form of motion.

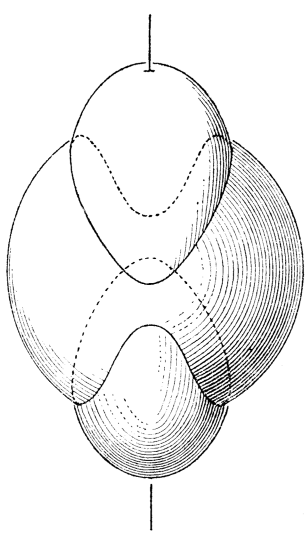

Fig. 19.

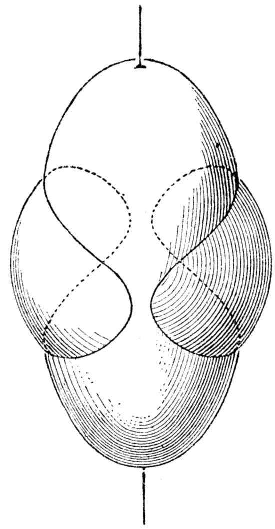

Fig. 20.

We also wish to consider briefly the forms of the polhode and herpolhode curves. The form of the polhode curve is, naturally, completely analogous to that of our impulse curve. In the case $G^2 = 2hB$, the polhode curve consists of a point, while the intersection of the ellipsoids 3) and 4) of page 124, on which the polhode runs, gives two congruent ellipses. (They result from the above intersection circles of the ellipsoid 1) by the deformation $p = L/A$, $q = M/B$, $r = N/C$.) If we now make $G^2 \gtrless 2hB$ by an external impact, the two ellipses dissolve into two ovals, which are removed from the point-shaped polhode of the case $G^2 = 2hB$ by a finite amount, however small the disturbing impact is.

From this behavior of the polhode curve, we immediately conclude that the herpolhode curve, which we have indeed obtained from the unrolling of the polhode curve, will also change its form discontinuously. While the herpolhode curve consists of a single point in the case $G^2 = 2hB$, it likewise attains, in the case $G^2 \gtrless 2hB$, a finite dimension that cannot be made arbitrarily small by the diminishment of the difference $G^2 - 2hB$. We cannot, in this place, go more deeply into the interesting details*) that appear here.

We now go over to the *symmetric top. For the symmetric top, there are obviously infinitely many permanent rotation axes; these are, in addition to the figure axis, the collected axes of the equatorial plane.* We also pose here the question of the *stability* of the considered form of motion.

For *rotation about the figure axis*, the question is settled if we conceive the symmetric top as the limiting case of the generalized top. For the generalized top, the figure axis corresponds in every case to the axis of the largest or smallest principal moment of inertia (according to whether the symmetric top is prolate or oblate). Thus we can say:

Rotation about the figure axis is a stable form of motion of the symmetric top.

For *rotation about an axis of the equatorial plane*, in contrast, the comparison with the generalized top forsakes us. Such an axis, namely, can be regarded equally well as either the limiting case of the intermediate principal axis or the limiting case of one of the two extreme principal axes. Correspondingly, the following specific investigation shows that the stability properties of the symmetric top for this rotation hold, in a certain sense, the middle ground between the complete stability of a rotation about an axis of an extremal principal moment of inertia and the complete instability of a rotation about the axis of the intermediate principal moment of inertia.

Here, we consider first the *herpolhode curve.* We rely on our previous theorem that the most general motion of the symmetric top is a regular precession about the axis of the impulse. To begin with, the impulse lies exactly on an equatorial axis. The herpolhode curve thus consists of a point that lies on this same axis. We then add a small additional impulse, by which the position of the impulse in space and in the body will be slightly altered instantaneously. The position of the rotation vector follows from our familiar construction. In every case, the altered vector is very slightly displaced in magnitude and direction from the

*) Under some circumstances, the herpolhode curve assumes a spiral form. This occurs if the impact is specifically chosen so that the previously mentioned intersection curve of the ellipsoids 3) and 4) (consisting of our two congruent ellipses) is not changed as such, and the rotation pole in the body is only displaced slightly from the double point of the intersection curve to one or the other of the ellipses.[67]

original. We now obtain the form of the herpolhode curve if we lead the endpoint of the thus constructed rotation vector in a small circle about the changed axis of the impulse. This circle always becomes smaller for a smaller disturbing impact. The herpolhode curve therefore changes its form arbitrarily little – in contrast to the herpolhode curve for the corresponding motion of the generalized top.

Nevertheless, however, we must designate the motion in question for the symmetric top as unstable according to our above definition of stability. Namely, we consider the *polhode curve.* For the original position of the impulse, the polhode curve consists of a single point. This statement appears to be in contradiction to the earlier result that the polhode curve is always a circle about the figure axis. The contradiction is resolved if we consider that in uniform rotation about an equatorial axis the endpoint of the rotation vector does have the tendency to progress in a circle about the figure axis, but does not leave its place here because the herpolhode curve consists of a single point. We now obtain the altered form of the polhode curve due to our impact if we let the altered endpoint of the rotation vector move in a circle about the figure axis. With the addition of our small impact, the polhode curve thus changes its form in a discontinuous way — in agreement with the polhode curve for the corresponding motion of the generalized top. The same holds for the curve that the impulse describes with respect to the body.

According to the preceding, we must say, in any case:

Uniform rotation of the top about an axis in its equatorial plane is an unstable form of motion.

Finally, we consider the *spherical top.* Here each axis is a principal axis of the ellipsoid of inertia, from which it follows:

For the spherical top, each axis through O is a permanent rotation axis.

The most general motion of the spherical top consists of a uniform rotation about an axis fixed in space.

One is easily convinced, moreover, that each such rotational motion has a *stable* character.

If we take up once again the comparison between the kinetics of the top and that of a single mass particle posed at the beginning of this section, then we can say:

The spherical top forms, not only with respect to the behavior of the impulse vector but also with respect to the velocity vector, an exact analogue to the single mass particle; namely, the magnitude and direction of this vector also remain constant in space for force-free motion of the spherical top. —

The word "stability" is frequently used, according to the precedent of Foucault, in a sense different from that given above. The word is used, namely, to mean the apparent tendency of a uniformly rotating body to retain the spatial direction of its rotation axis in the presence of external disturbances. The experiments that concern this phenomenon are sufficiently well known. We imagine, for example, a top such as our demonstration model depicted in the Introduction, which we may balance by upper weights so that the center of gravity lies at the support point. If we set the top in motion by unwinding a string, we impart to it a rather strong rotation that has the figure axis as the rotation axis. Without great exertion, it is attained that the top makes 20 revolutions per second. If we now wish to change the inclination of the figure axis discernibly, we must apply a considerable force. Smaller disturbances, such as a shaking of the pedestal or a light blow to the upper surface of the top, hardly cause a noticeable change in the state of motion. We compare this to the fact that the nonrotating top obviously reacts with an evident motion to any arbitrary impact. We will then, in fact, be inclined to believe in a certain capacity for resistance that the top acquires by virtue of its rotation.

Analogous phenomena are observed very frequently for freely moving bodies*). If one would strike a target with a thrown body, then one always hurls the body with as strong a rotation as possible. Through this alone can one produce a regular and predictable trajectory of the body. In other cases, all sorts of accidental circumstances, such as the effect of air resistance, incidental currents in the air, etc., would disturb the path considerably. This applies, for example, to the discus throwing of ancient times or the hoop game common today. The effect finds great employment in the construction of modern artillery and

*) Numerous examples of this type are found in the popularly received work of Perry: Spinning tops, London 1890; we wish to earnestly recommend this interesting and, through its amusing presentation, distinguished little book.[68]

infantry weapons; namely, in the use of *rifled* barrels, to which we will later return.

The explanation of all these phenomena is, for us who command the concept of the impulse, exceedingly simple. We speak in this respect of our top, but we could just as well think of any one of the given examples. Through the original impetus, we have created an impulse vector that lies nearly in the direction of the figure axis and has a considerable length. This impulse is composed with the impulse of the external disturbance according to the parallelogram of forces. If the latter is considerably smaller, as in the previous examples, then the original impulse will be changed only slightly in magnitude and direction. Thus the changed state of motion also differs only slightly from the original; the rotation axis retains its position in space approximately, and the figure axis also remains in the proximity of its earlier position.

To give a numerical example, we may take the previously cited number of 20 rotations in one second as a basis. The angular velocity then amounts to $2\pi \,.\, 20$ (sec^{-1}). It corresponds, according to our convention, to a rotation vector of length $2\pi \,.\, 20$ cm. To calculate the corresponding impulse vector, we must make a judgment of the magnitude of the moment of inertia about the figure axis. We imagine, for this purpose, the bell-shaped principal piece of our model to be replaced, for example, by a circular disk of 1 cm thickness and 10 cm radius. The density of the material is approximately 8 (gr. cm^{-3}), (namely, 7,6 for iron, 8,5 for copper). For the moment of inertia, one thus easily finds the value $4\pi \,.\, 10^4$ (gr. cm^2); the impulse vector thus receives, according to our earlier agreement, the length of $(4\pi)^2 10^5$, approximately equal to $15 \,.\, 10^6$ cm. The impulse vector is thus approximately 150 kilometers or 20 German miles long! We next calculate the magnitude of the additional impulse in a concrete example. We wish, for example, to let a body with mass one gram fall onto the edge of the circular disk of the top from a height of one-half meter. The velocity that our body attains is equal, according to the falling law, to $\sqrt{2gh}$; that is, for $h = 50$ cm and $g =$ ca. 900 (cm sec^{-2}), equal to 300. The applied impact thus amounts to 300 (gr. cm sec^{-1}). The turning moment itself, since the radius of the circular disk was taken to be 10 cm, is equal to 3000. Our additional impulse thus has a length of 30 m and is, if the figure axis stands vertical, directed horizontally. It is clear that this additional impulse, compared with the imposing length of the original impulse, signifies only a relatively insignificant change. The corresponding change in the state of motion will thus be hardly observable. In summary, we can say:

The phenomena in question are, in themselves, no more remarkable than the thoroughly self-evident fact that a rapidly moving body will be deflected in direction by a lateral impact ever more slightly as the original translational impulse, or, equivalently, its translational velocity, becomes larger. —

The situation may appear somewhat differently if we have, instead of a single or short disturbance, a continuous influence. Obviously, a quite small but persisting change in impulse can produce a considerable effect even compared to a very large initial value, if only we extend the observation over a sufficiently large time interval. This is in fact the case, as is shown in the sequel for the example of the gravitational influence.

After we have completely understood the phenomena themselves, we will, finally, criticize the expression *stability of the rotation axis* by which one frequently describes these phenomena.

First, we wish to reserve for the word "stability" the special sense that was given at the beginning of this section, and that forms the opposite of the word "lability." We will thus prefer to say, instead of stability of the rotation axis, *"conservation of the rotation axis."* Next, we prefer, in general, not to speak here of the rotation axis, but rather of the impulse axis. *It is not the rotation vector, but rather the impulse vector that takes the first place dynamically.* If we have determined the position of the impulse vector, the position of the rotation vector follows immediately, for example, on the basis of the construction of page 101. It is entirely false that an external cause, a force, is necessary for a displacement of the rotation vector. *The force is used not for the displacement of the rotation vector, but rather for the displacement of the impulse vector.* We think, for example, of the general motion for the force-free top. Here the rotation axis progressively changes its direction, in that it wanders along the herpolhode cone without the action of an external force, while, on the other hand, the impulse vector remains fixed. The change of the rotation axis occurs just because, and in just such a manner that, the impulse axis can remain unchanged.

And now (as we know), the "law of the conservation of the impulse axis" against external disturbances is not exactly valid. However small the added impulse compared to the original may be, it always produces a finite change. This is enough to separate the perhaps initially coinciding axes of the rotation and the impulse. In consequence, the rotation axis is no longer stationary in space, but rather describes (for the symmetric top) a thin circular cone about the impulse axis. The same holds for the motion of the figure axis.

We will therefore be obliged to correct the alleged law of the stability of the rotation axis; we say:

The impulse will, under certain conditions frequently present in practice, be changed relatively little by external disturbances. As a consequence, if the motion is initially about the figure axis, the herpolhode cone and the cone described by the figure axis in the altered motion retain a very small opening after the disturbance.

The Theory of the Top. Volume I
Introduction to the Kinematics and Kinetics of the Top
Klein, F.; Sommerfeld, A.
2008, XX, 280 p. 43 illus., Hardcover
ISBN: 978-0-8176-4720-9
A product of Birkhäuser Basel